INTERACTION NOTES
NOTE 483

Maximum RF Pickups by Wires Inside a Slotted Cavity*

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ABSTRACT

This report considers how to bound the power picked up by a wire inside a cavity with a slot in its wall. The consideration is based on equivalent circuits and power conservation, and is general enough for applications to real-world systems. The calculations compare favorably with measurement data.

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1.0 INTRODUCTION

Recently some progress has been made on calculating the bounds of RF coupling to the interior of a system [1-4]. The bounds are on the coupling cross section and the integral of it over all wavelengths. It has been realized that some of these bounds, although rigorous, can be too loose. Attempts have been made to tighten these bounds and it was found that several coupling loss mechanisms would play certain important roles in the search for a better bound [4]. In this report a similar study is taken up but, instead, it will focus on a "simple" canonical problem, a problem that involves a cavity, an aperture in one surface of the cavity, and a pickup wire inside the cavity. In Section 2 the problem will be defined precisely and the goal of the study spelled out explicitly. In Section 3 a circuit approach to the problem is explored and many related questions will be raised and answered. An approach based on power conservation law is discussed in Section 4 and the resulting bound calculation is compared with available experimental data. Two appendices are given, one on certain subtle differences and similarities of the reciprocity and optical theorems, and the other on a general integral-equation approach to the Babinet principle.

Before embarking on the present study it is perhaps worthwhile to point out one direct consequence of the bounding approach. Making use of the bound on the integral of the coupling cross section over all wavelengths, one can write down the exact solution to a boundary-value problem, which is not only a rarity in itself but also not easily achievable by other means. The problem is one of calculating the total energy (W_t) transmitted by a step-function plane wave, $\mathbf{E}_{inc} = \mathbf{E}_o\, u(t - x/c)$, through an aperture in an infinite, perfectly conducting ground plane. The solution is

$$W_t = \mu_o\, \mathbf{H}_o \cdot \boldsymbol{\alpha}_m \cdot \mathbf{H}_o - \varepsilon_o\, \mathbf{E}_o \cdot \boldsymbol{\alpha}_e \cdot \mathbf{E}_o \qquad (1)$$

where $\mathbf{E}_o$ and $\mathbf{H}_o$ are the electric and magnetic field amplitudes of the incident step-function wave, $\boldsymbol{\alpha}_e$ and $\boldsymbol{\alpha}_m$ are the electric and magnetic polarizability tensors of the aperture. It is interesting to note that the maximum energy penetration occurs for broadside incidence, as is evident from (1). By the Babinet principle, the total energy scattered (W_{sc}) by the complementary disk is

$$W_{sc} = \frac{1}{2}\,\varepsilon_o\,\mathbf{E}_o \cdot \mathbf{P} \cdot \mathbf{E}_o + \frac{1}{2}\,\mu_o\,\mathbf{H}_o \cdot \mathbf{M} \cdot \mathbf{H}_o \tag{2}$$

where **P** and **M** are the electric and magnetic polarizability tensors of the disk. Eq. (2) actually holds for scatterers of any shapes and any constitutions provided that W_{sc} is interpreted to be the sum of scattered and absorbed energy. Note that for a disk $\mathbf{P} = 4\,\boldsymbol{\alpha}_m$, $-\mathbf{M} = 4\,\boldsymbol{\alpha}_e$. From (1) it follows immediately that, among the principal components of $\boldsymbol{\alpha}_e$ and $\boldsymbol{\alpha}_m$, one has the inequality,

$$\frac{2}{\alpha_{e,aa}} > \frac{1}{\alpha_{m,bb}} + \frac{1}{\alpha_{m,cc}} \tag{3}$$

which is implied by the conjecture reported in [1], namely,

$$\frac{1}{\alpha_{e,aa}} \geq \frac{1}{\alpha_{m,bb}} + \frac{1}{\alpha_{m,cc}} \tag{4}$$

2.0 STATEMENT OF THE PROBLEM

Consider the problem of an electromagnetic wave incident on a cavity with a slot in its surface. Inside the cavity is a wire terminated at or leading to a resistive load. Such a situation is depicted in Fig. 1a. One wishes to calculate the power or energy picked up by the load Z_L.

In order to proceed with the calculation one usually asks for additional information such as the exact geometries of the wire, the slot and the cavity, the conductivity of the cavity walls and the wire, etc. One may then set up two coupled integral equations for the magnetic current in the slot and the electric current of the wire with $Z_L = 0$, and proceeds to solve them on an electronic computer. The next step is to calculate the input impedance or admittance looking out from the load Z_L. With the solutions of these two problems one then has a Thevenin or Norton equivalent circuit attaching to the load Z_L, from which the power or energy absorbed by Z_L can be determined. This is the conventional approach.

The conventional approach as just described, although tedious and difficult to obtain high accuracy, is nevertheless straightforwward. Since the objective of the present problem is to calculate power absorbed by the load, one has to make sure that the numerical results do not violate the power conservation law, that is, the net power flow through the slot into the cavity must equal the power absorbed by the load if the cavity walls and the wire are perfectly conducting. This power conservation requirement is such a delicate balance and can be easily violated by numerical solutions. One may argue that the requirement can be incorporated in the algorithm of the computer codes written for the two problems, one for computing the short-circuit current on the wire and the other the input driving-point impedance.

If, on the other hand, one puts the power conservation requirement up front in the formulation of the problem, one may choose a different line of attack rather than the conventional approach. If the problem asks only for the maximum power that could couple to the load for all possible wire configurations and locations, one may even abandon the conventional approach

