Interaction Notes

Note 490

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Approximation of Non-Uniform Multiconductor Transmission Lines
by Analytically Solvable Sections.

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Abstract

This paper considers the propagation of waves on non-uniform multiconductor transmission lines (NMTLs). Extending the results of previous papers, general NMTLs are considered as cascaded sections of shorter transmission lines, both uniform and non-uniform. The individual sections are considered in two forms, based on the diagonalization of the propagation matrix, or in the case of equal modal speeds based on the diagonalization of a normalized impedance matrix. Canonical forms of the variation of the appropriate eigenvalues are considered, giving closed-form expressions for the matrices (renormalized matrizants) describing the individual sections.

Contents

I. Introduction

In considering the propagation of signals along multiconductor transmission lines (MTL), the equations reduce to convenient forms if the MTL is uniform, i.e. has the impedance-per-unit-length matrix $\left(\tilde{Z}'_{n,m}(x,s)\right)$ and admittance-per-unit-length matrix $\left(\tilde{Y}'_{n,m}(x,s)\right)$ (as in fig. 1.1) independent of the spatial coordinate z (one-dimensional transmission-line approximation) [1]. In this case a certain linear combination of the voltage and current vectors and sources as

$$\left(\tilde{V}_n(z,s)\right)_q = \left(\tilde{V}_n(z,s)\right) + q\left(\tilde{Z}_{c_{n,m}}(s)\right)\cdot\left(\tilde{I}_n(z,s)\right)$$

$$\left(\tilde{V}'_{s_n}(z,s)\right)_q = \left(\tilde{V}_{s_n}(z,s)\right) + q\left(\tilde{Z}_{c_{n,m}}(s)\right)\cdot\left(\tilde{I}_{s_n}(z,s)\right)$$

$$n,m = 1,2,\cdots,N$$
$$q = \pm 1$$

$$\left(\tilde{Z}_{c_{n,m}}(s)\right) = \left(\tilde{\gamma}_{c_{n,m}}(s)\right)\cdot\left(\tilde{Y}'_{n,m}(s)\right)$$

$$= \left(\tilde{\gamma}_{c_{n,m}}(s)\right)^{-1}\cdot\left(\tilde{Z}'_{n,m}(s)\right) \tag{1.1}$$

$$\left(\tilde{\gamma}_{c_{n,m}}(s)\right) = \left[\left(\tilde{Z}'_{n,m}(s)\right)\cdot\left(\tilde{Y}'_{n,m}(s)\right)\right]^{\frac{1}{2}} \qquad \text{(positive real (p.r.) square root)}$$

where +1 corresponds to right-going $(+z)$ waves and -1 to left-going $(-z)$ waves gives a (relatively) simple form to the solution, due to the resulting first-order differential equation

$$\left[\left(1_{n,m}\right)\frac{d}{dz} + q\left(\tilde{\gamma}_{c_{n,m}}(s)\right)\right]\cdot\left(\tilde{V}_n(z,s)\right)_q = \left(\tilde{V}'_{s_n}(z,s)\right)_q \tag{1.2}$$

Note that reciprocity is assumed here so that the matrices in (1.1) are symmetric (equal to their transpose).

In more recent papers [4,5] special results are obtained by assuming that the per-unit-length matrices and their derivatives with respect to z all commute. This extends to nonuniform multiconductor transmission lines (NMTLs) for the case that the matrices are circulant, and the resulting second-order differential equation scalarizes in terms of the eigenvalues of the propagation matrix $\left(\tilde{\gamma}_{c_{n,m}}(z,s)\right)$. If one specializes to a high-frequency approximation [3] there are other simplifications which occur leading to a

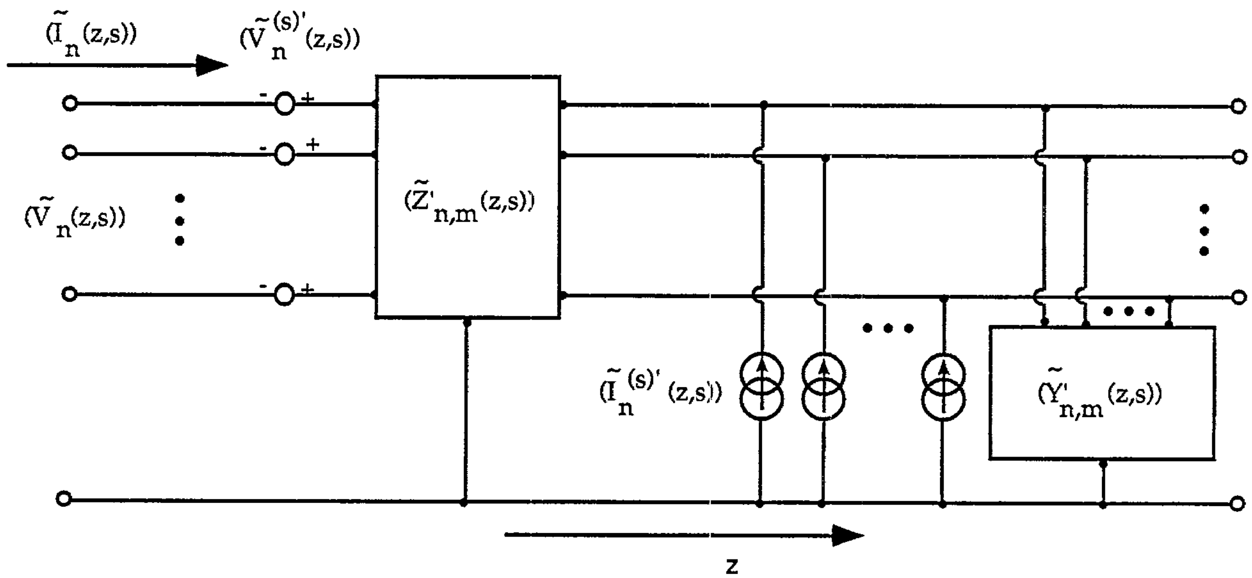

Fig. 1.1. Per-Unit-Length Equivalent Circuit of Multiconductor Transmission Line

generalized WKB approximation (for N-component vector waves) which can be solved in closed form in various special cases.

Consider now the general form of an NMTL, first from the telegrapher equations, for N conductors (plus reference), as

$$\frac{\partial}{\partial z}\left(\tilde{V}_n(z,s)\right) = -\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{I}_n(z,s)\right) + \left(\tilde{V}_n^{(s)'}(z,s)\right)$$
$$\frac{\partial}{\partial z}\left(\tilde{I}_n(z,s)\right) = -\left(\tilde{Y}'_{n,m}(z,s)\right)\cdot\left(\tilde{V}_n(z,s)\right) + \left(\tilde{I}_n^{(s)'}(z,s)\right) \tag{1.3}$$

These can be combined to give a single first-order differential equation of dimension $2N$ (vector components). Consider some NxN matrix which may be a function of frequency, but is not a function of the coordinate, and has dimension of impedance, which we use as

$$\frac{\partial}{\partial z}\begin{pmatrix}\left(\tilde{V}_n(z,s)\right)\\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{I}_n(z,s)\right)\end{pmatrix} = -\begin{pmatrix}\left(0_{n,m}\right) & \left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{Y}_{n,m}(s)\right)\\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right) & \left(0_{n,m}\right)\end{pmatrix}\cdot\begin{pmatrix}\left(\tilde{V}_n(z,s)\right)\\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{I}_n(z,s)\right)\end{pmatrix}$$
$$+\begin{pmatrix}\left(\tilde{V}_n^{(s)'}(z,s)\right)\\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{I}_n^{(s)'}(z,s)\right)\end{pmatrix} \tag{1.4}$$

$$\left(\tilde{Z}_{n,m(s)}\right) \equiv \left(\tilde{Y}_{n,m(s)}\right)^{-1} \equiv \text{normalizing impedance matrix}$$

$$\left(\tilde{Y}_{n,m(s)}\right) \equiv \text{normalizing admittance matrix}$$

Since we are dealing with symmetric matrices (reciprocity), the normalizing impedance and admittance matrices will be similarly constrained. Note the significance of the position independence of the normalizing matrix so that it passes through the spatial derivative. This first order equation is still inhomogeneous due to the presence of the sources. For our considerations here the sources will be set to zero to give a homogeneous equation. Of course, sources can still be inserted at discrete values of z in the form of boundary conditions.

In (1.4) we have a supervector/supermatrix form [1] of dimension $2N$. An alternate form that is also useful is the second-order differential equations of order N (without sources) for voltage as

$$\frac{\partial^2}{\partial z^2}\left(\tilde{V}_n(z,s)\right)-\left[\frac{\partial}{\partial z}\left(\tilde{Z}'_{n,m}(z,s)\right)\right]\cdot\left(\tilde{Z}'_{n,m}(z,s)\right)^{-1}\cdot\frac{\partial}{\partial z}\left(\tilde{V}_n(z,s)\right)-\left(\tilde{\gamma}_{c_{n,m}}(z,s)\right)^2\cdot\left(\tilde{V}_n(z,s)\right)=(0_n)$$

$$\left(\tilde{\gamma}_{c_{n,m}}(z,s)\right)^2\equiv\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right) \tag{1.5}$$

and for current as

$$\frac{\partial^2}{\partial z^2}\left(\tilde{I}_n(z,s)\right)-\left[\frac{\partial}{\partial z}\left(\tilde{Y}'_{n,m}(z,s)\right)\right]\cdot\left(\tilde{Y}'_{n,m}(z,s)\right)^{-1}\cdot\frac{\partial}{\partial z}\left(\tilde{I}_n(z,s)\right)-\left(\tilde{\gamma}_{c_{n,m}}(z,s)\right)_I^2\cdot\left(\tilde{I}_n(z,s)\right)=(0_n)$$

$$\left(\tilde{\gamma}_{c_{n,m}}(z,s)\right)_I^2\equiv\left(\tilde{Y}'_{n,m}(z,s)\right)\cdot\left(\tilde{Z}'_{n,m}(z,s)\right)=\left[\left(\gamma_{c_{n,m}}(z,s)\right)^2\right]^T \tag{1.6}$$

As discussed in [4] an interesting case has the various matrices and their derivatives appearing here all commuting pairwise, since the equations reduce to scalar ones involving eigenvalues. In that paper the important case of bicirculant (symmetric circulant) matrices is treated, for which the eigenmodes can be determined analytically.

II. First-Order Differential Equation: Solution in Terms of Solutions for Each Segment of NMTL

Consider the first-order differential matrix equation (2Nx2N)

$$\frac{d}{dz}\left(\tilde{u}_{n,m}(z,s)\right) = \left(\tilde{a}_{n,m}(z,s)\right)\cdot\left(\tilde{u}_{n,m}(z,s)\right)$$

$$\left(\tilde{u}_{n,m}(0,s)\right) = \left(1_{n,m}\right) \tag{2.1}$$

Each of these matrices can be partitioned into four NxN matrices in dimatrix form [2] as

$$\left(\tilde{u}_{n,m}(z,s)\right) = \left(\left(\tilde{u}_{n,m}(z,s)\right)_{v,v'}\right)$$

$$\left(\tilde{a}_{n,m}(z,s)\right) = \left(\left(\tilde{a}_{n,m}(z,s)\right)_{v,v'}\right) \tag{2.2}$$

$$v, v' = 1,2$$

Columns of $\left(\tilde{u}_{n,m}(x,s)\right)$ give a set of *2N* linearly independent vectors for expressing the *2N*-component combined voltage vectors as in (1.4), in the general form

$$\begin{pmatrix} \left(\tilde{V}_n(z,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{I}_n(z,s)\right) \end{pmatrix} = \left(\left(\tilde{u}_{n,m}(z,s)\right)_{v,v'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n(0,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{I}_n(0,s)\right) \end{pmatrix} \tag{2.3}$$

$\odot$ ≡ generalized dot product (for supermatrices and supervectors)

Thereby including the boundary condition at z=0 (a special case). In this case we also have

$$\left(\left(a_{n,m}(z,s)\right)_{v,v'}\right) = \begin{pmatrix} \left(0_{n,m}\right) & -\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{Y}_{n,m}(s)\right) \\ -\left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right) & \left(0_{n,m}\right) \end{pmatrix}$$

$$\left(\left(\tilde{u}_{n,m}(0,s)\right)_{v,v'}\right) = \left(\left(1_{n,m}\right)_{v,v'}\right) = \begin{pmatrix} \left(1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(1_{n,m}\right) \end{pmatrix} \tag{2.4}$$

The solution of (2.1) is called a matrizant [10,11,14]. By analogy with cascaded two-port networks (taken as differential lengths of the MTL) this is also thought of as a chain matrix [8]. Specifying both voltage and current vectors as boundary conditions at one end, one has the solution of

(2.1) with (2.3) giving the current vectors everywhere. The problem is that of solving (2.1). Some sufficient conditions can be found under which the matrizant can be written in closed form [3] as

$$\left(\tilde{u}_{n,m}(x,s)\right) = e^{\int_0^z \left(\tilde{a}_{n,m}(z',s)\right)dz'} \tag{2.5}$$

provided $\left(\tilde{a}_{n,m}\right)$ can be written as a sum of constant matrices times scalar functions of z with all these matrices pairwise commuting. The matrizant can also be written as an infinite series of repeated integrals (nth order for nth term) for the general case.

Noting that $\left(\tilde{a}_{n,m}\right)$ has all zeros on the diagonal we have the general results

$$\begin{aligned} tr\left(\left(\tilde{a}_{n,m}(z,s)\right)\right) &= \sum_{n=1}^{2N} \tilde{a}_{n,n}(z,s) = 0 \\ &= \sum_{\delta=1}^{2N} \tilde{X}_\delta\left(\left(\tilde{a}_{n,m}(z,s)\right)\right) \end{aligned} \tag{2.6}$$

$\tilde{X}_\delta \equiv$ eigenvalue of matrix for $\beta = 1,2,\cdots,2N$

$$\begin{aligned} \det\left(\left(\tilde{u}_{n,m}(z,s)\right)\right) &= e^{\int_0^z tr\left(\left(\tilde{a}_{n,m}(z',s)\right)\right)dz'} = 1 \\ &= \prod_{\delta=1}^{2N} \tilde{X}_\delta\left(\left(\tilde{u}_{n,m}(z,s)\right)\right) \end{aligned}$$

This constraint on the eigenvalues of $\left(\tilde{u}_{n,m}\right)$ effectively reduces by one the number to be calculated.

The choice of the normalizing impedance matrix $\left(\tilde{Z}_{n,m}(s)\right)$ is, in one sense, arbitrary. For convenience, however, let us consider the characteristic impedance matrix [1]

$$\left(\tilde{Z}_{c_{n,m}}(z,s)\right) \equiv \left(\tilde{Y}_{c_{n,m}}(z,s)\right)^{-1} = \left(\tilde{\gamma}_{c_{n,m}}(z,s)\right) \cdot \left(\tilde{Y}'_{n,m}(z,s)\right)^{-1}$$

$$= \left(\tilde{\gamma}_{c_{n,m}}(z,s)\right)^{-1} \cdot \left(\tilde{Z}'_{n,m}(z,s)\right) \tag{2.7}$$

$$\left(\tilde{\gamma}_{c_{n,m}}(z,s)\right) = \left[\left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(\tilde{Y}'_{n,m}(z,s)\right)\right]^{\frac{1}{2}} \quad \text{(p.r. square root)}$$

where this is a local definition, i.e. a function of position z. On a local basis this can be used to separate waves into $+z$ and $-z$ propagation directions, as in (1.1). This, however, conflicts with the construct in (1.4) which requires the normalizing impedance matrix to be independent of position. So an interesting choice is to choose the (matrix) value at some particular value of z, say z_ℓ. Then at and near this particular position one might expect the waves to similarly separate in an approximate sense.

