Interaction Notes

Note 487

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Properties of Eigenterms of the Impedance Integral Equation

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Abstract

This paper considers the eigenmodes and eigenimpedances (eigenvalues) of the impedance (E-field) integral equation for scatterers described by a closed surface. The modes are paired as electric and magnetic modes. The eigenimpedances are decomposed into the parallel combination of internal and external parts, these parts also being simply related between electric and magnetic modes.

Contents

I. Introduction

This paper delves further into the properties of the terms in the eigenmode expansion method (EEM), specifically as related to the eigenterms resulting from the E-field or impedance integral equation. The background is given in [2,5,6,15,26]. Our starting point is the impedance of E-field integral equation

$$\vec{1}_S(\vec{r}_s)\cdot\tilde{\vec{E}}^{(inc)}(\vec{r}_s,s)=\left\langle\tilde{\vec{\vec{Z}}}_t(\vec{r}_s,\vec{r}_s\,';s);\tilde{\vec{J}}_s(\vec{r}_s\,';s)\right\rangle$$

$$\tilde{\vec{\vec{Z}}}_t(\vec{r}_s,\vec{r}_s\,';s)=\vec{1}_S(\vec{r}_s)\cdot\tilde{\vec{\vec{Z}}}(\vec{r}_s,\vec{r}_s\,';s)\cdot\vec{1}_S(\vec{r}_s\,')$$

$\equiv$ impedance kernel (in time domain a convolution operator)

$$\tilde{\vec{\vec{Z}}}(\vec{r}_s,\vec{r}_s\,';s)=-s\mu_o\tilde{\vec{\vec{G}}}_o(\vec{r},\vec{r}\,';s)$$

$\tilde{\vec{\vec{G}}}_o(\vec{r},\vec{r}\,';s)\equiv$ dyadic Green's function of free space

$\vec{r}_s,\vec{r}_s\,'\in S$ (1.1)

$\vec{1}_S(\vec{r}_s)\equiv$ unit outward-pointing normal at $\vec{r}_s$ to closed surface S

$\vec{\vec{1}}_S(\vec{r}_s)\equiv\ \vec{\vec{1}}-\vec{1}_S(\vec{r}_s)\vec{1}_S(\vec{r}_s)$

$\equiv$ transverse dyad for S at $\vec{r}_s$

$\vec{\vec{1}}\equiv\vec{1}_x\vec{1}_x+\vec{1}_y\vec{1}_y+\vec{1}_z\vec{1}_z\equiv$ identity dyadic

$\vec{E}^{(inc)}(\vec{r},t)\equiv$ incident electric field from sources in V_{ex} or V_{in}, but not S

$\vec{J}_s(\vec{r}_s,t)\equiv$ surface current density on S

$<,>\equiv$ symmetric product

$\equiv$ integration over S with respect to common coordinates ($\vec{r}_s,\vec{r}_s\,'$, etc.) with sense of multiplication indicated above separating comma

$V_{ex}\bigcup S\bigcup V_{in}\equiv$ all three dimensional Euclidean space

$\sim\equiv$ two-sided Laplace transform over time t

$s\equiv\Omega+j\omega\equiv$ Laplace-transform variable or complex frequency

where our perfectly conducting object (scatterer) is illustrated in fig. 1.1. Note that the closed perfectly conducting surface S can be simply or multiply connected (e.g. a toroid).

The eigenvalues and eigenmodes are defined via

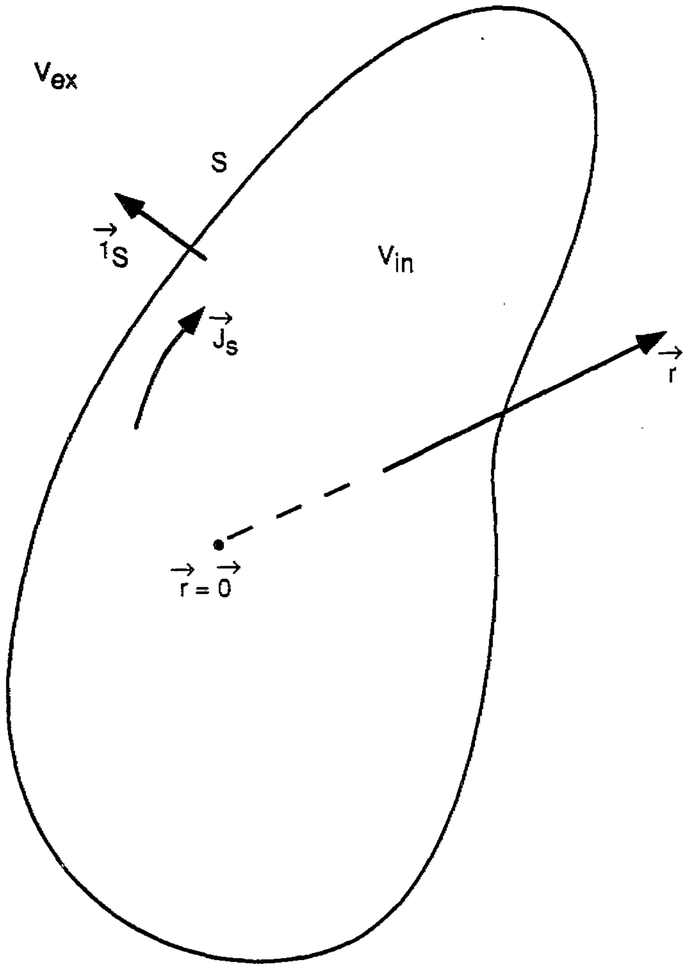

Fig. 1.1. Scatterer Consisting of Finite-Size Perfectly Conducting Closed Surface

$$\left\langle \tilde{\bar{\bar{Z}}}_t(\bar{r}_s,\bar{r}_s';s);\tilde{\bar{j}}_{s\beta}(\bar{r}_s',s)\right\rangle = \tilde{Z}_\beta(s)\tilde{\bar{j}}_{s\beta}(\bar{r}_s,s)$$
$$\left\langle \tilde{\bar{j}}_{s\beta}(\bar{r}_s,s);\tilde{\bar{\bar{Z}}}_t(\bar{r}_s,\bar{r}_s';s)\right\rangle = \tilde{Z}_\beta(s)\tilde{\bar{j}}_{s\beta}(\bar{r}_s',s) \tag{1.2}$$
$$\tilde{\bar{\bar{Z}}}_t(\bar{r}_s,\bar{r}_s';s) = \tilde{\bar{\bar{Z}}}_t^T(\bar{r}_s,\bar{r}_s';s) \text{ symmetric by reciprocity}$$
$$\left\langle \tilde{\bar{j}}_{s\beta_1}(\bar{r}_s,s);\tilde{\bar{j}}_{s\beta_2}(\bar{r}_s,s)\right\rangle = 1_{\beta_1,\beta_2} \text{ orthonormalization}$$

$\tilde{Z}_\beta(s) \equiv$ eigenimpedances

$\tilde{Y}_\beta \equiv \tilde{Z}_\beta^{-1}(s) \equiv$ eigenadmittances

$\tilde{z}_\beta(s) \equiv \dfrac{\tilde{Z}_\beta(s)}{Z_o} \equiv$ normalized eigenimpedances

$\tilde{y}_\beta(s) \equiv Z_o\tilde{Y}_\beta(s) \equiv$ normalized eigenadmittances

$$= \tilde{z}_\beta^{-1}(s)$$

$Z_o \equiv \sqrt{\dfrac{\mu_o}{\varepsilon_o}} \equiv$ wave impedance of free space

A sufficient condition (but not in general a necessary condition) for a complete set of eigenmodes spanning tangential vector fields on S is distinct eigenvalues. It is less clear what happens at discrete values of s when, say $\tilde{Z}_{\beta_1} = \tilde{Z}_{\beta_2}$ for $\beta_1 \neq \beta_2$ [17]. For example, the perfectly conducting sphere has no special problems when eigenvalues are equal [8]. In the case of a numerical approximation to the integral operator using the moment method (MoM) with N weighting functions the same as the N testing functions one obtains a symmetric NxN matrix. One can check that N separate eigenvalues are produced for any given s of interest, guaranteeing N mutually orthogonal eigenvectors [2].

Now write the kernel as

$$\tilde{\bar{\bar{Z}}}_t^{\upsilon}(\bar{r}_s,\bar{r}_s';s) = \tilde{\bar{\bar{Y}}}_t^{-\upsilon}(\bar{r}_s,\bar{r}_s';s) = \sum_\beta \tilde{Z}_\beta^{\upsilon}(s)\tilde{\bar{j}}_{s\beta}(\bar{r}_s,s)\tilde{\bar{j}}_{s\beta}(\bar{r}_s',s) \tag{1.3}$$

including the inverse as a special case. The identity kernel on S is

$$
\begin{aligned}
&\overset{\leftrightarrow}{1}_S(\vec{r}_s)\delta_s(\vec{r}-\vec{r}\,') = \tilde{\overset{\leftrightarrow}{Z}}{}_t^0(\vec{r}_s,\vec{r}_s{}';s) = \sum_\beta \tilde{\vec{j}}_{s\beta}(\vec{r}_s,s)\tilde{\vec{j}}_{s\beta}(\vec{r}_s{}',s) \\
&\int_S \delta_s(\vec{r}-\vec{r}\,')f(\vec{r}_s{}')dS' = f(\vec{r}_s) \\
&\int_S \left[\overset{\leftrightarrow}{1}_S(\vec{r}_s)\delta_s(\vec{r}_s-\vec{r}_s{}')\right]\cdot\vec{f}_t(\vec{r}_s{}')dS' = \vec{f}_t(\vec{r}_s) \\
&\quad \vec{f}_t(\vec{r}_s) \equiv \text{ tangential vector on } S
\end{aligned}
\tag{1.4}
$$

For later use note that we also have

$$
\begin{aligned}
\overset{\leftrightarrow}{1}_S(\vec{r}_s)\delta_s(\vec{r}_s-\vec{r}_s{}') &= -\overset{\leftrightarrow}{1}_S(\vec{r}_s)\times\left[\overset{\leftrightarrow}{1}_S(\vec{r}_s)\delta_s(\vec{r}_s-\vec{r}_s{}')\right]\times\overset{\leftrightarrow}{1}_S(\vec{r}_s{}') \\
&= \sum_\beta\left[\overset{\leftrightarrow}{1}_S(\vec{r}_s)\times\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s)\right]\left[\overset{\leftrightarrow}{1}_S(\vec{r}_s{}')\times\tilde{\vec{j}}_{s\beta}(\vec{r}_s{}',s)\right]
\end{aligned}
\tag{1.5}
$$

giving a representation in terms of what later will be used as complementary modes

$$
\begin{aligned}
&\tilde{\vec{j}}{}^{(c)}_{s\beta}(\vec{r}_s{}',s) \equiv \overset{\leftrightarrow}{1}_S(\vec{r}_s)\times\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s) \text{ complementary eigenmodes} \\
&\tilde{\vec{j}}{}^{(c)}_{s\beta}(\vec{r}_s,s)\cdot\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s) = 0 \text{ pointwise orthogonality} \\
&\left\langle\tilde{\vec{j}}{}^{(c)}_{s\beta_1}(\vec{r}_s{}',s);\tilde{\vec{j}}{}^{(c)}_{s\beta_2}(\vec{r}_s{}',s)\right\rangle = \left\langle\overset{\leftrightarrow}{1}_S(\vec{r}_s{}')\times\tilde{\vec{j}}_{s\beta_1}(\vec{r}_s{}',s);\overset{\leftrightarrow}{1}_S(\vec{r}_s{}')\times\tilde{\vec{j}}_{s\beta_2}(\vec{r}_s{}',s)\right\rangle \\
&\qquad = \left\langle\tilde{\vec{j}}_{s\beta_1}(\vec{r}_s{}',s);\tilde{\vec{j}}_{s\beta_2}(\vec{r}_s{}',s)\right\rangle \\
&\qquad = 1_{\beta_1,\beta_2} \quad \text{(orthonormal)}
\end{aligned}
\tag{1.6}
$$

Some auxiliary relations are [2]

$$
\begin{aligned}
&\tilde{Z}_\beta(s) = \left\langle\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s);\tilde{\overset{\leftrightarrow}{Z}}_t(\vec{r}_s,\vec{r}_s{}';s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s{}',s)\right\rangle \\
&\frac{d}{dS_s}\tilde{Z}_\beta(s) = \left\langle\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s);\frac{\partial}{\partial s}\tilde{\overset{\leftrightarrow}{Z}}_t(\vec{r}_s,\vec{r}_s{}';s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s{}',s)\right\rangle \\
&\tilde{Y}_\beta(s) = \left\langle\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s);\tilde{\overset{\leftrightarrow}{Y}}_t(\vec{r}_s,\vec{r}_s{}';s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s{}',s)\right\rangle \\
&\frac{d}{ds}\tilde{Y}_\beta(s) = \left\langle\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s);\frac{\partial}{\partial s}\tilde{\overset{\leftrightarrow}{Y}}_t(\vec{r}_s,\vec{r}_s{}';s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s{}',s)\right\rangle \\
&\left\langle\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s);\frac{\partial}{\partial s}\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s)\right\rangle = 0
\end{aligned}
\tag{1.7}
$$

II. Analytic Continuation from the Real Axis

In the integral equation we are dealing with quantities such as fields and currents which in time domain are real. As such in complex frequency (Laplace) domain they are conjugate symmetric, i.e.

$$\begin{aligned}
&\tilde{\vec{E}}(\vec{r}_s,s^*)=\tilde{\vec{E}}^*(\vec{r}_s,s) \quad , \quad \tilde{\vec{H}}(\vec{r}_s,s^*)=\tilde{\vec{H}}^*(\vec{r}_s,s)\\
&\tilde{\vec{J}}_S(\vec{r}_s,s^*)=\tilde{\vec{J}}_S^*(\vec{r}_s,s)\\
&\tilde{\tilde{Z}}_t(\vec{r}_s,\vec{r}_s\,';s^*)=\tilde{\tilde{Z}}_t^*(\vec{r}_s,\vec{r}_s\,';s)
\end{aligned} \tag{2.1}$$

for the various fields, etc. Then, in particular, for s real (i.e.$=\Omega$) these are real. Noting that the impedance kernel is symmetric, then on the Ω axis it is real and symmetric and therefore Hermitian (i.e. equals its conjugate transpose). The real symmetric NxN matrix approximating this is also Hermitian and has N orthogonal <u>real</u> eigenvectors whether or not the N <u>real</u> eigenvalues are all distinct or not [24]. For continuous operators the results carry through [27], so we have

$$\begin{aligned}
&s=\Omega \text{ (real)}\\
&\tilde{Z}_\beta(\Omega)= \text{ real scalar}\\
&\tilde{\vec{j}}_{s_\beta}(\vec{r}_s,\Omega)= \text{ real vector}\\
&\left\langle \tilde{\vec{j}}_{s_\beta}(\vec{r}_s,\Omega);\tilde{\vec{j}}_{s_\beta}(\vec{r}_s,\Omega)\right\rangle=\int_S\left|\tilde{\vec{j}}_{s_\beta}(\vec{r}_s,\Omega)\right|^2 dS=1
\end{aligned} \tag{2.2}$$

with the eigenmodes forming a complete orthonormal set on S.

As a passive object the eigenimpedances and eigenadmittances are positive-real (p.r.) functions, i.e.

$$\mathrm{Re}\left[\tilde{Z}_\beta(s)\right]\geq 0 \text{ for } \mathrm{Re}[s]=\Omega\geq 0 \tag{2.3}$$

where limiting cases of zero and infinity are possible on the $j\omega$ axis. Specifically the $\tilde{Z}_\beta(s)$ have no poles in the right half plane (RHP) excluding the $j\omega$ axis. Furthermore, the $\tilde{Y}_\beta$ are bounded in the RHP (since they are p.r. and we do not allow identically zero $\tilde{Z}_\beta(s)$) so the $\tilde{Z}_\beta(s)$ have no zeros in the RHP so we have

$$\mathrm{Re}\left[\tilde{Z}_\beta(s)\right]> 0 \text{ for } \mathrm{Re}[s]=\Omega> 0 \tag{2.4}$$

Note specifically on the real axis the impedance kernel is positive definite since

$$\tilde{Z}_\beta(\Omega)>0 \text{ for } \Omega>0 \tag{2.5}$$

From (1.7) the derivative of $\tilde{Z}_\beta(\Omega)$ on the real axis is real and bounded, this derivative applying in an analytic-function sense (i.e. independent of direction of ds).

As illustrated in fig. 2.1 let us analytically continue the eigenimpedances from some point on the positive Ω axis to some more general point s, say in the first or second quadrant via some path P_1, avoiding any singularities. Then repeat the process to s* by a path P_2 which is the reflection of P_1 through the Ω axis which we can indicate symbolically as

$$P_2 = P_1^* \tag{2.6}$$

i.e. conjugate symmetric paths. Following the parameters along the two paths allows us to define β such that

$$\begin{aligned} \tilde{Z}_\beta(s^*) &= \tilde{Z}_\beta^*(s) \\ \tilde{\vec{j}}_{s\beta}(\vec{r}_s, s^*) &= \tilde{\vec{j}}_{s\beta}^*(\vec{r}_s, s) \end{aligned} \tag{2.7}$$

Thus not only for each eigenparameter is there a conjugate parameter at s*, this is the conjugate of the same parameter (same β). If one is at some starting point on the Ω axis where say two eigenimpedances are equal, say for β_1 and β_2, then there is the question of degeneracy, i.e. how to properly separate the two orthogonal modes. In such a case move the starting point slightly along the Ω axis where this degeneracy no longer exists, uniquely identify the two (real) modes before analytic continuation.

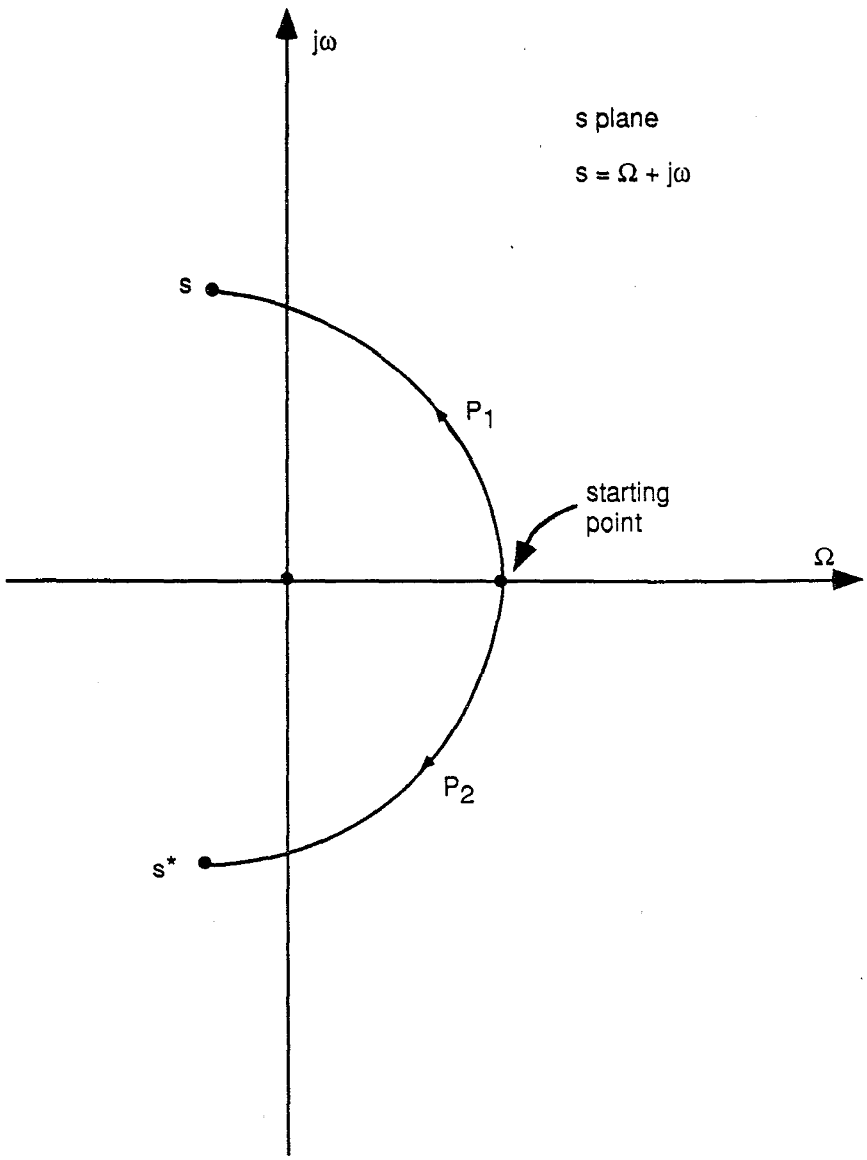

Fig. 2.1. Conjugate Paths from Positive Ω Axis

III. Boundary Conditions

In deriving the impedance integral equation we consider some scattered electric field

$$\tilde{\vec{E}}^{(sc)}(\vec{r},s) = -\left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}_S';s);\tilde{\vec{J}}_S(\vec{r}_S',s)\right\rangle \quad \vec{r} \notin S \tag{3.1}$$

which exists in both V_{ex} and V_{in}. The associated scattered magnetic field $\tilde{\vec{H}}^{(sc)}(\vec{r},s)$ related by the Maxwell equations, can also be found by an integral over the induced surface current density $\tilde{\vec{J}}_s(\vec{r}_S',s)$ using a different kernel related to the curl of the above kernel. Given some incident field we have the boundary condition on S that the tangential components of the total electric field are zero, i.e.

$$\vec{\vec{1}}_S(\vec{r}_S)\cdot\left[\tilde{\vec{E}}^{(inc)}(\vec{r}_S,s)+\tilde{\vec{E}}^{(sc)}(\vec{r}_S,s)\right]=\vec{0} \tag{3.2}$$

Taking the limit as $\vec{r}\to\vec{r}_s$ in (3.1) and applying (3.2) gives the impedance integral equation in (1.1). Note that taking tangential components makes the singularity in the kernel at $\vec{r}_S=\vec{r}_S'$ integrable [16].