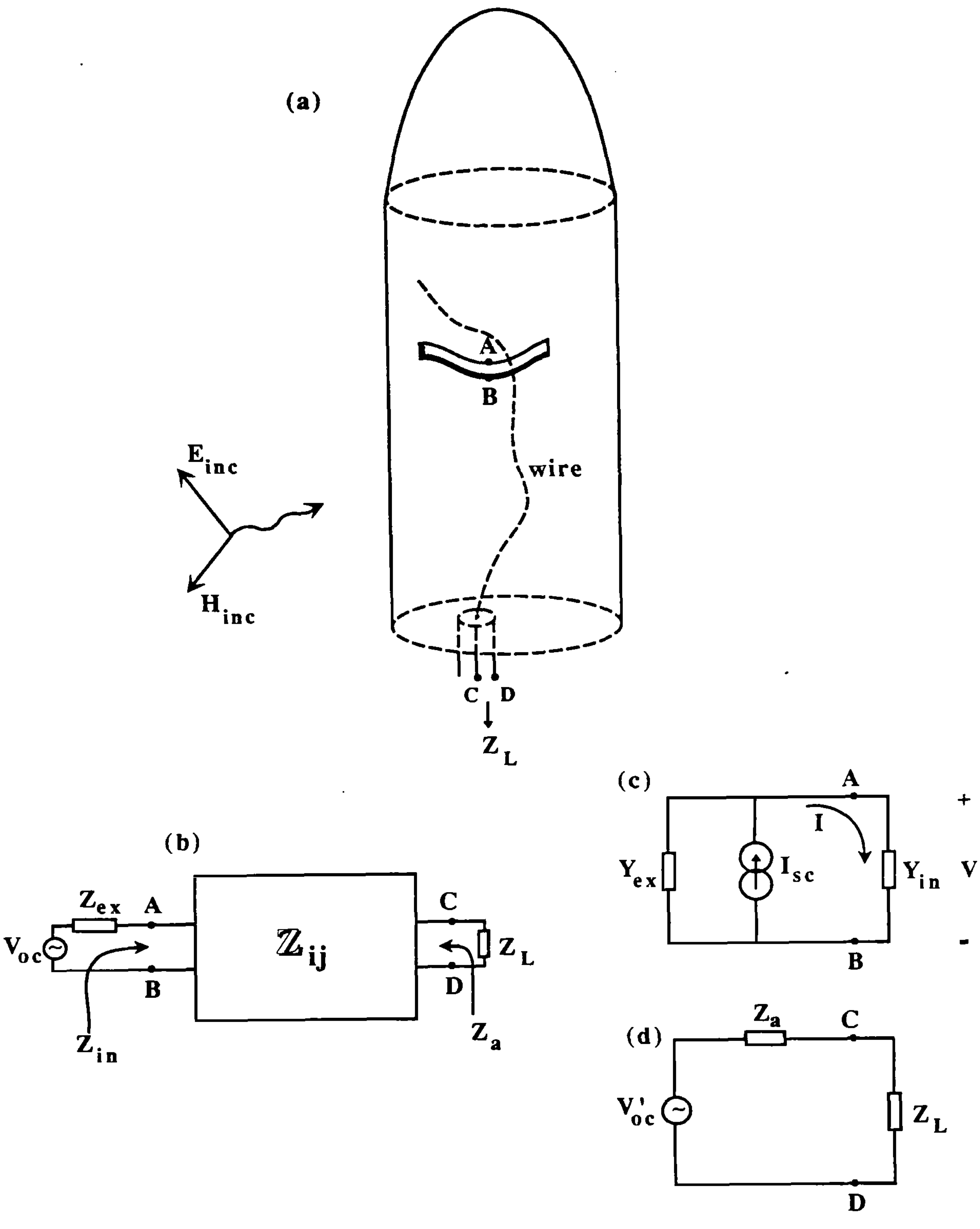

Figure 1. (a) Geometry of the problem, (b) equivalent circuit, (c) Norton equivalent circuit at A,B, and (d) Thévenin equivlent circuit at C, D.

altogether at the outset because the conventional approach becomes impractical in view of the infinitely many possibilities for these parameters, let alone the question of numerical accuracy imposed by the power conservation law.

One may raise the question as to why the problem under consideration is so loosely posed in the sense that the wire configuration and location are not precisely specified, nor are the dimensions and shapes of the cavity and slot. The reason is that in a real-world system one never has precise information on quesions of this sort. To make the results of the present study directly usable for real systems one must formulate the solution in such a way that it be independent of or insensitive to such "detailed" information. This means that a solution in terms of bound and trend is more useful than a numerical solution to a boundary-value problem with specific configuration and dimensions. To rephrase the problem to be considered in the following: *calculate the bound on the power or energy picked up by a wire inside a cavity with an aperture in its wall.*

3.0 CIRCUIT APPROACH

For the problem depicted in Fig. 1a the appropriate circuit for the power transfer from the slot to the load Z_L is given by Fig. 1b, and Fig. 1c is useful for estimating the total power penetrated through the slot.

From Fig. 1c one can immediately write down the real power, P, that couples to a load attached across A, B, namely,

$$P = \mathrm{Re}\left(V^{*}I\right) = \mathrm{Re}\left(\frac{|I_{sc}|^2}{|Y_{ex}+Y_{in}|^2}\, Y_{in}\right)$$

$$= \frac{|I_{sc}|^2}{|Y_{ex}+Y_{in}|^2}\, G_{in} \tag{5}$$

$$= \frac{|I_{sc}|^2}{4\,G_{ex}} \cdot \frac{4\,G_{in}G_{ex}}{(G_{in}+G_{ex})^2+(B_{in}+B_{ex})^2} \leq \frac{|I_{sc}|^2}{4G_{ex}}$$

where $Y_{ex} = G_{ex} - i\,B_{ex}$, $Y_{in} = Z_{in}^{-1} = G_{in} - i\,B_{in}$, and I_{sc} is the short circuit current that would flow through a perfectly conducitng wire connecting A and B (see Fig. 1a). Note that I_{sc} and G_{ex} contain the resonant properties of the slot and the external surfaces of the cavity. The question that comes up time and again is, Can one approximate (5) by the solution to the problem of the same slot in an infinite ground plane?

Consider for the moment the right-hand side of (5) and see what it will become for the case of an infinite ground plane. With the aid of the Babinet principle one has

$$\frac{1}{G_{ex}} = \frac{Z_o^2}{4\,R_r} \tag{6}$$

$$I_{sc} = 2H_{inc}\,h_{eff}$$

where $Z_o = 120\,\pi$ ohms, R_r and h_{eff} are respectively the radiation resistance and effective height of the complementary dipole and have the values

$$R_r \doteq 73\,\Omega$$

$$h_{eff} \doteq \frac{2}{\pi} L \tag{7}$$

for a half-wavelength dipole with length L. Substitution of (6) and (7) in (5) gives the familiar formula for the power absorption by a half-wavelength dipole or slot, viz.,

$$P \leq \frac{1}{2} L^2 S_{inc} \tag{8}$$

with $S_{inc} = E_{inc} H^*_{inc}$. The total power penetrating into the cavity is actually bounded by twice the value of the right-hand side of (8), namely $L^2 S_{inc}$, which follows from the duality principle. A rigorous consideration based on an integral equation formulation is given in Appendix B.

We now return to the original problem depicted in Fig. 1a and see which steps in the infinite ground plane problem can be generalized to fit our original problem. When the (narrow) slot is at half-wavelength resonance one may say that the field distribution in the slot is fixed and only the amplitude of the aperture field can be changed by the external and internal environments of the cavity. Thus one may say that the steps in (6) and (7) still hold approximately except that $2H_{inc}$ should be replaced by H_{sc}, the short-circuit field when the slot is completely covered with isotropic conductor. With this slight modification we have, in rationalized mks units,

$$\frac{|I_{sc}|^2}{4G_{ex}} \approx 200\,(H_{sc}\ell)^2 \tag{9}$$

where $\ell = L/2$. At the low-frequency limit, I_{sc} is well approximated by $H_{sc}\ell$. The *provisional* bound for the power through the slot into the cavity is approximately given by $400\,(H_{sc}\ell)^2$. If the only loss inside the cavity is due to the presence of the resistive part of Z_L, then the power absorption by Z_L is also bounded by 400 $(H_{sc}\ell)^2$. Obviously this bound can be far too loose because the wire load may not be able to absorb all the *upperbound* power transmitted through the slot. To make this point more explicit, recall that the absorption cross section of a wire averaged over all angles of incidence and polarization is given by

$$\sigma_w = \frac{\lambda^2}{8\pi} q \leq \frac{\lambda^2}{8\pi} = \frac{L^2}{2\pi} \tag{10}$$

where we have used the frequency that corresponds to a half-wavelength resonant slot. For a monopole of length L, σ_w should be twice the value given by (10) if S_{inc} is used for power calculation. Now, one has two cases to consider, namely, (a) all the power transmitted through the slot (L^2S_{inc}) will be absorbed by the wire load Z_L, and (b) part of L^2S_{inc} will be absorbed and the rest will be re-radiated through the slot into the space outside the cavity. This point will be picked up further for discussion in Section 4.0.

Since the problem at hand is to bound the power absorbed by the load Z_L, perhaps one should focus on the Thévenin equivalent circuit (Fig. 1d) directly attached to Z_L. The real power P_L absorbed by Z_L is given by

$$P_L = \left|V_{oc}'\right|^2 \cdot \frac{R_L}{|Z_a + Z_L|^2} = \frac{\left|V_{oc}'\right|^2}{4R_a} q \tag{11}$$

with

$$q = \frac{4R_a R_L}{|Z_a + Z_L|^2} \tag{12}$$

where R_a and R_L are repectively the real parts of Z_a and Z_L. Similarly, the real power P_a lost in Z_a is

$$P_a = \left|V_{oc}'\right|^2 \frac{R_a}{|Z_a + Z_L|^2} = \frac{\left|V_{oc}'\right|^2}{4R_L} q \tag{13}$$

and the ratio of P_L to P_a is the same as R_L to R_a. Without loss of generality for the discussion to follow, let the reactance part of Z_a tune out the reactance part of Z_L (i.e., $X_L = -X_a$), or vice versa. Then P_L and P_a depend only on the variables R_L and R_a apart from V_{oc}'. There are two distinct cases, namely, one with R_a fixed and varying R_L and the other with R_L fixed and varying R_a. These two cases lead to different values for maximum P_L, i.e.,

$$\max P_L = \frac{|V_{oc}'|^2}{4R_L} \qquad \text{(fixed } R_a\text{, varying } R_L\text{)} \tag{14a}$$

and

$$\max P_L = \frac{|V_{oc}'|^2}{R_L} \qquad \text{(fixed } R_L\text{, varying } R_a\text{)} \tag{14b}$$

Eq. (14a) is a familiar form and is obtained when $R_L = R_a$, whereas (14b) occurs at $R_a = 0$. However, it should be noted that if $R_a = 0$, P_L should also be identically zero because an antenna which does not radiate to far distances cannot, according to the reciprocity theorem, receive any power from distant radiation.