At this particular $z = z_\ell$ the characteristic impedance matrix is

$$\left(\tilde{Z}^{(\ell)}_{c_{n,m}}(s)\right) \equiv \left(\tilde{Y}^{(\ell)}_{c_{n,m}}(s)\right)^{-1} = \left(\tilde{Z}_{c_{n,m}}(z_\ell,s)\right)$$

$$= \left(\tilde{\gamma}^{(\ell)}_{c_{n,m}}(s)\right) \cdot \left(\tilde{Y}'^{(\ell)}_{n,m}(s)\right)^{-1}$$

$$= \left(\tilde{\gamma}^{(\ell)}_{c_{n,m}}(s)\right)^{-1} \cdot \left(\tilde{Z}'^{(\ell)}_{n,m}(s)\right)$$

$$\left(\tilde{Z}'^{(\ell)}_{n,m}(s)\right) \equiv \left(\tilde{Z}'_{n,m}(z_\ell,s)\right)$$

$$\left(\tilde{Y}'^{(\ell)}_{n,m}(s)\right) \equiv \left(\tilde{Y}'_{n,m}(z_\ell,s)\right) \tag{2.8}$$

$$\left(\tilde{\gamma}^{(\ell)}_{c_{n,m}}(s)\right) \equiv \left(\tilde{\gamma}_{c_{n,m}}(z_\ell,s)\right)$$

The coefficient supermatrix then takes the form

$$\left(\left(\tilde{a}^{(\ell)}_{n,m}(z,s)\right)_{\nu,\nu'}\right) = -\begin{pmatrix} \left(0_{n,m}\right) & \left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(\tilde{Z}'^{(\ell)}_{n,m}(s)\right)^{-1} \cdot \left(\tilde{\gamma}^{(\ell)}_{c_{n,m}}(s)\right) \\ \left(\tilde{\gamma}^{(\ell)}_{c_{n,m}}(s)\right) \cdot \left(\tilde{Y}'^{(\ell)}_{n,m}(s)\right)^{-1} \cdot \left(\tilde{Y}'_{n,m}(z,s)\right) & \left(0_{n,m}\right) \end{pmatrix} \tag{2.9}$$

indicating the symmetry induced in the expression due to the choice of the normalizing impedance matrix.

Carrying this a step further, apply this on a piecewise basis, i.e. consider a section of transmission line defined by

$$z_\ell \le z < z_{\ell+1} \tag{2.10}$$

Here in effect z_ℓ is the zero coordinate reference to which the foregoing results can be applied. Denote parameters for this section of the MTL by a superscript ℓ. Then we have

$$\begin{gathered}
\frac{d}{dz}\left(\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{v,v'}\right)=\left(\left(\tilde{a}_{n,m}^{(\ell)}(z,s)\right)_{v,v'}\right)\odot\left(\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{v,v'}\right)\\
\left(\left(\tilde{u}_{n,m}^{(\ell)}(z_\ell,s)\right)_{v,v'}\right)=\left(\left(1_{n,m}\right)_{v,v'}\right)
\end{gathered} \tag{2.11}$$

which applies to the $2N$-component combined voltage vectors as

$$\begin{pmatrix} \left(\tilde{V}_n(z,s)\right) \\ \left(\tilde{Z}_{c_{n,m}}^{(\ell)}(s)\right)\cdot\left(\tilde{I}_n(z,s)\right)\end{pmatrix}=\left(\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{v,v'}\right)\odot\begin{pmatrix} \left(\tilde{V}_n(z_\ell,s)\right) \\ \left(\tilde{Z}_{c_{n,m}}^{(\ell)}(s)\right)\cdot\left(\tilde{I}_n(z_\ell,s)\right)\end{pmatrix} \tag{2.12}$$

Now the matrizant is for the ℓth section of the line, i.e. from z_ℓ to $z_{\ell+1}$. At the end of this section one can write the combined voltage as

$$\begin{aligned}
\begin{pmatrix} \tilde{V}_n(z_{\ell+1},s) \\ \left(\tilde{Z}_{c_{n,m}}^{(\ell+1)}(s)\right)\cdot\left(\tilde{I}_n(z_{\ell+1},s)\right)\end{pmatrix}&=\begin{pmatrix} \left(1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(\tilde{Z}_{c_{n,m}}^{(\ell+1)}(s)\right)\cdot\left(\tilde{Z}_{c_{n,m}}^{(\ell)}(s)\right)^{-1}\end{pmatrix}\odot\begin{pmatrix} \tilde{V}_n(z_{\ell+1},s) \\ \left(\tilde{Z}_{c_{n,m}}^{(\ell)}(s)\right)\cdot\left(\tilde{I}_n(z_{\ell+1},s)\right)\end{pmatrix}\\
&=\begin{pmatrix} \left(1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(\tilde{Z}_{c_{n,m}}^{(\ell+1)}(s)\right)\cdot\left(\tilde{Z}_{c_{n,m}}^{(\ell)}(s)\right)^{-1}\end{pmatrix}\odot\left(\left(\tilde{u}_{n,m}^{(\ell)}(z_{\ell+1},s)\right)_{v,v'}\right)\\
&\quad\odot\begin{pmatrix} \left(\tilde{V}_n(z_\ell,s)\right) \\ \left(\tilde{Z}_{c_{n,m}}^{(\ell)}(s)\right)\cdot\left(\tilde{I}_n(z_\ell,s)\right)\end{pmatrix}
\end{aligned} \tag{2.13}$$

So let us define

$$\left(\tilde{U}_{n,m}^{(\ell+1)}(s)\right) \equiv \left(\left(\tilde{U}_{n,m}^{(\ell+1)}(s)\right)_{\nu,\nu'}\right)$$
$$\equiv \begin{pmatrix} (1_{n,m}) & (0_{n,m}) \\ (0_{n,m}) & \left(\tilde{Z}_{c_{n,m}}^{(\ell+1)}(s)\right)\cdot\left(\tilde{Z}_{c_{n,m}}^{(\ell)}(s)\right)^{-1} \end{pmatrix} \odot \left(\left(\tilde{u}_{n,m}^{(\ell)}(z_{\ell+1},s)\right)_{\nu,\nu'}\right) \tag{2.14}$$

This supermatrix relates the signals at $z_{\ell+1}$ as normalized there to those at z_ℓ with normalization at z_ℓ .

Extending this we can evaluate

$$\left(\left(\tilde{U}_{n,m}(z_\ell,s)\right)_{\nu,\nu'}\right) = \left(\left(\tilde{U}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{U}_{n,m}^{(\ell-1)}(s)\right)_{\nu,\nu'}\right) \odot \cdots \odot \left(\left(\tilde{U}_{n,m}^{(1)}(s)\right)_{\nu,\nu'}\right)$$
$$\equiv \bigodot_{\ell'=1}^{\ell} \left(\left(\tilde{U}_{n,m}^{(\ell+1-\ell')}(s)\right)_{\nu,\nu'}\right) \tag{2.15}$$

with the result

$$\begin{pmatrix} \tilde{V}_n(z_\ell,s) \\ \left(\tilde{Z}_{c_{n,m}}^{(\ell)}(s)\right)\cdot\left(\tilde{I}_n(z_\ell,s)\right) \end{pmatrix} = \left(\left(\tilde{U}_{n,m}(z_\ell,s)\right)_{\nu,\nu'}\right) \odot \begin{pmatrix} \tilde{V}_n(0,s) \\ \left(\tilde{Z}_{c_{n,m}}^{(0)}(s)\right)\cdot\left(\tilde{I}_n(0,s)\right) \end{pmatrix} \tag{2.16}$$

This can be used to extend the solution through any number of MTL sections, provided we have the solution for the matrizant of each section as in (2.12), renormalized as in (2.14).

III. Piecewise-Constant Approximation

The foregoing formulae can be readily applied to the case that the NMTL is approximated by a cascaded series of MTLs [9]. In this case one begins with the general (sourceless) equations as in (1.3) where the coefficient matrices are functions of position. To form a position-independent characteristic impedance matrix as in (1.4) or (2.10), let us take some particular z and use (2.7). The next consideration is which z to use. Consider the section of line defined by $z_\ell < z < z_{\ell+1}$. One can take the matrix value at z_ℓ or at some intermediate position which we can still take as $\left(\tilde{Z}^{(\ell)}_{c_{n,m}}(s)\right)$ in section II.

One would also like to have a propagation matrix independent of z, so from (2.7) one can constrain both by choosing the per-unit-length impedance and admittance matrices as constant in the interval, and use

$$
\begin{aligned}
\left(\tilde{\gamma}^{(\ell)}_{c_{n,m}}(s)\right) &= \left[\left(\tilde{Z}'^{(\ell)}_{n,m}(s)\right)\cdot\left(\tilde{Y}'^{(\ell)}_{n,m}(s)\right)\right]^{\frac{1}{2}} \\
\left(\tilde{Z}^{(\ell)}_{c_{n,m}}(s)\right) &= \left(\tilde{\gamma}^{(\ell)}_{c_{n,m}}(s)\right)\cdot\left(\tilde{Y}'^{(\ell)}_{n,m}(s)\right)^{-1} \\
&= \left(\tilde{\gamma}^{(\ell)}_{c_{n,m}}(s)\right)^{-1}\cdot\left(\tilde{Z}'^{(\ell)}_{n,m}(s)\right)
\end{aligned}
\tag{3.1}
$$

With this choice based on some average choice for the per-unit-length matrices, the problem considerably simplifies as

$$
\begin{aligned}
\left(\left(\tilde{a}^{(\ell)}_{n,m}(s)\right)_{v,v'}\right) &= -\begin{pmatrix} \left(0_{n,m}\right) & \left(\tilde{\gamma}^{(\ell)}_{c_{n,m}}(s)\right) \\ \left(\tilde{\gamma}^{(\ell)}_{c_{n,m}}(s)\right) & \left(0_{n,m}\right) \end{pmatrix} \\
\frac{d}{dz}\left(\left(\tilde{u}^{(\ell)}_{n,m}(z,s)\right)_{v,v'}\right) &= \left(\left(\tilde{a}^{(\ell)}_{n,m}(s)\right)_{v,v'}\right) \odot \left(\left(\tilde{u}^{(\ell)}_{n,m}(z,s)\right)_{v,v'}\right) \\
\left(\left(\tilde{u}^{(\ell)}_{n,m}(z_\ell,s)\right)_{v,v'}\right) &= \left(\left(1_{n,m}\right)_{v,v'}\right)
\end{aligned}
\tag{3.2}
$$

which has the solution (since $\left(\tilde{a}^{(\ell)}_{n,m}(s)\right)$ is independent of z)

$$\left(\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{\nu,\nu'}\right) = e^{\left(\left(\tilde{a}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right)[z-z_\ell]} \tag{3.3}$$

This can be used with (2.14) as

$$\left(\left(\tilde{U}_{n,m}^{(\ell+1)}(s)\right)_{\nu,\nu'}\right) = \begin{pmatrix} (1_{n,m}) & (0_{n,m}) \\ (0_{n,m}) & \left(\tilde{Z}_{c_{n,m}}^{(\ell+1)}(s)\right)\cdot\left(\tilde{Z}_{c_{n,m}}^{(\ell)}(s)\right)^{-1} \end{pmatrix} \odot e^{\left(\left(\tilde{a}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right)[z_{\ell+1}-z_\ell]} \tag{3.4}$$

to carry the solution to the beginning of the $\ell+1$st zone.

Noting the form of the matrix in (3.4) the exponential in (3.3) can be readily written in an alternate form as

$$\begin{aligned}
&e^{\left(\left(\tilde{a}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right)[z_{\ell+1}-z_\ell]} \\
&= \sum_{p=0}^{\infty} \frac{1}{p!}\left(\left(\tilde{a}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right)^p [z_{\ell+1}-z_\ell]^p \\
&= \sum_{p=0}^{\infty,2} \frac{1}{p!}\begin{pmatrix} \left(\tilde{\gamma}_{c_{n,m}}^{(\ell)}(s)\right) & (0_{n,m}) \\ (0_{n,m}) & \left(\tilde{\gamma}_{c_{n,m}}^{(\ell)}(s)\right) \end{pmatrix}^p [z_{\ell+1}-z_\ell]^p \\
&= -\sum_{p=1}^{\infty,2} \frac{1}{p!}\begin{pmatrix} (0_{n,m}) & \left(\tilde{\gamma}_{c_{n,m}}^{(\ell)}(s)\right) \\ \left(\tilde{\gamma}_{c_{n,m}}^{(\ell)}(s)\right) & (0_{n,m}) \end{pmatrix} \odot \begin{pmatrix} \left(\tilde{\gamma}_{c_{n,m}}^{(\ell)}(s)\right) & (0_{n,m}) \\ (0_{n,m}) & \left(\tilde{\gamma}_{c_{n,m}}^{(\ell)}(s)\right) \end{pmatrix}^{p-1} [z_{\ell+1}-z_\ell]^p \\
&= \begin{pmatrix} \cosh\left([z_{\ell+1}-z_\ell]\left(\tilde{\gamma}_{c_{n,m}}^{(\ell)}(s)\right)\right) & (0_{n,m}) \\ (0_{n,m}) & \cosh\left([z_{\ell+1}-z_\ell]\left(\tilde{\gamma}_{c_{n,m}}^{(\ell)}(s)\right)\right) \end{pmatrix} \\
&\quad - \begin{pmatrix} (0_{n,m}) & (1_{n,m}) \\ (1_{n,m}) & (0_{n,m}) \end{pmatrix} \odot \begin{pmatrix} \sinh\left([z_{\ell+1}-z_\ell]\left(\tilde{\gamma}_{c_{n,m}}^{(\ell)}(s)\right)\right) & (0_{n,m}) \\ (0_{n,m}) & \sinh\left([z_{\ell+1}-z_\ell]\left(\tilde{\gamma}_{c_{n,m}}^{(\ell)}(s)\right)\right) \end{pmatrix}
\end{aligned} \tag{3.5}$$

$$
= \begin{pmatrix} \cosh\left([z_{\ell+1} - z_\ell] \left(\tilde{\gamma}_{c_{n,m}}^{(\ell)}(s) \right) \right) & -\sinh\left([z_{\ell+1} - z_\ell] \left(\tilde{\gamma}_{c_{n,m}}^{(\ell)}(s) \right) \right) \\ -\sinh\left([z_{\ell+1} - z_\ell] \left(\tilde{\gamma}_{c_{n,m}}^{(\ell)}(s) \right) \right) & \cosh\left([z_{\ell+1} - z_\ell] \left(\tilde{\gamma}_{c_{n,m}}^{(\ell)}(s) \right) \right) \end{pmatrix}
$$

This gives analytic form to the blocks of size NxN.