Now the form of the incident field has not been specified. These come in two general forms

$$\begin{aligned}&\tilde{\vec{E}}^{(inc,ex)}(\vec{r},s)\,,\tilde{\vec{H}}^{(inc,ex)}(\vec{r},s) \Rightarrow \text{ external incidence, sources in } V_{ex} \text{ (including } \infty)\\ &\tilde{\vec{E}}^{(inc,in)}(\vec{r},s)\,,\tilde{\vec{H}}^{(inc,in)}(\vec{r},s) \Rightarrow \text{ internal incidence, sources in } V_{in}\end{aligned} \tag{3.3}$$

Sources on S are excluded from present consideration except in a limiting sense from a single specified side. In the case of external incidence we have

$$\left.\begin{aligned}\tilde{\vec{E}}^{(sc,in)}(\vec{r},s) &= -\tilde{\vec{E}}^{(inc,ex)}(\vec{r},s)\\ \tilde{\vec{H}}^{(sc,in)}(\vec{r},s) &= -\tilde{\vec{H}}^{(inc,ex)}(\vec{r},s)\end{aligned}\right\}\vec{r}\in V_{in} \tag{3.4}$$

with both satisfying the sourceless Maxwell equations. Note that for external incidence no power can penetrate through S to V_{in}. This is sometimes referred to an extended boundary condition [14,28]. Similarly for internal incidence no fields reach V_{ex} so we have

$$\left.\begin{aligned}\tilde{\vec{E}}^{(sc,ex)}(\vec{r},s) &= -\tilde{\vec{E}}^{(inc,in)}(\vec{r},s)\\ \tilde{\vec{H}}^{(sc,ex)}(\vec{r},s) &= -\tilde{\vec{H}}^{(inc,in)}(\vec{r},s)\end{aligned}\right\}\vec{r}\in V_{ex} \tag{3.5}$$

On S we also have the boundary (discontinuity) condition

$$\tilde{\vec{J}}_S(\vec{r}_S,s) = \vec{1}_S(\vec{r}_S) \times \left[\tilde{\vec{H}}^{(ex)}(\vec{r}_S,s) - \tilde{\vec{H}}^{(in)}(\vec{r}_S,s) \right] \tag{3.6}$$

where the exterior magnetic field is evaluated just outside S and the interior magnetic field just inside S in the limit $\vec{r} \to \vec{r}_S$. The incident field is continuous through S, so only the scattered field contributes to the surface current density as

$$\tilde{\vec{J}}_S(\vec{r}_S,s) = \vec{1}_S(\vec{r}_S) \times \left[\tilde{\vec{H}}^{(sc,ex)}(\vec{r}_S,s) - \tilde{\vec{H}}^{(sc,in)}(\vec{r}_S,s) \right] \tag{3.7}$$

This allows us to divide the surface current density into two parts as

$$\begin{aligned} \tilde{\vec{J}}_S(\vec{r}_S,s) &= \tilde{\vec{J}}_S^{(ex)}(\vec{r}_S,s) + \tilde{\vec{J}}_S^{(in)}(\vec{r}_S,s) \\ \tilde{\vec{J}}_S^{(ex)}(\vec{r}_S,s) &= \vec{1}_S(\vec{r}_S) \times \tilde{\vec{H}}^{(sc,ex)}(\vec{r}_S,s) \\ \tilde{\vec{J}}_S^{(in)}(\vec{r}_S,s) &= -\vec{1}_S(\vec{r}_S) \times \tilde{\vec{H}}^{(sc,in)}(\vec{r}_S,s) \end{aligned} \tag{3.8}$$

the sign reversal being accounted for by the convention of the unit surface normal for the closed surface pointing toward V_{ex} and away from V_{in}. Note that while the fields are considered scattered fields they can also be considered as radiated fields if one considers the surface current density as a distributed source on S. Such a source sends real power (for $s = j\omega$) into V_{ex} (radiation) and zero real power into V_{in} (lossless).

IV. Eigenadmittance Parts

With these preliminaries consider any incident field (external, internal, or a combination of both) and consider a single eigenterm of the impedance integral equation (by operating with $\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s)\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)$ as in the identity) as

$$
\begin{aligned}
&\tilde{Y}_\beta(s)\left\langle \tilde{\vec{E}}^{(inc)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s)=\left\langle \tilde{\vec{J}}_s(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s)\\
&\tilde{Y}_\beta(s)=\tilde{Z}_\beta^{-1}(s)=\frac{\left\langle \tilde{\vec{J}}_s(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle}{\left\langle \tilde{\vec{E}}^{(inc)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle}
\end{aligned}
\tag{4.1}
$$

Of course, our choice of the incident electric field should be one that is not orthogonal to the eigenmode of interest on S.

Now consider a case in which a wave is incident from V_{ex} (only). The internal scattered fields are the negative of the incident fields as in (3.4). Expand these internal fields on S via the identity (1.4) as

$$
\begin{aligned}
\vec{1}_S(\vec{r}_s)\cdot\tilde{\vec{E}}^{(sc,in)}(\vec{r}_s,s)&=-\vec{1}_S(\vec{r}_s)\cdot\tilde{\vec{E}}^{(inc,ex)}(\vec{r}_s,s)\\
&=-\sum_\beta\left\langle \tilde{\vec{E}}^{(inc,ex)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s)
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
\tilde{\vec{J}}_s^{(in)}(\vec{r}_s',s)&=-\vec{1}_S(\vec{r}_s)\times\tilde{\vec{H}}^{(sc,in)}(\vec{r}_s,s)=\vec{1}_S(\vec{r}_s)\times\tilde{\vec{H}}^{(inc,ex)}(\vec{r}_s,s)\\
&=\sum_\beta\left\langle \tilde{\vec{J}}^{(in)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s)\\
&=\sum_\beta\left\langle \vec{1}_S(\vec{r}_s')\times\tilde{\vec{H}}^{(inc,ex)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s)
\end{aligned}
$$

The expansion for the surface current density is equivalent to the expansion of the tangential incident or scattered magnetic field just inside S in terms of the complementary modes via the identity (1.5). This expansion is the same as originally used for the electric field and surface current density in the integral equation. Here we have identified the internal part of the surface current density.

Now as a gedankenexperiment let there be a single surface-current-density mode (βth) driven as a source. From the impedance integral equation this gives only one mode (the same β) for the tangential electric field on S, which we can consider as the scattered electric field in (4.2). In response to this "source" mode there is a unique distribution of surface magnetic field inside S which we take in the form of an internal surface current density as in (4.2). Identifying the βth mode as the internal part of $\tilde{\vec{J}}_s$ we can relate it to the scattered electric field by a coefficient $-\tilde{Y}_\beta^{(in)}(s)$ as

$$
\begin{aligned}
&\left\langle \tilde{\vec{J}}_s^{(in)}(\vec{r}_S{}',s); \tilde{\vec{j}}_{s\beta}(\vec{r}_S{}',s)\right\rangle = \left\langle \vec{1}_S(\vec{r}_S{}') \times \tilde{\vec{H}}^{(inc,ex)}(\vec{r}_S{}',s); \tilde{\vec{j}}_{s\beta}(\vec{r}_S{}',s)\right\rangle \\
&= -\tilde{Y}_\beta^{(in)}(s)\left\langle \tilde{\vec{E}}^{(sc,in)}(\vec{r}_S{}',s); \tilde{\vec{j}}_{s\beta}(\vec{r}_S{}',s)\right\rangle = \tilde{Y}_\beta^{(in)}(s)\left\langle \tilde{\vec{E}}^{(inc,ex)}(\vec{r}_S{}',s); \tilde{\vec{j}}_{s\beta}(\vec{r}_S{}',s)\right\rangle \\
&\tilde{Y}_\beta^{(in)}(s) \equiv \text{internal admittance} \\
&\qquad = Z_o^{-1}\tilde{y}_\beta^{(in)}(s) \\
&\tilde{y}_\beta^{(in)}(s) \equiv \text{normalized internal admittance}
\end{aligned}
\tag{4.3}
$$

With a general incident field (from V_{ex} or V_{in} or combination of both) we have

$$
\begin{aligned}
&\vec{1}_S(\vec{r}_s)\cdot\left[\tilde{\vec{E}}^{(sc,in)}(\vec{r}_s,s) + \tilde{\vec{E}}^{(inc)}(\vec{r}_s,s)\right] = \vec{0} \\
&\left\langle \tilde{\vec{J}}_s^{(in)}(\vec{r}_S{}',s); \tilde{\vec{j}}_{s\beta}(\vec{r}_S{}',s)\right\rangle = \tilde{Y}_\beta^{(in)}(s)\left\langle \tilde{\vec{E}}^{(inc)}(\vec{r}_S{}',s); \tilde{\vec{j}}_{s\beta}(\vec{r}_S{}',s)\right\rangle
\end{aligned}
\tag{4.4}
$$

The scattered field (including $\tilde{\vec{J}}_s^{(in)}$) is proportional to the strength of the assumed source surface-current-density mode $\tilde{\vec{j}}_{s\beta}$, i.e. with a coefficient $\left\langle \tilde{\vec{E}}^{(inc)}(\vec{r}_S{}',s); \tilde{\vec{j}}_{s\beta}(\vec{r}_S{}',s)\right\rangle$ which is just a complex number. The special choice of an external incident field is just to give a condition in which the extended boundary condition can be applied for relating the incident field to the interior surface current density (defined by interior scattered magnetic field).

The proportionality constant (constant with respect to space) is chosen in this form because it represents an admittance. It has various equivalent representations. From (4.3) we have

$$\tilde{y}_\beta^{(in)}(s) = \frac{Z_o \left\langle \vec{1}_S(\vec{r}_s') \times \tilde{\vec{H}}^{(inc,ex)}(\vec{r}_s', s) ; \tilde{\vec{j}}_{s_\beta}(\vec{r}_s', s) \right\rangle}{\left\langle \tilde{\vec{E}}^{(inc,ex)}(\vec{r}_s', s) ; \tilde{\vec{j}}_{s_\beta}(\vec{r}_s', s) \right\rangle} = -\frac{Z_o \left\langle \tilde{\vec{H}}^{(inc,ex)}(\vec{r}_s', s) ; \tilde{\vec{j}}_{s_\beta}^{(c)}(\vec{r}_s', s) \right\rangle}{\left\langle \tilde{\vec{E}}^{(inc,ex)}(\vec{r}_s', s) ; \tilde{\vec{j}}_{s_\beta}(\vec{r}_s', s) \right\rangle} \tag{4.5}$$

where the complementary mode enters in weighting the magnetic field. Note that this formula applies for an arbitrary external incident field (electric and magnetic), as long as it is not orthogonal to the mode. The weighting in effect converts the assumed incident field to one which is proportional to the eigenmode on S, and then takes the ratio of surface current density to incident electric field.

Now a dual of an incident field is also an incident field. The sources of this field are also dual (e.g. magnetic currents as dual of electric currents) but these are away from S and do not enter into the formulas. So let the external incident fields in (4.4) be chosen as dual fields giving

$$\tilde{y}_\beta^{(in)}(s) = -\frac{Z_o \left\langle \tilde{\vec{H}}^{(inc,ex,d)}(\vec{r}_s', s) ; \tilde{\vec{j}}_{s_\beta}^{(c)}(\vec{r}_s', s) \right\rangle}{\left\langle \tilde{\vec{E}}^{(inc,ex,d)}(\vec{r}_s', s) ; \tilde{\vec{j}}_{s_\beta}(\vec{r}_s', s) \right\rangle} \tag{4.6}$$

Now use the transformation formulas relating fields and their duals (Appendix A) to find

$$\tilde{y}_\beta^{(in)}(s) = \frac{\left\langle \tilde{\vec{E}}^{(inc,ex)}(\vec{r}_s', s) ; \tilde{\vec{j}}_{s_\beta}(\vec{r}_s', s) \right\rangle}{Z_o \left\langle \tilde{\vec{H}}^{(inc,ex)}(\vec{r}_s', s) ; \tilde{\vec{j}}_{s_\beta}^{(c)}(\vec{r}_s', s) \right\rangle} \tag{4.7}$$

as an alternate representation reversing the roles of the incident electric and magnetic fields. Noting that this internal admittance can be computed using any external incident wave consider a uniform TEM plane wave in which case (4.5) is transformed into (4.7) by a rotation of the polarization around the direction of incidence by an angle of $\pi/2$.

Similarly let there be a wave incident only from V_{in} where there are electric and/or magnetic sources to produce this wave. Now expand the external fields analogous to (4.2) as

$$
\begin{aligned}
\overset{\rightarrow}{1}_S(\vec{r}_s)\cdot\tilde{\vec{E}}^{(sc,ex)}(\vec{r}_s,s) &= -\overset{\rightarrow}{1}_S(\vec{r}_s)\cdot\tilde{\vec{E}}^{(inc,in)}(\vec{r}_s',s) \\
&= -\sum_\beta\left\langle\tilde{\vec{E}}^{(inc,in)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s)
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
\tilde{\vec{J}}_s^{(ex)}(\vec{r}_s,s) = \overset{\rightarrow}{1}_S(\vec{r}_s)\times\tilde{\vec{H}}^{(sc,ex)}(\vec{r}_s,s) &= -\overset{\rightarrow}{1}_S(\vec{r}_s)\times\tilde{\vec{H}}^{(inc,in)}(\vec{r}_s,s) \\
&= \sum_\beta\left\langle\tilde{\vec{J}}_s^{(ex)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s) \\
&= -\sum_\beta\left\langle\overset{\rightarrow}{1}_S(\vec{r}_s')\times\tilde{\vec{H}}^{(inc,in)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s)
\end{aligned}
$$

Note the change of sign on the magnetic field due to the orientation of the unit surface normal.

Again choose the surface current density as a source proportional to the βth eigenmode, giving thereby the scattered electric field as having only this mode in (4.7). Identifying the βth mode as the external part of $\tilde{\vec{J}}_s$, relate it to the scattered electric field by a coefficient $-\tilde{Y}_\beta^{(ex)}(s)$ as

$$
\begin{aligned}
&\left\langle\tilde{\vec{J}}_s^{(ex)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle = -\left\langle\overset{\rightarrow}{1}_S(\vec{r}_s')\times\tilde{\vec{H}}^{(inc,in)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle \\
&= -\tilde{Y}_\beta^{(ex)}(s)\left\langle\tilde{\vec{E}}^{(sc,ex)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle = \tilde{Y}_\beta^{(ex)}(s)\left\langle\tilde{\vec{E}}^{(inc,in)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle
\end{aligned}
\tag{4.9}
$$

$\tilde{Y}_\beta^{(ex)}(s) \equiv$ external admittance $= Z_o\tilde{y}_\beta^{(ex)}(s)$

$\tilde{y}_\beta^{(ex)}(s) \equiv$ normalized external admittance

$$
\begin{aligned}
&= -\frac{Z_o\left\langle\overset{\rightarrow}{1}_S(\vec{r}_s')\times\tilde{\vec{H}}^{(inc,in)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle}{\left\langle\tilde{\vec{E}}^{(inc,in)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle} \\
&= \frac{Z_o\left\langle\tilde{\vec{H}}^{(inc,in)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}^{(c)}(\vec{r}_s',s)\right\rangle}{\left\langle\tilde{\vec{E}}^{(inc,in)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle}
\end{aligned}
$$

Note the similarity of the external eigenadmittance to the internal eigenadmittance in (4.4) with a change of sign to account for the orientation of $\overleftrightarrow{1}_S(\vec{r}_s)$. With a general incident field (from V_{ex} or V_{in} or combination of both) we have

$$\overleftrightarrow{1}_S(\vec{r}_s)\cdot\left[\tilde{\vec{E}}^{(sc,ex)}(\vec{r}_s,s)+\tilde{\vec{E}}^{(inc)}(\vec{r}_s,s)\right]=\vec{0}$$
$$\left\langle\tilde{\vec{J}}_s^{(ex)}(\vec{r}_s{}',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s{}',s)\right\rangle=\tilde{Y}_\beta^{(ex)}(s)\left\langle\tilde{\vec{E}}^{(inc)}(\vec{r}_s{}',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s{}',s)\right\rangle \tag{4.10}$$

Again the scattered field (including $\tilde{\vec{J}}_s{}^{(ex)}$) is proportional to the strength of the assumed source surface-current-density mode $\tilde{\vec{j}}_{s\beta}$. The special choice of an internal incident field is again to allow the imposition of an extended boundary condition for relating the incident field to the exterior surface current density (defined by the exterior scattered magnetic field).

Again a dual incident field is as good as any other, so we have

$$\tilde{y}_\beta^{(ex)}(s)=\frac{Z_o\left\langle\tilde{\vec{H}}^{(inc,in,d)}(\vec{r}_s{}',s);\tilde{\vec{j}}_{s\beta}^{(c)}(\vec{r}_s{}',s)\right\rangle}{\left\langle\tilde{\vec{E}}^{(inc,in,d)}(\vec{r}_s{}',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s{}',s)\right\rangle} \tag{4.11}$$

Transforming these to the "unduals" we find

$$\tilde{y}_\beta^{(ex)}(s)=-\frac{\left\langle\tilde{\vec{E}}^{(inc,ex)}(\vec{r}_s{}',s);\tilde{\vec{j}}_{s\beta}^{(c)}(\vec{r}_s{}',s)\right\rangle}{Z_o\left\langle\tilde{\vec{H}}^{(inc,ex)}(\vec{r}_s{}',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s{}',s)\right\rangle} \tag{4.12}$$

Basically any internal incident field, such as from an electric or magnetic dipole, will work in (4.9) and (4.12) as long as the integrals do not give zero.

Now put the external and internal parts back together. Between (4.2) and (4.7) we have the complete expansion for the tangential electric field and surface current density on S. From (3.8) and (4.1) the coefficient of the βth mode is written as

$$\tilde{Y}_\beta\left\langle\tilde{\vec{E}}^{(inc)}(\vec{r}_s{}',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s{}',s)\right\rangle=\left\langle\tilde{\vec{J}}_s(\vec{r}_s{}',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s{}',s)\right\rangle$$
$$=\left\langle\tilde{\vec{J}}_s^{(ex)}(\vec{r}_s{}',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s{}',s)\right\rangle+\left\langle\tilde{\vec{J}}_s^{(in)}(\vec{r}_s{}',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s{}',s)\right\rangle \tag{4.13}$$

Now the external and internal surface current densities are associated with the scattered magnetic field, or equivalently the fields radiated by $\tilde{\vec{j}}_{s\beta}(\vec{r}_s, s)$ as a source. Look at this as a source with a coefficient $\left\langle \tilde{\vec{E}}^{(inc)}(\vec{r}_s\,', s); \tilde{\vec{j}}_{s\beta}(\vec{r}_s\,', s) \right\rangle$. Then using (4.4) and (4.10) we have

$$
\begin{aligned}
&\tilde{Y}_\beta \left\langle \tilde{\vec{E}}^{(inc)}(\vec{r}_s\,', s); \tilde{\vec{j}}_{s\beta}(\vec{r}_s\,', s) \right\rangle \\
&\quad = \tilde{Y}_\beta^{(ex)}(s) \left\langle \tilde{\vec{E}}^{(inc)}(\vec{r}_s\,', s); \tilde{\vec{j}}_{s\beta}(\vec{r}_s\,', s) \right\rangle + \tilde{Y}_\beta^{(in)} \left\langle \tilde{\vec{E}}^{(inc)}(\vec{r}_s\,', s); \tilde{\vec{j}}_{s\beta}(\vec{r}_s\,', s) \right\rangle \qquad (4.14) \\
&\tilde{Y}_\beta(s) = \tilde{Y}_\beta^{(ex)}(s) + \tilde{Y}_\beta^{(in)}(s)
\end{aligned}
$$

Knowing the modes one can now solve for not only the $\tilde{Y}_\beta$ from the impedance integral equation (1.1), but also the external part from (4.9) and/or (4.12) and the internal part from (4.5) and/or (4.7).

V. Mode Paring

Recall from (1.5) that the complementary modes also can be used to represent tangential fields on S. However, we do not yet know if they diagonalize the impedance kernel. Consider some surface-current-density complementary mode as

$$\tilde{\vec{J}}_{s_\beta}^{(c)}(\vec{r}_s,s) = \tilde{I}_\beta^{(c)}(s)\tilde{\vec{j}}_{s_\beta}^{(c)}(\vec{r}_s,s) = \tilde{I}_\beta^{(c)}(s)\vec{1}_S(\vec{r}_s)\times\tilde{\vec{j}}_{s_\beta}(\vec{r}_s,s) \tag{5.1}$$

with a coefficient $\tilde{I}_\beta^{(c)}(s)$ (units As). In order to satisfy the Maxwell equations and boundary conditions let us consider certain appropriate interior and exterior fields, similar to the approach in section IV. Note that if some electric and magnetic fields satisfy the Maxwell equations, then the dual fields (away from sources) necessarily satisfy the Maxwell equations, and conversely. On S, however we will still require that the surface current density be electric (no actual magnetic currents).