A general procedure to calculate the receiving property of an antenna, especially an electrically small antenna, is to assume, for convenience, $R_a = 0$ (or more precisely, $R_L >> R_a$) and to proceed with the calculation without encountering any fundamental difficulty. In this way, the maximum real power that any load can extract from the incident wave is also given by (14b). It must be borne in mind that, although R_a can be very small in comparison to R_L, it is not zero; otherwise $P_L = 0$. This also means that R_L cannot be zero in formula (14b), i.e., the maximum power absorbed can never be infinite no matter what wire configurations and slot dimensions one may choose.

Eq. (11) is plotted in Figs. 2a and 2b for the two distinct cases when the reactive parts of Z_L and Z_a are tuned out of each other. Case (a) is for the situation where the configuration of the geometry is fixed and the load can be varied, whereas case (b) is for fixed load and changing

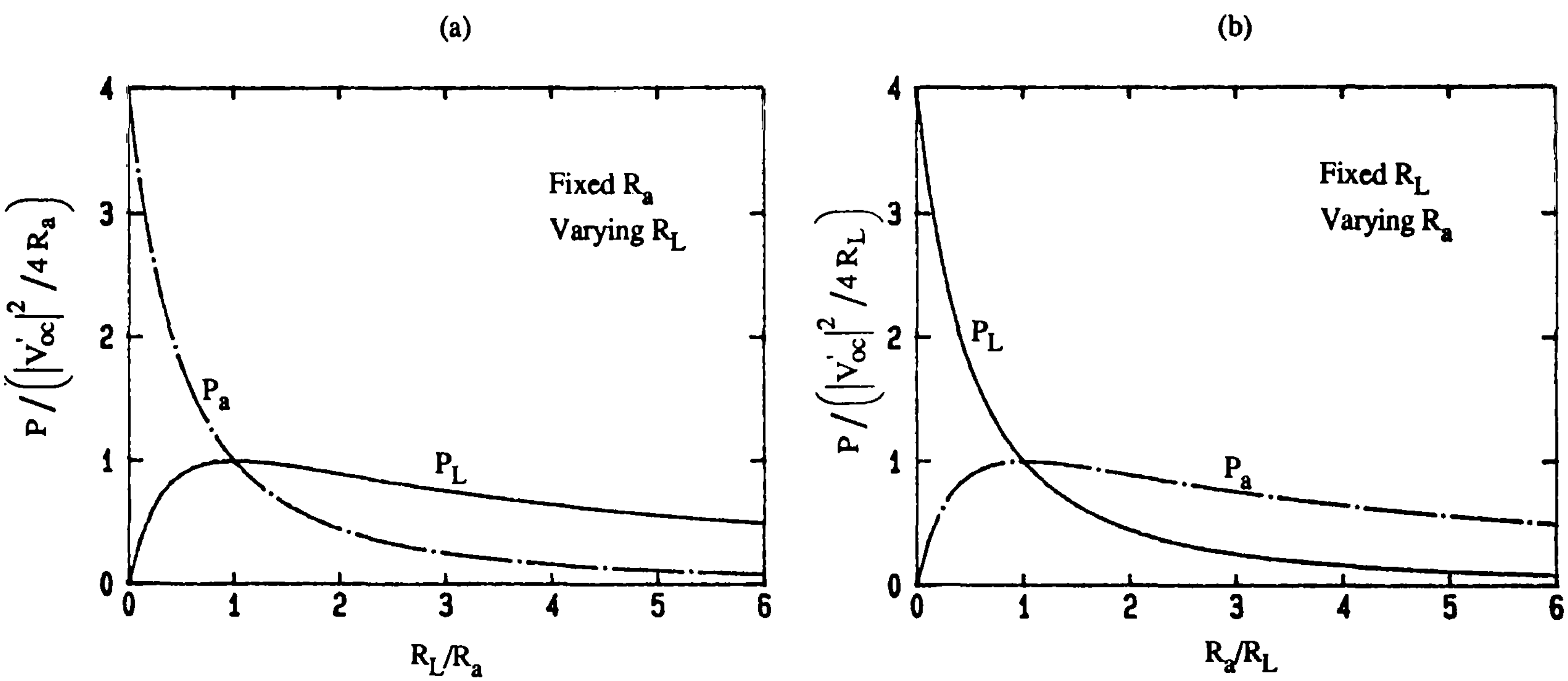

Figure 2. Power absorption P_L as a function of (a) R_L/R_a and (b) R_a/R_L.

configuration. In the latter case, however, V_{oc}' is changing as well. Thus, one cannot say that (14b) contradicts (14a). But the basic question still remains as to which of the two expressions gives the maximum possible response, assuming that one has the freedom of either varying Z_L or Z_a or both. Does one really have to go through all these cases in order to find out which gives the maximum?

4.0 POWER CONSERVATION APPROACH

In a recent paper [5] it was suggested that a decomposition approach may be a good candidate for estimating the trends and bounds of the back-door coupling into a system. For coupling through only one layer of shield (on which there are POE's) the approach makes use of the following approximate formula for the absorption cross section (σ_p) of the pin:

$$\sigma_p \approx \left[\sum_i \sigma\,(\mathrm{POE}_i)\right] \cdot \left[\frac{\lambda\, Q}{2\,\pi\, V}\right] \cdot [\sigma_w] \tag{15}$$

where $\sigma(\mathrm{POE}_i)$ is the penetration cross section of the i-th POE on the shield, excluding the re-radiation out from the cavity, σ_w is the absorption cross section of the wire in the absence of the shield with random angle of incidence, λ is the wavelength, and V and Q are the volume and quality factor of the cavity within which the wire is located (cf. Fig. 1a).

Formula (15) can also be derived from a power conservation consideration. Notice that in obtaining (15), the following relations have been used:

$$Q = \frac{\omega W}{P_T} \tag{16}$$

$$W = \frac{S V}{c} \tag{17}$$

where W and P_T are, respectively, the total energy and power dissipated in the cavity (including the loss in the shield and the loading of the wire) after the steady state is reached, S is the average power density within the cavity, $\omega = 2\pi c/\lambda$, and c is the speed of light. With (16) and (17), (15) can be re-written as

$$\sigma_p \approx \left[\sum_i \sigma(POE_i)\right] \frac{S \cdot \sigma_w}{P_T}$$

$$= \left[\sum_i \sigma(POE_i)\right] \frac{\sigma_w}{\sigma_T} \tag{18}$$

where σ_T is the total absorption cross section including the losses inside the cavity, in the shield walls, and re-radiation through the POE's into the outer space of the cavity. Eq. (18) also holds if one includes re-radiation effect for $\sigma(POE_i)$, while excluding re-radiation effect for σ_T. There are various ways to look at (18). For example, the factor σ_w/σ_T can be considered as the fraction of penetrant power absorbed by the wire. On the other hand, the factor $[\sum_i \sigma(POE_i)]/\sigma_T$ can be thought of as the shielding of the cavity.