Going a step further one can diagonalize the propagation matrix as [1]

$$
\left(\tilde{\gamma}_{c_{n,m}}^{(\ell)}(s) \right) = \sum_{\beta=1}^{N} \tilde{\gamma}_{\beta}^{(\ell)}(s) \left(\tilde{v}_{c_n}^{(\ell)}(s) \right)_\beta \left(\tilde{i}_{c_n}^{(\ell)}(s) \right)_\beta
$$

$$
\left(\tilde{v}_{c_n}^{(\ell)}(s) \right)_{\beta_1} \cdot \left(\tilde{i}_{c_n}^{(\ell)}(s) \right)_{\beta_2} = 1_{\beta_1,\beta_2} \qquad \text{(biorthonormal)} \tag{3.6}
$$

$$
\left(\tilde{v}_{c_n}^{(\ell)}(s) \right)_\beta = \left(\tilde{Z}_{c_{n,m}}^{(\ell)}(s) \right) \cdot \left(\tilde{i}_{c_n}^{(\ell)}(s) \right)_\beta = \left(\tilde{i}_{c_n}^{(\ell)}(s) \right)_\beta \cdot \left(\tilde{Z}_{c_{n,m}}^{(\ell)}(s) \right)
$$

noting reciprocity. This also gives

$$
\left(\tilde{Z}_{c_{n,m}}^{(\ell)}(s) \right) = \sum_{\beta=1}^{N} \left(\tilde{v}_{c_n}^{(\ell)}(s) \right)_\beta \left(\tilde{v}_{c_n}^{(\ell)}(s) \right)_\beta
$$

$$
\left(\tilde{Y}_{c_{n,m}}^{(\ell)}(s) \right) = \sum_{\beta=1}^{N} \left(\tilde{i}_{c_n}^{(\ell)}(s) \right)_\beta \left(\tilde{i}_{c_n}^{(\ell)}(s) \right)_\beta
$$

$$
\left(\tilde{Z}'^{(\ell)}_{n,m}(s) \right) = \sum_{\beta=1}^{N} \tilde{\gamma}_{\beta}^{(\ell)}(s) \left(\tilde{v}_{c_n}^{(\ell)}(s) \right)_\beta \left(\tilde{v}_{c_n}^{(\ell)}(s) \right)_\beta \tag{3.7}
$$

$$
\left(\tilde{Y}'^{(\ell)}_{n,m}(s) \right) = \sum_{\beta=1}^{N} \tilde{\gamma}_{\beta}^{(\ell)}(s) \left(\tilde{i}_{c_n}^{(\ell)}(s) \right)_\beta \left(\tilde{i}_{c_n}^{(\ell)}(s) \right)_\beta
$$

These are here applied to the case that the parameters are independent of the coordinate z. One could let these be functions of z but this would complicate things via the derivatives with respect to z, particularly with respect to the (right and left) eigenvectors of the propagation matrix. For present purposes this and the matrices in (3.7) are all constant.

In terms of these eigenvectors we have

$$\left(\left(\tilde{a}_{n,m}^{(\ell)}(s)\right)_{v,v'}\right) = -\begin{pmatrix} \left(0_{n,m}\right) & \sum_{\beta=1}^{N} \tilde{\gamma}_{\beta}^{(\ell)}(s) \left(\tilde{v}_{c_n}^{(\ell)}(s)\right)_{\beta} \left(\tilde{i}_{c_n}^{(\ell)}(s)\right)_{\beta} \\ \sum_{\beta=1}^{N} \tilde{\gamma}_{\beta}^{(\ell)}(s) \left(\tilde{v}_{c_n}^{(\ell)}(s)\right)_{\beta} \left(\tilde{i}_{c_n}^{(\ell)}(s)\right)_{\beta} & \left(0_{n,m}\right) \end{pmatrix}$$

$$e^{\left(\left(\tilde{a}_{n,m}^{(\ell)}(s)\right)_{v,v'}\right)\left[z_{\ell+1}-z_{\ell}\right]}$$

$$= \begin{pmatrix} \sum_{\beta=1}^{N} \cosh\left(\left[z_{\ell+1}-z_{\ell}\right]\tilde{\gamma}_{\beta}^{(\ell)}(s)\right)\left(\tilde{v}_{c_n}^{(\ell)}(s)\right)_{\beta}\left(\tilde{i}_{c_n}^{(\ell)}(s)\right)_{\beta} & -\sum_{\beta=1}^{N} \sinh\left(\left[z_{\ell+1}-z_{\ell}\right]\tilde{\gamma}_{\beta}^{(\ell)}(s)\right)\left(\tilde{v}_{c_n}^{(\ell)}(s)\right)_{\beta}\left(\tilde{i}_{c_n}^{(\ell)}(s)\right)_{\beta} \\ -\sum_{\beta=1}^{N} \sinh\left(\left[z_{\ell+1}-z_{\ell}\right]\tilde{\gamma}_{\beta}^{(\ell)}(s)\right)\left(\tilde{v}_{c_n}^{(\ell)}(s)\right)_{\beta}\left(\tilde{i}_{c_n}^{(\ell)}(s)\right)_{\beta} & \sum_{\beta=1}^{N} \cosh\left(\left[z_{\ell+1}-z_{\ell}\right]\tilde{\gamma}_{\beta}^{(\ell)}(s)\right)\left(\tilde{v}_{c_n}^{(\ell)}(s)\right)_{\beta}\left(\tilde{i}_{c_n}^{(\ell)}(s)\right)_{\beta} \end{pmatrix}$$

(3.8)

These matrices then have dimension $2N$ eigenvectors as

$$\left(\left(\tilde{a}_{n,m}^{(\ell)}(s)\right)_{v,v'}\right) \odot \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\tilde{v}_{c_n}^{(\ell)}(s)\right)_\beta \\ \pm\frac{1}{\sqrt{2}}\left(\tilde{v}_{c_n}^{(\ell)}(s)\right)_\beta \end{pmatrix} = \mp\tilde{\gamma}_\beta^{(\ell)}(s) \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\tilde{v}_{c_n}^{(\ell)}(s)\right)_\beta \\ \pm\frac{1}{\sqrt{2}}\left(\tilde{v}_{c_n}(s)\right)_\beta \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}}\left(\tilde{i}_{c_n}^{(\ell)}(s)\right)_\beta \\ \pm\frac{1}{\sqrt{2}}\left(\tilde{i}_{c_n}^{(\ell)}(s)\right)_\beta \end{pmatrix} \odot \left(\left(\tilde{a}_{n,m}^{(\ell)}(s)\right)_{v,v'}\right) = \mp\tilde{\gamma}_\beta^{(\ell)}(s) \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\tilde{i}_{c_n}^{(\ell)}(s)\right)_\beta \\ \pm\frac{1}{\sqrt{2}}\left(\tilde{i}_{c_n}^{(\ell)}(s)\right)_\beta \end{pmatrix}$$

$$e^{\left(\left(\tilde{a}_{n,m}^{(\ell)}(s)\right)_{v,v'}\right)[z_{\ell+1}-z_\ell]} \odot \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\tilde{v}_{c_n}^{(\ell)}(s)\right)_\beta \\ \pm\frac{1}{\sqrt{2}}\left(\tilde{v}_{c_n}^{(\ell)}(s)\right)_\beta \end{pmatrix} = e^{\mp\tilde{\gamma}^{(\ell)}(s)[z_{\ell+1}-z_\ell]} \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\tilde{v}_{c_n}^{(\ell)}(s)\right)_\beta \\ \pm\frac{1}{\sqrt{2}}\left(\tilde{v}_{c_n}^{(\ell)}(s)\right)_\beta \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}}\left(\tilde{i}_{c_n}^{(\ell)}(s)\right)_\beta \\ \pm\frac{1}{\sqrt{2}}\left(\tilde{i}_{c_n}^{(\ell)}(s)\right)_\beta \end{pmatrix} \odot e^{\left(\left(\tilde{a}_{n,m}^{(\ell)}(s)\right)_{v,v'}\right)[z_{\ell+1}-z_\ell]} = e^{\mp\tilde{\gamma}_\beta^{(\ell)}(s)[z_{\ell+1}-z_\ell]} \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\tilde{i}_{c_n}^{(\ell)}(s)\right)_\beta \\ \pm\frac{1}{\sqrt{2}}\left(\tilde{i}_{c_n}^{(\ell)}(s)\right)_\beta \end{pmatrix} \tag{3.9}$$

The biorthornormality condition is

$$\begin{pmatrix} \frac{1}{\sqrt{2}}\left(\tilde{v}_{c_n}^{(\ell)}(s)\right)_{\beta_1} \\ \pm\frac{1}{\sqrt{2}}\left(\tilde{v}_{c_n}^{(\ell)}(s)\right)_{\beta_1} \end{pmatrix} \odot \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\tilde{i}_{c_n}^{(\ell)}(s)\right)_{\beta_2} \\ \pm\frac{1}{\sqrt{2}}\left(\tilde{i}_{c_n}^{(\ell)}(s)\right)_{\beta_2} \end{pmatrix} = 1_{\beta_1,\beta_2}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}}\left(\tilde{v}_{c_n}^{(\ell)}(s)\right)_{\beta_1} \\ \pm\frac{1}{\sqrt{2}}\left(\tilde{v}_{c_n}^{(\ell)}(s)\right)_{\beta_1} \end{pmatrix} \odot \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\tilde{i}_{c_n}^{(\ell)}(s)\right)_{\beta_2} \\ \mp\frac{1}{\sqrt{2}}\left(\tilde{i}_{c_n}^{(\ell)}(s)\right)_{\beta_2} \end{pmatrix} = 0 \tag{3.10}$$

There are $2N$ of these eigenvalues for the N values of β and two choices of the $\pm$ sign, and similarly for the eigenvectors. In effect one can think of β running from 1 to N with the upper sign, and then from $N+1$ to $2N$ with the lower sign.

In terms of these eigenvectors one can construct matrices with these as rows (first index) or columns (second index) respectively as

$$\left(\tilde{\ell}_{n,m}^{(\ell)}(s)\right) \equiv \begin{pmatrix} \left(\tilde{i}_{c_n}^{(\ell)}(s)\right)_1 \\ \vdots \\ \left(\tilde{i}_{c_n}^{(\ell)}(s)\right)_N \end{pmatrix} \quad \text{(vectors as rows)}$$

$$\left(\tilde{r}_{n,m}^{(\ell)}(s)\right) \equiv \left(\left(\tilde{v}_{c_n}^{(\ell)}(s)\right)_1, \cdots, \left(\tilde{v}_{c_n}^{(\ell)}(s)\right)_N\right) \quad \text{(vectors as columns)} \tag{3.11}$$

$$\left(\tilde{\ell}_{n,m}^{(\ell)}(s)\right) \cdot \left(\tilde{r}_{n,m}^{(\ell)}(s)\right) = \left(1_{n,m}\right) \quad \text{(NxN identity)}$$

In 2Nx2N form we have supermatrices of eigenvectors as

$$\left(\left(\tilde{\ell}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right) \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \left(\tilde{\ell}_{n,m}^{(\ell)}(s)\right) & \left(\tilde{\ell}_{n,m}^{(\ell)}(s)\right) \\ \left(\tilde{\ell}_{n,m}^{(\ell)}(s)\right) & -\left(\tilde{\ell}_{n,m}^{(\ell)}(s)\right) \end{pmatrix}$$

$$\left(\left(\tilde{r}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right) \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \left(\tilde{r}_{n,m}^{(\ell)}(s)\right) & \left(\tilde{r}_{n,m}^{(\ell)}(s)\right) \\ \left(\tilde{r}_{n,m}^{(\ell)}(s)\right) & -\left(\tilde{r}_{n,m}^{(\ell)}(s)\right) \end{pmatrix} \tag{3.12}$$

$$\left(\left(\tilde{\ell}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{r}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right) = \left(\left(1_{n,m}\right)_{\nu,\nu'}\right)$$

Applying this to the previous gives

$$\left(\tilde{\ell}_{n,m}^{(\ell)}(s)\right) \cdot \left(\tilde{\gamma}_{c_{n,m}}^{(\ell)}(s)\right) \cdot \left(\tilde{r}_{n,m}^{(\ell)}(s)\right) = \begin{pmatrix} \tilde{\gamma}_1(s) & & & O \\ & \tilde{\gamma}_1(s) & & \\ & & \ddots & \\ O & & & \tilde{\gamma}_N(s) \end{pmatrix}$$

$$= \left(\tilde{\gamma}_n(s) 1_{n,m}\right)$$

$$\left(\left(\tilde{\ell}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{a}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{r}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right) = \begin{pmatrix} -\left(\tilde{\gamma}_n(s) 1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(\tilde{\gamma}_n(s) 1_{n,m}\right) \end{pmatrix} \tag{3.13}$$

$$\left(\left(\tilde{\ell}_{n,m}^{(\ell)}(s)\right)_{v,v'}\right) \odot e^{\left(\left(\tilde{a}_{n,m}^{(\ell)}(s)\right)_{v,v'}\right)[z_{\ell+1}-z_\ell]} \odot \left(\left(\tilde{r}_{n,m}^{(\ell)}(s)\right)_{v,v'}\right)$$

$$= \begin{pmatrix} \left(e^{-\tilde{\gamma}_n(s)[z_{\ell+1}-z_\ell]} 1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(e^{\tilde{\gamma}_n(s)[z_{\ell+1}-z_\ell]} 1_{n,m}\right) \end{pmatrix}$$

So in terms of the eigenvalues and eigenvectors of the NxN representation, the full 2Nx2N supermatrices and supervectors can also be conveniently calculated.

IV. Second-Order-Differential-Equation Solution for Matrizant: Position-Independent Common Eigenvectors for NxN Matrices

In [5] it is shown that the NMTL equations in NxN form can be diagonalized if appropriate matrices and their derivatives with respect to z all commute. A special case considered there was that of bicirculant (symmetric circulant) matrices which gave special forms to the eigenvectors and eigenvalues. Here let us consider the more general case with

$$\left(\tilde{Z}'_{n,m}(z,s)\right)=\sum_{\beta=1}^{N}\tilde{Z}'_{\beta}(z,s)\;(\tilde{x}_n(s))_{\beta}\;(\tilde{x}_n(s))_{\beta}$$

$$\left(\tilde{Y}'_{n,m}(z,s)\right)=\sum_{\beta=1}^{N}\tilde{Y}'_{\beta}(z,s)\;(\tilde{x}_n(s))_{\beta}\;(\tilde{x}_n(s))_{\beta} \qquad (4.1)$$

$$(\tilde{x}_n(s))_{\beta_1}\cdot(\tilde{x}_n(s))_{\beta_2}=1_{\beta_1,\beta_2} \qquad \text{(orthogonality)}$$

so that the eigenvectors are constrained to be independent of position z. The matrices are symmetric by reciprocity. Except for the normalization between the $(\tilde{v}_n)_\beta$ and $(\tilde{i}_n)_\beta$ in (3.5), they both reduce to the $(x_n)_\beta$ used here where the commutation property makes the equations more symmetric. With this constraint on the form of the eigenvectors we also have

$$\left(\tilde{\gamma}_{c_{n,m}}(z,s)\right)=\left(\tilde{\gamma}_{c_{n,m}}(z,s)\right)^T=\sum_{\beta=1}^{N}\tilde{\gamma}_{c_\beta}(z,s)\;(\tilde{x}_n(s))_{\beta}\;(\tilde{x}_n(s))_{\beta}$$

$$\tilde{\gamma}_{c_\beta}(z,s)=\left[\tilde{Z}'_{\beta}(z,s)\;\tilde{Y}'_{\beta}(z,s)\right]^{\frac{1}{2}} \qquad \text{(p.r. square root)}$$

$$\left(\tilde{Z}_{c_{n,m}}(z,s)\right)=\sum_{\beta=1}^{N}\tilde{Z}_{c_\beta}(z,s)\;(\tilde{x}_n(s))_{\beta}\;(\tilde{x}_n(s))_{\beta}$$

$$\left(\tilde{Y}_{c_{n,m}}(z,s)\right)=\sum_{\beta=1}^{N}\tilde{Y}_{c_\beta}(z,s)\;(\tilde{x}_n(s))_{\beta}\;(\tilde{x}_n(s))_{\beta} \qquad (4.2)$$

$$\tilde{Z}_{c_\beta}(z,s)=\tilde{Y}_{c_\beta}^{-1}(z,s)=\left[\frac{\tilde{Z}'_{\beta}(z,s)}{\tilde{Y}'_{\beta}(z,s)}\right]^{\frac{1}{2}} \qquad \text{(p.r. square root)}$$

All the matrices in (4.1) and (4.2) and their z derivatives commute with all other matrices in this set.