Considering first the exterior scattering problem let there be some external incident field which we take as a dual field. In V_{in} the extended boundary condition gives

$$\begin{aligned}\tilde{\vec{E}}^{(sc,in,d)}(\vec{r},s) &= -\tilde{\vec{E}}^{(inc,ex,d)}(\vec{r},s)\\ \tilde{\vec{H}}^{(sc,in,d)}(\vec{r},s) &= -\tilde{\vec{H}}^{(inc,ex,d)}(\vec{r},s)\end{aligned} \tag{5.2}$$

Now expand these fields on S (just inside S) in terms of the eigenmodes as (βth term)

$$\begin{aligned}\tilde{\vec{E}}_\beta^{(sc,in,d)}(\vec{r}_s,s) &\equiv \left\langle \tilde{\vec{E}}^{(sc,in,d)}(\vec{r}_s{}',s);\tilde{\vec{j}}_{s_\beta}(\vec{r}_s{}',s)\right\rangle \tilde{\vec{j}}_{s_\beta}(\vec{r}_s,s)\\ \tilde{\vec{J}}_{s_\beta}^{(in,d)}(\vec{r}_s,s) &\equiv -\left\langle \vec{1}_S(\vec{r}_s{}')\times\tilde{\vec{H}}^{(sc,in,d)}(\vec{r}_s{}',s);\tilde{\vec{j}}_{s_\beta}(\vec{r}_s{}',s)\right\rangle \tilde{\vec{j}}_{s_\beta}(\vec{r}_s,s)\\ \tilde{\vec{H}}^{(sc,in,d)}(\vec{r}_s,s) &\equiv -\left\langle \vec{1}_S(\vec{r}_s{}')\times\tilde{\vec{H}}^{(sc,in,d)}(\vec{r}_s{}',s);\tilde{\vec{j}}_{s_\beta}(\vec{r}_s{}',s)\right\rangle \vec{1}_S(\vec{r}_s)\times\tilde{\vec{j}}_{s_\beta}(\vec{r}_s,s)\end{aligned} \tag{5.3}$$

Here we keep the actual distributions (modes) on S together with the coefficients. Note that the dual electric field and dual interior surface current density are expanded in terms of the $\tilde{\vec{j}}_{s_\beta}$ modes, making the dual magnetic field take the form of the complementary modes $\tilde{\vec{j}}_{s_\beta}^{(c)}$.

Now transform the dual electric and magnetic fields back (to the "unduals") giving

$$\tilde{\bar{H}}_{\beta}^{(sc,in)}(\vec{r}_s,s) \equiv \left\langle \tilde{\bar{H}}^{(sc,in)}(\vec{r}_s{}',s);\tilde{\bar{j}}_{s\beta}(\vec{r}_s{}',s)\right\rangle \tilde{\bar{j}}_{s\beta}^{(c)}(\vec{r}_s,s)$$

$$\tilde{\bar{E}}^{(sc,in)}(\vec{r}_s,s) = -\left\langle \bar{1}_S(\vec{r}_s{}')\times\tilde{\bar{E}}^{(sc,in)}(\vec{r}_s{}',s);\tilde{\bar{j}}_{s\beta}(\vec{r}_s{}',s)\right\rangle \bar{1}_S(\vec{r}_s)\times\tilde{\bar{j}}_{s\beta}(\vec{r}_s,s) \tag{5.4}$$

The internal part of the associated surface current density is just

$$\begin{aligned}\tilde{\bar{J}}_{s\beta}^{(in)}(\vec{r}_s,s) &\equiv -\bar{1}_S(\vec{r}_s)\times\tilde{\bar{H}}^{(sc,in)}(\vec{r}_s,s)\\ &= -\left\langle \tilde{\bar{H}}^{(sc,in)}(\vec{r}_s{}',s);\tilde{\bar{j}}_{s\beta}(\vec{r}_s{}',s)\right\rangle \bar{1}_S(\vec{r}_s)\times\tilde{\bar{j}}_{s\beta}(\vec{r}_s,s)\end{aligned} \tag{5.5}$$

Now that the electric field and internal surface current density are both expanded in terms of complementary modes, let us replace the scattered fields by external incident fields giving

$$\begin{aligned}\tilde{\bar{E}}_{\beta}^{(inc,ex)}(\vec{r}_s,s) &\equiv -\left\langle \bar{1}_S(\vec{r}_s{}')\times\tilde{\bar{E}}^{(inc,ex)}(\vec{r}_s{}',s);\tilde{\bar{j}}_{s\beta}(\vec{r}_s{}',s)\right\rangle \tilde{\bar{j}}_{s}^{(c)}(\vec{r}_s,s)\\ &= \left\langle \tilde{\bar{E}}^{(inc,ex)}(\vec{r}_s{}',s);\tilde{\bar{j}}_{s\beta}^{(c)}(\vec{r}_s{}',s)\right\rangle \tilde{\bar{j}}_{s}^{(c)}(\vec{r}_s,s)\\ \tilde{\bar{J}}_{s\beta}^{(in)}(\vec{r}_s,s) &= \left\langle \tilde{\bar{H}}^{(inc,ex)}(\vec{r}_s{}',s);\tilde{\bar{j}}_{s\beta}(\vec{r}_s{}',s)\right\rangle \tilde{\bar{j}}_{s\beta}^{(c)}(\vec{r}_s,s)\end{aligned} \tag{5.6}$$

Relating these defines what we might call the complementary internal admittance (βth) as

$$\begin{aligned}\tilde{\bar{J}}_{s\beta}^{(in)}(\vec{r}_s,s) &\equiv -\tilde{Y}_{\beta}^{(in,c)}(s)\tilde{\bar{E}}^{(in,sc)}(\vec{r}_s,s) = \tilde{Y}_{\beta}^{(in,c)}(s)\tilde{\bar{E}}^{(inc,ex)}(\vec{r}_s,s)\\ \tilde{Y}_{\beta}^{(in,c)}(s) &\equiv Z_o^{-1}\tilde{y}_{\beta}^{(in,c)} \equiv \text{complementary internal admittance}\\ \tilde{y}_{\beta}^{(in,c)}(s) &= \frac{Z_o\left\langle \tilde{\bar{H}}^{(inc,ex)}(\vec{r}_s{}',s);\tilde{\bar{j}}_{s\beta}(\vec{r}_s{}',s)\right\rangle}{\left\langle \tilde{\bar{E}}^{(inc,ex)}(\vec{r}_s{}',s);\tilde{\bar{j}}_{s\beta}^{(c)}(\vec{r}_s{}',s)\right\rangle}\\ &= \tilde{y}_{\beta}^{(in)^{-1}}(s)\end{aligned} \tag{5.7}$$

where the relation to internal admittance comes from (4.7). Analogous to (4.4) let us have an arbitrary incident field (from V_{ex} or V_{in} or a combination of both) giving in terms of complementary modes

$$\left\langle \tilde{\vec{J}}_s^{(in)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}^{(c)}(\vec{r}_s',s)\right\rangle \tilde{\vec{j}}_{s\beta}^{(c)}(\vec{r}_s,s) = -\tilde{Y}_\beta^{(in,c)}(s)\left\langle \tilde{\vec{E}}^{(sc,in)}(\vec{r}_s,s);\tilde{\vec{j}}_{s\beta}^{(c)}(\vec{r}_s',s)\right\rangle \tilde{\vec{j}}_{s\beta}^{(c)}(\vec{r}_s,s)$$
$$= \tilde{Y}_\beta^{(in,c)}(s)\left\langle \tilde{\vec{E}}^{(inc)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}^{(c)}(\vec{r}_s',s)\right\rangle \tilde{\vec{j}}_{s\beta}^{(c)}(\vec{r}_s,s) \tag{5.8}$$

The previous use of an external incident field was merely to determine an admittance using the extended boundary condition. A general incident electric field still produces a scattered electric field via tangential field on S.

Considering second the interior scattering problem begin with the dual fields associated with internal incidence as

$$\tilde{\vec{E}}^{(sc,ex,d)}(\vec{r}_s,s) = -\tilde{\vec{E}}^{(inc,in,d)}(\vec{r}_s,s)$$
$$\tilde{\vec{H}}^{(sc,ex,d)}(\vec{r}_s,s) = -\tilde{\vec{H}}^{(inc,in,d)}(\vec{r}_s,s) \tag{5.9}$$

Expanding these on S in terms of the eigenmodes gives the βth term as

$$\tilde{\vec{E}}_\beta^{(sc,ex,d)}(\vec{r}_s,s) \equiv \left\langle \tilde{\vec{E}}^{(sc,ex,d)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle \tilde{\vec{j}}_{s\beta}(\vec{r}_s,s)$$
$$\tilde{\vec{J}}^{(ex,d)}(\vec{r}_s,s) \equiv \left\langle \vec{1}_S(\vec{r}_s')\times\tilde{\vec{H}}^{(sc,ex,d)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle \tilde{\vec{j}}_{s\beta}(\vec{r}_s,s) \tag{5.10}$$
$$\tilde{\vec{H}}_\beta^{(sc,ex,d)}(\vec{r}_s,s) \equiv -\left\langle \vec{1}_S(\vec{r}_s')\times\tilde{\vec{H}}^{(sc,ex,d)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle \vec{1}_S(\vec{r}_s)\times\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s)$$

Again the dual magnetic field ends up expanded in terms of the complementary modes $\tilde{\vec{j}}_{s\beta}^{(c)}$ because the dual interior surface current density is expanded in terms of the $\tilde{\vec{j}}_{s\beta}$ modes.

Transforming the dual surface fields back (to the "unduals") gives

$$\tilde{\vec{H}}_\beta^{(sc,ex)}(\vec{r}_s,s) \equiv \left\langle \tilde{\vec{H}}^{(sc,ex)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle \tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)$$
$$\tilde{\vec{E}}_\beta^{(sc,ex)}(\vec{r}_s,s) = -\left\langle \vec{1}_S(\vec{r}_s')\times\tilde{\vec{E}}^{(sc,ex)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle \vec{1}_S(\vec{r}_s)\times\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s) \tag{5.11}$$

while the external part of the associated surface current density is just

$$
\begin{aligned}
\tilde{\vec{j}}_{s\beta}^{(ex)}(\vec{r}_s,s) &\equiv \vec{1}_S(\vec{r}_s)\times\tilde{\vec{H}}^{(sc,ex)}(\vec{r}_s',s) \\
&= \left\langle \tilde{\vec{H}}^{(sc,ex)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle \vec{1}_S(\vec{r}_s)\times\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s)
\end{aligned}
\tag{5.12}
$$

Replacing scattered fields by internal incident fields gives

$$
\begin{aligned}
\tilde{\vec{E}}_\beta^{(inc,in)}(\vec{r}_s',s) &= -\left\langle \vec{1}_S(\vec{r}_s)\times\tilde{\vec{E}}^{(inc,in)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle \vec{1}_S(\vec{r}_s)\times\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s) \\
&= \left\langle \tilde{\vec{E}}^{(inc,in)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}^{(c)}(\vec{r}_s',s)\right\rangle \tilde{\vec{j}}_{s\beta}^{(c)}(\vec{r}_s,s) \\
\tilde{\vec{j}}_{s\beta}^{(in)}(\vec{r}_s,s) &= -\left\langle \tilde{\vec{H}}^{(inc,in)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle \tilde{\vec{j}}_{s\beta}^{(c)}(\vec{r}_s,s)
\end{aligned}
\tag{5.13}
$$

Relating these gives the complementary internal admittance βth as

$$
\begin{aligned}
&\tilde{\vec{j}}_{s\beta}^{(ex)}(\vec{r}_s,s) \equiv -\tilde{Y}_\beta^{(ex,c)}(s)\tilde{\vec{E}}_\beta^{(ex,sc)}(\vec{r}_s,s) = Y_\beta^{(ex,c)}(s)\tilde{\vec{E}}_\beta^{(inc,ex)}(\vec{r}_s,s) \\
&\tilde{Y}_\beta^{(ex,c)}(s) \equiv Z_o^{-1}\tilde{y}_\beta^{(ex,c)}(s) \equiv \text{ complementary external admittance} \\
&\tilde{y}_\beta^{(ex,c)} = -\frac{Z_o\left\langle \tilde{\vec{H}}^{(inc,in)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle}{\left\langle \tilde{\vec{E}}^{(inc,in)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}^{(c)}(\vec{r}_s',s)\right\rangle} \\
&\qquad = \tilde{y}_\beta^{(ex)^{-1}}(s)
\end{aligned}
\tag{5.14}
$$

where the relation to the external admittance comes from (4.12) Analogous to (4.10) now consider an arbitrary incident field giving in terms of complementary modes

$$
\begin{aligned}
\left\langle \tilde{\vec{J}}_s^{(ex)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}^{(c)}(\vec{r}_s',s)\right\rangle \tilde{\vec{j}}_{s\beta}^{(c)}(\vec{r}_s,s) &= -\tilde{Y}_\beta^{(ex,c)}(s)\left\langle \tilde{\vec{E}}^{(sc,ex)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}^{(c)}(\vec{r}_s',s)\right\rangle \tilde{\vec{j}}_{s\beta}^{(c)}(\vec{r}_s,s) \\
&= \tilde{Y}_\beta^{(ex,c)}(s)\left\langle \tilde{\vec{E}}^{(inc)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}^{(c)}(\vec{r}_s',s)\right\rangle \tilde{\vec{j}}_{s\beta}^{(c)}(\vec{r}_s,s)
\end{aligned}
\tag{5.15}
$$

The previous use of an internal incident field was merely for application of the extended boundary condition in computing an admittance. A general incident electric field still gives a scattered field via the boundary condition of zero tangential field on S.

Now combine the external and internal parts from (5.8) and (5.15) as

$$
\begin{aligned}
&\left[\tilde{Y}_{\beta}^{(ex,c)}(s)+\tilde{Y}_{\beta}^{(in,c)}(s)\right]\left\langle \tilde{\vec{\vec{E}}}^{(inc)}(\vec{r}_S',s);\vec{\vec{j}}_{s\beta}^{(c)}(\vec{r}_S',s)\right\rangle \vec{\vec{j}}_{s\beta}^{(c)}(\vec{r}_S,s) \\
&= \tilde{Y}_{\beta}^{(c)}(s)\left\langle \tilde{\vec{\vec{E}}}^{(inc)}(\vec{r}_S',s);\vec{\vec{j}}_{s\beta}^{(c)}(\vec{r}_S',s)\right\rangle \vec{\vec{j}}_{s\beta}^{(c)}(\vec{r}_S,s) \\
&= \left\langle \tilde{\vec{\vec{J}}}_s^{(ex)}(\vec{r}_S',s)+\tilde{\vec{\vec{J}}}_s^{(in)}(\vec{r}_S',s);\vec{\vec{j}}_{s\beta}^{(c)}(\vec{r}_S',s)\right\rangle \vec{\vec{j}}_{s\beta}^{(c)}(\vec{r}_S,s) \\
&= \left\langle \tilde{\vec{\vec{J}}}_s(\vec{r}_S',s);\vec{\vec{j}}_{s\beta}^{(c)}(\vec{r}_S',s)\right\rangle \vec{\vec{j}}_{s\beta}^{(c)}(\vec{r}_S,s)
\end{aligned}
\tag{5.16}
$$

where we have identified

$$
\begin{aligned}
&Y_{\beta}^{(c)} = Z_o^{-1}\tilde{y}_{\beta}^{(c)}(s) \equiv \tilde{Y}_{\beta}^{(ex,c)}(s)+\tilde{Y}_{\beta}^{(in,c)}(s) \equiv Z_{\beta}^{(c)^{-1}}(s) \\
&\tilde{y}_{\beta}^{(c)}(s) = \tilde{y}_{\beta}^{(ex,c)}(s)+\tilde{y}_{\beta}^{(in,c)}(s) = \tilde{y}_{\beta}^{(ex)^{-1}}(s) + \tilde{y}_{\beta}^{(in)^{-1}}(s)
\end{aligned}
\tag{5.17}
$$

So from (5.16) we see that the boundary conditions on S relate the tangential part of the incident electric field to the surface current density via the complementary modes. Summing over β and noting the identity (1.5) we have

$$
\begin{aligned}
\vec{\vec{1}}_S(\vec{r}_S)\cdot\tilde{\vec{\vec{E}}}^{(inc)}(\vec{r}_S,s) &= \sum_{\beta}\left\langle \tilde{\vec{\vec{E}}}^{(inc)}(\vec{r}_S',s);\vec{\vec{j}}_{s\beta}^{(c)}(\vec{r}_S',s)\right\rangle \vec{\vec{j}}_{s\beta}^{(c)}(\vec{r}_S,s) \\
&= \sum_{\beta}\tilde{Z}_{\beta}^{(c)}(s)\left\langle \tilde{\vec{\vec{J}}}_s(\vec{r}_S',s);\vec{\vec{j}}_{s\beta}^{(c)}(\vec{r}_S',s)\right\rangle \vec{\vec{j}}_{s\beta}^{(c)}(\vec{r}_S,s) \\
&= \left\langle \tilde{\vec{\vec{Z}}}_t(\vec{r}_S,\vec{r}_S';s);\tilde{\vec{\vec{J}}}_s(\vec{r}_S',s)\right\rangle
\end{aligned}
\tag{5.18}
$$

$$
\tilde{\vec{\vec{Z}}}_t(\vec{r}_S,\vec{r}_S';s) = \sum_{\beta}\tilde{Z}_{\beta}^{(c)}(s)\vec{\vec{j}}_{s\beta}^{(c)}(\vec{r}_S,s)\vec{\vec{j}}_{s\beta}^{(c)}(\vec{r}_S',s)
$$

$$
\tilde{\vec{\vec{Y}}}_t(\vec{r}_S,\vec{r}_S';s) = \sum_{\beta}\tilde{Y}_{\beta}^{(c)}(s)\vec{\vec{j}}_{c\beta}^{(c)}(\vec{r}_S,s)\vec{\vec{j}}_{c\beta}^{(c)}(\vec{r}_S',s)
$$

So now we have a diagonalization of the impedance integral equation in terms of complementary eigenimpedances and complementary eigenmodes.

For simplicity consider the case of distinct eigenvalues so that we can easily compare the two diagonalizations. Consider a surface current density with a spatial distribution of one of the original eigenmodes $\tilde{\vec{j}}_{s_\beta}$; then the associated electric field (tangential part) is proportional to the same mode, i.e.

$$
\begin{aligned}
&\tilde{\vec{J}}_s(\vec{r}_s,s) = \tilde{I}_s(s)\tilde{\vec{j}}_{s_\beta}(\vec{r}_s,s) \\
&\overleftrightarrow{1}_S(\vec{r}_s)\cdot\tilde{\vec{E}}^{(inc)}(\vec{r}_s,s) = \tilde{Z}_\beta(s)\tilde{\vec{J}}_s(\vec{r}_s,s) = \tilde{Z}_\beta(s)\tilde{I}_s(s)\tilde{\vec{j}}_{s_\beta}(\vec{r}_s,s)
\end{aligned}
\tag{5.19}
$$

Using our new diagonal representation from (5.18) we have

$$
\begin{aligned}
\overleftrightarrow{1}_S(\vec{r}_s)\cdot\tilde{\vec{E}}^{(inc)}(\vec{r}_s,s) &= \left\langle \tilde{\overleftrightarrow{Z}}_t(\vec{r}_s,\vec{r}_s{}',s);\tilde{\vec{J}}_s(\vec{r}_s{}',s)\right\rangle \\
&= \tilde{I}_s(s)\sum_{\beta'}\tilde{Z}^{(c)}_{\beta'}(s)\left\langle \tilde{\vec{j}}^{(c)}_{s_{\beta'}}(\vec{r}_s{}',s);\tilde{\vec{j}}_{s_\beta}(\vec{r}_s{}',s)\right\rangle\tilde{\vec{j}}^{(c)}_{s_{\beta'}}(\vec{r}_s,s) \\
&= \tilde{I}_s(s)\tilde{Z}_\beta(s)\tilde{\vec{j}}_{s_\beta}(\vec{r}_s,s)
\end{aligned}
\tag{5.20}
$$

For one β' operate (multiply and integrate over S) by $\tilde{\vec{j}}^{(c)}_{s_{\beta'}}$ giving

$$
\tilde{Z}^{(c)}_{\beta'}(s)\left\langle \tilde{\vec{j}}^{(c)}_{s_{\beta'}}(\vec{r}_s{}',s);\tilde{\vec{j}}_{s_\beta}(\vec{r}_s{}',s)\right\rangle = \tilde{Z}_\beta(s)\left\langle \tilde{\vec{j}}^{(c)}_{s_{\beta'}}(\vec{r}_s{}',s);\tilde{\vec{j}}_{s_\beta}(\vec{r}_s{}',s)\right\rangle
\tag{5.21}
$$

Assuming for some β' that

$$
\left\langle \tilde{\vec{j}}^{(c)}_{s_{\beta'}}(\vec{r}_s{}',s);\tilde{\vec{j}}_{s_\beta}(\vec{r}_s{}',s)\right\rangle \neq 0
\tag{5.22}
$$

then

$$
\tilde{Z}^{(c)}_{\beta'}(s) = \tilde{Z}_\beta(s)
\tag{5.23}
$$

and since the eigenvalue $\tilde{Z}^{(c)}_{\beta'}(s)$ is assumed distinct, i.e.

$$
\tilde{Z}^{(c)}_{\beta''}(s) \neq \tilde{Z}^{(c)}_{\beta'}(s) \text{ for each } \beta'' \neq \beta'
\tag{5.24}
$$

where we restrict our range of s for this to be correct (i.e. avoid points of eigenvalue degeneracy). Then we have

$$
\left\langle \tilde{\vec{j}}^{(c)}_{s_{\beta''}}(\vec{r}_s{}',s);\tilde{\vec{j}}_{s_\beta}(\vec{r}_s{}',s)\right\rangle = 0 \text{ for each } \beta'' \neq \beta'
\tag{5.25}
$$

i.e. $\tilde{\vec{j}}_{s_\beta}$ is orthogonal to all the $\tilde{\vec{j}}^{(c)}_{s_{\beta''}}$ except $\tilde{\vec{j}}^{(c)}_{s_\beta}$. Then $\tilde{\vec{j}}_{s_\beta}$ must be representable by $\tilde{\vec{j}}^{(c)}_{s_{\beta'}}$ alone, which allows us to identify

$$\tilde{\vec{j}}^{(c)}_{s_{\beta''}}(\vec{r}_s,s) = \pm\tilde{\vec{j}}_{s_\beta}(\vec{r}_s,s) \tag{5.26}$$

due to orthonormality.