By writing $\sigma_T = \sigma_w + \sigma_r$, one has from (18)

$$\sigma_p = \left[\sum_i \sigma(POE_i)\right] \frac{\sigma_w}{\sigma_w + \sigma_r} \tag{19}$$

Furthermore, since by definition

$$\sigma_w Q_w = \sigma_T Q_T = \sigma_r Q_r = \frac{\omega W}{S} \tag{20}$$

one has

$$\sigma_p \approx \left[\sum_i \sigma(POE_i)\right] \frac{Q_w^{-1}}{Q_w^{-1} + Q_r^{-1}}$$

$$= \left[\sum_i \sigma(POE_i)\right] \frac{Q_r}{Q_w + Q_r}$$

$$= \left[\sum_i \sigma(POE_i)\right] \left(1 + \frac{Q_w}{Q_r}\right)^{-1} \tag{21}$$

where the Q's are the corresponding quality factors.

To calculate σ_r, Q_r, etc. is generally difficult, especially for a real-world system. However, under certain circumstances this is possible. For example, consider a wire inside a perfectly conducting cavity with a thin slot in its surface (see Fig. 1a). If the length of the wire (ℓ_w) is much greater than the slot length (ℓ_s), then at frequencies below the wire's fundamental resonance ($f < c/(4\ell_w)$), $\sigma_T \approx \sigma_w$. That is, from (18), one has

$$\sigma_p \approx \sigma(\text{slot}) \tag{22}$$

On the other hand, at frequencies greater that the slot's fundamental resonance ($f > c/(2\ell_s)$) one has

$$\begin{aligned} \sigma_T &= \sigma_w + \sigma_{slot} \\ &\approx 2\,\sigma_w \end{aligned} \tag{23}$$

in the trend sense, where σ_{slot} is the slot cross section when the polarization and the angle of incidence are randomized. In other words, σ_{slot} is the same as σ_r introduced in (19), which is different from $\sigma(\text{slot})$, the slot penetration cross section excluding re-radiation out of the cavity. From (18) and (23), one then has

$$\sigma_p \approx \tfrac{1}{2}\,\sigma(\text{slot}) \tag{24}$$

When the upperbound of $\sigma(\text{slot})$ is used, Eq. (24) provides a good estimate on the upperbound of the pin absorption cross section, as indicated in Fig. 3 which shows the comparison between the estimate and a measured result taken from Ref. 8 for $\ell_w \approx 44$ cm and $\ell_s \approx 10$ cm. One thus arrives at the following estimates:

$$\begin{aligned} \text{upperbound of } \sigma(\text{slot}) &\approx \ell_s^2, && \text{at } f = f_s = c/(2\ell_s) \approx 1.5 \text{ GHz} \\ &\approx \ell_s^2\,(f/f_s)^4, && \text{for } f \le f_s \\ &\approx \ell_s^2\,(f_s/f)^2, && \text{for } f \ge f_s \end{aligned} \tag{25}$$

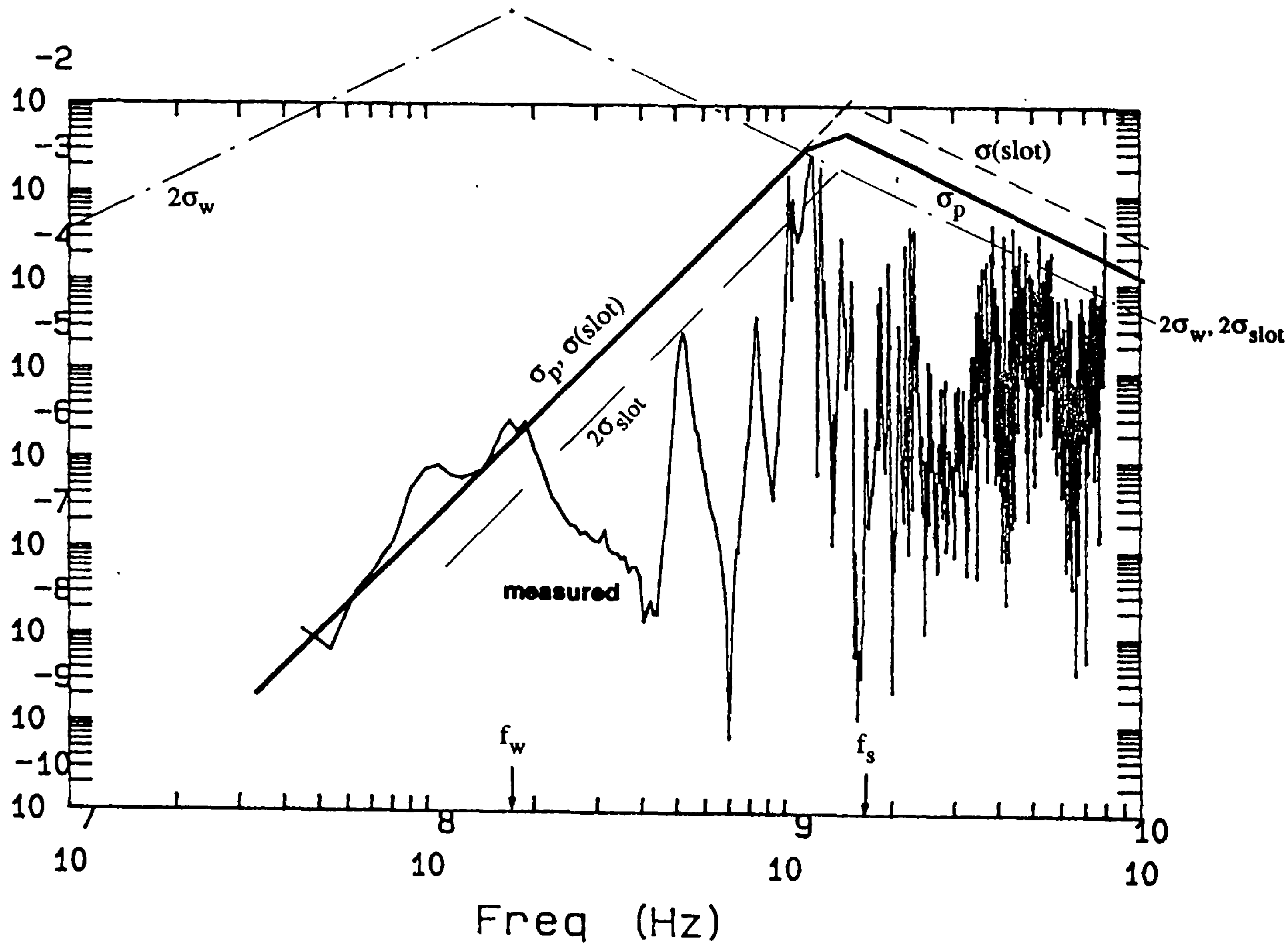

Figure 3. Comparison of measured data with a bounding estimate (σ_p) for the absorption cross section of Fig. 1a with a 10 cm thin slot. In the figure, various cross-section trends used in the estimate calculation are also given (see Eqs. (19), (25)-(27)).

$$\text{upperbound of } \sigma_{slot} \approx \ell_s^2 / (4\pi), \quad \text{at } f = f_s$$
$$\approx \ell_s^2 (f / f_s)^4 / (4\pi), \quad \text{for } f \le f_s \tag{26}$$
$$\approx \ell_s^2 (f_s / f)^2 / (4\pi), \quad \text{for } f \ge f_s$$

$$\text{upperbound of } \sigma_w \approx \ell_w^2 / \pi, \quad \text{at } f = f_w = c / (4\ell_w) \approx 170 \text{ MHz}$$
$$\approx \ell_w^2 (f / f_w)^2 / \pi, \quad \text{for } f \le f_w \tag{27}$$
$$\approx \ell_w^2 (f_w / f)^2 / \pi, \quad \text{for } f \ge f_w$$

The bounding calculation curve given in Fig. 3 makes use of the approximate equations (18), (25), (26) and (27) which appear to be violated only around 100 MHz and 8 GHz. The violation around 100 MHz is due to the neglect of the body resonance, which can give a few dB enhancement. The violation near 8 GHz may be due to the possibility that $\sigma_w >> \sigma_{slot}$ at that particular frequency, so that $\sigma_p \approx \sigma(\text{slot})$ instead of $\sigma \approx \sigma(\text{slot})/2$ (notice that the underestimate is about 3 dB, a factor of 2). From the figure, it is also observed that over-estimations appear in several frequency ranges. These over-estimations could have been avioded if the formulas given for $\sigma(\text{slot})$, σ_{slot} and σ_w had allowed for more detailed frequency dependence than in the oversimplified equations (25), (26) and (27).