Expand voltage and current in terms of these eigenvectors as

$$\tilde{v}_\beta(z,s) = (\tilde{x}_n(s))_\beta \cdot (\tilde{V}_n(z,s))$$

$$\tilde{i}_\beta(z,s) = \tilde{Z}_{c_\beta}^{(\ell)}(s) \; (\tilde{x}_n(s))_\beta \cdot (\tilde{I}_n(z,s))$$

$$(\tilde{V}_n(z,s)) = \sum_{\beta=1}^{N} \tilde{v}_\beta(z,s)(\tilde{x}_n(s))_\beta \tag{4.3}$$

$$(\tilde{I}_n(z,s)) = \sum_{\beta=1}^{N} \tilde{Y}_{c_\beta}^{(\ell)}(s) \; \tilde{i}_\beta(z,s)(\tilde{x}_n(s))_\beta$$

The second-order differential equations (1.5) and (1.6) then scalarize as

$$\left\{\frac{\partial^2}{\partial z^2} - \left[\frac{\partial}{\partial z}\ell n\left(\tilde{Z}'_\beta(z,s)\right)\right]\frac{\partial}{\partial z} - \tilde{\gamma}_\beta^2(z,s)\right\} \tilde{v}_\beta(z,s) = 0$$

$$\left\{\frac{\partial^2}{\partial z^2} - \left[\frac{\partial}{\partial z}\ell n\left(\tilde{Y}'_\beta(z,s)\right)\right]\frac{\partial}{\partial z} - \tilde{\gamma}_\beta^2(z,s)\right\} \tilde{i}_\beta(z,s) = 0 \tag{4.4}$$

with the relations between them (from (1.3) without sources) as

$$\frac{\partial}{\partial z}\tilde{v}_\beta(z,s) = -\tilde{Z}'_\beta(z,s) \; \tilde{Y}_{c_\beta}^{(\ell)}(s) \; \tilde{i}_\beta(z,s)$$

$$\frac{\partial}{\partial z}\tilde{i}_\beta(z,s) = -\tilde{Y}'_\beta(z,s) \; \tilde{Z}_{c_\beta}^{(\ell)}(s) \; \tilde{v}_\beta(z,s) \tag{4.5}$$

For convenience these βth eigenterms have been normalized via the characteristic impedance/admittance eigenvalues so that $\tilde{v}_\beta$ and $\tilde{i}_\beta$ have the same dimensions. As before, the superscript ℓ refers to some convenient z_ℓ as

$$\tilde{Z}_{c_\beta}^{(\ell)}(s) \equiv \tilde{Z}_{c_\beta}(z_\ell,s) \; , \; \tilde{Y}_{c_\beta}^{(\ell)} \equiv \tilde{Y}_{c_\beta}^{(\ell)}(z_\ell,s) \tag{4.6}$$

Noting that homogeneous second-order linear differential equations have two linearly independent solutions, denote these by $\tilde{v}_\beta^{(\ell,s)}$ and $\tilde{i}_\beta^{(\ell,s)}$ for $\delta = 1,2$. These are related by

$$\frac{\partial}{\partial z}\tilde{v}_\beta^{(\ell,\delta)}(z,s) = -\tilde{Z}'_\beta(z,s)\ \tilde{Y}_{c_\beta}^{(\ell)}(s)\ \tilde{i}_\beta^{(\ell,\delta)}(z,s)$$

$$\frac{\partial}{\partial z}\tilde{i}_\beta^{(\ell,\delta)}(z,s) = -\tilde{Y}'_\beta(z,s)\ \tilde{Z}_{c_\beta}^{(\ell)}(s)\ \tilde{v}_\beta^{(\ell,\delta)}(z,s) \tag{4.7}$$

Note that there is a superscript ℓ, since there may be other special functions for other sections of the NMTL. Then we can write the voltage and current vectors in (4.3) using linear combinations of these functions as

$$\left(\tilde{V}_n(z,s)\right) = \sum_{\beta=1}^{N}\left[\tilde{d}_\beta^{(1)}(s)\ \tilde{v}_\beta^{(\ell,1)}(z,s) + \tilde{d}_\beta^{(2)}(s)\ \tilde{v}_\beta^{(\ell,2)}(z,s)\right]\left(\tilde{x}_n(s)\right)_\beta$$

$$\left(\tilde{I}_n(z,s)\right) = \sum_{\beta=1}^{N}\tilde{Y}_{c_\beta}^{(\ell)}(s)\left[\tilde{d}_\beta^{(1)}(s)\ \tilde{i}_\beta^{(\ell,1)}(z,s) + \tilde{d}_\beta^{(2)}(z,s)\ \tilde{i}_\beta^{(\ell,2)}(z,s)\right]\left(\tilde{x}_n(s)\right)_\beta \tag{4.8}$$

The coefficients depend on boundary conditions, say at $z = z_\ell$.

Now go to the matrizant form as in section II where

$$\frac{\partial}{\partial z}\left(\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{\nu,\nu'}\right) = \left(\left(\tilde{a}_{n,m}^{(\ell)}(z,s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{\nu,\nu'}\right)$$

$$\left(\left(\tilde{a}_{n,m}^{(\ell)}(z,s)\right)_{\nu,\nu'}\right) = -\begin{pmatrix} \left(0_{n,m}\right) & \sum_{\beta=1}^{N}\tilde{Z}'_\beta(z,s)\ \tilde{Y}_{c_\beta}^{(\ell)}(s)\ \left(\tilde{x}_n(s)\right)_\beta\left(\tilde{x}_n(s)\right)_\beta \\ \sum_{\beta=1}^{N}\tilde{Y}'_\beta(z,s)\ \tilde{Z}_{c_\beta}^{(\ell)}(s)\ \left(\tilde{x}_n(s)\right)_\beta\left(\tilde{x}_n(s)\right)_\beta & \left(0_{n,m}\right) \end{pmatrix} \tag{4.9}$$

$$\left(\left(\tilde{u}_{n,m}^{(\ell)}(z_\ell,s)\right)_{\nu,\nu'}\right) = \left(\left(1_{n,m}\right)_{\nu,\nu'}\right) \qquad \text{(boundary condition)}$$

This can be written out as four equations for the NxN matrices (blocks) as

$$\frac{\partial}{\partial z}\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{1,1} = -\left[\sum_{\beta=1}^{N}\tilde{Z}'_{\beta}(z,s)\ \tilde{Y}_{c_{\beta}}^{(\ell)}(s)\ \left(\tilde{x}_n(s)\right)_{\beta}\ \left(\tilde{x}_n(s)\right)_{\beta}\right]\cdot\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{2,1}$$

$$\frac{\partial}{\partial z}\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{1,2} = -\left[\sum_{\beta=1}^{N}\tilde{Z}'_{\beta}(z,s)\ \tilde{Y}_{c_{\beta}}^{(\ell)}(s)\ \left(\tilde{x}_n(s)\right)_{\beta}\ \left(\tilde{x}_n(s)\right)_{\beta}\right]\cdot\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{2,2}$$

$$\frac{\partial}{\partial z}\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{2,1} = -\left[\sum_{\beta=1}^{N}\tilde{Y}'_{\beta}(z,s)\ \tilde{Z}_{c_{\beta}}^{(\ell)}(s)\ \left(\tilde{x}_n(s)\right)_{\beta}\ \left(\tilde{x}_n(s)\right)_{\beta}\right]\cdot\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{1,1} \qquad (4.10)$$

$$\frac{\partial}{\partial z}\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{2,2} = -\left[\sum_{\beta=1}^{N}\tilde{Y}'_{\beta}(z,s)\ \tilde{Z}_{c_{\beta}}^{(\ell)}(s)\ \left(\tilde{x}_n(s)\right)_{\beta}\ \left(\tilde{x}_n(s)\right)_{\beta}\right]\cdot\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{1,2}$$

These are coupled first order equations.

Write the second order differential equations for these blocks in the same way as done for the voltage and current vectors (section I), giving

$$\left\{\left(1_{n,m}\right)\frac{\partial^2}{\partial z^2} - \left[\sum_{\beta=1}^{N}\frac{\partial}{\partial z}\ell n\left(\tilde{Z}'_{\beta}(z,s)\right)\ \left(\tilde{x}_n(s)\right)_{\beta}\ \left(\tilde{x}_n(s)\right)_{\beta}\right]\frac{\partial}{\partial z} - \left[\sum_{\beta=1}^{N}\tilde{\gamma}_{\beta}^{2}(z,s)\ \left(\tilde{x}_n(s)\right)_{\beta}\ \left(\tilde{x}_n(s)\right)_{\beta}\right]\right\}$$
$$\cdot\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{1,1} = \left(0_{n,m}\right)$$

$$\left\{\left(1_{n,m}\right)\frac{\partial^2}{\partial z^2} - \left[\sum_{\beta=1}^{N}\frac{\partial}{\partial z}\ell n\left(\tilde{Z}'_{\beta}(z,s)\right)\ \left(\tilde{x}_n(s)\right)_{\beta}\ \left(\tilde{x}_n(s)\right)_{\beta}\right]\frac{\partial}{\partial z} - \left[\sum_{\beta=1}^{N}\tilde{\gamma}_{\beta}^{2}(z,s)\ \left(\tilde{x}_n(s)\right)_{\beta}\ \left(\tilde{x}_n(s)\right)_{\beta}\right]\right\}$$
$$\cdot\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{1,2} = \left(0_{n,m}\right)$$

$$\left\{\left(1_{n,m}\right)\frac{\partial^2}{\partial z^2} - \left[\sum_{\beta=1}^{N}\frac{\partial}{\partial z}\ell n\left(\tilde{Y}'_{\beta}(z,s)\right)\ \left(\tilde{x}_n(s)\right)_{\beta}\ \left(\tilde{x}_n(s)\right)_{\beta}\right]\frac{\partial}{\partial z} - \left[\sum_{\beta=1}^{N}\tilde{\gamma}_{\beta}^{2}(z,s)\ \left(\tilde{x}_n(s)\right)_{\beta}\ \left(\tilde{x}_n(s)\right)_{\beta}\right]\right\}$$
$$\cdot\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{2,1} = \left(0_{n,m}\right)$$

$$\left\{(1_{n,m})\frac{\partial^2}{\partial z^2}-\left[\sum_{\beta=1}^{N}\frac{\partial}{\partial z}\ell n\left(\tilde{Y}_\beta'(z,s)\right)\,(\tilde{x}_n(s))_\beta\,(\tilde{x}_n(s))_\beta\right]\frac{\partial}{\partial z}-\left[\sum_{\beta=1}^{N}\tilde{\gamma}_\beta^2(z,s)\,(\tilde{x}_n(s))_\beta\,(\tilde{x}_n(s))_\beta\right]\right\}$$

$$\cdot\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{2,2}=\left(0_{n,m}\right) \tag{4.11}$$

All four of these equations have the same form. Clearly the $(\tilde{x}_n(s))_\beta$ are N eigenvectors for each of the above equations. Consider an individual eigenterm which we take of the form

$$\tilde{b}_{\beta;\ell}^{(\nu,\nu')}(z,s)\,(\tilde{x}_n(s))_\beta\,(\tilde{x}_n(s))_\beta \tag{4.12}$$

where the coefficients (eigenvalues) are scalar functions of z on which the derivatives act. Then (4.11) separates into the scalar equations

$$\left\{\frac{\partial^2}{\partial z^2}-\frac{\partial}{\partial z}\ell n\left(\tilde{Z}_\beta'(z,s)\right)\frac{\partial}{\partial z}-\tilde{\gamma}_\beta^2(z,s)\right\}\tilde{b}_{\beta;\ell}^{(1,1)}(z,s)=0$$

$$\left\{\frac{\partial^2}{\partial z^2}-\frac{\partial}{\partial z}\ell n\left(\tilde{Z}_\beta'(z,s)\right)\frac{\partial}{\partial z}-\tilde{\gamma}_\beta^2(z,s)\right\}\tilde{b}_{\beta;\ell}^{(1,2)}(z,s)=0$$

$$\left\{\frac{\partial^2}{\partial z^2}-\frac{\partial}{\partial z}\ell n\left(\tilde{Y}_\beta'(z,s)\right)\frac{\partial}{\partial z}-\tilde{\gamma}_\beta^2(z,s)\right\}\tilde{b}_{\beta;\ell}^{(2,1)}(z,s)=0 \tag{4.13}$$

$$\left\{\frac{\partial^2}{\partial z^2}-\frac{\partial}{\partial z}\ell n\left(\tilde{Y}_\beta'(z,s)\right)\frac{\partial}{\partial z}-\tilde{\gamma}_\beta^2(z,s)\right\}\tilde{b}_{\beta;\ell}^{(2,2)}(z,s)=0$$

Solving for the $\tilde{b}_\beta^{(\nu,\nu')}$ we construct the blocks of the matrizant as

$$\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{\nu,\nu'}=\sum_{\beta=1}^{N}\tilde{b}_{\beta;\ell}^{(\nu,\nu')}(z,s)\,(\tilde{x}_n(s))_\beta\,(\tilde{x}_n(s))_\beta \tag{4.14}$$

The $\tilde{b}_{\beta;\ell}^{(\nu,\nu')}$ are mutually related via (4.10) as

$$\frac{\partial}{\partial z}\tilde{b}_{\beta;\ell}^{(1,1)}(z,s) = -\tilde{Z}'_{\beta}(z,s)\ \tilde{Y}_{c_\beta}^{(\ell)}(s)\ \tilde{b}_{\beta;\ell}^{(2,1)}(z,s)$$

$$\frac{\partial}{\partial z}\tilde{b}_{\beta;\ell}^{(1,2)}(z,s) = -\tilde{Z}'_{\beta}(z,s)\ \tilde{Y}_{c_\beta}^{(\ell)}(s)\ \tilde{b}_{\beta;\ell}^{(2,2)}(z,s)$$

$$\frac{\partial}{\partial z}\tilde{b}_{\beta;\ell}^{(2,1)}(z,s) = -\tilde{Y}'_{\beta}(z,s)\ \tilde{Z}_{c_\beta}^{(\ell)}(s)\ \tilde{b}_{\beta;\ell}^{(1,1)}(z,s) \tag{4.15}$$

$$\frac{\partial}{\partial z}\tilde{b}_{\beta;\ell}^{(2,2)}(z,s) = -\tilde{Y}'_{\beta}(z,s)\ \tilde{Z}_{c_\beta}^{(\ell)}(s)\ \tilde{b}_{\beta;\ell}^{(1,2)}(z,s)$$

Noting in (4.13) which coefficients satisfy a voltage or a current differential equation, let us write

$$\tilde{b}_{\beta;\ell}^{(1,1)}(z,s) = \tilde{d}_{\beta;1}^{(1,1)}(s)\ \tilde{v}_{\beta}^{(\ell,1)}(z,s) + \tilde{d}_{\beta;2}^{(1,1)}(s)\ \tilde{v}_{\beta}^{(\ell,2)}(z,s)$$

$$\tilde{b}_{\beta;\ell}^{(1,2)}(z,s) = \tilde{d}_{\beta;1}^{(1,2)}(s)\ \tilde{v}_{\beta}^{(\ell,1)}(z,s) + \tilde{d}_{\beta;2}^{(1,2)}(s)\ \tilde{v}_{\beta}^{(\ell,2)}(z,s)$$

$$\tilde{b}_{\beta;\ell}^{(2,1)}(z,s) = \tilde{d}_{\beta;1}^{(2,1)}(s)\ \tilde{i}_{\beta}^{(\ell,1)}(z,s) + \tilde{d}_{\beta;2}^{(2,1)}(s)\ \tilde{i}_{\beta}^{(\ell,2)}(z,s) \tag{4.16}$$

$$\tilde{b}_{\beta;\ell}^{(2,2)}(z,s) = \tilde{d}_{\beta;1}^{(2,2)}(s)\ \tilde{i}_{\beta}^{(\ell,1)}(z,s) + \tilde{d}_{\beta;2}^{(2,2)}(s)\ \tilde{i}_{\beta}^{(\ell,2)}(z,s)$$