Viewed another way the diagonalization of the impedance kernel is unique for distinct eigenvalues. A distinct eigenvalue in one representation must equal an eigenvalue in the other representation, and similarly for the corresponding eigenmodes. There is then a one-to-one pairwise correspondence between the terms in the two representations, i.e.

$$\tilde{Z}^{(c)}_{\beta'}(s)\tilde{\vec{j}}^{(c)}_{s_{\beta'}}(\vec{r}_s,s)\tilde{\vec{j}}^{(c)}_{s_{\beta'}}(\vec{r}_s{}',s) = \tilde{Z}_\beta(s)\tilde{\vec{j}}_{s_\beta}(\vec{r}_s,s)\tilde{\vec{j}}_{s_\beta}(\vec{r}_s{}',s) \tag{5.27}$$

Note also from (5.22) that $\beta' \neq \beta$. So now partition the β index into two parts which will be later associated with electric (e) and magnetic (h) eigenmodes with

$$\begin{aligned}
&\beta = \left({}^{e}_{h},n\right)\\
&\tilde{\vec{j}}_{s_{h,n}}(\vec{r}_s,s) \equiv \tilde{\vec{j}}^{(c)}_{s_{e,n}}(\vec{r}_s,s) = \vec{1}_S(\vec{r}_s)\times\tilde{\vec{j}}_{s_{e,n}}(\vec{r}_s,s)\\
&\tilde{\vec{j}}_{s_{e,n}}(\vec{r}_s,s) \equiv -\tilde{\vec{j}}^{(c)}_{s_{h,n}}(\vec{r}_s,s) = -\vec{1}_S(\vec{r}_s)\times\tilde{\vec{j}}_{s_{h,n}}(\vec{r}_s,s)\\
&\tilde{Z}_{h,n}(s) = \tilde{Z}^{(c)}_{e,n}(s) = Z_o\tilde{z}_{h,n}(s) = Z_o\tilde{y}^{-1}_{h,n}(s)\\
&\tilde{Z}_{e,n}(s) = \tilde{Z}^{(c)}_{h,n}(s) = Z_o\tilde{z}_{e,n}(s) = Z_o\tilde{y}^{-1}_{e,n}(s)\\[1ex]
&\tilde{y}_{e,n}(s) = \tilde{y}^{(ex)}_{e,n}(s) + \tilde{y}^{(in)}_{e,n}(s) = \tilde{y}^{(ex)^{-1}}_{h,n}(s) + \tilde{y}^{(in)^{-1}}_{h,n}(s)\\
&\tilde{y}_{h,n}(s) = \tilde{y}^{(ex)}_{h,n}(s) + \tilde{y}^{(in)}_{h,n}(s) = \tilde{y}^{(ex)^{-1}}_{e,n}(s) + \tilde{y}^{(in)^{-1}}_{e,n}(s)
\end{aligned} \tag{5.28}$$

One thus only needs half the modes and half the eigenadmittance parts, the remaining terms being expressed in terms of these.

VI. Interior Eigenadmittances: Reactance Functions

Consider the Poynting vector theorem in complex-frequency domain without sources in V_- [7]

$$\int_{S_-}\left[\tilde{\vec{E}}(\vec{r}_s,s)\times\tilde{\vec{H}}(\vec{r}_s,-s)\right]\cdot \vec{1}_S(\vec{r}_s)dS$$
$$=\int_{V_-}\left[\varepsilon_o s\tilde{\vec{E}}(\vec{r},s)\cdot\tilde{\vec{E}}(\vec{r},-s)-\mu_o s\tilde{\vec{H}}(\vec{r},s)\cdot\tilde{\vec{H}}(\vec{r},-s)\right]dV \tag{6.1}$$

Here S_- is taken as just inside S with the limit taken as $S_-\to S$ for present purposes. Similarly V_- is the volume inside the closed surface S_-, and $V_-\to V_{in}$ in the limit. With no sources in V_- the volume integral in (6.1) is over terms which are manifestly odd functions of s. So we have for the interior problem

$$\int_{S_-}\left[\tilde{\vec{E}}(\vec{r}_s,s)\times\tilde{\vec{H}}(\vec{r}_s,-s)\right]\cdot\vec{1}_S(\vec{r}_s)dS$$
$$=-\int_{S_-}\tilde{\vec{E}}({}_s\vec{r},s)\cdot\left[\vec{1}_S(\vec{r}_s)\times\tilde{\vec{H}}(\vec{r},-s)\right]dS$$
$$=\int_{S_-}\tilde{\vec{E}}(\vec{r}_s,-s)\cdot\left[\vec{1}_S(\vec{r}_s)\times\tilde{\vec{H}}(\vec{r},s)\right]dS \tag{6.2}$$
$$=\text{ odd function of } s$$

If one sets $s=j\omega$ then conjugate symmetry replaces functions of -s by their conjugates and the above implies zero real power into V_- (lossless).

Applying this result to the case of an external incident field (no sources in V_{in}) we have

$$\left\langle\tilde{\vec{E}}^{(inc,ex)}(\vec{r}_s,-s);\left[\vec{1}_S(\vec{r}_s)\times\tilde{\vec{H}}^{(inc,ex)}(\vec{r}_s,s)\right]\right\rangle$$
$$=-\left\langle\tilde{\vec{E}}^{(inc,ex)}(\vec{r}_s,-s);\left[\vec{1}_S(\vec{r}_s)\times\tilde{\vec{H}}^{(sc,ex)}(\vec{r}_s,s)\right]\right\rangle \tag{6.3}$$
$$=\left\langle\tilde{\vec{E}}^{(inc,ex)}(\vec{r}_s,-s);\tilde{\vec{J}}_s^{(in)}(\vec{r}_s,s)\right\rangle$$
$$=\text{ odd function of } s$$

Now using the representations of the identity

$$
\begin{aligned}
\overleftrightarrow{1}_S(\vec{r}_s)\delta_s(\vec{r}_s-\vec{r}_s') &= \sum_\beta \tilde{\vec{j}}_{s\beta}(\vec{r}_s,s)\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s) \\
&= \sum_\beta \tilde{\vec{j}}_{s\beta}(\vec{r}_s,-s)\tilde{\vec{j}}_{s\beta}(\vec{r}_s',-s)
\end{aligned}
\tag{6.4}
$$

we have

$$
\begin{aligned}
&\overleftrightarrow{1}_S(\vec{r}_s)\cdot\tilde{\vec{E}}^{(inc,ex)}(\vec{r}_s,-s) = \sum_\beta \left\langle \tilde{\vec{E}}^{(inc,ex)}(\vec{r}_s',-s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle \tilde{\vec{j}}_{s\beta}(\vec{r}_s,s) \\
&\tilde{\vec{J}}_s^{(in)}(\vec{r}_s,s) = \sum_\beta \left\langle \overleftrightarrow{1}_S(\vec{r}_s)\times\tilde{\vec{H}}^{(inc,ex)}(\vec{r}_s,s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle \tilde{\vec{j}}_{s\beta}(\vec{r}_s,s) \\
&\left\langle \tilde{\vec{E}}^{(inc,ex)}(\vec{r}_s',-s);\tilde{\vec{J}}_s^{(in)}(\vec{r}_s',s)\right\rangle = \sum_\beta \left\langle \tilde{\vec{E}}^{(inc,ex)}(\vec{r}_s',-s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle \left\langle \overleftrightarrow{1}_S(\vec{r}_s')\times\tilde{\vec{H}}^{(inc,ex)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle \\
&= \text{ odd function of } s
\end{aligned}
\tag{6.5}
$$

So let us consider that the external electric field is chosen to take the particular form

$$
\begin{aligned}
&\tilde{\vec{E}}^{(inc,ex)}(\vec{r}_s,s) \equiv \tilde{\vec{E}}_\beta(\vec{r}_s,s) = \tilde{E}_\beta(s)\tilde{\vec{j}}_{s\beta}(\vec{r}_s',-s) \\
&\tilde{E}_\beta(s) = \left\langle \tilde{\vec{E}}^{(inc,ex)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',-s)\right\rangle
\end{aligned}
\tag{6.6}
$$

i.e. with spatial form of the eigenmode of -s. With this choice (6.5) becomes

$$
\begin{aligned}
&\left\langle \tilde{\vec{E}}_\beta^{(inc,ex)}(\vec{r}_s',-s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle \left\langle \overleftrightarrow{1}_S(\vec{r}_s')\times\tilde{\vec{H}}^{(inc,ex)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle \\
&= \tilde{E}_\beta(-s)\left\langle \overleftrightarrow{1}_S(\vec{r}_s')\times\tilde{\vec{H}}^{(inc,ex)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle \\
&= \tilde{E}_\beta(-s)\left\langle \tilde{\vec{E}}_\beta(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle \tilde{Y}_\beta^{(in)}(s) \\
&= \tilde{E}_\beta(-s)\tilde{E}_\beta(s)\left\langle \tilde{\vec{j}}_{s\beta}(\vec{r}_s',-s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle \tilde{Y}_\beta^{(in)}(s) \\
&= \text{ odd function of } s
\end{aligned}
\tag{6.7}
$$

where we have used (4.5) for the eigenadmittance. Evidently we have

$$\tilde{\tilde{E}}_\beta(-s)\tilde{\tilde{E}}_\beta(s) = \text{ even function of } s$$
$$\left\langle \tilde{\tilde{j}}_{s\beta}(\vec{r}_s{}',-s); \tilde{\tilde{j}}_{s\beta}(\vec{r}_s{}',s) \right\rangle = \text{ even function of } s \tag{6.8}$$

and furthermore on the $j\omega$ axis

$$\left\langle \tilde{\tilde{j}}_{s\beta}(\vec{r}_s{}',j\omega); \tilde{\tilde{j}}_{s\beta}(\vec{r}_s{}',j\omega) \right\rangle = 1 \text{ (orthonormality)}$$
$$\left\langle \tilde{\tilde{j}}_{s\beta}(\vec{r}_s{}',-j\omega); \tilde{\tilde{j}}_{s\beta}(\vec{r}_s{}',j\omega) \right\rangle = \left\langle \tilde{\tilde{j}}^*_{s\beta}(\vec{r}_s{}',j\omega); \tilde{\tilde{j}}_{s\beta}(\vec{r}_s{}',j\omega) \right\rangle \tag{6.9}$$
$$= \text{ real number } \geq 1$$

So (6.7) gives

$$\tilde{Y}_\beta^{(in)}(s) = -\tilde{Y}_\beta^{(in)}(-s) = \text{ odd function of } s \tag{6.10}$$

Since the internal eigenadmittance is a p.r. function (section II), it has no singularities in the RHP, and since it is an odd function of s it has no singularities in the LHP. This is a reactance function [22,23] (Foster's theorem) with properties:

1. It is an odd p.r. function
2. All poles are first order and lie on the $j\omega$ axis.
3. All pole residues are real and positive.
4. All zeros are first order and lie on the $j\omega$ axis.
5. All coefficients of zeros are real and positive.
6. Poles and zeros alternate on the $j\omega$ axis.
7. s=0 is either a zero or a pole.
8. $\mathrm{Re}\left[y_\beta^{(in)}(s)\right] > 0$ in RHP

$$\mathrm{Re}\left[y_\beta^{(in)^{-1}}(s)\right] > 0 \text{ in RHP}$$
$$\mathrm{Re}\left[y_\beta^{(in)}(s)\right] < 0 \text{ in LHP} \tag{6.11}$$
$$\mathrm{Re}\left[y_\beta^{(in)^{-1}}(s)\right] < 0 \text{ in LHP}$$

Note that for this electromagnetic problem (as distinguished from a finite lumped-element circuit problem) $s = \infty$ is neither a zero nor a pole.

Now take the formula for the normalized internal admittance (any of the several being suitable for present purposes) as

$$y_\beta^{(in)}(s) = \frac{Z_o \left\langle \vec{1}_S(\vec{r}_s') \times \tilde{\vec{H}}^{(inc,ex)}(\vec{r}_s', s) ; \tilde{\vec{j}}_{s\beta}(\vec{r}_s', s) \right\rangle}{\left\langle \tilde{\vec{E}}^{(inc,ex)}(\vec{r}_s', s) ; \tilde{\vec{j}}_{s\beta}(\vec{r}_s', s) \right\rangle} \tag{6.12}$$

An acceptable form of external incident wave is a plane wave

$$\begin{aligned}
&\tilde{\vec{E}}^{(p)}(\vec{r}, s) = E_o \tilde{f}(s) \vec{1}_p e^{-\gamma \vec{1}_1 \cdot \vec{r}} \\
&\tilde{\vec{H}}^{(p)}(\vec{r}, s) = \frac{E_o}{Z_o} \tilde{f}(s) \vec{1}_1 \times \vec{1}_p e^{-\gamma \vec{1}_1 \cdot \vec{r}} \\
&\gamma \equiv \frac{s}{c} \equiv \text{ propagation constant} \\
&c = [\mu_o \varepsilon_o]^{-\frac{1}{2}} \equiv \text{ speed fo light} \\
&\vec{1}_1 \equiv \text{ direction of propagation} \\
&\vec{1}_p \equiv \text{ polarization} \\
&\vec{1}_1 \cdot \vec{1}_p = 0 \\
&f(t) = \text{ incident waveform}
\end{aligned} \tag{6.13}$$

As indicated in fig. 6.1 let $\vec{r}_f$ be the first point the wave touches the scatterer and $\vec{r}_b$ the last point with distances $-L_f$ and L_b respectively the distances from the time reference plane through the origin [8]. As $\text{Re}[s] \to \infty$ in the RHP the contribution to the integrals is dominated by the contribution from a region S_f centered on $\vec{r}_f$ where the exponential is largest as

$$\begin{aligned}
&Z_o \left\langle \vec{1}_S(\vec{r}_s') \times \tilde{\vec{H}}^{(p)}(\vec{r}_s', s); \tilde{\vec{j}}_{s\beta}(\vec{r}_s', s) \right\rangle \\
&\to E_o \tilde{f}(s) \int_{S_f} \left[\vec{1}_S(\vec{r}_s') \times \left[\vec{1}_1 \times \vec{1}_p \right] e^{-\gamma \vec{1}_1 \cdot \vec{r}_s'} \right] \cdot \tilde{\vec{j}}_{s\beta}(\vec{r}_s', s) dS' \\
&\to E_o \tilde{f}(s) \int_{S_f} \left[-\vec{1}_1 \times \left[\vec{1}_1 \times \vec{1}_p \right] \right] \cdot \tilde{\vec{j}}_{s\beta}(\vec{r}_s', s) e^{-\gamma \vec{1}_1 \cdot \vec{r}_s'} dS' \\
&= E_o \tilde{f}(s) \vec{1}_p \cdot \int_{S_f} \tilde{\vec{j}}_{s\beta}(\vec{r}_s', s) e^{-\gamma \vec{1}_1 \cdot \vec{r}_s'} dS' \quad \text{in RHP}
\end{aligned} \tag{6.14}$$

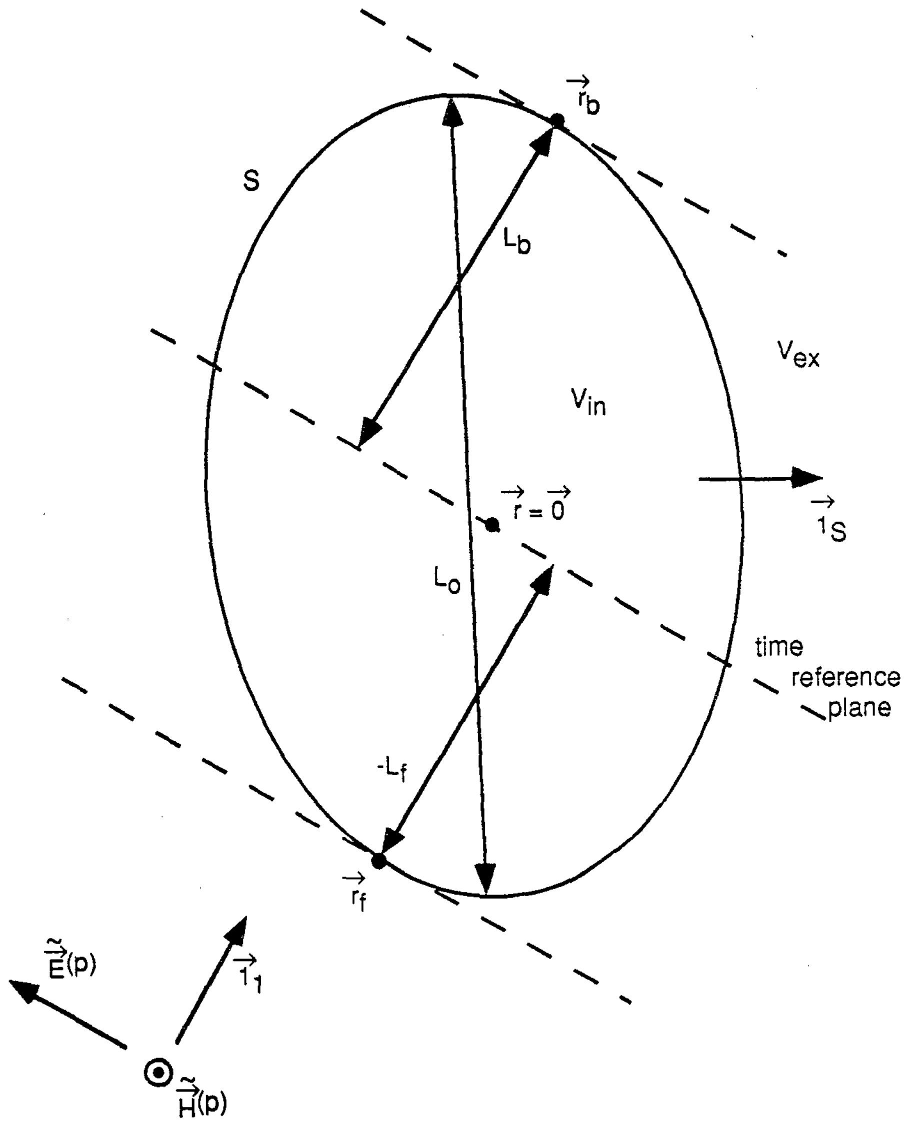

Fig. 6.1. Scatterer Illuminated by Plane Wave (External Incidence)

$$\left\langle \tilde{\bar{\bar{E}}}^{(p)}(\vec{r}_s',s);\tilde{\bar{\bar{j}}}_{s\beta}(\vec{r}_s',s)\right\rangle$$
$$\rightarrow E_o\tilde{f}(s)\vec{1}_p\cdot\int_{S_f}\tilde{\bar{\bar{j}}}_{s\beta}(\vec{r}_s',s)e^{-\gamma\vec{1}_1\cdot\vec{r}_s'}dS' \quad \text{in RHP}$$

As in [6] the behavior of the exponential decay away from $\vec{r}_f$ for $\mathrm{Re}[s]\rightarrow+\infty$ dominates the assumed slower variation of the mode around $\vec{r}_f$. So we have

$$\tilde{y}_\beta^{(in)}(s)\rightarrow 1 \text{ in RHP} \tag{6.15}$$

Similarly as $\mathrm{Re}[s]\rightarrow-\infty$ in the LHP the contribution to the integrals is dominated by the contribution from a region S_b centered on $\vec{r}_b$ where the exponential is now largest as

$$\begin{aligned}
&Z_o\left\langle \vec{1}_S(\vec{r}_s')\times\tilde{\bar{\bar{H}}}^{(p)}(\vec{r}_s',s);\tilde{\bar{\bar{j}}}_{s\beta}(\vec{r}_s',s)\right\rangle\\
&\rightarrow E_o\tilde{f}(s)\int_{Sb}\left[\vec{1}_S(\vec{r}_s')\times\left[\vec{1}_1\times\vec{1}_p\right]e^{-\gamma\vec{1}_1\cdot\vec{r}_s'}\right]\cdot\tilde{\bar{\bar{j}}}_{s\beta}(\vec{r}_s',s)dS'\\
&\rightarrow E_o\tilde{f}(s)\int_{S_b}\left[\vec{1}_p\times\left[\vec{1}_1\times\vec{1}_p\right]\right]\cdot\tilde{\bar{\bar{j}}}_{s\beta}(\vec{r}_s',s)e^{-\gamma\vec{1}_1\cdot\vec{r}_s'}dS'\\
&=-E_o\tilde{f}(s)\vec{1}_p\cdot\int_{S_b}\tilde{\bar{\bar{j}}}_{s\beta}(\vec{r}_s',s)e^{-\gamma\vec{1}_1\cdot\vec{r}_s'}dS' \quad \text{in LHP}
\end{aligned} \tag{6.16}$$

$$\left\langle \tilde{\bar{\bar{E}}}^{(p)}(\vec{r}_s',s);\tilde{\bar{\bar{j}}}_{s\beta}(\vec{r}_s',s)\right\rangle$$
$$\rightarrow E_o\tilde{f}(s)\vec{1}_p\cdot\int_{S_b}\tilde{\bar{\bar{j}}}_{s\beta}(\vec{r}_s',s)e^{-\gamma\vec{1}_1\cdot\vec{r}_s'}dS' \quad \text{in LHP}$$

So we have

$$\tilde{y}_\beta^{(in)}(s)\rightarrow -1 \text{ in LHP} \tag{6.17}$$

with this result and (6.15) consistent with an odd function of s in (6.10).