The good agreement described above may be an indication that the effect of the cavity loading of the aperture on the upperbound estimate is negligible. If this is true, the upperbound estimate based on a power conservation consideration will be very useful. This is because the estimates of penetration cross sections without the cavity loading effects are readily available for many apertures. For example, in the same cylindrical structure as that in Fig. 1a with a circumferential slot of diameter d instead of a 10 cm slot, one then has approximately

$$\text{upperbound of } \sigma(\text{circum. slot}) \approx \frac{3}{4}\pi d^2, \qquad \text{at } f = f_a = c/(d\pi) \approx 315 \text{ MHz}$$

$$\approx \frac{3}{4}\pi d^2 (f/f_a)^4, \qquad \text{for } f \leq f_a \tag{28}$$

$$\approx \frac{3}{4}\pi d^2 (f_a/f)^2; \qquad \text{for } f \geq f_a$$

$$\text{upperbound of } \sigma_{\text{circum.slot}} \approx \frac{1}{16}\pi d^2, \qquad \text{at } f = f_a$$

$$\approx \frac{\pi}{16} d^2 (f/f_a)^4, \qquad \text{for } f \leq f_a \tag{29}$$

$$\approx \frac{\pi}{16} d^2 (f_a/f)^2, \qquad \text{for } f \geq f_a$$

Applying (27), (28), and (29) to (18) one obtains the bounding estimate curve of Fig. 4 , which again is in good agreement with the measured data [8] except for a few disagreement points similar to those shown in Fig. 3.

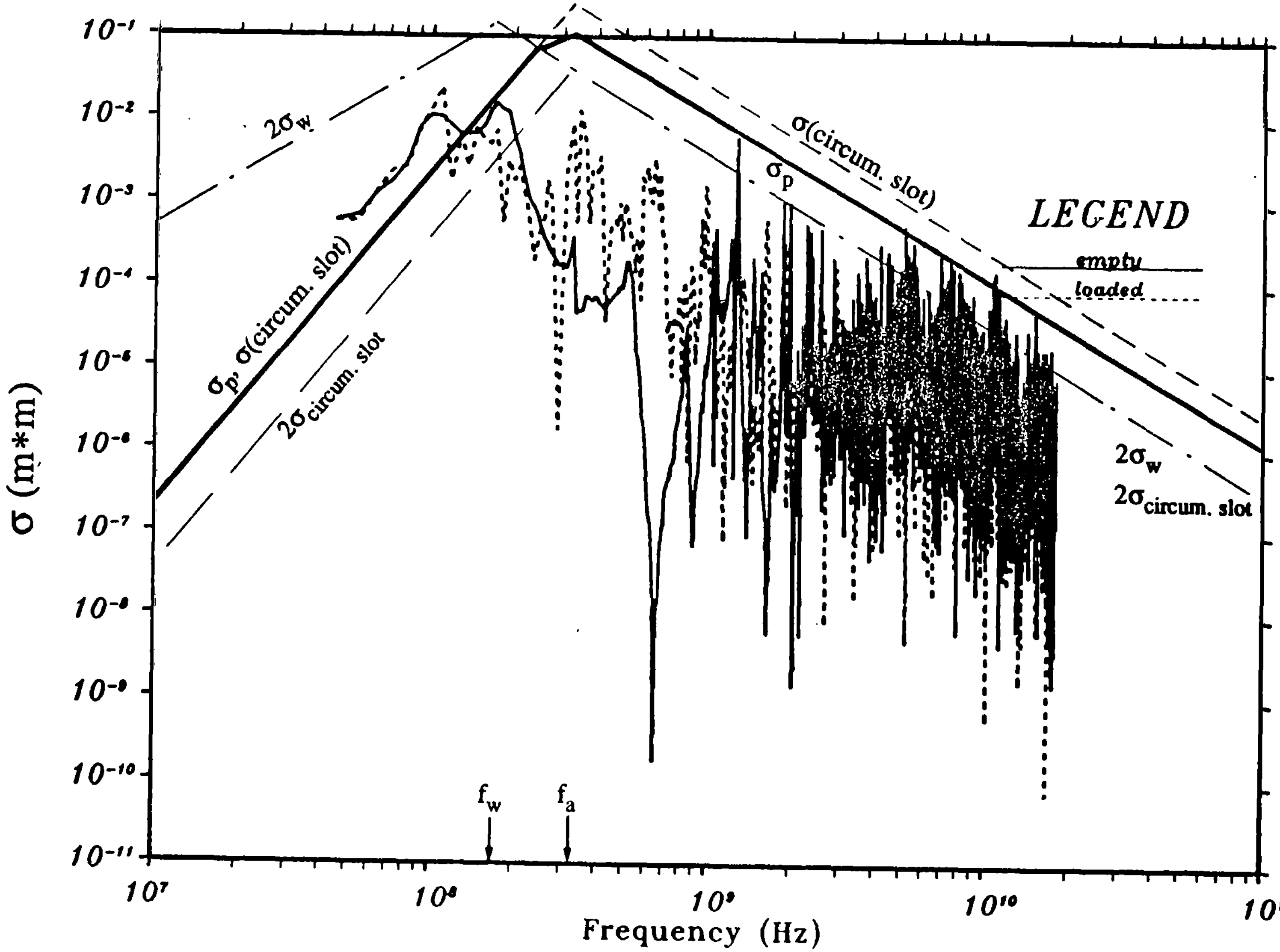

Figure 4. Comparison of measured data and a bounding estimate (σ_p) for the absorption cross-section of Fig. 1a with a circumferential slot. In the figure, various cross-section trends used in the estimate calculation are also given (see Eqs. (19), (27)-(29)). The measusred data for a lightly loaded cavity is included.

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APPENDIX A

RECIPROCITY AND OPTICAL THEOREMS

There is some striking similarity in the derivations of the reciprocity and optical theorems in field theory. The former leads to

$$\sigma_a = \frac{\lambda^2}{4\pi} G\, p\, q \tag{A·1}$$

while the latter yields

$$\sigma_a + \sigma_s = \frac{\lambda^2}{\pi} \mathrm{Re}\,(\mathbf{1}_1 \cdot \mathbf{A}), \qquad \text{with } \mathbf{E}_s = \mathbf{A}\,\frac{e^{ikr}}{4\pi r},\ r \to \infty \tag{A·2}$$

where σ_a and σ_s are the absorption and scattering cross sections. In particle physics σ_a and σ_s are referred to, respectively, as elastic and inelastic scattering cross sections.