Using (4.15) to give

$$\frac{\partial}{\partial z}\tilde{b}_{\beta;\ell}^{(1,1)}(z,s) = -\tilde{Z}'_{\beta}(s)\ \tilde{Y}_{c_\beta}^{(\ell)}(s)\left[\tilde{d}_{\beta;1}^{(1,1)}(s)\ \tilde{i}_{\beta}^{(\ell,1)}(z,s) + \tilde{d}_{\beta;2}^{(1,1)}(s)\ \tilde{i}_{\beta}^{(\ell,2)}(z,s)\right]$$

$$= -\tilde{Z}'_{\beta}(s)\ \tilde{Y}_{c_\beta}^{(\ell)}(s)\left[\tilde{d}_{\beta;1}^{(2,1)}(s)\ \tilde{i}_{\beta}^{(\ell,1)}(z,s) + \tilde{d}_{\beta;2}^{(2,1)}(s)\ \tilde{i}_{\beta}^{(\ell,2)}(z,s)\right]$$

$$\frac{\partial}{\partial z}\tilde{b}_{\beta;\ell}^{(2,2)}(z,s) = -\tilde{Y}'_{\beta}(s)\ \tilde{Z}_{c_\beta}^{(\ell)}(s)\left[\tilde{d}_{\beta;1}^{(1,2)}(s)\ \tilde{v}_{\beta}^{(\ell,1)}(z,s) + \tilde{d}_{\beta;2}^{(1,2)}(s)\ \tilde{v}_{\beta}^{(\ell,2)}(z,s)\right] \tag{4.17}$$

$$= -\tilde{Y}'_{\beta}(s)\ \tilde{Z}_{c_\beta}^{(\ell)}(s)\left[\tilde{d}_{\beta;1}^{(2,2)}(s)\ \tilde{v}_{\beta}^{(\ell,1)}(z,s) + \tilde{d}_{\beta;2}^{(2,2)}(s)\ \tilde{v}_{\beta}^{(\ell,2)}(z,s)\right]$$

the coefficients are reduced to four unknowns (for each β) as

$$\tilde{d}_{\beta;1}^{(1,1)}(s)=\tilde{d}_{\beta;1}^{(2,1)}(s)\ ,\ \ \tilde{d}_{\beta;2}^{(1,1)}(s)=\tilde{d}_{\beta;2}^{(2,1)}(s)$$

$$\tilde{d}_{\beta;1}^{(2,2)}(s)=\tilde{d}_{\beta;1}^{(1,2)}(s)\ ,\ \ \tilde{d}_{\beta;2}^{(2,2)}(s)=\tilde{d}_{\beta;2}^{(2,1)}(s) \qquad (4.18)$$

Applying the boundary conditions to (4.16) as

$$\tilde{b}_{\beta;\ell}^{(1,1)}(z_\ell,s)=1=\tilde{d}_{\beta;1}^{(1,1)}(s)\ \tilde{v}_\beta^{(\ell,1)}(z_\ell,s)+\tilde{d}_{\beta;2}^{(1,1)}(s)\ \tilde{v}_\beta^{(\ell,2)}(z_\ell,s)$$

$$\tilde{b}_{\beta;\ell}^{(1,2)}(z_\ell,s)=0=\tilde{d}_{\beta;1}^{(2,2)}(s)\ \tilde{v}_\beta^{(\ell,1)}(z_\ell,s)+\tilde{d}_{\beta;2}^{(2,2)}(s)\ \tilde{v}_\beta^{(\ell,2)}(z_\ell,s)$$

$$\tilde{b}_{\beta;\ell}^{(2,1)}(z_\ell,s)=0=\tilde{d}_{\beta;1}^{(1,1)}(s)\ \tilde{i}_\beta^{(\ell,1)}(z_\ell,s)+\tilde{d}_{\beta;2}^{(1,1)}(s)\ \tilde{i}_\beta^{(\ell,2)}(z_\ell,s) \qquad (4.19)$$

$$\tilde{b}_{\beta;\ell}^{(2,2)}(z_\ell,s)=1=\tilde{d}_{\beta;1}^{(2,2)}(s)\ \tilde{i}_\beta^{(\ell,1)}(z_\ell,s)+\tilde{d}_{\beta;2}^{(2,2)}(s)\ \tilde{i}_\beta^{(\ell,2)}(z_\ell,s)$$

this gives matrix equations for the coefficients as

$$\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\tilde{v}_\beta^{(\ell,1)}(z_\ell,s) & \tilde{v}_\beta^{(\ell,2)}(z_\ell,s)\\ \tilde{i}_\beta^{(\ell,1)}(z_\ell,s) & \tilde{i}_\beta^{(\ell,2)}(z_\ell,s)\end{pmatrix}\cdot\begin{pmatrix}\tilde{d}_{\beta;1}^{(1,1)}(s)\\ \tilde{d}_{\beta;2}^{(1,1)}(s)\end{pmatrix}$$

$$\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\tilde{i}_\beta^{(\ell,1)}(z_\ell,s) & \tilde{i}_\beta^{(\ell,2)}(z_\ell,s)\\ \tilde{v}_\beta^{(\ell,1)}(z_\ell,s) & \tilde{v}_\beta^{(\ell,2)}(z_\ell,s)\end{pmatrix}\cdot\begin{pmatrix}\tilde{d}_{\beta;1}^{(2,2)}(s)\\ \tilde{d}_{\beta;2}^{(2,2)}(s)\end{pmatrix} \qquad (4.20)$$

These are solved as

$$\begin{pmatrix}\tilde{d}_{\beta;1}^{(1,1)}(s)\\ \tilde{d}_{\beta;2}^{(1,1)}(s)\end{pmatrix}=\tilde{D}_\beta^{(\ell)}(s)\begin{pmatrix}\tilde{i}_\beta^{(\ell,2)}(z_\ell,s) & -\tilde{v}_\beta^{(\ell,2)}(z_\ell,s)\\ -\tilde{i}_\beta^{(\ell,1)}(z_\ell,s) & \tilde{v}_\beta^{(\ell,1)}(z_\ell,s)\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}$$

$$\begin{pmatrix}\tilde{d}_{\beta;1}^{(2,2)}(s)\\ \tilde{d}_{\beta;2}^{(2,2)}(s)\end{pmatrix}=-\tilde{D}_\beta^{(\ell)}(s)\begin{pmatrix}\tilde{v}_\beta^{(\ell,2)}(z_\ell,s) & -\tilde{i}_\beta^{(\ell,2)}(z_\ell,s)\\ -\tilde{v}_\beta^{(\ell,1)}(z_\ell,s) & \tilde{i}_\beta^{(\ell,1)}(z_\ell,s)\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} \qquad (4.21)$$

where the common coefficient has the form of a Wronskian

$$
\begin{aligned}
\tilde{D}_{\beta}^{(\ell)^{-1}}(s) &= \left[\tilde{v}_{\beta}^{(\ell,1)}(z_{\ell},s)\ \tilde{i}_{\beta}^{(\ell,2)}(z_{\ell},s) - \tilde{v}_{\beta}^{(\ell,2)}(z_{\ell},s)\ \tilde{i}_{\beta}^{(\ell,1)}(z_{\ell},s)\right] \\
&= -\frac{\tilde{Z}_{c_{\beta}}^{(\ell)}(s)}{\tilde{Z}_{\beta}'(z,s)}\left[\tilde{v}_{\beta}^{(\ell,1)}(z,s)\frac{\partial}{\partial z}\tilde{v}_{\beta}^{(\ell,2)}(z,s) - \tilde{v}_{\beta}^{(\ell,2)}(z,s)\frac{\partial}{\partial z}\tilde{v}_{\beta}^{(\ell,1)}(z,s)\right]\Bigg|_{z=z_{\ell}} \\
&= \frac{\tilde{Y}_{c_{\beta}}^{(\ell)}(s)}{\tilde{Y}_{\beta}'(z,s)}\left[\tilde{i}_{\beta}^{(\ell,1)}(z,s)\frac{\partial}{\partial z}\tilde{i}_{\beta}^{(\ell,2)}(z,s) - \tilde{i}_{\beta}^{(\ell,2)}(z,s)\frac{\partial}{\partial z}\tilde{i}_{\beta}^{(\ell,1)}(z,s)\right]\Bigg|_{z=z_{\ell}}
\end{aligned}
\tag{4.22}
$$

So now collect together the various terms to give the matrizant

$$
\left(\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{\nu,\nu'}\right) =
$$

$$
\begin{pmatrix}
\sum_{\beta=1}^{N}\tilde{D}_{\beta}^{(\ell)}(s)\left[\tilde{i}_{\beta}^{(\ell,2)}(z_{\ell},s)\tilde{v}_{\beta}^{(\ell,1)}(z,s) - \tilde{i}_{\beta}^{(\ell,1)}(z_{\ell},s)\tilde{v}_{\beta}^{(\ell,2)}(z,s)\right](\tilde{x}_n(s))_{\beta}\ (\tilde{x}_n(s))_{\beta} & \sum_{\beta=1}^{N}\tilde{D}_{\beta}^{(\ell)}(s)\left[-\tilde{v}_{\beta}^{(\ell,2)}(z_{\ell},s)\tilde{v}_{\beta}^{(\ell,1)}(z,s) + \tilde{v}_{\beta}^{(\ell,1)}(z_{\ell},s)\tilde{v}_{\beta}^{(\ell,2)}(z,s)\right](\tilde{x}_n(s))_{\beta}\ (\tilde{x}_n(s))_{\beta} \\
\sum_{\beta=1}^{N}\tilde{D}_{\beta}^{(\ell)}(s)\left[\tilde{i}_{\beta}^{(\ell,2)}(z_{\ell},s)\tilde{i}_{\beta}^{(\ell,1)}(z,s) - \tilde{i}_{\beta}^{(\ell,1)}(z_{\ell},s)\tilde{i}_{\beta}^{(\ell,2)}(z,s)\right](\tilde{x}_n(s))_{\beta}\ (\tilde{x}_n(s))_{\beta} & \sum_{\beta=1}^{N}\tilde{D}_{\beta}^{(\ell)}(s)\left[-\tilde{v}_{\beta}^{(\ell,2)}(z_{\ell},s)\tilde{i}_{\beta}^{(\ell,1)}(z,s) + \tilde{v}_{\beta}^{(\ell,1)}(z_{\ell},s)\tilde{i}_{\beta}^{(\ell,2)}(z,s)\right](\tilde{x}_n(s))_{\beta}\ (\tilde{x}_n(s))_{\beta}
\end{pmatrix}
\tag{4.23}
$$

This can be evaluated at $z = z_{\ell} + 1$ and renormalized to the characteristic impedance matrix there via (2.14).

If we further assume that the eigenvectors $(\tilde{x}_n(s))_{\beta}$, for the ℓth section are not changed on going into the $\ell+1$st section, then the renormalization can be accomplished in terms of the same modes as

$$\left(\left(\tilde{U}_{n,m}^{(\ell+1)}(s)\right)_{v,v'}\right)=$$

$$\begin{pmatrix} \sum_{\beta=1}^{N}\tilde{D}_{\beta}^{(\ell)}(s)\left[\tilde{i}_{\beta}^{(\ell,2)}(z_{\ell},s)\tilde{v}_{\beta}^{(\ell,1)}(z_{\ell+1},s)-\tilde{i}_{\beta}^{(\ell,1)}(z_{\ell},s)\tilde{v}_{\beta}^{(\ell,2)}(z_{\ell+1},s)\right](\tilde{x}_n(s))_{\beta}\ (\tilde{x}_n(s))_{\beta} & \sum_{\beta=1}^{N}\tilde{D}_{\beta}^{(\ell)}(s)\left[-\tilde{v}_{\beta}^{(\ell,2)}(z_{\ell},s)\tilde{v}_{\beta}^{(\ell,1)}(z_{\ell+1},s)+\tilde{v}_{\beta}^{(\ell,1)}(z_{\ell},s)\tilde{v}_{\beta}^{(\ell,2)}(z_{\ell+1},s)\right](\tilde{x}_n(s))_{\beta}\ (\tilde{x}_n(s))_{\beta} \\ \sum_{\beta=1}^{N}\tilde{D}_{\beta}^{(\ell)}(s)\frac{\tilde{Z}_{c_\beta}^{(\ell+1)}(s)}{\tilde{Z}_{c_\beta}^{(\ell)}(s)}\left[\tilde{i}_{\beta}^{(\ell,2)}(z_{\ell},s)\tilde{i}_{\beta}^{(\ell,1)}(z_{\ell+1},s)-\tilde{i}_{\beta}^{(\ell,1)}(z_{\ell},s)\tilde{i}_{\beta}^{(\ell,2)}(z_{\ell+1},s)\right](\tilde{x}_n(s))_{\beta}\ (\tilde{x}_n(s))_{\beta} & \sum_{\beta=1}^{N}\tilde{D}_{\beta}^{(\ell)}(s)\frac{\tilde{Z}_{c_\beta}^{(\ell+1)}(s)}{\tilde{Z}_{c_\beta}^{(\ell)}(s)}\left[-\tilde{v}_{\beta}^{(\ell,2)}(z_{\ell},s)\tilde{i}_{\beta}^{(\ell,1)}(z_{\ell+1},s)+\tilde{v}_{\beta}^{(\ell,1)}(z_{\ell},s)\tilde{i}_{\beta}^{(\ell,2)}(z_{\ell+1},s)\right](\tilde{x}_n(s))_{\beta}\ (\tilde{x}_n(s)) \end{pmatrix}$$

(4.24)

If, however, there are new modes $\left(\tilde{x}_n^{(\ell+1)}(s)\right)_{\beta}$ on the ℓ+1st section, then the matrix $\left(\tilde{Z}_{c_{n,m}}^{(\ell+1)}(s)\right)$ appearing in (2.14) in the renormalization cannot be placed inside the summations in (4.24) in terms of its eigenvalues. Rather $\left(\tilde{Z}_{c_{n,m}}^{(\ell+1)}(s)\right)\cdot$ appears in front of the summations in two places (the (2,1) and (2,2) positions).

V. Limitations of Foregoing Solutions for NMTL Segments

In our search for analytically convenient forms of the NMTL parameters section IV has shown the general result for the case that $\left(\tilde{Z}'_{n,m}(z,s)\right)$, $\left(\tilde{Y}'_{n,m}(z,s)\right)$, $\left(\tilde{\gamma}'_{c_{n,m}}(z,s)\right)$, and $\left(\tilde{Z}_{c_{n,m}}(z,s)\right)$ are all symmetric. and diagonalized by the same set of N eigenvectors $(\tilde{x}_n(s))_\beta$ with each matrix having in general N eigenvalues. In the case that the NMTL has the propagation eigenvalues $\tilde{\gamma}_\beta(z,s)$ not all the same, this diagonalization is based on the physical modes which propagate with different speeds along the NMTL. As in section III one can have a biorthonormal set of eigenmodes $\left(\tilde{v}_{c_n}(s)\right)_\beta$ and $\left(\tilde{i}_{c_n}(s)\right)_\beta$ for a nonsymmetric diagonalization of the (nonsymmetric) propagation matrix in (3.5). This gives a representation of the other matrices (in general non diagonal) as in (3.6). Provided these $2N$ eigenvectors are all position-independent, except perhaps for position-dependent scalar multipliers, then the results of section IV can be extended to this case, since one does not need to take the z derivatives of the various eigenvectors into account (which would otherwise be of different orientation (not parallel) with respect to the original eigenvectors).