Considering a pole expansion of the internal eigenadmittance write a pole expansion as

$$\tilde{y}_\beta^{(in)}(s) = a_\infty s + a_o + \sum_{m=1}^{N} a_m [s - s_m]^{-1}$$

$$\mathrm{Re}[s_m] = 0 \qquad (6.18)$$

If we assume N finite then there is some largest $|s_m|$ and we have

$$\tilde{y}_\beta^{(in)}(s) = a_\infty s + a_o + 0\left(s^{-1}\right) \text{ as } s \to \infty \qquad (6.19)$$

From (6.15) and (6.17) there is then no pole at ∞ and a_o cannot be both +1 and -1. So we conclude that the internal eigenadmittance has the properties we can append to (6.11):

9. $\tilde{y}_\beta^{(in)}(s) \to +1$ in RHP

   $\tilde{y}_\beta^{(in)}(s) \to -1$ in LHP

10. Number of poles is infinite. (6.20)

11. Number of zeros is infinite.

At s=0 the internal eigenadmittance must have either a zero or a pole. If it is a zero write

$$\tilde{Y}_\beta^{(in)}(s) = \tilde{Y}_{e,n}^{(in)}(s) = sC_n^{(in)} + 0\left(s^3\right) \text{ as } s \to 0 \qquad (6.21)$$

where β is now taken as (e,n) i.e. the nth electric eigenmode. Similarly if there is a pole at s=0 we have

$$Y_\beta^{(in)}(s) = Y_{h,n}^{(in)}(s) = \frac{1}{sL_n^{(in)}} + 0(s) \text{ as } s \to 0 \qquad (6.22)$$

where the next terms in the expansion are limited by (6.10) giving odd terms in Taylor and Laurent series. For the same n the two modes are complementary giving

$$Z_o \tilde{Y}_{e,n}^{(in)}(s) = \left[Z_o \tilde{Y}_{h,n}^{(in)}(s) \right]^{-1}$$

$$\frac{L_n^{(in)}}{C_n^{(in)}} = Z_o^2 = \frac{\mu_o}{\varepsilon_o} \qquad (6.23)$$

In order to evaluate the coefficient as $s \to 0$ we need consider only an E mode. Rewrite (6.12) in the form

$$\tilde{y}_{e,n}^{(in)}(s) = \frac{\left\langle \tilde{\vec{H}}^{(inc,ex)}(\vec{r}_s', s); \tilde{\vec{j}}_{s_{h,n}}(\vec{r}_s', s) \right\rangle}{\left\langle \tilde{\vec{E}}^{(inc,ex)}(\vec{r}_s', s); \tilde{\vec{j}}_{s_{e,n}}(\vec{r}_s', s) \right\rangle} \tag{6.24}$$
$$= sC_n^{(in)} + 0\left(s^3\right) \text{ as } s \to 0$$

Choosing the plane wave in (6.13) gives

$$C_n^{(in)} = \lim_{s \to 0} \frac{1}{Z_o s} \frac{\left[\vec{1}_1 \times \vec{1}_p\right] \cdot \left\langle e^{-\gamma \vec{1}_1 \cdot \vec{r}_s'}, \tilde{\vec{j}}_{s_{h,n}}(\vec{r}_s', s) \right\rangle}{\vec{1}_p \cdot \left\langle e^{-\gamma \vec{1}_1 \cdot \vec{r}_s'}; \tilde{\vec{j}}_{s_{e,n}}(\vec{r}_s', s) \right\rangle} \tag{6.25}$$

Considering the denominator integral as $s \to 0$

$$\vec{1}_p \cdot \left\langle e^{-\gamma \vec{1}_1 \cdot \vec{r}_s'}; \tilde{\vec{j}}_{s_{e,n}}(\vec{r}_s', s) \right\rangle \to \vec{1}_p \cdot \int_S \tilde{\vec{j}}_{s_{e,n}}(\vec{r}_s', 0) dS'$$
$$= -\vec{1}_p \cdot \int_S \vec{r}_s' \nabla_s' \cdot \tilde{\vec{j}}_{s_{e,n}}(\vec{r}_s', 0) dS' \tag{6.26}$$
$$= -\vec{1}_p \cdot \text{ (real vector)}$$

where integration by parts casts the result in terms of the divergence of the eigenmode, giving an electric-dipole-like term.

One can take as the definition of an E mode a not-identically zero divergence at s=0, consistent with the capacitive behavior in (6.21). As discussed in Appendix B, tangential vector fields on S can be represented by two parts, one with zero surface divergence and one with zero normal surface curl. In transforming a mode to the complementary mode, operation by $\vec{1}_S(\vec{r}_s') \times$ interchanges the roles of these two parts, converting electric to magnetic and conversely. The numerator integral as $s \to 0$ should be $0(s)$ as $s \to 0$. As a magnetic mode then this is the case provided

$$\nabla_s \cdot \tilde{\vec{j}}_{s_{h,n}}(\vec{r}_s', 0) = 0$$
$$\vec{1}_S \cdot \left[\nabla_s \times \tilde{\vec{j}}_{s_{h,n}}(\vec{r}_s, 0)\right] = -\nabla_s \cdot \left[\vec{1}_S(\vec{r}_s) \times \tilde{\vec{j}}_{s_{h,n}}(\vec{r}_s, 0)\right] \not\equiv 0 \tag{6.27}$$

and consistent with this

$$\nabla_s \cdot \bar{\bar{j}}_{s_{e,n}}(\bar{r}_s,0) \neq 0$$

$$\bar{1}_S(\bar{r}) \cdot \left[\nabla_s \times \bar{\bar{j}}_{s_{e,n}}(\bar{r}_s,0) \right] = -\nabla_s \cdot \left[\bar{1}_S(\bar{r}_s) \times \bar{\bar{j}}_{s_{e,n}}(\bar{r}_s,0) \right] = 0 \tag{6.28}$$

Expanding the exponential in the numerator of (6.25) then gives as $s \to 0$

$$\begin{aligned} &\frac{1}{Z_o s}\left[\bar{1}_1 \times \bar{1}_p\right] \cdot \left\langle e^{-\gamma \bar{1}_1 \cdot \bar{r}_s'}, \bar{\bar{j}}_{s_{h,n}}(\bar{r}_s', s)\right\rangle \to \frac{1}{Z_o s}\left[\bar{1}_1 \times \bar{1}_p\right] \cdot \left\langle -\gamma \bar{1}_1 \cdot \bar{r}_s', \bar{\bar{j}}_{s_{h,n}}(\bar{r}_s', 0)\right\rangle \\ &\qquad = -\varepsilon_o\left[\bar{1}_1 \times \bar{1}_p\right] \cdot \left\langle \bar{1}_1 \cdot \bar{r}_s', \bar{\bar{j}}_{s_{h,n}}(\bar{r}_s', 0)\right\rangle \end{aligned} \tag{6.29}$$

Writing the integral as

$$\begin{aligned} &\left\langle \bar{1}_1 \cdot \bar{r}_s', \bar{\bar{j}}_{s_{h,n}}(\bar{r}_s', 0)\right\rangle \\ &\qquad = \frac{1}{2}\left[\left\langle \bar{1}_1 \cdot \bar{r}_s', \bar{\bar{j}}_{s_{h,n}}(\bar{r}_s', 0)\right\rangle + \left\langle \bar{1}_1 \cdot \bar{\bar{j}}_{s_{h,n}}(\bar{r}_s', 0), \bar{r}_s'\right\rangle\right] \\ &\qquad = \frac{1}{2}\left[\left\langle \bar{1}_1 \cdot \bar{r}_s', \bar{\bar{j}}_{s_{h,n}}(\bar{r}_s', 0)\right\rangle - \left\langle \bar{1}_1 \cdot \bar{\bar{j}}_{s_{h,n}}(\bar{r}_s', 0), \bar{r}_s'\right\rangle\right] \end{aligned} \tag{6.30}$$

the first part (symmetric) gives an electric quadruple term [25] which is zero since the H mode is divergenceless at s=0. The second part is a magnetic-dipole-like term which is rearranged [27] as

$$\left\langle \bar{1}_1 \cdot \bar{r}_s', \bar{\bar{j}}_{s_{h,n}}(\bar{r}_s', 0)\right\rangle = -\frac{1}{2}\bar{1}_1 \times \left\langle \bar{r}'_s \overset{\times}{,} \bar{\bar{j}}_{s_{h,n}}(\bar{r}_s', 0)\right\rangle \tag{6.31}$$

Then the result of (6.29) becomes

$$\begin{aligned} &-\varepsilon_o\left[\bar{1}_1 \times \bar{1}_p\right] \cdot \left\langle \bar{1}_1 \cdot \bar{r}_s', \bar{\bar{j}}_{s_{h,n}}(\bar{r}_s', 0)\right\rangle \\ &= \varepsilon_o \bar{1}_p \cdot \left[\bar{1}_1 \times \left\langle \bar{1}_1 \cdot \bar{r}_s', \bar{\bar{j}}_{s_{h,n}}(\bar{r}_s', 0)\right\rangle\right] \\ &= \frac{\varepsilon_o}{2}\bar{1}_p \cdot \left\langle \bar{r}_s' \overset{\times}{,} \bar{\bar{j}}_{s_{h,n}}(\bar{r}_s', 0)\right\rangle \\ &= \varepsilon_o \bar{1}_p \cdot \text{(real vector)} \end{aligned} \tag{6.32}$$

Combining (6.26) and (6.32) gives the interior modal capacitance as

$$C_n^{(in)} = \varepsilon_o \frac{\frac{1}{2}\vec{1}_p \cdot \left\langle \vec{r}_s' ; \times \vec{\vec{j}}_{s_{h,n}}(\vec{r}_s', 0) \right\rangle}{\vec{1}_p \cdot \int_S \vec{\vec{j}}_{s_{e,n}}(\vec{r}_s', 0) dS'} > 0 \tag{6.33}$$

Since this capacitance must be positive (from p.r. condition) this ratio of integrals must be positive. Varying over all possible $\vec{1}_p$ we have

$$\begin{aligned} \int_S \vec{\vec{j}}_{s_{e,n}}(\vec{r}_s', 0) dS' &= -\left\langle \vec{r}_s', \nabla_s' \cdot \vec{\vec{j}}_{s_{e,n}}(\vec{r}_s', 0) \right\rangle \\ &= \frac{\varepsilon_o}{C_n^{(in)}} \frac{1}{2} \left\langle \vec{r}_s' ; \times \vec{\vec{j}}_{s_{h,n}}(\vec{r}_s', 0) \right\rangle \end{aligned} \tag{6.34}$$

so that the electric-dipole-like and magnetic-dipole-like vectors are parallel (have the same orientation). Note that (6.23) gives the interior modal inductance for the complementary magnetic mode from (6.34) as well.

Note that the above results are for the case that the appropriate dipole moments for the modes are non zero. This is not always the case as the example of the sphere illustrates [1,8]. In this example the dipole terms correspond to $n = 1$, and $n \geq 2$ corresponds to higher order multipoles. In such a case that a given n has no dipolar parts, one can extend the analysis by continuing the expansion of the exponentials until a non zero term (for small s) is obtained. Other forms of external incident waves can also be used in the formula for the internal eigenadmittance.

So now index the internal poles and zeros as indicated in fig. 6.2. Since the poles correspond to internal natural frequencies let us start with them. Since s=0 is a zero for E-mode eigenadmittances, we have

$$\begin{aligned} & s_{e,n,n'}^{(in)} \equiv E\text{-mode poles} = H\text{-mode zeros} \\ & s_{e,n,-n'}^{(in)} \equiv -s_{e,n,n'}^{(in)} = \text{imaginary} \\ & n' \neq 0 \end{aligned} \tag{6.35}$$

The natural frequencies are ordered by their distance from the origin, with $\pm n'$ designating conjugate pairs on the $j\omega$ axis. For H modes $s = 0$ is a pole, so we have

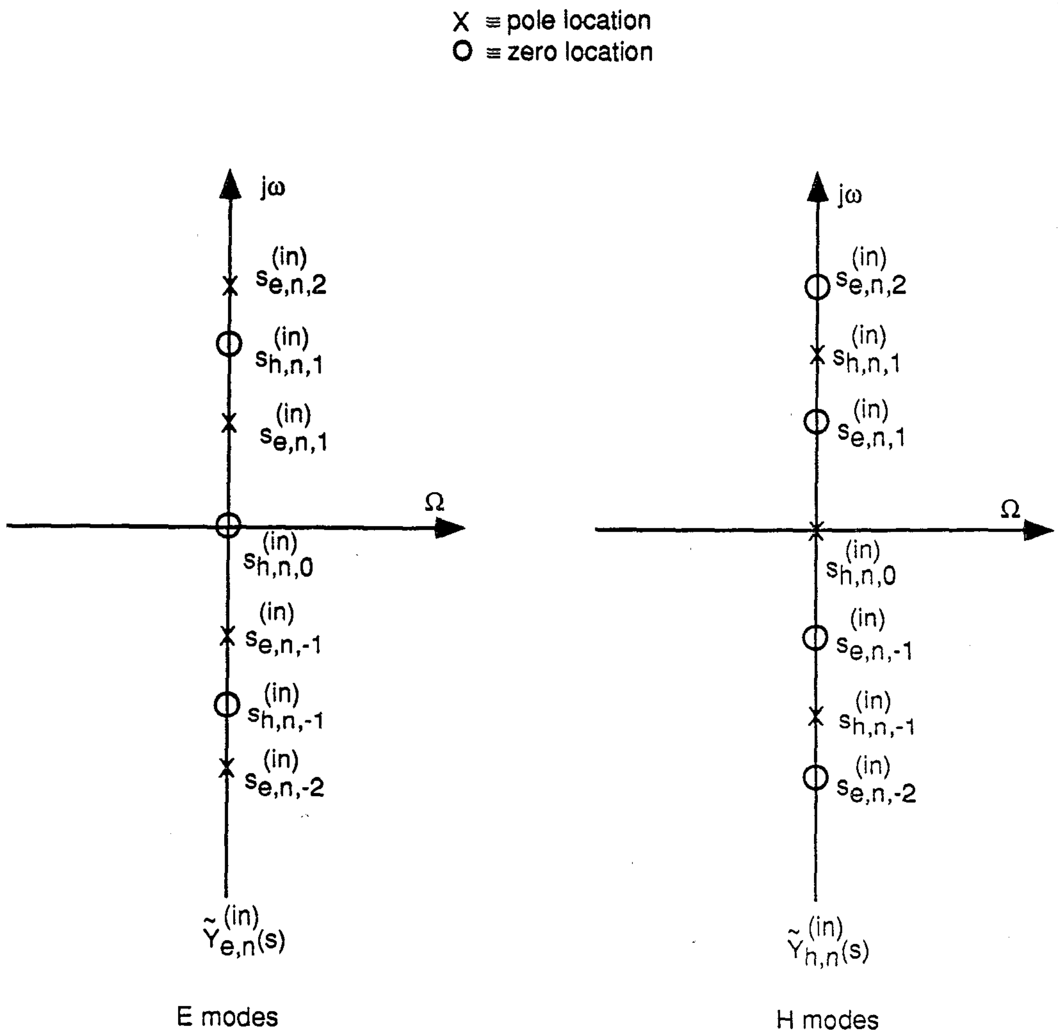

Fig. 6.2. Poles and Zeros of Internal Eigenadmittances

$$
\begin{aligned}
&s^{(in)}_{h,n,n'} \equiv H\text{-mode poles} = E\text{-mode zeros} \\
&s^{(in)}_{h,n,-n'} \equiv s^{(in)*}_{h,n,n'} = \text{imaginary} \\
&s^{(in)}_{h,n,0} = 0
\end{aligned} \tag{6.36}
$$

Noting the reciprocal nature of the internal eigenadmittances, the poles of one are the zeros of the other, thereby indexing both poles and zeros.

The poles of the interior eigenadmittances are zeros of the denominator in (6.12), i.e.

$$
\left\langle \tilde{\vec{E}}^{(inc,ex)}\left(\vec{r}_S', s_{\beta,n'}\right); \tilde{\vec{j}}_{s_\beta}\left(\vec{r}_S', s_{\beta,n'}\right)\right\rangle = 0 \tag{6.37}
$$

In effect the content of the external incident electric field for the particular mode is zero at an internal natural frequency. This corresponds to an interior or cavity natural mode with tangential electric field zero on S. Consider the fields of such a mode in V_{in} which satisfy the wave equation (following [20])

$$
\begin{aligned}
&\left[\nabla^2 + \left(\frac{s^{(in)}_{\beta,n'}}{c}\right)^2\right]\tilde{\vec{E}}\left(\vec{r}\,', s^{(in)}_{\beta,n'}\right) = \vec{0} \text{ in } V_{in} \\
&\nabla \cdot \tilde{\vec{E}}\left(\vec{r}\,', s^{(in)}_{\beta,n'}\right) = 0 \text{ in } V_{in} \\
&\vec{1}_S(\vec{r}_s) \cdot \tilde{\vec{E}}\left(\vec{r}_s, s^{(in)}_{\beta,n'}\right) = \vec{0} \text{ on } S \\
&\nabla \times \tilde{\vec{E}}\left(\vec{r}\,', s^{(in)}_{\beta,n'}\right) = -s^{(in)}_{\beta,n'}\mu_0\, \tilde{\vec{H}}\left(\vec{r}, s^{(in)}_{\beta,n'}\right) \text{ in } V_{in}
\end{aligned} \tag{6.38}
$$

The solution of this with imaginary natural frequencies admits a purely imaginary electric field with purely real magnetic field. The boundary condition on S (inside) has

$$
-\vec{1}_S(\vec{r}_s) \times \tilde{\vec{H}}\left(\vec{r}_s, s^{(in)}_{\beta,n'}\right) = \vec{J}^{(in)}_s\left(\vec{r}_s, s^{(in)}_{\beta,n'}\right) \tag{6.39}
$$

this surface current density being proportional to the eigenmode. Noting the orthonormalization condition we have

$$
\begin{aligned}
&\tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s, s_{\beta,n'}^{(in)}\right) = \text{ real vector function} \\
&\left\langle \tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s\,', s_{\beta,n'}^{(in)}\right); \tilde{\vec{j}}_{s_\beta}^{\,*}\left(\vec{r}_s\,', s_{\beta,n'}^{(in)}\right)\right\rangle = 1
\end{aligned}
\tag{6.40}
$$

and noting the conjugate symmetry of the modes from (2.7) we have

$$
\tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s\,', s_{\beta,-n'}^{(in)}\right) = \tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s\,', s_{\beta,n'}^{(in)}\right)
\tag{6.41}
$$

Using the complementary-mode relationship we have from (5.28)

$$
\begin{aligned}
&\tilde{\vec{j}}_{s_\beta}^{(c)}\left(\vec{r}_s\,', s\right) = \vec{1}_S\left(\vec{r}_s\right) \times \tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s, s\right) = \pm\tilde{\vec{j}}_{s_{\beta'}}\left(\vec{r}_s, s\right) \\
&\beta = \left({}^{e}_{h}, n\right) \\
&\beta' = \left({}^{e}_{h}, n\right)
\end{aligned}
\tag{6.42}
$$

So at zeros of $\tilde{y}_\beta^{(in)}(s)$ which are poles of $\tilde{y}_\beta^{(in,c)}(s)$ the modes are also real functions as

$$
\begin{aligned}
&\tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s\,', s_{\beta',n'}^{(in)}\right) = \tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s\,', s_{\beta',-n'}^{(in)}\right) \\
&\qquad\qquad = \text{real vector function} \\
&\left\langle \tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s\,', s_{\beta',n'}^{(in)}\right); \tilde{\vec{j}}_{s_\beta}^{\,*}\left(\vec{r}_s\,', s_{\beta',n'}\right)\right\rangle = 1
\end{aligned}
\tag{6.43}
$$

Summarizing, the eigenmodes are real at both poles (natural frequencies) and zeros (natural frequencies for complementary eigenmodes) of internal eigenadmittances.

VII. Exterior Eigenadmittances

The normalized external eigenadmittances take the form (one of several suitable forms)

$$\tilde{y}_\beta^{(ex)}(s) = -\frac{Z_o\left\langle \bar{1}_S(\vec{r}_s') \times \tilde{\bar{H}}^{(inc,in)}(\vec{r}_s',s); \tilde{\bar{j}}_{s_\beta}(\vec{r}_s',s)\right\rangle}{\left\langle \tilde{\bar{E}}^{(inc,ex)}(\vec{r}_s',s); \tilde{\bar{j}}_{s_\beta}(\vec{r}_s',s)\right\rangle} \tag{7.1}$$

Unlike the interior eigenadmittances in Section VI these are not lossless, so the Poynting vector theorem, allowing for sources (electric and magnetic) in V_{in}, does not lead to an odd function (reactance function) as discussed there. However, this is a p.r. function associated with the lossy problem of the energy transported by the external scattered fields.

Considering the incident fields on S from sources in V_{in}, note that the radiation condition makes these fields as in (7.1) represent positive outgoing real power for $s = j\omega$, except at $s = 0$ since finite-size antennas cannot radiate power at D.C. with finite currents and voltages driving the antenna. Given some tangential E field on S_+ (i.e. just outside S), say in the form (spatial distribution) of the βth eigenmode this will radiate a finite, non-zero real power indicating that the external eigenadmittance has no poles or zeros, except at $s = 0$. As will be computed later this is a zero for an E mode and a pole for an H mode. As a p.r. function these must be first order with real and positive residue (for a pole) or coefficient (for a zero). All other poles and zeros are in the LHP excluding the $j\omega$ axis for this assumed meromorphic function.

For our incident wave let there be a small source at $\vec{r}_o$ as indicated in fig. 7.1. For high-frequency asymptotic evaluation we have the usual results for the far field of some antenna as

$$\begin{aligned} &\tilde{\bar{E}}^{(inc,in)}(\vec{r},s) \to \tilde{\bar{V}}(\theta,\phi;s)\frac{e^{-\gamma|\vec{r}-\vec{r}_o|}}{|\vec{r}-\vec{r}_o|} \text{ as } |s| \to \infty \\ &\tilde{\bar{H}}^{(inc,in)}(\vec{r},s) \to \frac{1}{Z_o}\left[\frac{\vec{r}-\vec{r}_o}{|\vec{r}-\vec{r}_o|}\right] \times \tilde{\bar{E}}^{(inc,in)}(\vec{r}_s,s) \text{ as } |s| \to \infty \\ &\tilde{\bar{V}}(\theta,\phi;s) \equiv \text{ antenna pattern} \end{aligned} \tag{7.2}$$

This might be a simple electric or magnetic dipole, or some more elaborate source.