Consider Fig. A·1 where S_A is the surface enclosing an antenna or scatterer and S_∞ is the surface of a sphere with a very large radius. Then the Lorentz reciprocity theorem gives, for the fields of the transmitting and receiving problem,

$$\int_{S_\infty + S_A} (\mathbf{E}_r \times \mathbf{H}_t - \mathbf{E}_t \times \mathbf{H}_r) \cdot \mathbf{n}\, dS = 0 \tag{A·3}$$

while the power conservation gives, for the total field, $\mathbf{E}_{inc} + \mathbf{E}_{sc}$,

$$\int_{S_\infty + S_A} \mathrm{Re}\left[(\mathbf{E}_{inc} + \mathbf{E}_{sc}) \times (\mathbf{H}^*_{inc} + \mathbf{H}^*_{sc})\right] \cdot \mathbf{n}\, dS = 0 \tag{A·4}$$

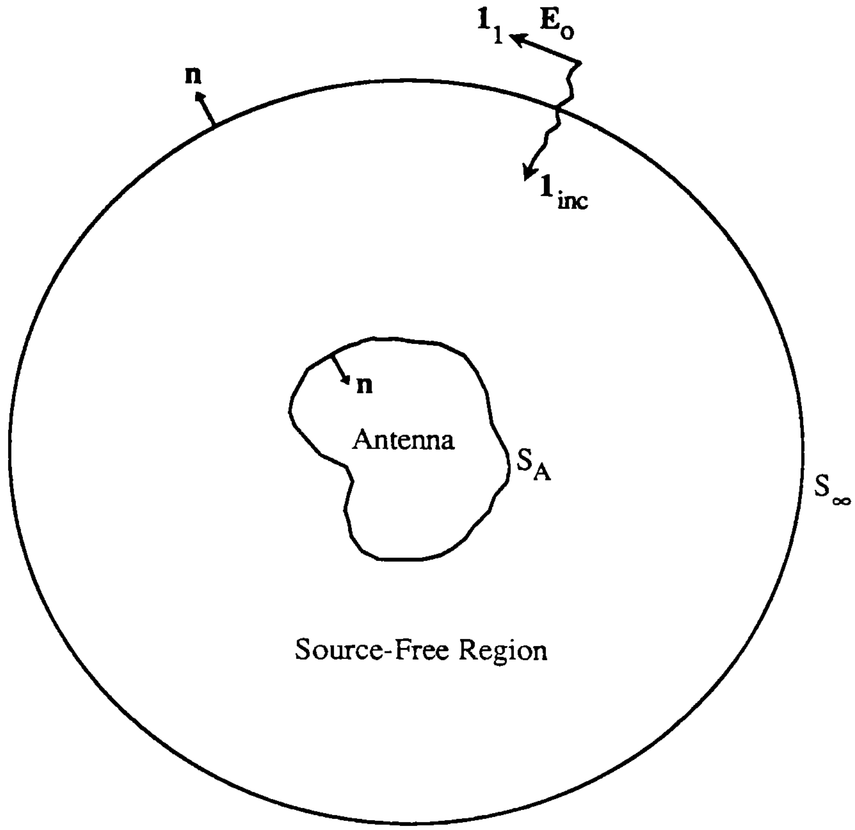

Figure A·1. The source-free region where the Gauss theorem is applied.

Eqs. (A·3) and (A·4) can be reduced to [4]

$$V_t\, I_r + V_r\, I_t = -\int_{S_\infty} (\mathbf{E}_r \times \mathbf{H}_t - \mathbf{E}_t \times \mathbf{H}_r)\cdot \mathbf{n}\, dS \tag{A·5}$$

$$P_a + P_s = -\int_{S_\infty} \mathrm{Re}\left[(\mathbf{E}_{inc} \times \mathbf{H}^*_{sc} + \mathbf{E}_{sc} \times \mathbf{H}^*_{inc})\right] \cdot \mathbf{n}\, dS \tag{A·6}$$

where (V_t, I_t) and (V_r, I_r) are the voltages and currents at the antenna's terminals of the transmitting and receiving problem, P_a and P_s are the power absorbed and power scattered. To evalutate the surface integrals over S_∞ the method of stationary phase is employed. There are two stationary points: one is along the propagation direction of the incident wave $\mathbf{1}_{inc}$ (forward scattering direction) and the other along the specular direction of the incident wave $-\mathbf{1}_{inc}$ (backward scattering direction). It is interesting to note that only the stationary point at $-\mathbf{1}_{inc}$ contributes to the integral (A·5), whereas only the stationary point at $\mathbf{1}_{inc}$ contributes to the integral (A·6). At the stationary point $\mathbf{1}_{inc}$ on S_∞, $(\mathbf{E}_r \times \mathbf{H}_t - \mathbf{E}_t \times \mathbf{H}_r)\cdot\mathbf{1}_{inc} \equiv 0$ [6]. At the stationary point $-\mathbf{1}_{inc}$ on S_∞, $\mathbf{E}_{inc} \times \mathbf{H}^*_{sc} + \mathbf{E}_{sc} \times \mathbf{H}^*_{inc}$ has no real part [7]. Hence, Eqs. (A·5) and (A·6) give

$$V_t\, I_r + V_r\, I_t = \frac{\lambda^2}{\pi}\, S_{inc}\, \mathbf{1}_l\cdot\mathbf{F}(-\mathbf{1}_{inc}) \tag{A·7}$$

$$P_a + P_s = \frac{\lambda^2}{\pi}\, S_{inc}\, \mathrm{Re}\left[\mathbf{1}_l\cdot\mathbf{A}(\mathbf{1}_{inc})\right] \tag{A·8}$$

where S_{inc} $(= E_0^2 / Z_0)$ is the Poynting vector of the incident wave, and the amplitudes $\mathbf{F}$ and $\mathbf{A}$ are defined as follows:

$$\mathbf{E}_t \sim i\, E_o\, F\, \frac{e^{ikr}}{kr} \tag{A·9}$$

$$\mathbf{E}_{sc} \sim i\, E_o\, A\, \frac{e^{ikr}}{kr} \tag{A·10}$$

with E_o being the amplitude of the incident plane wave.

The right-hand side of Eq. (A·6) can be transformed to another form. Since

$$\nabla \times \mathbf{E}_{sc} = i\omega\mu\, \mathbf{H}_{sc} - \mathbf{J}_m \tag{A·11}$$

$$\nabla \times \mathbf{H}_{sc} = -\, i\omega\varepsilon\, \mathbf{E}_{sc} + \mathbf{J} \tag{A·12}$$

everywhere throughout all space, where **J** and $\mathbf{J}_m$ are the electric and magnetic currents that give rise to $\mathbf{E}_{sc}$ and $\mathbf{H}_{sc}$, one has

$$\mathrm{Re}\, \nabla \cdot (\mathbf{E}_{inc} \times \mathbf{H}^*_{sc} + \mathbf{E}_{sc} \times \mathbf{H}^*_{inc}) = -\, \mathrm{Re}\, (\mathbf{J}^* \cdot \mathbf{E}_{inc} + \mathbf{J}_m \cdot \mathbf{H}^*_{inc}) \tag{A·13}$$

By virtue of the Gauss theorem Eq. (A·6) can also be written as

$$P_a + P_s = \int \mathrm{Re}\, (\mathbf{J}^* \cdot \mathbf{E}_{inc} + \mathbf{J}_m \cdot \mathbf{H}^*_{inc})\, dV \tag{A·14}$$

If P_a and P_s are the time-average power, the right-hand side should be halved.

APPENDIX B

INTEGRAL-EQUATION APPROACH TO THE BABINET PRINCIPLE

In this appendix we will derive using an integral equation formalism the receiving characteristics of a slot antenna in an infinite planar screen from those of its complementary antenna, simply called the strip dipole antenna in the following.