A limitation in the foregoing approach concerns the smooth matching of the solution at z_ℓ to that at $z_{\ell+1}$. If the characteristic impedance matrix $\left(\tilde{Z}_{c_{n,m}}(z_\ell,s)\right)$ has a representation in terms of the $(\tilde{x}_n(s))_\beta$ or $\left(\tilde{v}_{c_n}(s)\right)_\beta$ at $z=z_\ell$, one cannot in general represent $\left(\tilde{Z}_{c_{n,m}}(z_{\ell+1},s)\right)$ in terms of sums over N dyadic terms like $(\tilde{x}_n(s))_\beta\ (\tilde{x}_n(s))_\beta$ or $\left(\tilde{v}_{c_n}(s)\right)_\beta\ \left(\tilde{v}_{c_n}(s)\right)_\beta$. One can use the dyads to construct intermediate values of $\left(\tilde{Z}_{c_{n,m}}(z,s)\right)$ for $z_\ell < z < z_{\ell+1}$ which are continuously varying from the value at z_ℓ. However, as one reaches $z_{\ell+1}$ there will in general be a discontinuity in the matrix as one tries to approximate a pre-specified value $\left(\tilde{Z}_{c_{n,m}}(z_{\ell+1},s)\right)$. Associate with such discontinuities there are reflections, including in the high-frequency limit.

In segmenting an NMTL we would like to use the general solution (2.16) for the line as discussed in section II. However, one would like to be able to construct the solutions for the individual segments (the $\left(\left(\tilde{u}^{(\ell)}_{n,m}(z_{\ell+1},s)\right)_{\nu,\nu'}\right)$) in a form which was not limited to the piecewise-constant form in section III, or other forms which contain discontinuities at the segment boundaries. This would remove the high-frequency reflections produced at the boundaries by approximation of the NMTL in terms of these segments, thereby giving a better approximate solution. Instead of a piecewise-constant approximation let us look for continuous approximation.

VI. Continuous Characteristic-Impedance-Matrix Approximation for the Case of the Propagation Matrix as a Scalar Function Times the Identity Matrix

So now let us make some assumption which will allow a continuous variation from $\left(\tilde{Z}_{c_{n,m}}(z_\ell,s)\right)$ to $\left(\tilde{Z}_{c_{n,m}}(z_{\ell+1},s)\right)$. One way to do this is to constrain the propagation matrix to the form (as in [3])

$$\left(\tilde{\gamma}_{c_{n,m}}(z,s)\right)=\left[\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right)\right]^{\frac{1}{2}}\equiv\tilde{\gamma}(z,s)\ \left(1_{n,m}\right) \tag{6.1}$$

i.e. all modes propagate with the same speed. Furthermore the identity commutes with everything, and can be described by a complete set of eigenvectors (orthogonal or biorthogonal) that are generated by any NxN matrix with such a complete set.

Basically (6.1) assumes that, except for scalar factors, the per-unit-length impedance and admittance matrices are mutually inverse. These can then be represented as

$$\left(\tilde{Z}'_{n,m}(z,s)\right)=\tilde{Z}'(z,s)\ \left(\tilde{f}_{g_{n,m}}(z,s)\right) \qquad \text{(symmetric)}$$

$$\left(\tilde{Y}'_{n,m}(z,s)\right)=\tilde{Y}'(z,s)\ \left(\tilde{f}_{g_{n,m}}(z,s)\right)^{-1} \qquad \text{(symmetric)} \tag{6.2}$$

$$\tilde{Z}'(z,s)\ \tilde{Y}'(z,s)=\tilde{\gamma}^2(z,s)$$

which also gives

$$\left(\tilde{Z}_{c_{n,m}}(z,s)\right)=\tilde{Z}_c(z,s)\ \left(\tilde{f}_{g_{n,m}}(z,s)\right)$$

$$\left(\tilde{Y}_{c_{n,m}}(z,s)\right)=\tilde{Y}_c(z,s)\ \left(\tilde{f}_{g_{n,m}}(z,s)\right)^{-1} \tag{6.3}$$

$$\tilde{Z}_c(z,s)=\tilde{Y}_c^{-1}(z,s)=\left[\frac{\tilde{Z}'(z,s)}{\tilde{Y}'(z,s)}\right]^{\frac{1}{2}}$$

So now we have the matrix $\left(\tilde{f}_{g_{n,m}}(z,s)\right)$ to deal with.

Physically this form includes the case that the electric parameters (ε, σ) and magnetic parameter (μ) of the medium surrounding the N perfectly conducting wires (plus reference) are uniform on each cross section (constant z). So, while other special cases may fall under the form of (6.2) let us further assume that

$$\left(\tilde{f}_{g_{n,m}}(z,s)\right) \equiv \left(f_{g_{n,m}}(z)\right) \equiv \text{geometrical} - \text{impedance} - \text{factor matrix} \tag{6.4}$$

so that it is frequency independent as well as dimensionless. Then we have

$$\begin{aligned} &\tilde{Z}'(z,s) = s\mu \quad , \quad \tilde{Y}'(z,s) = \sigma + s\varepsilon \\ &\tilde{\gamma}(z,s) = [s\mu(\sigma + s\varepsilon)]^{\frac{1}{2}}, \ \tilde{Z}_c(z,s) = \tilde{Y}_c^{-1}(z,s) = \left[\frac{s\mu}{\sigma + s\varepsilon}\right]^{\frac{1}{2}} \end{aligned} \tag{6.5}$$

where $\mu, \varepsilon,$ and σ can be functions of z and s.

Substituting this form in (2.11) for the ℓth section we have

$$\frac{d}{dz}\left(\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{\nu,\nu'}\right) = \left(\left(\tilde{a}_{n,m}^{(\ell)}(z,s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{\nu,\nu'}\right)$$

$$\left(\left(\tilde{u}_{n,m}^{(\ell)}(z_\ell,s)\right)_{\nu,\nu'}\right) = \left(\left(1_{n,m}\right)_{\nu,\nu'}\right)$$

$$\begin{aligned} \left(\left(\tilde{a}_{n,m}^{(\ell)}(z,s)\right)_{\nu,\nu'}\right) &= -\begin{pmatrix} \left(0_{n,m}\right) & \left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(\tilde{Y}'^{(\ell)}_{c_{n,m}}(s)\right) \\ \left(\tilde{Z}^{(\ell)}_{c_{n,m}}(s)\right) \cdot \left(\tilde{Y}'_{n,m}(z,s)\right) & \left(0_{n,m}\right) \end{pmatrix} \\ &= -\tilde{\gamma}(z,s) \begin{pmatrix} \left(0_{n,m}\right) & \tilde{Z}_c(z,s) \ \tilde{Y}_c(z_\ell,s) \left(f_{n,m}^{(\ell)}(z)\right) \\ \tilde{Y}_c(z,s) \ \tilde{Z}_c(z_\ell,s) \left(f_{n,m}^{(\ell)}(z)\right) & \left(0_{n,m}\right) \end{pmatrix} \end{aligned} \tag{6.6}$$

$$\left(f_{n,m}^{(\ell)}(z)\right) \equiv \left(f_{g_{n,m}}(z)\right) \cdot \left(f_{g_{n,m}}(z_\ell)\right)^{-1}$$

Thus, after factoring out $-\tilde{\gamma}(z,s)$ the supermatrix $\left(\left(\tilde{a}_{n,m}^{(\ell)}(z,s)\right)_{\nu,\nu'}\right)$ has the off-diagonal blocks as mutually inverse. Furthermore at $z = z_\ell$ this is

$$\left(\left(\tilde{a}_{n,m}^{(\ell)}(z_\ell,s)\right)_{\nu,\nu'}\right) = -\tilde{\gamma}(z_\ell,s)\begin{pmatrix} (0_{n,m}) & (1_{n,m}) \\ (1_{n,m}) & (0_{n,m}) \end{pmatrix} \tag{6.7}$$

By a change of the coordinate z in the general form

$$\tilde{\gamma}(z,s)\,dz = \tilde{\gamma}(z_\ell,s)\,d\varsigma$$
$$\int_{z_\ell}^{z} \tilde{\gamma}(z',s)\,dz' = \tilde{\gamma}(z_\ell,s)\left[\varsigma - \varsigma_\ell\right] \tag{6.8}$$

the first-order differential equation in (6.6) has the propagation constant $\tilde{\gamma}(z,s)$ moved to the derivative (now with respect to ς) on the left hand side, leaving $\tilde{\gamma}(z_\ell,s)$ on the right hand side. Furthermore the coefficient $\tilde{Z}_c(z,s)\ \tilde{Y}_c(z_\ell,s)$ of $\left(f_{n,m}^{(\ell)}(z)\right)$ can be absorbed into this matrix (which can allowed the more general form as a function of frequency if desired). Then the coefficient matrix has the canonical form

$$\left(\left(\tilde{a}_{n,m}(z,s)\right)_{\nu,\nu'}\right) = -\gamma\begin{pmatrix} (0_{n,m}) & \left(f_{n,m}^{(\ell)}(z)\right) \\ \left(f_{n,m}^{(\ell)}(z)\right)^{-1} & (0_{n,m}) \end{pmatrix}$$
$$\gamma \equiv s\sqrt{\mu\varepsilon} \tag{6.9}$$

which by the above discussion can be changed to the more general dependence on z and s indicated in (6.6). Interestingly enough the form in (6.9) is for constant μ and ε with $\sigma = 0$. This corresponds to the case of N perfectly conducting wires (plus reference) in a uniform lossless/dispersionless medium.

Now we come to the important advantage for this form of the NMTL equations (all modal velocities the same). Suppose that we have some functional form of $\left(f_{g_{n,m}}(z)\right)$ specified to us along the line. This matrix, while symmetric (reciprocity), may have eigenvectors which are functions of z. However, consider the values of this matrix at the various z_ℓ as some specified set of $\left(f_{g_{n,m}}(z_\ell)\right)$, taken essentially as samples along the NMTL. Next consider the normalized form of this matrix in (6.9) which is in general nonsymmetric. In particular note that at the endpoints of the interval $z_\ell \le z \le z_{\ell+1}$ we have

$$\left(f_{n,m}^{(\ell)}(z_\ell)\right)=\left(1_{n,m}\right) \quad , \quad \left(f_{n,m}^{(\ell)}(z_{\ell+1})\right)=\left(f_{g_{n,m}}(z_{\ell+1})\right)\cdot\left(f_{g_{n,m}}(z_\ell)\right)^{-1} \tag{6.10}$$

These two matrices commute, so the idea is to interpolate between these two matrices on the same interval by a matrix function of z which commutes with other values of the matrix for all pairs of points on the interval.

To form this interpolation matrix diagonalize as

$$\left(f_{n,m}^{(\ell)}(z_{\ell+1})\right)=\sum_{\beta=1}^{N} f_\beta^{(\ell)}(z_\ell+1)\left(g_n^{(\ell)}\right)_\beta\left(h_n^{(\ell)}\right)_\beta$$

$$\left(g_n\right)_{\beta_1}\cdot\left(h_n\right)_{\beta_2}=1_{\beta_1,\beta_2} \qquad \text{(biorthonormal)} \tag{6.11}$$

where we have assumed that this is not an atypical case and a complete set of eigenvectors exists. Clearly this matrix commutes with the identity which we write as

$$\left(1_{n,m}\right)=\left(f_{n,m}^{(\ell)}(z_\ell)\right)=\sum_{\beta=1}^{N}\left(g_n^{(\ell)}\right)_\beta\left(h_n^{(\ell)}\right)_\beta \tag{6.12}$$

This is of the same form as (6.11) except that the eigenvalues are all 1. This suggests that we try the form (as an approximation)

$$\left(f_{n,m}^{(\ell)'}(z)\right)\equiv\sum_{\beta=1}^{N} f_\beta^{(\ell)}(z)\left(g_n^{(\ell)}\right)_\beta\left(h_n^{(\ell)}\right)_\beta$$

$$\simeq\left(f_{n,m}^{(\ell)}(z)\right) \tag{6.13}$$

with equality assured at both z_ℓ and $z_{\ell+1}$ provided

$$f_\beta^{(\ell)}(z_\ell)=1 \tag{6.14}$$

So let us require that all $f_\beta^{(\ell)}(z)$ be continuous from z_ℓ (in 6.14) to $z_{\ell+1}$ (from (6.11)). The interpolating functions can be chosen for our convenience (e.g. a polynomial or an exponential in z), perhaps with an eye to minimizing the error in the approximation in (6.13).

So assuming the form in (6.13) as our $\left(f_{n,m}^{(\ell)}(z)\right)$ we need to find the matrizant $\left(\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{v,v'}\right)$. Following a procedure similar to that in Section IV the coefficient matrix is written as

$$\left(\left(\tilde{a}_{n,m}^{(\ell)}(z,s)\right)_{v,v'}\right) = -\gamma \begin{pmatrix} \left(0_{n,m}\right) & \sum_{\beta=1}^{N} f_\beta^{(\ell)}(z)\left(g_n^{(\ell)}\right)_\beta \left(h_n^{(\ell)}\right)_\beta \\ \sum_{\beta=1}^{N} f_\beta^{(\ell)^{-1}}(z)\left(g_n^{(\ell)}\right)_\beta \left(h_n^{(\ell)}\right)_\beta & \left(0_{n,m}\right) \end{pmatrix} \tag{6.15}$$

The matrizant blocks in (4.14) are replaced by

$$\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{v,v'} = \sum_{\beta=1}^{N} \tilde{b}_{\beta;\ell}^{(v,v')}(z,s)\left(g_n^{(\ell)}\right)_\beta \left(h_n^{(\ell)}\right)_\beta \tag{6.16}$$

and we replace

$$\begin{aligned} \tilde{Z}'_\beta(z,s)\ \tilde{Y}_{c_\beta}^{(\ell)}(s) &\to \gamma\ f_\beta^{(\ell)}(z) \\ \tilde{Y}'_\beta(z,s)\ \tilde{Z}_{c_\beta}^{(\ell)}(s) &\to \gamma\ f_\beta^{(\ell)^{-1}}(z) \end{aligned} \tag{6.17}$$

The second-order differential equations for the eigenvalues are

$$\left\{\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial z}\ell n\left(f_\beta^{(\ell)}(z)\right)\frac{\partial}{\partial z} - \gamma^2\right\}\ \tilde{b}_{\beta;\ell}^{(1,1)}(z,s) = 0$$

$$\left\{\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial z}\ell n\left(f_\beta^{(\ell)}(z)\right)\frac{\partial}{\partial z} - \gamma^2\right\}\ \tilde{b}_{\beta;\ell}^{(1,2)}(z,s) = 0$$

$$\left\{\frac{\partial^2}{\partial z^2}+\frac{\partial}{\partial z}\ell n\left(f_\beta^{(\ell)}(z)\right)\frac{\partial}{\partial z}-\gamma^2\right\}\ \tilde{b}_{\beta;\ell}^{(2,1)}(z,s)=0$$

$$\left\{\frac{\partial^2}{\partial z^2}+\frac{\partial}{\partial z}\ell n\left(f_\beta^{(\ell)}(z)\right)\frac{\partial}{\partial z}-\gamma^2\right\}\ \tilde{b}_{\beta;\ell}^{(2,2)}(z,s)=0 \tag{6.18}$$

Use the $\tilde{v}_\beta^{(\ell,\delta)}(z,s)$ (voltage-like) for the first two of these equations and the $\tilde{i}_\beta^{(\ell,\delta)}(z,s)$ (current-like) for the second two, and note that

$$\frac{\partial}{\partial z}\tilde{v}_\beta^{(\ell,\delta)}(z,s)=-\gamma\ f_\beta^{(\ell)}(z)\ \tilde{i}_\beta^{(\ell,\delta)}(z,s)$$

$$\frac{\partial}{\partial z}\tilde{i}_\beta^{(\ell,\delta)}(z,s)=-\gamma\ f_\beta^{(\ell)^{-1}}(z)\ \tilde{v}_\beta^{(\ell,\delta)}(z,s)$$

$$\begin{aligned}\tilde{D}_\beta^{(\ell)^{-1}}(s)&=\left[\tilde{v}_\beta^{(\ell,1)}(z_\ell,s)\ \tilde{i}_\beta^{(\ell,2)}(z_\ell,s)-\tilde{v}_\beta^{(\ell,2)}(z_\ell,s)\ \tilde{i}_\beta^{(\ell,1)}(z_\ell,s)\right]\\ &=-\frac{1}{\gamma f_\beta^{(\ell)}(z)}\left[\tilde{v}_\beta^{(\ell,1)}(z,s)\frac{\partial}{\partial z}\tilde{v}_\beta^{(\ell,2)}(z,s)-\tilde{v}_\beta^{(\ell,2)}(z,s)\frac{\partial}{\partial z}\tilde{v}_\beta^{(\ell,1)}(z,s)\right]\Bigg|_{z=z_\ell}\\ &=\frac{f_\beta^{(\ell)}(z)}{\gamma}\left[\tilde{i}_\beta^{(\ell,1)}(z,s)\frac{\partial}{\partial z}\tilde{i}_\beta^{(\ell,2)}(z,s)-\tilde{i}_\beta^{(\ell,2)}(z,s)\frac{\partial}{\partial z}\tilde{i}_\beta^{(\ell,1)}(z_\ell,s)\right]\Bigg|_{z=z_\ell}\end{aligned} \tag{6.19}$$

Here one can note that the ℓth section of the NMTL will in general have different special functions to describe voltages and currents depending on the particular $f_\beta^{(\ell)}(z)$ used for each section.