Consider $\mathrm{Re}[s] \to \infty$ in the RHP. The contribution to the integrals is dominated by the contribution from a region S_1 centered on $\vec{r}_1$, the point on S closest to $\vec{r}_o$ where the exponential is largest giving

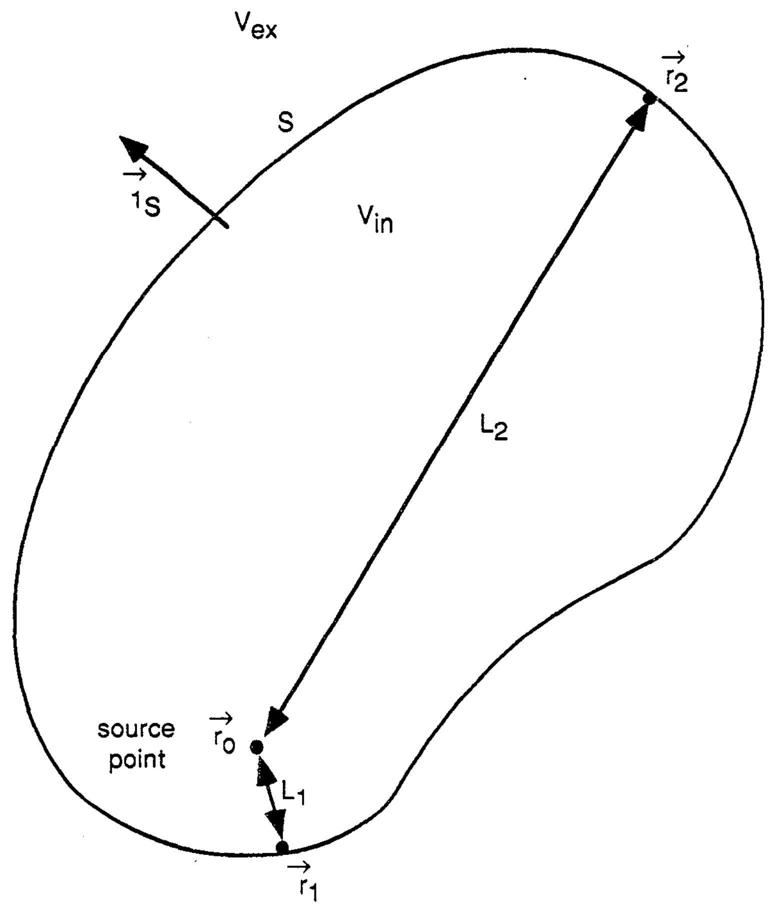

Fig. 7.1. Source Illuminated by Small Antenna in V_{in} (Internal Incidence)

$$Z_o\left\langle \vec{1}_S(\vec{r}_s')\times\tilde{\vec{H}}^{(inc,in)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle$$

$$\rightarrow\int_{S_1}\left[\vec{1}_S(\vec{r}_s')\times\left[\left[\frac{\vec{r}_s-\vec{r}_o}{|\vec{r}_s-\vec{r}_o|}\right]\times\tilde{\vec{V}}(\theta,\phi;s)\right]\frac{e^{-\gamma|\vec{r}_s-\vec{r}_o|}}{|\vec{r}_s-\vec{r}_o|}\right]\cdot\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)dS' \tag{7.3}$$

$$\rightarrow\int_{S_1}\left[\vec{1}_S(\vec{r}_1)\times\left[\vec{1}_S(\vec{r}_1)\times\tilde{\vec{V}}(\theta,\phi;s)\right]\right]\cdot\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\frac{e^{-\gamma|\vec{r}_s-\vec{r}_o|}}{|\vec{r}_s-\vec{r}_o|}dS'$$

$$=-\frac{\tilde{\vec{V}}(\theta_1,\phi_1;s)}{|\vec{r}_1-\vec{r}_o|}\cdot\int_{S_1}\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)e^{-\gamma|\vec{r}_s-\vec{r}_o|}dS' \text{ in RHP}$$

$$\left\langle \tilde{\vec{E}}^{(inc,in)}(\vec{r}_s',s);\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)\right\rangle$$

$$\rightarrow\frac{\tilde{\vec{V}}(\theta_1,\phi_1;s)}{|\vec{r}_1-\vec{r}_o|}\cdot\int_{S_1}\tilde{\vec{j}}_{s\beta}(\vec{r}_s',s)e^{-\gamma|\vec{r}_s-\vec{r}_o|}dS' \text{ in RHP}$$

The pattern function now is assumed slowly varying as compared with the exponential and the subscripts on the angles indicate the direction to $\vec{r}_1$ from $\vec{r}_o$ (taken as the origin of spherical coordinates for this purpose). So we have

$$\tilde{y}_\beta^{(ex)}(s)\rightarrow 1 \text{ in RHP} \tag{7.4}$$

Similarly let $\mathrm{Re}[s]\rightarrow-\infty$ in the LHP where the contribution to the integrals is dominated by the region S_2 centered on $\vec{r}_2$, the point on S farthest from $\vec{r}_o$ where the exponential is now largest. The results are as in (7.3) with 1 subscripts replaced by 2 subscripts. The signs are also the same since $\vec{1}_S(\vec{r}_s)$ is aligned parallel to $\vec{r}_s-\vec{r}_o$ (outward) in both cases. Hence we have

$$\tilde{y}_\beta^{(ex)}(s)\rightarrow 1 \text{ in LHP} \tag{7.5}$$

Note that this implies that there is some $\Omega_{left}<0$ such that there are no poles or zeros for $\Omega\le\Omega_{left}$. Adding this to the information for no poles or zeros in the RHP gives a strip $\Omega_{left}<\Omega\le 0$ containing all the poles and zeros of the ßth exterior eigenadmittance.

Further refining the asymptotics for large $|s|$, let Ω be in the above strip and let $\omega\rightarrow+\infty$. Then the exponentials in the integrals do not grow or decay with distance (thereby isolating $\vec{r}_1$ and $\vec{r}_2$), but they do oscillate more and more rapidly as $\omega\rightarrow\infty$ suggesting an evaluation by the common stationary-phase approach. In this case we look for positions on S where the surface is locally perpendicular to a ray from $\vec{r}_o$ to $\vec{r}_s$. In the vicinity of such points $\vec{r}_\ell$ the phase of $\tilde{\tilde{V}}(\theta_\ell,\phi_\ell;\Omega+j\omega)$ is approximately uniform as compared to other parts of S for a suitably chosen source (e.g. an electric or magnetic dipole). Note that such stationary points have the property that $\vec{1}_S(\vec{r}_\ell)$ is parallel to $\vec{r}_\ell-\vec{r}_o$. We have already encountered two such points $\vec{r}_1$ and $\vec{r}_2$. There can be in principle more such points depending on the shape of S and choice of $\vec{r}_o$. For reentrant shapes we can even have $\vec{1}_S(\vec{r}_\ell)$ antiparallel to $\vec{r}_\ell-\vec{r}_o$.

Assuming a finite set of stationary points, then for each one we obtain integrals in both numerator and denominator which are evaluated as in (7.3), now for $\omega\rightarrow+\infty$. Adding the contributions from all the stationary points the numerator will again be asymptotically the same as the denominator giving

$$\tilde{y}_\beta^{(ex)}(s)\rightarrow 1 \text{ in strip as } \omega\rightarrow+\infty \tag{7.6}$$

Noting the conjugate symmetry the stationary phase also works for $\omega\rightarrow-\infty$ giving

$$\tilde{y}_\beta^{(ex)}(s)\rightarrow 1 \text{ in strip as } \omega\rightarrow-\infty \tag{7.7}$$

Note that there is some $\omega_+>0$ such that there are no poles or zeros for $\omega\geq\omega_+$, and similarly for $\omega<\omega_-\equiv-\omega_+$.

One can approach this stationary phase result a little differently by making $\vec{r}_o$ close to S, say near $\vec{r}_1$ as in fig. 7.1. Then the denominator $|\vec{r}_\ell-\vec{r}_o|$ is minimized for the stationary point at $\vec{r}_1$ as in (7.3). As $\vec{r}_o$ is made closer to $\vec{r}_1$ this contribution will get larger, dominating the contribution from the other stationary points. Then one only has the two integrals as in (4.3) which are asymptotically the same. Note that one first takes $\omega\rightarrow+\infty$ for each choice of $\vec{r}_o$ before taking $\vec{r}_o\rightarrow\vec{r}_1$. In the limit one could also consider a point electric dipole just inside S and evaluate the integrals for $\omega\rightarrow+\infty$ over what is approximately a circular disk.

Assume that $\tilde{y}_{s_\beta}^{(ex)}$ is a meromorphic function of s and thereby has only a finite number of poles in any finite region of the complex plane [21]. Here we have found a rectangle defined by

$$\begin{aligned} &\mathrm{Re}[s]=0,\Omega_{left} \\ &\omega=\pm\omega_+ \end{aligned} \tag{7.8}$$

outside of which there are no poles. Then the number of poles of the exterior eigenadmittance must be finite, say N. Consider a closed contour enclosing this rectangle. On this contour the function is 1 (asymptotically). The change in phase around the contour is then zero so that by the argument-number theorem

Number of poles (including multiplicity)

-Number of zeros (including multiplicity)

=0 (7.9)

and there are also N zeros (exactly). Recognizing the reciprocal relationship between the electric and magnetic exterior eigenimpedances we now have

$$N(n) = \begin{cases} \text{number of zeros of } \tilde{y}_{e,n}^{(ex)}(s) \\ \text{number of poles of } \tilde{y}_{e,n}^{(ex)}(s) \\ \text{number of zeros of } \tilde{y}_{h,n}^{(ex)}(s) \\ \text{number of poles of } \tilde{y}_{h,n}^{(ex)}(s) \end{cases} \tag{7.10}$$

This number is then only a function of the index n. For the sphere one can observe that N is just n+1 where n is now the index on the spherical Bessel functions [1,15].

Looking at fig. 7.2 there is an illustration of what the pattern of poles and zeros might look like, consistent with the foregoing. As in the previous section (for the internal eigenadmittances) we have

$$\begin{aligned} s_{e,n,n'}^{(ex)} &\equiv E-\text{mode poles } = H-\text{mode zeros} \\ s_{h,n,n'}^{(ex)} &\equiv H-\text{mode poles } = E-\text{mode zeros} \\ s_{e,n,-n'}^{(ex)} &= s_{e,n,n'}^{(ex)^*} \\ s_{h,n,-n'}^{(ex)} &= s_{h,n,n'}^{(ex)^*} \end{aligned} \tag{7.11}$$

At s=0 we can define, consistent with the results for the internal eigenadmittance,

$$s_{h,n,0}^{(ex)} = 0 \tag{7.12}$$

However, there can in general be other points on the negative real axis $(\omega = 0, \Omega < 0)$ with natural frequencies for E and H modes, as the sphere exhibits [1,8,15]. This requires (in some cases) another index as

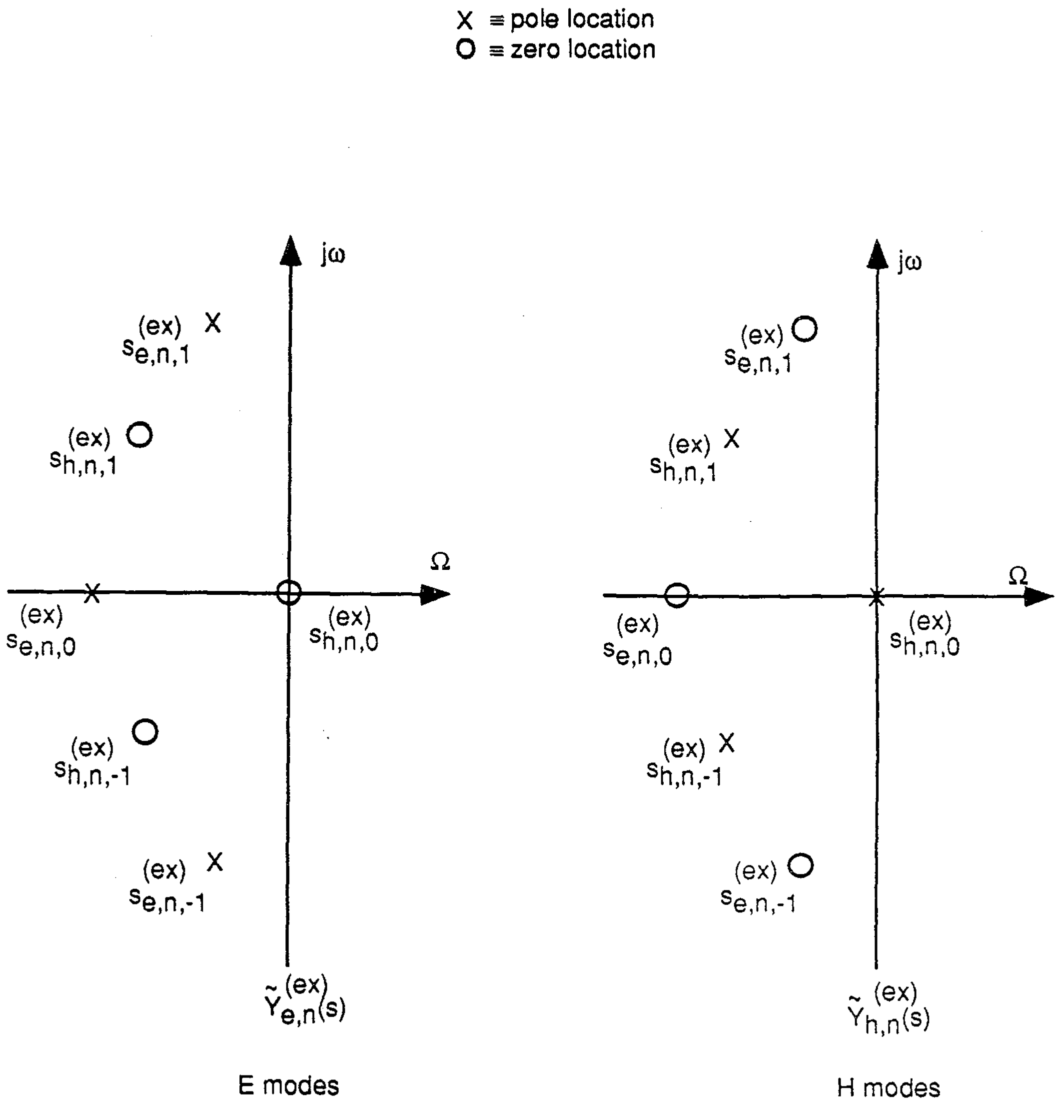

Fig. 7.2. Poles and Zeros of External Eigenadmittances

$$
\begin{aligned}
&s^{(ex)}_{h,n,0,n''} = \text{real } \leq 0 \,,\, s^{(ex)}_{h,n,0,0} = 0 \\
&s^{(ex)}_{h,n,0,n''} = \text{real } < 0
\end{aligned}
\tag{7.13}
$$

with say $n'' = 1,2,\cdots$ with the one exception above.

Now look more closely at the natural frequency at s=0. For our source point at $\vec{r}_o$ (fig. 7.1) let us take an electric dipole of moment $\tilde{\vec{p}}$ as [9]

$$
\begin{aligned}
&\tilde{\vec{p}}(s) = \vec{1}_p \tilde{f}(s) \\
&\tilde{\vec{E}}^{(di)}(\vec{r},s) = e^{-\gamma|\vec{r}-\vec{r}_o|}\left\{\frac{\mu_o}{4\pi|\vec{r}-\vec{r}_o|}s^2\left[\frac{(\vec{r}-\vec{r}_o)(\vec{r}-\vec{r}_o)}{|\vec{r}-\vec{r}_o|^2} - \overleftrightarrow{1}\right]\right. \\
&\qquad \left. + \left[\frac{Z_o s}{4\pi|\vec{r}-\vec{r}_o|^2} + \frac{1}{4\pi\varepsilon_o|\vec{r}-\vec{r}_o|^3}\right]\left[\frac{3(\vec{r}-\vec{r}_o)(\vec{r}-\vec{r}_o)}{|\vec{r}-\vec{r}_o|^2} - \overleftrightarrow{1}\right]\right\}\cdot\tilde{\vec{p}}(s) \\
&\tilde{\vec{H}}^{(di)}(\vec{r},s) = -e^{-\gamma|\vec{r}-\vec{r}_o|}\left\{\frac{s^2}{4\pi c|\vec{r}-\vec{r}_o|} + \frac{s}{4\pi|\vec{r}-\vec{r}_o|^2}\right\}\frac{\vec{r}-\vec{r}_o}{|\vec{r}-\vec{r}_o|}\times\tilde{\vec{p}}(s)
\end{aligned}
\tag{7.14}
$$

Then for small s the internal eigenadmittance for an E mode is

$$
\begin{aligned}
Y^{(ex)}_{e,n}(s) &= \frac{\left\langle \tilde{\vec{H}}^{(d)}(r'_s,s);\tilde{\vec{j}}_{s_{h,n}}(r'_s,s)\right\rangle}{\left\langle \tilde{\vec{E}}^{(d)}(r'_s,s);\tilde{\vec{j}}_{s_{e,n}}(r'_s,s)\right\rangle} \\
&= sC^{(ex)}_n + 0\left(s^2\right) \text{ as } s \to 0
\end{aligned}
\tag{7.15}
$$

Considering the denominator integral as $s \to 0$ we have

$$
\left\langle \tilde{\vec{E}}^{(di)}(\vec{r}'_s,s);\tilde{\vec{j}}_{s_{e,n}}(\vec{r}'_s,s)\right\rangle \to \frac{1}{4\pi\varepsilon_o}\tilde{\vec{p}}(s)\cdot\int_S \frac{1}{|\vec{r}'_s-\vec{r}_o|^3}\left[\frac{3(\vec{r}'_s-\vec{r}_o)(\vec{r}'_s-\vec{r}_o)}{|\vec{r}'_s-\vec{r}_o|^2} - \overleftrightarrow{1}\right]\cdot\tilde{\vec{j}}_{s_{e,n}}(\vec{r}'_s,0)dS'
\tag{7.16}
$$

The numerator integral similarly gives

$$
\left\langle \tilde{\vec{H}}^{(di)}(\vec{r}'_s,s);\tilde{\vec{j}}_{s_{h,n}}(\vec{r}'_s,s)\right\rangle \to \frac{s}{4\pi}\tilde{\vec{p}}(s)\cdot\int_S \frac{1}{|\vec{r}'_s-\vec{r}_o|^2}\frac{(\vec{r}'_s-\vec{r}_o)}{|\vec{r}'_s-\vec{r}_o|}\times\tilde{\vec{j}}_{s_{h,n}}(\vec{r}'_s,0)dS'
\tag{7.17}
$$

Then we have

$$C_n^{(ex)} = \varepsilon_o \frac{\bar{\bar{p}}(s)\cdot\int_S \frac{1}{|\vec{r}_s' - \vec{r}_o|^2}\frac{(\vec{r}_s' - \vec{r}_o)}{|\vec{r}_s' - \vec{r}_o|}\times \bar{\bar{j}}_{s_{h,n}}(\vec{r}_s',0)dS'}{\bar{\bar{p}}(s)\cdot\int_S \frac{1}{|\vec{r}_s' - \vec{r}_o|^3}\left[\frac{3(\vec{r}_s' - \vec{r}_o)(\vec{r}_s' - \vec{r}_o)}{|\vec{r}_s' - \vec{r}_o|^2} - \bar{1}\right]\cdot \bar{\bar{j}}_{s_{e,n}}(\vec{r}_s',0)dS'} > 0 \tag{7.18}$$

Since the capacitance must be positive (p.r. condition) the ratio of integrals must be positive. Varying over all possible orientations of the dipole gives

$$\begin{aligned} &\int_S \frac{1}{|\vec{r}_s' - \vec{r}_o|^3}\left[\frac{3(\vec{r}_s' - \vec{r}_o)(\vec{r}_s' - \vec{r}_o)}{|\vec{r}_s' - \vec{r}_o|^2} - \bar{1}\right]\cdot \bar{\bar{j}}_{s_{e,n}}(s,0)dS' \\ &= \frac{\varepsilon_o}{C_n^{(ex)}}\int_S \frac{1}{|\vec{r}_s' - \vec{r}_o|^2}\frac{(\vec{r}_s' - \vec{r}_o)}{|\vec{r}_s' - \vec{r}_o|}\times \bar{\bar{j}}_{s_{h,n}}(s,0)dS' \end{aligned} \tag{7.19}$$

which can be compared to (6.34). Note that this result is independent of $\vec{r}_o$ as long as it is in V_{in}. Furthermore, we have for the complementary magnetic mode

$$\frac{L_n^{(ex)}}{\mu_o} = \frac{C_n^{(ex)}}{\varepsilon_o} \tag{7.20}$$

While in (7.15) the error is easily shown to be $0(s^2)$ by use of Taylor expansions, the next term in the expansion of the external admittance can be seen to be $0(s^3)$. This is based on consideration of low-frequency scattered power. It is the $1/r$ terms in the dipole fields which contribute to the radiated power. The $1/r^3$ and $1/r^2$ terms should contribute nothing to the real radiated power. In an incident wave (unit amplitude (s independent)) the induced electric dipole moment is proportional to the field. The associated scattered electric and magnetic far fields are both proportional to s^2. The real power in each dipole term is like s^4 (for $s = j\omega$). So a real power which appears in the even powers of s in the admittance expansion does not appear in s^2 but in higher powers.