B-1. The Slot Problem

First, let us consider the slot antenna depicted in Fig. B.la. We are interested in finding the open-circuit voltage, V_{oc}^{s} , across a, b which may be connected to a two-wire line or a coaxial cable. In addition, we want the input impedance Z_{in}^{s} , of the antenna across a,b. Let $\mathbf{E}_{sc}$, $\mathbf{H}_{sc}$ be the scattered field satisfying the radiation condition at infinity. Then the Helmholtz representation theorem says

$$\mathbf{H}_{sc} = \nabla \times \int \mathbf{n}' \times \mathbf{H}_{sc}\, G\, dS' + \frac{1}{i\omega\mu}\, \nabla \times \nabla \times \int \mathbf{n}' \times \mathbf{E}_{sc}\, G\, dS' \tag{B.1}$$

where the time factor $e^{-i\omega t}$ has been suppressed, $\mathbf{n}'$ is the unit normal pointing into the region containing the observation point; $G = e^{ik|\underline{r}-\underline{r}'|} / 4\pi|\underline{r} - \underline{r}'|$. Equation (B·1) can be derived most easily from the vector potentials, **A** and **A***, corresponding to the current sources, **K** and **K***, where $\mathbf{K} \equiv \mathbf{n}' \times \mathbf{H}_{sc}$ and $\mathbf{K}^* \equiv \mathbf{n}' \times \mathbf{E}_{sc}$. The corresponding equation for $\mathbf{E}_{sc}$ will be given below when treating the strip dipole antenna. Now we have

$$\mathbf{n} \times [\nabla G \times (\mathbf{n}' \times \mathbf{H})] = \mathbf{n} \times \mathbf{H}\, \frac{\partial G}{\partial n'} + \mathbf{n} \times \mathbf{n}'\, (\mathbf{H} \cdot \nabla G) = \mathbf{n} \times \mathbf{H}\, \frac{\partial G}{\partial n'} ,$$

since $\mathbf{n} \times \mathbf{n}' = 0$ for a planar screen. Thus, by choosing $(\partial G/\partial n') = 0$ at the plane of the screen $(z = 0)$, we have

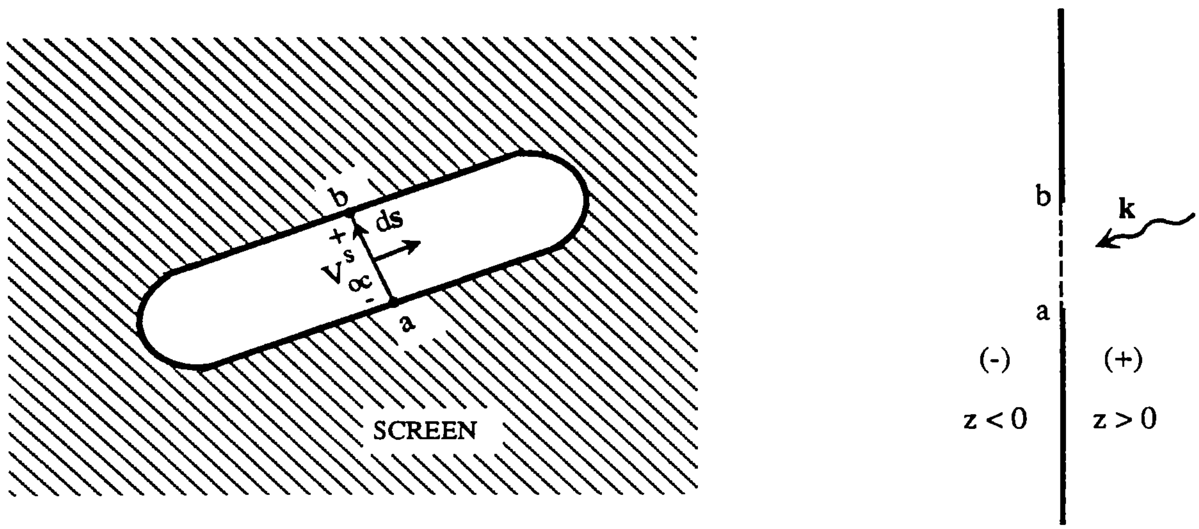

Figure B.1a. The Slot Problem.

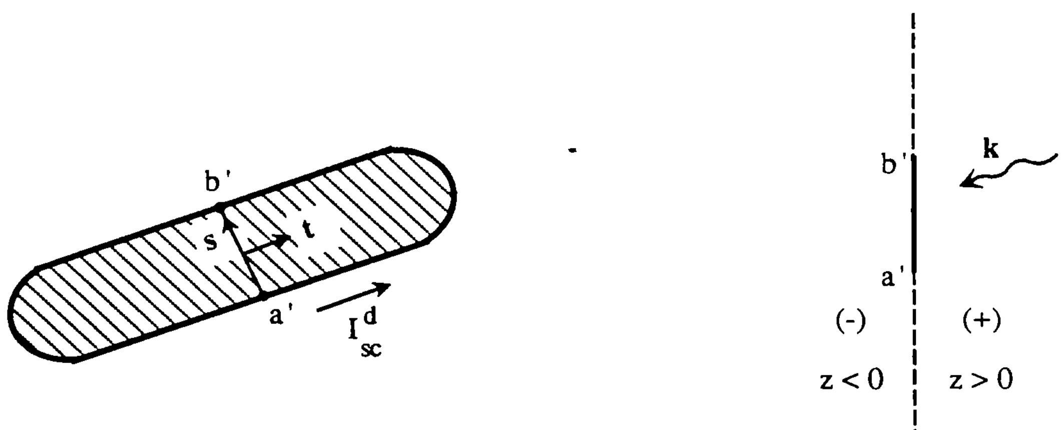

Figure B.1b. The Strip Problem.

$$\mathbf{n}\times\mathbf{H}_{sc} = \frac{1}{2\pi i\omega\mu}\,\mathbf{n}\times\nabla\times\nabla\times\int_{slot}\mathbf{n}'\times\mathbf{E}_{sc}\;\varphi\;dS' \tag{B·2}$$

with $\varphi = 4\pi G$. On the illuminated side of the screen ($z > 0$) $\mathbf{H}^+_{total} = \mathbf{H}_{inc} + \mathbf{H}_{ref} + \mathbf{H}^+_{sc}$, and for $z < 0$ $\mathbf{H}^-_{total} = \mathbf{H}^-_{sc}$, where $\mathbf{H}_{ref}$ is the reflected field by the infinite screen. Matching $\mathbf{n}\times\mathbf{H}^+_{total} = \mathbf{n}\times\mathbf{H}^-_{total}$ and $\mathbf{n}\times\mathbf{E}^+_{total} = \mathbf{n}\times\mathbf{E}^-_{total}$ in the slot, and noting that $\mathbf{n}\times(\mathbf{E}_{inc} + \mathbf{E}_{ref}) = 0$ at $z = 0$ we obtain

$$\begin{aligned}\frac{1}{2\pi i\omega}\,\mathbf{n}\times\nabla\times\nabla\times\int_{slot}\mathbf{n}'\times\mathbf{E}_{sc}\;\varphi\;dS' &= -\,\frac{1}{2}\,\mu\mathbf{n}\times(\mathbf{H}_{inc} + \mathbf{H}_{ref})\\ &= -\,\mu\mathbf{n}\times\mathbf{H}_{inc}\end{aligned} \tag{B·3}$$

Next, we introduce V^S_{oc} and the effective height of the slot $\mathbf{h}^S$ as follows (Fig. B·1a).

$$\begin{aligned}V^s_{oc} &= -\int_a^b \mathbf{E}_{total}\cdot d\mathbf{s} = -\int_a^b(\mathbf{E}_{sc} + \mathbf{E}_{inc})\cdot d\mathbf{s}\\ &\equiv \mathbf{h}^s\cdot\mathbf{E}_{inc} = Z_0\mathbf{H}_{inc}\cdot(\mathbf{k}\times\mathbf{h}^s)\end{aligned} \tag{B·4}$$

where $\mathbf{k}$ is the unit vector in the propagation direction of the incident wave and $Z_o = (\mu/\varepsilon)^{\frac{1}{2}}$.