The matrizant now has the form

$$\left(\left(\tilde{u}_{n,m}^{(\ell)}(z,s)\right)_{\nu,\nu'}\right)=$$

$$\begin{pmatrix} \sum_{\beta=1}^{N}\tilde{D}_{\beta}^{(\ell)}(s)\left[\tilde{i}_{\beta}^{(\ell,2)}(z_{\ell},s)\ \tilde{i}_{\beta}^{(\ell,1)}(z,s)-\tilde{i}_{\beta}^{(\ell,1)}(z_{\ell},s)\ \tilde{v}_{\beta}^{(\ell,2)}(z,s)\right]\left(g_n^{(\ell)}\right)_{\beta}\left(h_n^{(\ell)}\right)_{\beta} & \sum_{\beta=1}^{N}\tilde{D}_{\beta}^{(\ell)}(s)\left[-\tilde{v}_{\beta}^{(\ell,2)}(z_{\ell},s)\ \tilde{v}_{\beta}^{(\ell,1)}(z,s)+\tilde{v}_{\beta}^{(\ell,1)}(z_{\ell},s)\ \tilde{v}_{\beta}^{(\ell,2)}(z,s)\right]\left(g_n^{(\ell)}\right)_{\beta}\left(h_n^{(\ell)}\right)_{\beta} \\ \sum_{\beta=1}^{N}\tilde{D}_{\beta}^{(\ell)}(s)\left[\tilde{i}_{\beta}^{(\ell,2)}(z_{\ell},s)\ \tilde{i}_{\beta}^{(\ell,1)}(z,s)-\tilde{i}_{\beta}^{(\ell,1)}(z_{\ell},s)\ \tilde{i}_{\beta}^{(\ell,2)}(z,s)\right]\left(g_n^{(\ell)}\right)_{\beta}\left(h_n^{(\ell)}\right)_{\beta} & \sum_{\beta=1}^{N}\tilde{D}_{\beta}^{(\ell)}(s)\left[-\tilde{v}_{\beta}^{(\ell,2)}(z_{\ell},s)\ \tilde{i}_{\beta}^{(\ell,1)}(z,s)+\tilde{v}_{\beta}^{(\ell,1)}(z_{\ell},s)\ \tilde{i}_{\beta}^{(\ell,2)}(z,s)\right]\left(g_n^{(\ell)}\right)_{\beta}\left(h_n^{(\ell)}\right)_{\beta} \end{pmatrix} \tag{6.20}$$

This can be readily evaluated at $z=z_{\ell+1}$. Then from (2.14) we have

$$\left(\left(\tilde{U}_{n,m}^{(\ell+1)}(s)\right)_{\nu,\nu'}\right)=\begin{pmatrix}(1_{n,m}) & (0_{n,m}) \\ (0_{n,m}) & \left(\tilde{Z}_{c_{n,m}}^{(\ell+1)}(s)\right)\cdot\left(\tilde{Z}_{c_{n,m}}^{(\ell)}(s)\right)^{-1}\end{pmatrix}\left(\left(\tilde{u}_{n,m}^{(\ell)}(z_{\ell+1},s)\right)_{\nu,\nu'}\right)$$

$$\left(\tilde{Z}_{c_{n,m}}^{(\ell+1)}(s)\right)\cdot\left(\tilde{Z}_{c_{n,m}}^{(\ell)}(s)\right)^{-1}=\left(f_{g_{n,m}}(z_{\ell+1})\right)\cdot\left(f_{g_{n,m}}(z_{\ell})\right)^{-1} \tag{6.21}$$

$$=\left(f_{n,m}^{(\ell)}(z_{\ell+1})\right)=\sum_{\beta=1}^{N}f_{\beta}^{(\ell)}(z_{\ell+1})\left(g_n^{(\ell)}\right)_{\beta}\left(h_n^{(\ell)}\right)_{\beta}$$

This can be combined with (6.20) to give

$$\left(\left(\tilde{U}_{n,m}^{(\ell+1)}(s)\right)_{v,v'}\right) =$$

$$\begin{pmatrix} \sum_{\beta=1}^{N} \tilde{D}_\beta^{(\ell)}(s)\left[\tilde{i}_\beta^{(\ell,2)}(z_\ell,s)\tilde{v}_\beta^{(\ell,1)}(z_{\ell+1},s) - \tilde{i}_\beta^{(\ell,1)}(z_\ell,s)\tilde{v}_\beta^{(\ell,2)}(z_{\ell+1},s)\right]\left(g_n^{(\ell)}\right)_\beta\left(h_n^{(\ell)}\right)_\beta \\ \sum_{\beta=1}^{N} \tilde{D}_\beta^{(\ell)}\left[-\tilde{v}_\beta^{(\ell,2)}(z_\ell,s)\tilde{v}_\beta^{(\ell,1)}(z_{\ell+1},s) + \tilde{v}_\beta^{(\ell,1)}(z_\ell,s)\tilde{v}_\beta^{(\ell,2)}(z_{\ell+1},s)\right]\left(g_n^{(\ell)}\right)_\beta\left(h_n^{(\ell)}\right)_\beta \\ \sum_{\beta=1}^{N} \tilde{D}_\beta^{(\ell)} f_\beta^{(\ell)}(z_{\ell+1})\left[\tilde{i}_\beta^{(\ell,2)}(z_\ell,s)\tilde{i}_\beta^{(\ell,1)}(z_{\ell+1},s) - \tilde{i}_\beta^{(\ell,1)}(z_\ell,s)\tilde{i}_\beta^{(\ell,2)}(z_{\ell+1},s)\right]\left(g_n^{(\ell)}\right)_\beta\left(h_n^{(\ell)}\right)_\beta \\ \sum_{\beta=1}^{N} \tilde{D}_\beta^{(\ell)} f_\beta^{(\ell)}(z_{\ell+1})\left[-\tilde{v}_\beta^{(\ell,2)}(z_\ell,s)\tilde{i}_\beta^{(\ell,1)}(z_{\ell+1},s) + \tilde{v}_\beta^{(\ell,1)}(z_\ell,s)\tilde{i}_\beta^{(\ell,2)}(z_{\ell+1},s)\right]\left(g_n^{(\ell)}\right)_\beta\left(h_n^{(\ell)}\right)_\beta \end{pmatrix} \quad (6.22)$$

This can now be used directly in (2.15) and (2.16) to obtain the matrizant for as many sections ($\ell' = 0, \ldots, \ell$) as desired.

VII. Case of $N=1$ for Piecewise Constant Approximation

The special case of $N=1$ reduces our matrizant to 2x2 form where β=1 is the only value allowed, i.e. only one eigenvalue for the matrix blocks which are reduced to scalars in this case. The piecewise constant approximation in section III then has the solution for the ℓth segment (including renormalization) in (3.4) take the form

$$\left(U_{\nu,\nu'}^{(\ell+1)}(s)\right)=\begin{pmatrix}1 & 0\\ 0 & \tilde{Z}_c^{(\ell+1)}(s)\ \tilde{Z}_c^{(\ell)^{-1}}(s)\end{pmatrix}\cdot e^{\left(\tilde{a}_{\nu,\nu'}(s)\right)[z_{\ell+1}-z_\ell]}$$

$$\left(a_{\nu,\nu'}^{(\ell)}(s)\right)=-\tilde{\gamma}^{(\ell)}(s)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \tag{7.1}$$

Here the propagation constant and characteristic impedance are scalars, constant for each section of the line. The exponential matrix takes the form (from (3.4)) as

$$e^{\left(\tilde{a}_{\nu,\nu'}(s)\right)[z_{\ell+1}-z_\ell]}=\begin{pmatrix}\cosh\left([z_{\ell+1}-z_\ell]\tilde{\gamma}^{(\ell)}(s)\right) & -\sinh\left([z_{\ell+1}-z_\ell]\tilde{\gamma}^{(\ell)}(s)\right)\\ -\sinh\left([z_{\ell+1}-z_\ell]\tilde{\gamma}^{(\ell)}(s)\right) & \cosh\left([z_{\ell+1}-z_\ell]\tilde{\gamma}^{(\ell)}(s)\right)\end{pmatrix} \tag{7.2}$$

Combining, we have

$$\left(\tilde{U}_{\nu,\nu'}^{(\ell+1)}(s)\right)=\begin{pmatrix}\cosh\left([z_{\ell+1}-z_\ell]\tilde{\gamma}^{(\ell)}(s)\right) & -\sinh\left([z_{\ell+1}-z_\ell]\tilde{\gamma}^{(\ell)}(s)\right)\\ -\tilde{Z}_c^{(\ell+1)}(s)\ \tilde{Z}_c^{(\ell)^{-1}}(s)\sinh\left([z_{\ell+1}-z_\ell]\tilde{\gamma}^{(\ell)}(s)\right) & -\tilde{Z}_c^{(\ell+1)}(s)\ \tilde{Z}_c^{(\ell)^{-1}}(s)\cosh\left([z_{\ell+1}-z_\ell]\tilde{\gamma}^{(\ell)}(s)\right)\end{pmatrix} \tag{7.3}$$

as the solution for the ℓth line section. This is a comparatively simple 2x2 matrix which, of course, only applies to the case of a single-conductor (plus reference) transmission line.

Referring back to (2.15) and (2.16) this gives the solution in terms of a product of 2x2 matrices for the various sections of the transmission line. This solution has, of course, the discontinuities in going from sections of characteristic impedance $\tilde{Z}_c^{(\ell)}(s)$ *to* $\tilde{Z}_c^{(\ell+1)}(s)$ in approximating a more general continuous $\tilde{Z}_c(z,s)$.

VIII. Case of $N=1$ for Continuous Characteristic Impedance at Section Boundaries

Sections IV and VI present two different approaches to the problem of the variation of the characteristic impedance within a section. Section IV allows different speeds for the eigenmodes, but has limitations concerning the continuity of the characteristic-impedance matrix at the section boundaries. Section VI makes the characteristic-impedance matrix continuous at the section boundaries, but at the price of having all modes propagate at the same speed.

For the case of N=1, however, the problems simplify considerably. Now there is only one propagation constant (± for two directions) at each z. Furthermore, there is only one mode (β=1) so that we have

$$\begin{aligned} \left(\tilde{x}_n(s)\right)_\beta &= 1 \qquad \text{(section IV)} \\ \left(\tilde{g}_n^{(\ell)}\right)_\beta = \left(h_n^{(\ell)}\right)_\beta &= 1 \qquad \text{(section VI)} \end{aligned} \tag{8.1}$$

and the index β can be suppressed on all the eigenvalues. The formalisms in the two sections then reduce to the same formalism.

The representation for the ℓth section in (4.24) and (6.22) now becomes a 2x2 matrix as

$$\left(\tilde{U}_{v,v'}^{(\ell)}(s)\right) =$$

$$\begin{pmatrix} \tilde{D}^{(\ell)}(s)\left[\tilde{i}^{(\ell,2)}(z_\ell,s)\tilde{v}^{(\ell,1)}(z_{\ell+1},s) - \tilde{i}^{(\ell,1)}(z_\ell,s)\tilde{v}^{(\ell,2)}(z_{\ell+1},s)\right] & \tilde{D}^{(\ell)}(s)\left[-\tilde{v}^{(\ell,2)}(z_\ell,s)\tilde{v}^{(\ell,1)}(z_{\ell+1},s) - \tilde{v}^{(\ell,1)}(z_\ell,s)\tilde{v}^{(\ell,2)}(z_{\ell+1},s)\right] \\ \tilde{D}^{(\ell)}(s)\dfrac{\tilde{Z}_c^{(\ell+1)}(s)}{\tilde{Z}_c^{(\ell)}(s)}\left[\tilde{i}^{(\ell,2)}(z_\ell,s)\tilde{i}^{(\ell,1)}(z_{\ell+1},s) - \tilde{i}^{(\ell,1)}(z_\ell,s)\tilde{i}^{(\ell,2)}(z_{\ell+1},s)\right] & \tilde{D}^{(\ell)}(s)\dfrac{\tilde{Z}_c^{(\ell+1)}(s)}{\tilde{Z}_c^{(\ell)}(s)}\left[\tilde{v}^{(\ell,2)}(z_\ell,s)\tilde{i}^{(\ell,1)}(z_{\ell+1},s) - \tilde{v}^{(\ell,1)}(z_\ell,s)\tilde{i}^{(\ell,2)}(z_{\ell+1},s)\right] \end{pmatrix} \tag{8.2}$$

Using (2.15) and (2.16) one can construct the representation of the entire transmission line (now single conductor plus reference). The characteristic impedance $\tilde{Z}_c(z,s)$ can now be approximated in a continuous manner through the section boundaries and take the prescribed $\tilde{Z}_c^{(\ell)}(s)$ there.

IX. Concluding Remarks

The analysis of NMTLs has now led to some interesting extensions involving canonical forms of the variation of the eigenvalues of an appropriately normalized characteristic impedance matrix. Here explicit formulas are given for the cases of linear and exponential variation. Using various solutions in the literature for nonuniform transmission lines, these can now be applied to NMTLs as well. Such cases include arbitrary (real) powers of the coordinate z as well as other cases described by linear second-order differential equations such as may be found in various books (e.g. [12]).

The results here depend on the symmetries in the matrices characterizing the NMTL. These symmetries include reciprocity, the relation between the electric and magnetic parameters of the media in which the wires are embedded, and the geometry of the multiconductor cross section. Such symmetries are used here to construct canonical forms of analytically solvable NMTLs which can be used to approximate actual NMTLs, at least on a section-by-section basis. Another potential use of such canonical forms, and the associated symmetries, is for the design of NMTLs with various desirable properties for special applications.