VIII. Eigenmodes

Having discussed the eigenadmittance decomposition let us go back to the eigenmodes themselves. In section V the modes were observed to come in pairs which we have termed "electric" and "magnetic". In sections VI and VII we have seen that one of these leads to a capacitive eigenadmittance for both the interior and exterior parts as $s \to 0$, and a corresponding inductive eigenadmittance for the other mode of the pair (the complementary mode). Identifying the capacitive case as the electric mode we have for small s

$$
\begin{aligned}
&\nabla_s \cdot \tilde{\vec{j}}_{s_{e,n}}(\vec{r}_s,0) \neq 0, \quad \vec{1}_S(\vec{r}_s)\cdot\left[\nabla_s \times \tilde{\vec{j}}_{s_{e,n}}(\vec{r}_s,0)\right] = 0 \\
&\tilde{Y}_{e,n}^{(in)}(s) \to sC_n^{(in)}, \quad \tilde{Y}_{e,n}^{(ex)}(s) \to sC_n^{(ex)} \\
&\tilde{Y}_{e,n}(s) \to sC_n, \quad C_n = C_n^{(in)} + C_n^{(ex)}
\end{aligned}
\tag{8.1}
$$

Similarly for the complementary magnetic mode we have

$$
\begin{aligned}
&\nabla_s \cdot \tilde{\vec{j}}_{s_{h,n}}(\vec{r}_s,0) = 0, \quad \vec{1}_S(\vec{r}_s)\cdot\left[\nabla_s \times \tilde{\vec{j}}_{s_{e,n}}(\vec{r}_s,0)\right] \neq 0 \\
&\tilde{Y}_{h,n}^{(in)}(s) \to \left[sL_n^{(in)}\right]^{-1}, \quad \tilde{Y}_{h,n}^{(ex)}(s) \to \left[sL_n^{(ex)}\right]^{-1} \\
&\tilde{Y}_{h,n}(s) \to \left[sL_n\right]^{-1}, \quad L_n^{-1} = L_n^{(in)^{-1}} + L_n^{(ex)^{-1}}
\end{aligned}
\tag{8.2}
$$

These are generally related as

$$
\begin{aligned}
&\tilde{\vec{j}}_{s_{h,n}}(\vec{r}_s,s) = \vec{1}_S(\vec{r}_s)\times\tilde{\vec{j}}_{s_{e,n}}(\vec{r}_s,s), \tilde{\vec{j}}_{s_{e,n}}(\vec{r}_s,s) = -\vec{1}_S(\vec{r}_s)\times\tilde{\vec{j}}_{s_{h,n}}(\vec{r}_s,s) \\
&\tilde{y}_{e,n}^{(in)}(s) = \tilde{y}_{h,n}^{(in)^{-1}}(s) \ , \tilde{y}_{e,n}^{(ex)}(s) = \tilde{y}_{h,n}^{(ex)^{-1}}(s) \\
&\frac{C_n^{(in)}}{\varepsilon_o} = \frac{L_n^{(in)}}{\mu_o}, \frac{C_n^{(ex)}}{\varepsilon_o} = \frac{L_n^{(ex)}}{\mu_o}
\end{aligned}
\tag{8.3}
$$

While for certain geometries (such as a sphere [1,8]) the modes are frequency-independent, purely real-vector, and pure electric or magnetic, there is reason to expect that these properties do not all extend to arbitrarily shaped scatterers. As discussed in Appendices B and C, tangential vector fields and hence, surface current density on S can be expanded in terms of solenoidal and irrotational parts. Apply this to each eigenmode in the form

$$
\begin{aligned}
&\tilde{\vec{j}}_{s_{e,n}}(\vec{r}_s,s) = \nabla_s\tilde{\Phi}_{e,n}(\vec{r}_s,s) - \vec{1}_S(\vec{r}_s)\times\nabla_s\tilde{\Phi}_{h,n}(\vec{r}_s,s) \\
&\tilde{\vec{j}}_{s_{h,n}}(\vec{r}_s,s) = \nabla_s\tilde{\Phi}_{h,n}(\vec{r}_s,s) + \vec{1}_S(\vec{r}_s)\times\nabla_s\tilde{\Phi}_{e,n}(\vec{r}_s,s)
\end{aligned}
\tag{8.4}
$$

So, there are only two scalar functions needed to characterize the mode pair. These satisfy the Poisson equation on S

$$\nabla_s^2 \tilde{\Phi}_{\substack{e\\h},n}(\vec{r}_s,s) = -\rho_{s_{\substack{e\\h},n}}(\vec{r}_s,s) \equiv \nabla_s \cdot \tilde{\vec{j}}_{s_{\substack{e\\h},n}}(\vec{r}_s,s) = \mp \nabla_s \cdot \left[\vec{1}_S(\vec{r}_s) \times \tilde{\vec{j}}_{s_{\substack{h\\e},n}}(\vec{r}_s,s) \right] \tag{8.5}$$

which can be solved as in Appendix B. Note that this decomposition applies as well in combination with operations that commute with the operations in (8.4), such as real and imaginary parts, even and odd parts (with respect to s), etc. For convenience we have defined

$$\begin{aligned} \rho_{s_{e,n}}(\vec{r}_s,s) &\equiv -\nabla_s \cdot \tilde{\vec{j}}_{s_{e,n}} \equiv \text{surface-charge-density eigenmodes} \\ \rho_{s_{h,n}}(\vec{r}_s,s) &\equiv -\nabla_s \cdot \tilde{\vec{j}}_{s_{h,n}} \equiv \text{equivalent-magnetic-charge-density eigenmodes} \end{aligned} \tag{8.6}$$

Noting that

$$\int_S \tilde{\Phi}_{e,n}(\vec{r}_s,s)dS = 0 = \int_S \tilde{\Phi}_{h,n}(\vec{r}_s,s)dS \tag{8.7}$$

together with the low-frequency properties of the eigenmodes in (8.1) and (8.2) we have

$$\begin{aligned} &\nabla_s \cdot \tilde{\vec{j}}_{s_{h,n}}(\vec{r}_s,0) = 0 = \nabla_s^2 \tilde{\Phi}_{h,n}(\vec{r}_s,0) \\ &\tilde{\Phi}_{h,n}(\vec{r}_s,0) = 0 \end{aligned} \tag{8.8}$$

With the orthonormalization condition and Green's Theorem we have

$$\begin{aligned} \left\langle \tilde{\vec{j}}_{s_{e,n}}(\vec{r}_s,0);\tilde{\vec{j}}_{s_{e,n}}(\vec{r}_s,0)\right\rangle = 1 &= \left\langle \nabla_s\tilde{\Phi}_{e,n}(\vec{r}_s,0);\nabla_s\tilde{\Phi}_{e,n}(\vec{r}_s,0)\right\rangle \\ &= -\left\langle \nabla_s^2\tilde{\Phi}_{e,n}(\vec{r}_s,0);\tilde{\Phi}_{e,n}(\vec{r}_s,0)\right\rangle \\ &= \left\langle \tilde{\Phi}_{e,n}(\vec{r}_s,0);\tilde{\rho}_{s_{e,n}}(\vec{r}_s,0)\right\rangle \end{aligned}$$

as a kind of normalization for the electrostatic potential mode. Note also that

$$\nabla_s \cdot \left[\vec{1}_S(\vec{r}_s) \times \tilde{\vec{j}}_{s_{h,n}}(\vec{r}_s,0) \right] = -\nabla_s \cdot \tilde{\vec{j}}_{s_{e,n}}(\vec{r}_s,0) \tag{8.9}$$

and is thereby described by the electric potential above.

For general complex frequencies the orthonormalization equations mix the electric and magnetic potentials. For both modes of the same type (electric, magnetic) but with different indices (n_1,n_2) we have

$$
\begin{aligned}
&\left\langle \tilde{\vec{j}}_{s_{e,n_1}}(\vec{r}_s,s);\tilde{\vec{j}}_{s_{e,n_2}}(\vec{r}_s,s)\right\rangle = \left\langle \tilde{\vec{j}}_{s_{h,n_1}}(\vec{r}_s,s);\tilde{\vec{j}}_{s_{h,n_2}}(\vec{r}_s,s)\right\rangle = 1_{n_1,n_2} \\
&= \left\langle \nabla_s\tilde{\Phi}_{e,n_1}(\vec{r}_s,s);\nabla_s\tilde{\Phi}_{e,n_2}(\vec{r}_s,s)\right\rangle + \left\langle \vec{1}_S(\vec{r}_s)\times\nabla_s\tilde{\Phi}_{h,n_1}(\vec{r}_s,s);\vec{1}_S(\vec{r}_s)\times\nabla_s\tilde{\Phi}_{h,n_2}(\vec{r}_s,s)\right\rangle \\
&+\left\langle \nabla_s\tilde{\Phi}_{e,n_1}(\vec{r}_s,s);\vec{1}_S(\vec{r}_s)\times\nabla_s\tilde{\Phi}_{e,n_2}(\vec{r}_s,s)\right\rangle - \left\langle \vec{1}_S(\vec{r}_s)\times\nabla_s\tilde{\Phi}_{h,n_1}(\vec{r}_s,s);\nabla_s\tilde{\Phi}_{h,n_2}(\vec{r}_s,s)\right\rangle \\
&= \left\langle \nabla_s\tilde{\Phi}_{e,n_1}(\vec{r}_s,s);\nabla_s\tilde{\Phi}_{e,n_2}(\vec{r}_s,s)\right\rangle + \left\langle \nabla_s\tilde{\Phi}_{h,n_1}(\vec{r}_s,s);\nabla_s\tilde{\Phi}_{h,n_2}(\vec{r}_s,s)\right\rangle \\
&= \left\langle \tilde{\Phi}_{e,n_1}(\vec{r}_s,s);\tilde{\rho}_{s_{e,n_2}}(\vec{r}_s,s)\right\rangle + \left\langle \tilde{\Phi}_{h,n_1}(\vec{r}_s,s);\tilde{\rho}_{s_{h,n_2}}(\vec{r}_s,s)\right\rangle
\end{aligned}
\tag{8.10}
$$

Here we have also used the identities [27] for the divergence of a scalar times a vector and note that the divergence of $\vec{1}_S$ cross the surface gradient is zero. For a mixture of electric and magnetic eigenmodes the result is zero instead of the $1_{n_1,n_2}$ (Kronecker delta).

Also for each eigenmode the integral of surface divergence and normal surface curl is zero as

$$
\int_S \tilde{\rho}_{s_{\substack{e\\h},n}}(\vec{r}_s,s) = 0 \tag{8.11}
$$

Stated another way, mode by mode the net surface charge and equivalent magnetic charge/meter is zero, i.e. these parameters are conserved.

Besides the division of the eigenmodes into E and H modes, there is the question of assigning particular values of n to particular modes. One approach concerns the capacitance/inductance associated with the eigenadmittances as $s \to 0$. Noting for the sphere [8] that $C_n^{(in)}$ decreases with increasing n we might choose

$$
\begin{aligned}
&C_1^{(in)} \geq C_2^{(in)} \geq C_3^{(in)} \geq \ldots \geq 0 \\
&\frac{C_n^{(in)}}{\varepsilon_o} = \frac{L_n^{(in)}}{\mu_o}
\end{aligned}
\tag{8.12}
$$

Similarly the sphere has $C_n^{(ex)}$ decreasing so we might use

$$
\begin{aligned}
&C_1^{(ex)} \geq C_2^{(ex)} \geq C_3^{(ex)} \geq \ldots \geq 0 \\
&\frac{C_n^{(ex)}}{\varepsilon_o} = \frac{L_n^{(ex)}}{\mu_o}
\end{aligned}
\tag{8.13}
$$

We may be fortunate in practical problems (or even in general) if these two ways to assign n give the same results, but this is not obvious. One might also use

$$C_n = C_n^{(ex)} + C_n^{(in)} \tag{8.14}$$

or

$$L_n = \left[L_n^{(ex)} + L_n^{(in)} \right]^{-1} \tag{8.15}$$

for this purpose. A more traditional approach might count the number of "halfwaves" in the eigenmodes to assign n, as is the traditional approach for separable geometries of sphere, circular cylinder, etc. Whether this would exactly correspond to the previous ordering schemes is not clear. Then, more than one eigenmode may have the same number of "wiggles", but look different as a spatial distribution. The case of the sphere illustrates such a situation and the index is the pair (n,m). Perhaps one should adopt something similar here. Further research, both analytical and numerical, may help us here. For the present the modal indexing scheme is perhaps best left at least partly open.

IX. Eigenimpedances with Uniform Sheet Impedance Loading on S

Now it is appropriate to revisit an earlier result concerning the loading of S by an impedance sheet of the form

$$\begin{aligned}\overleftrightarrow{1}_S(\vec{r}_s)\cdot\tilde{\vec{E}}(\vec{r}_s,s) &= \overleftrightarrow{1}_S(\vec{r}_s)\cdot\left[\tilde{\vec{E}}^{(inc)}(\vec{r}_s,s)+\tilde{\vec{E}}^{(sc)}(\vec{r}_s,s)\right]\\ &= \tilde{Z}_l(s)\tilde{\vec{J}}(\vec{r}_s,s) \qquad (9.1)\\ \tilde{Z}_l(s) &\equiv \text{ sheet impedance}\end{aligned}$$

where this is taken in the simple scalar and position-independent form. Then the E-field integral equation is modified to the form [2,15,26]

$$\overleftrightarrow{1}_S(\vec{r}_s)\cdot\tilde{\vec{E}}^{(inc)}(\vec{r}_s,s)=\left\langle\tilde{\overleftrightarrow{Z}}_t(\vec{r}_s,\vec{r}_s{}';s)+\tilde{Z}_l(s)\overleftrightarrow{1}\delta_s(\vec{r}_s-\vec{r}_s{}');\tilde{\vec{J}}_s(\vec{r}_s{}',s)\right\rangle \qquad (9.2)$$

which has eigenmodes and eigenvalues

$$\begin{aligned}\tilde{\vec{j}}_{s_\beta}(\vec{r}_s,s) &\xrightarrow[\text{loading}]{\text{impedance}} \tilde{\vec{j}}_{s_\beta}(\vec{r}_s,s)\\ \tilde{Z}_\beta(s) &\xrightarrow[\text{loading}]{\text{impedance}} \tilde{Z}_\beta(s)+\tilde{Z}_l(s) \qquad (9.3)\end{aligned}$$

i.e. the eigenmodes are unchanged, but the new eigenvalues (also being eigenimpedances) are changed by the simple addition of the loading sheet impedance. As discussed previously one can use this result for synthesizing a $\tilde{Z}_l(s)$ which makes $\tilde{Z}_\beta(s)+\tilde{Z}_l(s)$ have prescribed zeros of various orders, thereby giving prescribed poles (including higher order poles) in the response of the scatterer.

With our present results $\tilde{Z}_\beta(s)$ has been decomposed into exterior and interior parts. Together with the impedance loading the new eigenimpedances have the series/parallel circuit representation in fig. 9.1. This is an interesting representation of the modal decomposition of the scattering problem, remembering the modes also come in pairs (electric and magnetic).

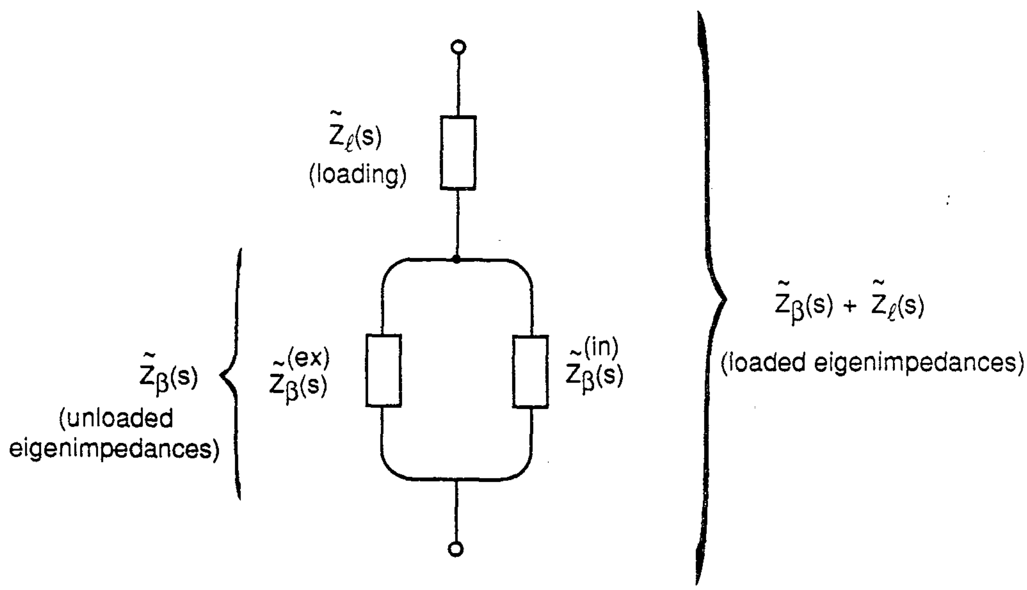

Fig. 9.1. Eigenimpedance Decomposition

X. Concluding Remarks

The diagonalization of the impedance (E-field) integral equation has yielded a rich structure involving mode pairing (electric and magnetic) and decomposition of the eigenimpedances into internal and external parts. Some similar results can also be found for the pseudosymmetric H-field integral equation [3,4].

Various extensions of the results can be pursued. Central to the discussion is the closed surface S dividing V_{in} from V_{out}. This has allowed the separate representation of the eigenadmittance parts as integrals involving exterior and interior incident waves. If S is opened this complicates matters. However, one can consider an open S as a limit of a closed S which has been deformed so that it consists of two surfaces which are some small distance apart and joined at the edges. Alternately, think of an open S as having some thickness to separate the two sides for application of boundary conditions. Finally, of course, one would let the thickness tend to zero.

It would be helpful to have more information on the spatial and frequency properties of the eigenmodes. By letting $s \to \infty$ in the RHP one may be able to convert the integral equation to a differential equation on S due to the concentration of the contribution from the kernel to positions where $\vec{r}_S'$ is near $\vec{r}_S$. Numerical experiments involving various shapes of S should be able to shed more light on the properties of these modes.

Appendix A. Duality

Duality is the symmetry in the Maxwell equations with respect to interchanging electric and magnetic parameters [11]. This is compactly represented by the combined field

$$\vec{E}_q(\vec{r},t) = \vec{E}(\vec{r},t) + qjZ_o\vec{H}(\vec{r},t)$$
$$q = \pm 1 \equiv \text{ separation index} \tag{A.1}$$

and combined current density

$$\vec{J}_q(\vec{r},t) = \vec{J}(\vec{r},t) + \frac{qj}{Z_o}\vec{J}_m(\vec{r},t)$$
$$\vec{J}(\vec{r},t) \equiv \text{ electric current density}$$
$$\vec{J}_m(\vec{r},t) \equiv \text{ magnetic current density, in general fictitious but useful as an "equivalent" magnetic current density in various problems} \tag{A.2}$$

The combined Maxwell equations become

$$\left[\nabla \times \frac{-qj}{c}\frac{\partial}{\partial t}\right]\vec{E}_q(\vec{r},t) = qjZ_o\vec{J}_q(\vec{r},t) \tag{A.3}$$

which in Laplace form is

$$[\nabla \times \; -qj\gamma]\tilde{\vec{E}}_q(\vec{r},s) = qjZ_o\tilde{\vec{J}}_q(\vec{r},s) \tag{A.4}$$

Note now that the combined quantities are not conjugate symmetric since the time-domain quantities are complex. Now we have

$$\tilde{\vec{E}}_q(\vec{r},s^*) = \tilde{\vec{E}}^*_{-q}(\vec{r},s) \tag{A.5}$$

and similarly for other parameters.

Dual parameters are found by multiplication by -qj as

$$\vec{E}_q^{(d)}(\vec{r},t) = -qj\vec{E}(\vec{r},t)$$
$$\vec{E}^{(d)}(\vec{r},t) = Z_o\vec{H}(\vec{r},t), \quad \vec{H}^{(d)}(\vec{r},t) = -\frac{1}{Z_o}\vec{E}(\vec{r},t)$$
$$\vec{J}_q^{(d)}(\vec{r},t) = -\frac{qj}{Z_o}\vec{J}_q(\vec{r},t) \tag{A.6}$$
$$\vec{J}^{(d)}(\vec{r},t) = \frac{1}{Z_o}\vec{J}_m(\vec{r},t), \quad \vec{J}_m^{(d)}(\vec{r},t) = -Z_o\vec{J}(\vec{r},t)$$

These satisfy the Maxwell equations just as well as the original (or "undual") parameters. Note that in transforming to the dual fields the dual current densities are also involved. However, in regions away from current densities no current densities are involved, and both sets of fields satisfy the free-space Maxwell equations (and radiation condition where appropriate).

Appendix B. Decomposition of Tangential Fields on a Surface

The Helmholtz theorem [9,12,19,27] allows one to decompose a volume current density into a solenoidal (divergenceless) and an irrotational (curlless) part. Less well known is the analogous decomposition on a surface. In [27] this is worked out for a sphere. Following this procedure we see that the result applies for a simply connected closed surface and can readily be extended to a multiply connected closed surface.

Consider a tangential vector field $\vec{F}(\vec{r}_S)$ on S (fig. B.1) with suitable smoothness (differentiability). Then we wish too express $\vec{F}$ in the form

$$\begin{aligned} &\vec{F}(\vec{r}_S) = \nabla_s U(\vec{r}_S) + \vec{1}_S(\vec{r}_S) \times \nabla_s V(\vec{r}_S) \\ &U, V \equiv \text{ scalar potentials} \end{aligned} \tag{B.1}$$

with the usual definition of the ∇_s operator [18,27]. One can impose the condition

$$\int_S U(\vec{r}_s) dS = 0 = \int_V V(\vec{r}_s) dS \tag{B.2}$$

which merely fixes the additive constant. Adding an arbitrary constant to these potentials in no way affects the surface gradients. To derive this, following [27], form

$$\begin{aligned} &\nabla_s \cdot \vec{F}(\vec{r}_S) = \nabla_s^2 U(\vec{r}_s) \\ &\nabla_s \cdot \left[\vec{F}(\vec{r}_s) \times \vec{1}_S(\vec{r}_s)\right] = \nabla_s^2 V(\vec{r}_s) = \vec{1}_S(\vec{r}_s) \cdot \left[\nabla_s \times \vec{F}(\vec{r}_s)\right] \end{aligned} \tag{B.3}$$

using various relations on a surface, here and in the following. Note that

$$\begin{aligned} &\int_S \nabla_s^2 U(\vec{r}_s) dS = \int_S \nabla_s \cdot \vec{F}(\vec{r}_S) dS = -\oint_{C\to 0} \vec{F}(\vec{r}_S) \cdot \vec{1}_m dl = 0 \\ &\int_S \nabla_2^2 V(\vec{r}_s) dS = \int_S \nabla_s \cdot \left[\vec{F}(\vec{r}_s) \times \vec{1}_S(\vec{r}_s)\right] dS = -\oint_{C\to 0} \vec{F}(\vec{r}_s) \cdot \vec{1}_t d\ell = 0 \end{aligned} \tag{B.4}$$

which admits constant solutions for U and V (and which are constrained by (B.2)). Note that this last result assumes that the closed contour on S can be shrunk to a contour of zero radius $(C \to 0)$ on S with $\vec{F}$ not too singular. This we can do provided S is simply connected.