B-2. The Strip Problem

Now, we move on to the complementary strip dipole antenna shown in Fig. B·1b. The Helmholtz representation theorem gives

$$\mathbf{E}_{sc} = \nabla\times\int\mathbf{n}'\times\mathbf{E}_{sc}G\;dS' - \frac{1}{i\omega\varepsilon}\nabla\times\nabla\times\int\mathbf{n}'\times\mathbf{H}_{sc}G\;dS' \tag{B·5}$$

Again, we take $(\partial G/\partial n) = 0$ at $z = 0$, so that

$$\mathbf{n}\times\mathbf{E}^{sc}_+ = -\frac{1}{2\pi i\omega\varepsilon}\,\mathbf{n}\times\nabla\times\nabla\times\int_{strip}\mathbf{n}'\times\mathbf{H}^+_{sc}\;\varphi\;dS' \tag{B·6}$$

since, from symmetry, $\mathbf{n} \times \mathbf{H}_{sc} \equiv 0$ in the aperture. At $z = 0+$, (B·6) gives

$$\frac{1}{2\pi i\omega} \mathbf{n} \times \nabla \times \nabla \times \int_{strip} \mathbf{n}' \times \mathbf{H}_{sc}^{+} \varphi \, dS' = -\varepsilon \mathbf{n} \times \mathbf{E}_{sc}^{+}$$

$$= \varepsilon \mathbf{n} \times \mathbf{E}_{inc} \qquad \text{(B·7)}$$

The total current, I_{sc}^{d}, induced across a', b' is given by

$$I_{sc}^{d} = \int_{a'}^{b'} \left(\mathbf{H}_{total}^{+} - \mathbf{H}_{total}^{-}\right) \cdot ds = 2\int_{a'}^{b'} \left(\mathbf{H}_{sc}^{+} + \mathbf{H}_{inc}\right) \cdot ds \qquad \left(\text{since } \mathbf{H}_{+}^{sc} = \mathbf{H}_{-}^{sc}\right)$$

$$\equiv \frac{1}{Z_{in}^{d}} V_{oc}^{d} = \frac{1}{Z_{in}^{d}} \mathbf{h}^{d} \cdot \mathbf{E}_{inc} \qquad \text{(B·8)}$$

where we have introduced the effective height $\mathbf{h}^{d}$ for the strip dipole. Comparing (B·3) and (B·7) one can observe that if

$$\underset{\text{(slot)}}{\mu \mathbf{H}_{inc}} \quad \Leftrightarrow \quad \underset{\text{(strip)}}{\varepsilon \mathbf{E}_{inc}}$$

then

$$\underset{\text{(slot)}}{-\int_{a}^{b} \mathbf{E}_{sc} \cdot ds} \quad \Leftrightarrow \quad \underset{\text{(strip)}}{\int_{a'}^{b'} \mathbf{H}_{sc}^{+} \cdot ds}$$

which means

$$2V_{oc}^{s} \quad \Leftrightarrow \quad I_{sc}^{d}$$

that is,

$$\mathbf{k} \times \mathbf{h}^{s} = \frac{1}{2} \frac{Z_o}{Z_{in}^{d}} \mathbf{h}^{d} \qquad \text{(B·9)}$$

which is one of the two results we have set out to seek.

B-3. The Transmitting Problem

To find the relationship between the input impedance Z_{in}^{s} of the slot and Z_{in}^{d} of the strip dipole we consider Fig. B-2. From symmetry, $\mathbf{H}^{+} = -\mathbf{H}^{-}$ and $\mathbf{E}^{+} = \mathbf{E}^{-}$ for both problems. The aperture electric field $\mathbf{E}^{s}$ in the slot problem satisfies (c.f. Eq. (B-2))

$$\frac{1}{2\pi i\omega}\nabla\times\nabla\times\int_{slot}\mathbf{n}'\times\mathbf{E}^{s}\,\varphi\,dS' = \mu\mathbf{H}^{-} = \frac{1}{2}\mu I_{o}\,\mathbf{t}\delta(t-t_{o}) \tag{B-10}$$

On the other hand, the tangential magnetic field in the strip dipole problem satisfies (c.f. Eq.(B-7))

$$\frac{1}{2\pi i\omega}\nabla\times\nabla\times\int_{strip}\mathbf{n}'\times\mathbf{H}_{+}^{d}\,\varphi\,dS' \equiv \varepsilon V_{o}\mathbf{t}\delta(t-t_{o}) \tag{B-11}$$

Upon comparing (B-10) and (B-11) we get

$$\frac{2}{\mu I_{o}}\int_{a}^{b}\mathbf{E}^{s}\cdot ds = \frac{1}{\varepsilon V_{o}}\int_{a'}^{b'}\mathbf{H}_{+}^{d}\cdot ds \tag{B-12}$$

Let us recall the usual definition for the input impedance:

$$Z_{in}^{s} = \frac{1}{I_{o}}\int_{a}^{b}\mathbf{E}^{s}\cdot ds, \qquad Z_{in}^{d} = \frac{V_{o}}{2\int_{a'}^{b'}\mathbf{H}_{+}^{d}\cdot ds}$$

where $\mathbf{E}^{s}$ and $\mathbf{H}_{+}^{d}$ are evaluated at $t = t_{o}$. Using (B-12) we immediately have

$$\frac{2}{\mu}Z_{in}^{s} \equiv \frac{1}{\varepsilon}\,\frac{1}{2Z_{in}^{d}}$$

or

$$Z_{in}^{s}\,Z_{in}^{d} = \frac{1}{4}Z_{o}^{2} \tag{B-13}$$

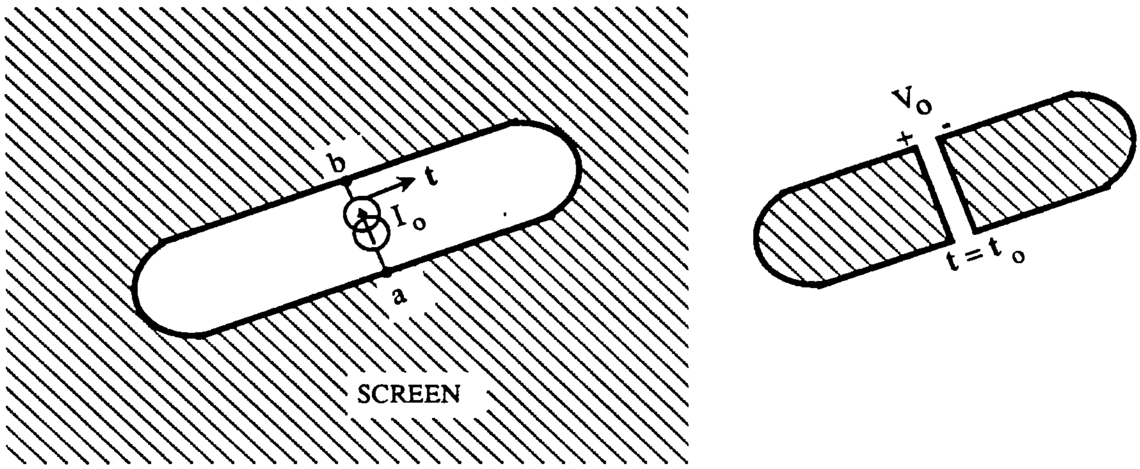

Figure B.2. The Transmitting Problem.

When the slot antenna (Fig. B·1a) is used as a receiving antenna, it is physically more appealing to think of the induced short-circuit current I_{sc}^{s} across a,b. Using (B·4), (B·9) and (B·13) we get

$$I_{sc}^{s} = \frac{1}{Z_{in}^{s}} V_{oc}^{s} = \frac{Z_o}{Z_{in}^{s}} \mathbf{H}_{inc} \cdot (\mathbf{k} \times \mathbf{h}^{s}) = \frac{Z_o^2}{2Z_{in}^{s} Z_{in}^{d}} \mathbf{h}^{d} \cdot \mathbf{H}_{inc}$$

$$= 2\mathbf{H}_{inc} \cdot \mathbf{h}^{d} \qquad \text{(B·14)}$$

which is the dual to $V_{oc}^{d} = \mathbf{E}_{inc} \cdot \mathbf{h}^{d}$.