Appendix A. Linear Variation of Eigenvalues

In the form used in section VI the differential equations (second order) for the voltage-like and current-like modes take the form (from (6.18) and (6.19))

$$\left\{\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial z}\ell n\left(f_\beta^{(\ell)}(z)\right)\frac{\partial}{\partial z} - \gamma^2\right\}\tilde{v}_\beta^{(\ell,\delta)}(z,s) = 0$$

$$\left\{\frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial z}\ell n\left(f_\beta^{(\ell)}(z)\right)\frac{\partial}{\partial z} - \gamma^2\right\}\tilde{i}_\beta^{(\ell,\delta)}(z,s) = 0$$

$$\frac{\partial}{\partial z}\tilde{v}_\beta^{(\ell,\delta)}(z,s) = -\gamma\ f_\beta^{(\ell)}(z)\ \tilde{i}_\beta^{(\ell,\delta)}(z,s)$$

$$\frac{\partial}{\partial z}\tilde{i}_\beta^{(\ell,\delta)}(z,s) = -\gamma\ f_\beta^{(\ell)^{-1}}(z)\ \tilde{v}_\beta^{(\ell,\delta)}(z,s) \qquad \text{(A.1)}$$

$$\delta = 1,2$$

$$z_\ell < z < z_{\ell+1}$$

These equations in terms of the eigenvalues $f_\beta^{(\ell)}(z)$ of $\left(f_{n,m}^{(\ell)}(z)\right)$ can also be applied to section IV (specifically (4.4) and (4.5), or (4.13) and (4.15)) provided one reinterprets the eigenvalues in terms of the impedance - and admittance-per-unit-length eigenvalues appearing there. By an appropriate scaling the form in (A.1) is achieved.

Now $f_\beta^{(\ell)}(z)$ is constrained as

$$f_\beta^{(\ell)}(z_\ell) = 1$$

$$\text{(A.2)}$$

$$f_\beta^{(\ell)}(z_{\ell+1}) \equiv eigenvalue\ \ of\ \left(f_{n,m}^{(\ell)}(z_{\ell+1})\right) = \left(f_{g_{n,m}}^{(\ell)}(z_{\ell+1})\right)\cdot\left(f_{g_{n,m}}(z_\ell)\right)^{-1}$$

One can choose various forms for $f_\beta^{(\ell)}(z)$ matching these boundary conditions. In particular one can choose some power of (z + some constant). For simplicity one can choose the first power (linear variation) as

$$f_\beta^{(\ell)}(z) = 1 + \frac{z - z_\ell}{z_{\ell+1} - z_\ell}\left[f_\beta^{(\ell)}(z_{\ell+1}) - 1\right]$$

$$= \frac{z_{\ell+1} - f_\beta^{(\ell)}(z_{\ell+1})z_\ell + \left[f_\beta^{(\ell)}(z_{\ell+1}) - 1\right]z}{z_{\ell+1} - z_\ell}$$

$$= p_\beta^{(\ell)}\left[z - z_{0,\beta}^{(\ell)}\right]$$

$$z_{0,\beta} = \frac{f_\beta^{(\ell)}(z_{\ell+1})\ z_\ell - z_{\ell+1}}{f_\beta^{(\ell)}(z_{\ell+1}) - 1} = \frac{z_\ell - z_{\ell+1}}{f_\beta^{(\ell)}(z_{\ell+1}) - 1} + z_\ell \tag{A.3}$$

$$p_\beta^{(\ell)} = \frac{f_\beta^{(\ell)}(z_{\ell+1}) - 1}{z_{\ell+1} - z_\ell}$$

Since we are considering only positive definite $\left(f_{g_{n,m}}(z)\right)$ the interpolation of the $f_\beta^{(\ell)}(z)$ is strictly positive in the interval $z_\ell \le z \le z_{\ell+1}$. The zeros $z_{0,\beta}$ lie outside this interval. Note that the first-power form in (A.3) is for an impedance which applies to the $\tilde{v}_\beta^{(\ell,\delta)}$ in (A.1). For the $\tilde{i}_\beta^{(\ell,\delta)}$ this is a minus-one-power form corresponding to an admittance, with no singularity in the line section under consideration.

Substituting in (A.1) we now have

$$\left\{\frac{\partial^2}{\partial(\gamma z)^2} - \frac{1}{\gamma\left[z - z_{0,\beta}^{(\ell)}\right]}\ \frac{\partial}{\partial(\gamma z)} - 1\right\}\tilde{v}_\beta^{(\ell,\delta)}(z,s) = 0$$

$$\left\{\frac{\partial^2}{\partial(\gamma z)^2} + \frac{1}{\gamma\left[z - z_{0,\beta}^{(\ell)}\right]}\ \frac{\partial}{\partial(\gamma z)} - 1\right\}\tilde{i}_\beta^{(\ell,\delta)}(z,s) = 0$$

$$\frac{\partial}{\partial(\gamma z)}\tilde{v}_\beta^{(\ell,\delta)}(z,s) = -p_\beta^{(\ell)}\left[z - z_{0,\beta}^{(\ell)}\right]\ \tilde{i}_\beta^{(\ell,\delta)}(z,s)$$

$$\frac{\partial}{\partial(\gamma z)}\tilde{i}_\beta^{(\ell,\delta)}(z,s) = -\frac{1}{p_\beta^{(\ell)}\left[z - z_{0,\beta}^{(\ell)}\right]}\ \tilde{v}_\beta^{(\ell,\delta)}(z,s) \tag{A.4}$$

Here we recognize the Bessel equation of order zero for the current modes with two independent solutions [6,13] which we take for convenience as

$$\tilde{i}_{\beta}^{(\ell,1)}(z,s) = \mathrm{K}_o\left(\gamma\left[z - z_{o,\beta}^{(\ell)}\right]\right)$$

$$\tilde{i}_{\beta}^{(\ell,2)}(z,s) = \mathrm{I}_o\left(\gamma\left[z - z_{o,\beta}^{(\ell)}\right]\right) \tag{A.5}$$

From this the voltage modes are constructed as

$$\tilde{v}_{\beta}^{(\ell,1)}(z,s) = p_{\beta}^{(\ell)}\left[z - z_{o,\beta}^{(\ell)}\right]\ \mathrm{K}_1\left(\gamma\left[z - z_{o,\beta}^{(\ell)}\right]\right)$$

$$\tilde{v}_{\beta}^{(\ell,2)}(z,s) = -p_{\beta}^{(\ell)}\left[z - z_{o,\beta}^{(\ell)}\right]\ \mathrm{I}_1\left(\gamma\left[z - z_{o,\beta}^{(\ell)}\right]\right) \tag{A.6}$$

The Wronskien relationship used in sections IV and VI is [13]

$$\begin{aligned}
D_{\beta}^{(\ell)^{-1}}(s) &= \left[\tilde{v}_{\beta}^{(\ell,1)}(z,s)\ \tilde{i}_{\beta}^{(\ell,2)}(z,s) - \tilde{v}_{\beta}^{(\ell,2)}(z,s)\ \tilde{i}_{\beta}^{(\ell,1)}(z,s)\right] \\
&= f_{\beta}^{(\ell)}(z)\left[\tilde{i}_{\beta}^{(\ell,1)}(z,s)\frac{\partial}{\partial(\gamma z)}\tilde{i}_{\beta}^{(\ell,2)}(z,s) - \tilde{i}_{\beta}^{(\ell,2)}(z,s)\frac{\partial}{\partial(\gamma z)}\tilde{i}_{\beta}^{(\ell,1)}(z,s)\right]\Bigg|_{z=z_\ell} \\
&= f_{\beta}^{(\ell)}(z)\ W\left\{\mathrm{K}_o\left(\gamma\left[z - z_{o,\beta}^{(\ell)}\right]\right),\ \mathrm{I}_o\left(\gamma\left[z - z_{o,\beta}\right]\right)\right\} \\
&= \frac{f_{\beta}^{(\ell)}(z)}{\gamma\left[z - z_{o,\beta}^{(\ell)}\right]} = \frac{1}{\gamma}p_{\beta}^{(\ell)}(s)
\end{aligned} \tag{A.7}$$

For section IV replace γ by $\gamma_{\beta}^{(\ell)}$ and note that the $f_{\beta}^{(\ell)}(z)$ represents the variation of the eigenvalues there as well, except for normalized impedance per unit length (and reciprocal admittance per unit length) since the eigenmodes are the same for the ℓ+1st section as for the ℓth.

As the line becomes uniform, i.e. $f_{\beta}^{(\ell)}(z_{\ell+1}) \to 1$, note that $D_{\beta}^{(\ell)}(s) \to \infty$, so that appropriate limits are needed. Alternately the solution for such a uniform section can be treated as in section III.

Appendix B. Exponential Variation of Eigenvalues

An alternate approach uses exponential interpolation for the eigenvalues as

$$f_\beta^{(\ell)}(z) = e^{2\alpha_\beta^{(\ell)}[z-z_\ell]}$$

$$\alpha_\beta^{(\ell)} = \frac{1}{2}\,\frac{\ell n\left(f_\beta^{(\ell)}(z_{\ell+1})\right)}{z_{\ell+1}-z_\ell} \tag{B.1}$$

where the factor of 2 is included for later convenience. Then (A.1) takes the form

$$\left\{\frac{\partial^2}{\partial z^2} - 2\alpha_\beta^{(\ell)}\frac{\partial}{\partial z} - \gamma^2\right\}\tilde{v}_\beta^{(\ell,\delta)}(z,s) = 0$$

$$\left\{\frac{\partial^2}{\partial z^2} + 2\alpha_\beta^{(\ell)}\frac{\partial}{\partial z} - \gamma^2\right\}\tilde{i}_\beta^{(\ell,\delta)}(z,s) = 0$$

$$\frac{\partial}{\partial z}\tilde{v}_\beta^{(\ell,\delta)}(z,s) = -\gamma e^{2\alpha_\beta^{(\ell)}[z-z_\ell]}\tilde{i}_\beta^{(\ell,\delta)}(z,s) \tag{B.2}$$

$$\frac{\partial}{\partial z}\tilde{i}_\beta^{(\ell,\delta)}(z,s) = -\gamma e^{-2\alpha_\beta^{(\ell)}[z-z_\ell]}\tilde{v}_\beta^{(\ell,\delta)}(z,s)$$

This is solved in the usual way by substitution of an exponential [7]. Starting with the voltage modes we have

$$\tilde{v}_\beta^{(\ell,\delta)}(z,s) = e^{\tilde{r}_\beta^{(\ell,\delta)}(s)[z-z_\ell]}$$

$$\tilde{r}_\beta^{(\ell,1)}(s) = \alpha_\beta^{(\ell)} - \left[\alpha_\beta^{(\ell)^2} + \gamma^2\right]^{\frac{1}{2}} \qquad (+z \text{ propagation}) \tag{B.3}$$

$$\tilde{r}_\beta^{(\ell,2)}(s) = \alpha_\beta^{(\ell)} + \left[\alpha_\beta^{(\ell)^2} + \gamma^2\right]^{\frac{1}{2}} \qquad (-z \text{ propagation})$$

with due attention to the positive square root to keep things analytic (causal) in the right half s plane. The current modes are then constructed as

$$\tilde{i}_{\beta}^{(\ell,1)}(z,s) = -\frac{\tilde{r}_{\beta}^{(\ell,1)}(s)}{\gamma} \; e^{-\tilde{r}_{\beta}^{(\ell,2)}(s)[z-z_{\ell}]}$$

$$\tilde{i}_{\beta}^{(\ell,2)}(z,s) = -\frac{\tilde{r}_{\beta}^{(\ell,2)}(s)}{\gamma} \; e^{-\tilde{r}_{\beta}^{(\ell,1)}(s)[z-z_{\ell}]} \qquad \text{(B.4)}$$

The Wronskian relationship is now

$$D_{\beta}^{(\ell)^{-1}}(s) = \left[\tilde{v}_{\beta}^{(\ell,1)}(z,s)\;\tilde{i}_{\beta}^{(\ell,2)}(z,s) - \tilde{v}_{\beta}^{(\ell,2)}(z,s)\;\tilde{i}_{\beta}^{(\ell,1)}(z,s)\right]$$

$$= -\frac{\tilde{r}_{\beta}^{(\ell,2)}}{\gamma} + \frac{\tilde{r}_{\beta}^{(\ell,1)}}{\gamma} \qquad \text{(B.5)}$$

$$= -\frac{2}{\gamma}\left[\alpha_{\beta}^{(\ell)^2} + \gamma^2\right]^{\frac{1}{2}}$$

Due to the exponential nature of the solutions there are simplifications that appear in the combination of functions that give the renormalized matrizant as in (6.22) and (4.24). In the form of (6.22) this becomes

$$\left(\left(\tilde{U}_{n,m}^{(\ell+1)}(s)\right)_{\nu,\nu'}\right)=$$

$$\left(\begin{array}{ll}\sum_{\beta=1}^{N} e^{\alpha_\beta^{(\ell)}[z_{\ell+1}-z_\ell]}\left[\cosh\left(\left[\alpha_\beta^{(\ell)^2}+\gamma^2\right]^{\frac{1}{2}}[z_{\ell+1}-z_\ell]\right)-\frac{\alpha_\beta^{(\ell)}}{\left[\alpha_\beta^{(\ell)^2}+\gamma^2\right]^{\frac{1}{2}}}\sinh\left(\left[\alpha_\beta^{(\ell)^2}+\gamma^2\right]^{\frac{1}{2}}[z_{\ell+1}-z_\ell]\right)\left(g_n^{(\ell)}\right)_\beta\left(h_n^{(\ell)}\right)_\beta\right] & -\sum_{\beta=1}^{N} e^{\alpha_\beta^{(\ell)}[z_{\ell+1}-z_\ell]}\frac{\gamma}{\left[\alpha_\beta^{(\ell)^2}+\gamma^2\right]^{\frac{1}{2}}}\sinh\left(\left[\alpha_\beta^{(\ell)^2}+\gamma^2\right]^{\frac{1}{2}}[z_{\ell+1}-z_\ell]\right)\left(g_n^{(\ell)}\right)_\beta\left(h_n^{(\ell)}\right)_\beta \\ -\sum_{\beta=1}^{N} f_\beta^{(\ell)}(z_{\ell+1})e^{-\alpha_\beta^{(\ell)}[z_{\ell+1}-z_\ell]}\frac{\gamma}{\left[\alpha_\beta^{(\ell)^2}+\gamma^2\right]^{\frac{1}{2}}}\sinh\left(\left[\alpha_\beta^{(\ell)^2}+\gamma^2\right]^{\frac{1}{2}}[z_{\ell+1}-z_\ell]\right)\left(g_n^{(\ell)}\right)_\beta\left(h_n^{(\ell)}\right)_\beta & \sum_{\beta=1}^{N} f_\beta^{(\ell)}(z_{\ell+1})e^{-\alpha_\beta^{(\ell)}[z_{\ell+1}-z_\ell]}\left[\cosh\left(\left[\alpha_\beta^{(\ell)^2}+\gamma^2\right]^{\frac{1}{2}}[z_{\ell+1}-z_\ell]\right)-\frac{\alpha_\beta^{(\ell)}}{\left[\alpha_\beta^{(\ell)^2}+\gamma^2\right]^{\frac{1}{2}}}\sinh\left(\left[\alpha_\beta^{(\ell)^2}+\gamma^2\right]^{\frac{1}{2}}[z_{\ell+1}-z_\ell]\right)\right]\left(g_n^{(\ell)}\right)_\beta\left(h_n^{(\ell)}\right)_\beta\end{array}\right)$$

(B.4)

Comparing this to the uniform MTL in (3.8) one can note that (B.4) achieves this form by letting $\alpha_\beta^{(\ell)} \to 0$, noting also the impedance renormalization in (B.4). As an interesting observation note that the supermatrix in (B.4) has no branch cut in the γ plane. This is found by noting that the cosh is represented by a series involving only even powers of the argument, and the combination $\left[\alpha^{(\ell)^2}+\gamma^2\right]^{-\frac{1}{2}} \sinh\left(\left[\alpha_\beta^{(\ell)}+\gamma^2\right]^{\frac{1}{2}}[z_{\ell+1}-z_\ell]\right)$ similarly removes the square root by the presence of only even powers of the square root.

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