Note that ∇_s^2 is self adjoint on S and we have the Green's theorems

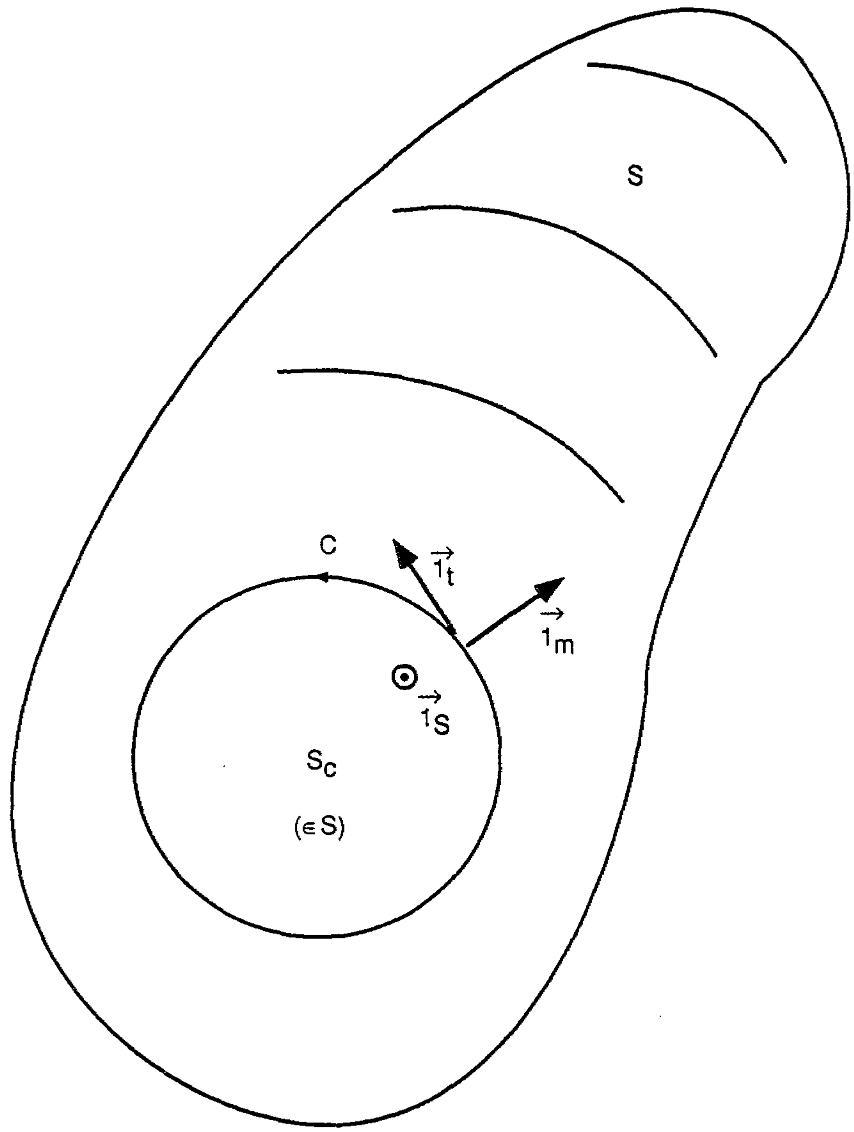

Fig. B.1. Simple Closed Surface with Small Portion S_c Bounded by Simple Closed Contour C

$$\int_S \left[U_1(\vec{r}_s)\nabla_s^2 U_2(\vec{r}_s) - U_2(\vec{r}_s)\nabla_s^2 U_1(\vec{r}_s)\right] dS = 0$$
$$\int_S \left[U_1(\vec{r}_s)\nabla_s^2 U_2(\vec{r}_s) + [\nabla_s U_1(\vec{r}_s)]\cdot[\nabla_s U_2(\vec{r}_s)]\right] dS = 0 \tag{B.5}$$

So solutions of the eigenvalue equation

$$\nabla_s^2 \Phi_\ell(\vec{r}_s) = \chi_\ell \Phi_n(\vec{r}_s) \tag{B.6}$$

form a complete orthonormal set with real eigenvalues and real eigenfunctions. These are not then the same as the modes being considered in this paper, except in special cases

Solutions of the homogeneous equation

$$\nabla_s^2 U_o(\vec{r}_s) = 0 \tag{B.7}$$

must be orthogonal to the forcing function as

$$\int_S U_o(\vec{r}_s)\nabla_s \cdot \vec{F}_s(\vec{r}_s) dS = 0 \tag{B.8}$$

Hence we have

$$\int_S \nabla_s [U_o(\vec{r}_s)\vec{F}(\vec{r}_s) dS] - \int_S [\nabla_s U_o(\vec{r}_s)]\cdot\vec{F}(\vec{r}_s) dS = 0 \tag{B.9}$$

But the first term being zero, we have

$$\int_S [\nabla_s U_o(\vec{r}_s)]\cdot\vec{F}(\vec{r}_s) dS = 0 \tag{B.10}$$

Now $\vec{U}_o$ does not depend of $\vec{F}$, so chose $\vec{F}$ arbitrarily, say as $\vec{0}$ outside of some small region S_o, and a constant tangential vector in S_o. This requires within S_o

$$\nabla_s U_o(\vec{r}_s) = \vec{1}_a \frac{1}{h_a}\frac{\partial U_o(\vec{r}_s)}{\partial u_a} + \vec{1}_b \frac{1}{h_b}\frac{\partial U_o(\vec{r}_s)}{\partial u_b} = 0 \tag{B.11}$$

$u_a, u_b \equiv$ local orthogonal curvilinear coordinates on S

However, S_o can be chosen as any small region of S, so that the gradient must be everywhere zero on S. The only solution is

$$U_o(\vec{r}_s) = \text{constant} \tag{B.12}$$

which is zero if (B.2) is imposed. The same applies for V_0 by using $\vec{F} \times \vec{1}_S$.

Besides the potentials U and V one needs to know if any other terms are needed to express a general $\vec{F}$ (tangential on S). Define

$$\vec{F}'(\vec{r}_s) \equiv \vec{F}(\vec{r}_s) - \nabla_s U(\vec{r}_s) - \vec{1}_S(\vec{r}_s) \times \nabla_s V(\vec{r}_s) \tag{B.13}$$

which implies

$$\nabla_s \cdot \vec{F}'(\vec{r}_s) = 0 = \nabla_s \cdot [\vec{F}'(\vec{r}_s) \times \vec{1}_S(\vec{r}_s)] \tag{B.14}$$

However, we have

$$\begin{aligned} 0 &= \int_{S_c} \nabla_s \cdot [\vec{F}'(\vec{r}_s) \times \vec{1}_S(\vec{r}_s)] dS = \int_C [\vec{F}'(\vec{r}_s) \times \vec{1}_S(\vec{r}_s)] \cdot \vec{1}_m(\vec{r}_s) dS \\ &= \int_C \vec{F}'(\vec{r}_s) \cdot \vec{1}_t(\vec{r}_s) dS \end{aligned} \tag{B.15}$$

which implies that $\vec{F}'$ is the surface gradient of some scalar function [18], say U', but then

$$\nabla_s^2 U'(\vec{r}_s) = \nabla_s \cdot \vec{F}'(\vec{r}_s) = 0 \tag{B.16}$$

with the only solution

$$\begin{aligned} U'(\vec{r}_s) &= \text{ constant} \\ \nabla U'(\vec{r}_s) &= \vec{F}'(\vec{r}_s) = \vec{0} \end{aligned} \tag{B.17}$$

implying that the solution (B.1) is unique.

The problem is then reduced to finding the solution to problems of the form

$$\nabla_s^2 \Phi(\vec{r}_s) = \xi(\vec{r}_s) \tag{B.18}$$

which can be done via the eigenvalue equation (B.6) with

$$\left\langle \Phi_{l_1}(\vec{r}_s), \Phi_{l_2}(\vec{r}_s) \right\rangle = \vec{1}_{l_1, l_2} \tag{B.19}$$

A Green's function can be formed

$$g(\vec{r}_s,\vec{r}_s') = \sum_l \chi_l^{-1}\Phi_l(\vec{r}_s)\Phi_l(\vec{r}_s')$$
$$\nabla_s^2 g(\vec{r}_s,\vec{r}_s') = \vec{1}_S(\vec{r}_s)\delta_s(\vec{r}_s-\vec{r}_s') = \sum_l \Phi_l(\vec{r}_s)\Phi_l(\vec{r}_s') \tag{B.20}$$

by which (B.18) is solved as

$$\Phi(\vec{r}_s) = \sum_l \chi_l^{-1}\langle\Phi_l(\vec{r}_s');\xi(\vec{r}_s')\rangle\Phi_l(\vec{r}_s) \tag{B.21}$$

Note for all eigenfunctions by (B.2) that

$$\int_S \Phi_l(\vec{r}_s') = 0 \tag{B.22}$$

It has also been shown [13] that

$$g(\vec{r}_s,\vec{r}_s') = \frac{1}{2\pi}\ln\left(\frac{|\vec{r}_s-\vec{r}_s'|}{l'}\right) \text{ as } \vec{r}_s' \to \vec{r}_s \tag{B.23}$$
$l' \equiv$ reference length

which is a logarithmic singularity as one would expect for a scalar problem in two dimensions.

If one considers a multiply connected surface as in fig. B.2, some new characteristics enter for tangential vector fields on S. Now we have contours, such as C_{a1} which are closed but cannot be shrunk to a point without leaving S. As such the conditions of (B.4) springing from $C \to 0$ are perhaps not fulfilled. At a minimum we can consider a term in the potential given by a jump $U_{a1}^{(0)}$ in crossing the contour C_{b1} as one goes around C_{a1}. This corresponds to a vector $\vec{F}_{a1}$ parallel to the orientation of C_{a1} with

$$\oint_{C_{a1}} \vec{F}_{a1}(\vec{r}_s)\cdot d\vec{l} = \oint_{C_{a1}} \nabla_s U_{a1}(\vec{r}_s)\cdot d\vec{l} = U_{a1}^{(o)}$$

$$\vec{F}_{a1}(\vec{r}_s) = \nabla_s U_{a1}(\vec{r}_s) \tag{B.24}$$

A simple example of this is the static current flowing in a simple perfectly conducting loop driven from a source at a gap, or immersed in a magnetic field. Then one can have cases with

$$\nabla_s\cdot\vec{F}_{a1}(\vec{r}_s) = 0 = \nabla_s^2 U_{a1}(\vec{r}_s) \tag{B.25}$$

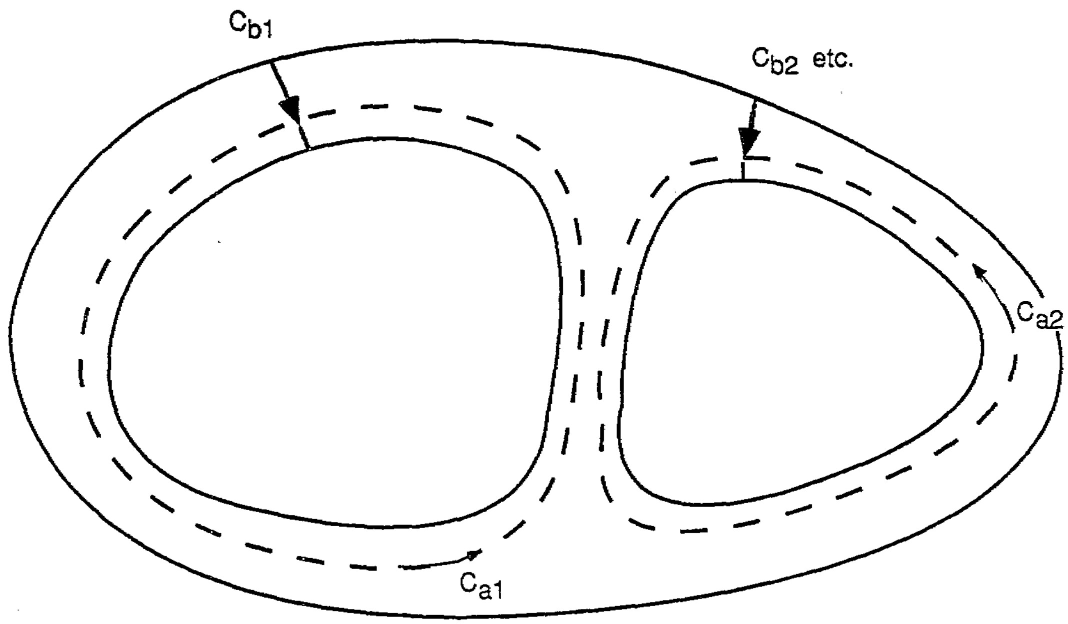

Fig. B.2. Multiply-Connected Closed Surface so that the potential satisfies a homogeneous (i.e. Laplace) equation, subject to the jump boundary condition at C_{b1}. In fact we can also have

$$\nabla_s \cdot [\vec{1}_S(\vec{r}_s) \times \vec{F}_{a1}(\vec{r}_s)] = -\vec{1}_S(\vec{r}_s) \cdot [\nabla_s \times \vec{F}_{a1}(\vec{r}_s)] \tag{B.26}$$

i.e. the tangential vector field can have both zero surface divergence and zero normal surface curl. Physically this corresponds to a circulating current with no charge, as in the case of the perfectly conducting ring in an incident static magnetic field.

We can also have a tangential vector field $\vec{F}_{b1}$ parallel to C_{b1} and crossing C_{a1}. If one wishes this can be identified with a U or V potential in (B.1). If one uses

$$\vec{F}_{b1}(\vec{r}_s) = \nabla_s U_{b1}(\vec{r}_s) \tag{B.27}$$

one can also think of this vector field as orthogonal (complementary) to $\vec{F}_{a1}$, but with potential jump $V_{b1}^{(0)}$ across C_{a1}. This is like the current in a toroidal coil, bringing current into and out of the page in fig. B.2, instead of parallel to the page. So associated with a handle in a multiply connected surface one has these two homogeneous solutions (harmonic functions) to include.

One can also have a closed surface which is not connected (not all in one piece) as in fig. B.3. In this case the closed contour C can exist on only one of the closed pieces. Physically these correspond to electrostatic modes with perhaps various potentials and total charges on each piece. Tangential vector fields on each piece can be treated by the regular procedures here. See also [1 (App. A)].

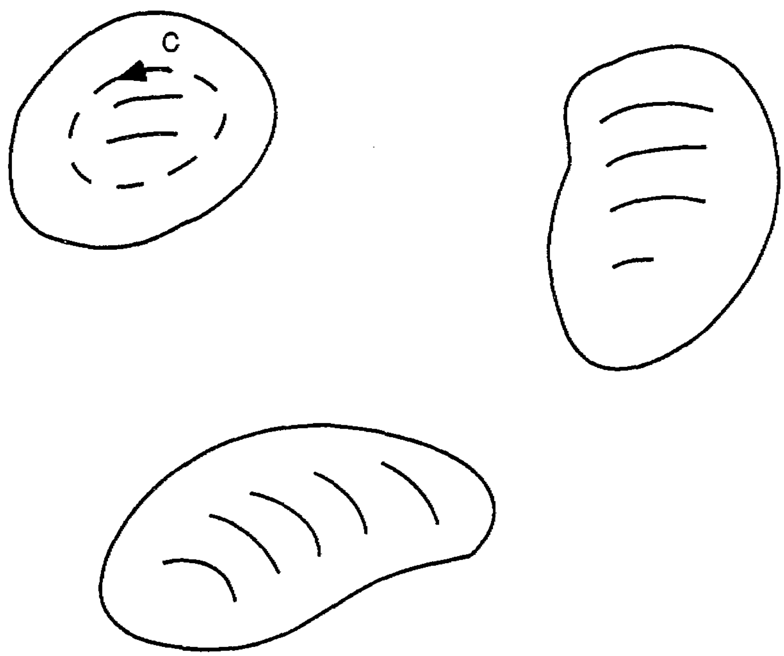

Fig. B.3. Unconnected Closed Surface

Appendix C. Decomposition of Surface Current Density

Apply the decomposition procedure in Appendix B to a general surface current density on S as

$$
\begin{aligned}
\tilde{\vec{J}}_s(\vec{r}_s,s) &= s\nabla_s\tilde{\Phi}_e(\vec{r}_s,s) - \frac{1}{\mu_o}\vec{1}_S(\vec{r}_s)\times\nabla_s\tilde{\Phi}_h(\vec{r}_s,s) \\
\vec{J}_s(\vec{r}_s,t) &= \frac{\partial}{\partial t}\nabla_s\Phi_e(\vec{r}_s,t) - \frac{1}{\mu_o}\vec{1}_S(\vec{r}_s)\times\nabla_s\tilde{\Phi}_h(\vec{r}_s,t)
\end{aligned}
\tag{C.1}
$$

Then we have

$$
\begin{aligned}
&\nabla_s^2\Phi_e(\vec{r}_s,t) = -\rho_s(\vec{r}_s,t) \equiv \text{ surface charge density} \\
&\qquad\qquad\qquad\qquad\text{(dimension Coulombs / meter}^2\text{)} \\
&\nabla_s\cdot\vec{J}_s(\vec{r}_s,t) = -\frac{\partial}{\partial t}\rho_s(\vec{r}_s,t) \text{ (electric continuity equation)} \\
&\Phi_e(\vec{r}_s,t) \equiv \text{ surface - charge - density potential} \\
&\qquad\text{(dimension Coulombs)} \\
&\nabla_s^2\Phi_h(\vec{r}_s,t) = -\rho_h(\vec{r}_s,t) \equiv \text{ equivalent magnetic - charge - density} \\
&\qquad\qquad\qquad\qquad\text{(dimension Webers / meter}^3\text{)} \\
&\nabla_s\cdot[\vec{1}_S(\vec{r}_s)\times\vec{J}_s(\vec{r}_s,t)] = -\frac{1}{\mu_o}\rho_h(\vec{r}_s,t) \quad \text{(magnetic continuity equation)} \\
&\Phi_h(\vec{r}_s,t) \equiv \text{ eqivalent - magnetic - charge - density potential} \\
&\qquad\text{(dimension Webers / meter)}
\end{aligned}
\tag{C.2}
$$

Note that these potentials are defined on the two-dimensional space of S, and are not the usual scalar and vector potentials, They are not formulated in a retarded time, but are static like. While they are defined via the surface Laplacian, and have real solutions for real forcing functions, the more general complex forcing functions give complex potentials. The electric potential is associated wit the usual surface charge density. The magnetic potential is associated with the equivalent magnetic charge density [5,10] which is related to the normal derivative of the normal magnetic field at S. With dimensions Webers/m^3 its integral over any portion of S has dimension of Webers/m, not Webers corresponding to magnetic charge.

Note that both potentials satisfy the same equation

$$
\nabla_s^2\Phi(\vec{r}_s,t) = -u(\vec{r}_s,t) \tag{C.3}
$$

with, of course, their respective electric and magnetic sources specifying u. The solutions for the electric and magnetic potentials then take the same form with the same Green's function as in (B.20) and (B.21).

A consequence of (B.2) is the convenient result

$$\int_S \Phi_e(\vec{r}_s,t)dS = 0$$
$$\int_S \Phi_h(\vec{r}_s,t)dS = 0 \tag{C.4}$$

although a constant shift in potential is not important. Of greater significance (B.4) implies

$$Q(t) = \int_S \rho(\vec{r}_s,t)dS = 0 \text{ (electric charge with zero initial conditions)}$$
$$\int_S \nabla_s \cdot \vec{J}_s(\vec{r}_s,t)dS = 0 \tag{C.5}$$

which is just conservation of charge. Similarly we have

$$q_h(t) = \int_S \rho_h(\vec{r}_s,t)dS = -\mu_o \int_S \nabla_s \cdot [\vec{1}_S(\vec{r}_s)\times\vec{J}_s(\vec{r}_s,t)]dS = 0 \tag{C.6}$$
(equivalent magnetic charge / meter)

This magnetic parameter is then conserved just like electric charge.

Defining the complement of the surface current density we have

$$\tilde{\vec{J}}_s^{(c)}(\vec{r}_s,s) \equiv \vec{1}_S(\vec{r}_s)\times\tilde{\vec{J}}_s(\vec{r}_s,s) = \frac{1}{\mu_o}\nabla_s\tilde{\Phi}_h(\vec{r}_s,s) + s\vec{1}_S(\vec{r}_s)\times\tilde{\Phi}_e(\vec{r}_s,s) \tag{C.7}$$

Here note the interchange of the role of the electric and magnetic potentials as

$$\begin{aligned}
\tilde{\vec{J}}_s(\vec{r}_s,s) &\longrightarrow \tilde{\vec{J}}_s^{(c)}(\vec{r}_s,s)\\
s\tilde{\Phi}_e(\vec{r}_s,s) &\longrightarrow \frac{1}{\mu_o}\tilde{\Phi}_h(\vec{r}_s,s)\\
-\frac{1}{\mu_o}\tilde{\Phi}_h(\vec{r}_s,s) &\longrightarrow s\tilde{\Phi}_e(\vec{r}_s,s)
\end{aligned} \tag{C.8}$$

corresponding to the interchange of the roles of surface divergence and normal component of surface curl.

As discussed in Appendix B, multiply connected bodies as in fig. B.2 need special consideration. One can have surface-current -density distributions on S with both zero surface divergence and zero normal surface curl. Such can be described by the surface gradient of a scalar with a jump condition in going around a closed path which cannot be shrunk to zero on S.

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