INTERACTION NOTES

Note 491

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On the Bounds of RF Coupling by Impulse and Step-Function Waveforms

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ABSTRACT

This report is an attempt to treat the problem of bounding the energy coupled to or scattered by a system by an impulsive or a step-function incident wave. For an impulse incident wave the high-frequency limit of the forward scattering amplitude is needed, whereas its low-frequency limit is required for a step-function incident wave. The report starts with some circuit representation of a generic coupling problem, in which the positive-real properties of driving impedance or admittance can be immediately invoked. A particular case, namely, a thin wire antenna, is then treated, for which the induced wire current can be represented approximately by an analytical expression, allowing for a detailed demonstration of some of the ideas involved. Finally, a general approach is discussed for obtaining the forward scattering amplitude at the low- and high-frequency limit. The latter case encounters certain fundamental difficulties, which are discussed in some detail.

PREFACE

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INTRODUCTION

In a previous effort [1], the total extinction cross section, σ_t, is shown to be related to the real part of the forward scattering amplitude by the following integral relationship

$$\int_o^\infty \frac{\sigma_t(x)}{x^2 - \omega^2}\, dx = \frac{\pi c}{2\omega^2}\, A_1'(\omega) \tag{1}$$

where

$$A_1' = \mathrm{Re}\,(\mathbf{1}_1 \cdot \mathbf{A}) \qquad \text{in the forward scattering direction}$$

and

$$\mathbf{E}_{sc} = \mathbf{A}\frac{e^{ikr}}{4\pi r}$$

with **A** normalized to an incident electric field of unit amplitude in the direction of unit vector $\mathbf{1}_1$.

For low frequencies (1) gives

$$\int_o^\infty \sigma_t(\lambda)d\lambda = \pi^2 c^2 (A_1' / \omega^2)_{\omega\to 0} \tag{2}$$

(λ = wavelength), and for high frequencies it yields

$$\int_o^\infty \sigma_t(\omega)\, d\omega = -\frac{\pi c}{2}\left(A_1'\right)_{\omega\to\infty} \tag{3}$$

The right side of (2) is related to the polarizabilities of the scatterer, since **A** for dipole radiation is proportional to ω^2. Recognizing that the absorption cross section never exceeds the total cross section and that the left side of (2) is directly related to the total energy absorbed and scattered by a step-function plane wave, one can then bound the energy absorbed by any object for a step-function incident wave [2, 3].

In this report we will focus on (3), i.e., the integral of the total cross section over all frequencies. We must bear in mind that if σ_t has a limiting value, say σ_∞, for $\omega \rightarrow \infty$, the integrand in (3) should be replaced by $\sigma_t(\omega) - \sigma_\infty$.

There is an analogue to (1) in circuit theory. Let Z and Y be driving impedance and admittance, and Z = R + jX and Y= G + jB.

Then [4]

$$X(\omega) = \frac{2}{\pi}\,\omega \int_0^\infty \frac{R(x)}{x^2 - \omega^2}\,dx, \qquad R(\omega) = \frac{2}{\pi} \int_0^\infty \frac{xX(x)}{x^2 - \omega^2}\,dx \tag{4}$$

and similarly for G and B.

When $\omega \to \infty$, one has

$$-\frac{2}{\pi}\,\frac{1}{\omega} \int_0^\infty R(x)\,dx = \lim_{\omega\to\infty} X(\omega) \to -\frac{1}{\omega C}$$

or

$$\int_0^\infty R(\omega)\,d\omega = \frac{\pi}{2C} \tag{5a}$$

which is known as the resistance-integral theorem [3].

At the low-frequency limit when $\omega \to 0$, one gets

$$\frac{2}{\pi}\,\omega \int_0^\infty \frac{R(x)}{x^2}\,dx = \lim_{\omega\to 0} X(\omega) \to \omega L$$

or

$$\int_0^\infty \frac{R(\omega)}{\omega^2}\,d\omega = \frac{\pi}{2}\,L \tag{5b}$$

which is equivalent to

$$\int_0^\infty R(\lambda)\,d\lambda = \pi^2 c\, L$$

Similarly for G and B we have

$$\int_{0}^{\infty} G(\omega)\, d\omega = \frac{\pi}{2L} \qquad \omega \to \infty \tag{6a}$$

$$\int_{0}^{\infty} \frac{G(\omega)}{\omega^2}\, d\omega = \frac{\pi}{2} C \qquad \omega \to 0 \tag{6b}$$

It goes without saying that the necessary condition for (6a) and (6b) to hold is the existence of the integrals.

Let us illustrate how equations (5) and (6) can be used, first in the case of a transmitting antenna and then coupling to cavity through a slot.

Suppose an antenna is excited by a voltage source V(t). If V is an impulse, i.e., $V(t) = V_\delta\ \delta(t)$, then from (6a) we have

$$\text{Total Radiated Energy} = 2\int_{0}^{\infty} |V(\omega)|^2 G(\omega)\, df = \frac{V_\delta^2}{2L} \tag{7a}$$

($\omega = 2\pi f$) meaning that the total radiated energy is limited by the inductance. If, on the other hand, the voltage source is a step function, i.e., $V(t) = V_u\, u(t)$, then (6b) gives

$$\text{Total Radiated Energy} = 2\int_{0}^{\infty} V_u^2 \frac{G(\omega)}{\omega^2}\, df = \frac{1}{2} C\, V_u^2 \tag{7b}$$

implying that the capacitance sets the limit for the total radiated energy.

If the antenna is excited by a current source such as opening a switch, equations (5a) and (5b) can be utilized to derive limits on the total radiated energy for impulse and step-function sources, with C and L being the capacitance and inductance of the driving-point impedance.

Let us now consider the problem of coupling to a wire inside a slotted cavity, which has been studied in a previous report [5]. Supposing the voltage induced across the slot at A, B as shown in Fig. 1 is either an impulse or a step-function pulse, one asks for the upperbound energy picked up by the resistive load of the wire in either case. When the cavity and the wire are perfectly conducting, there are two kinds of irreversible

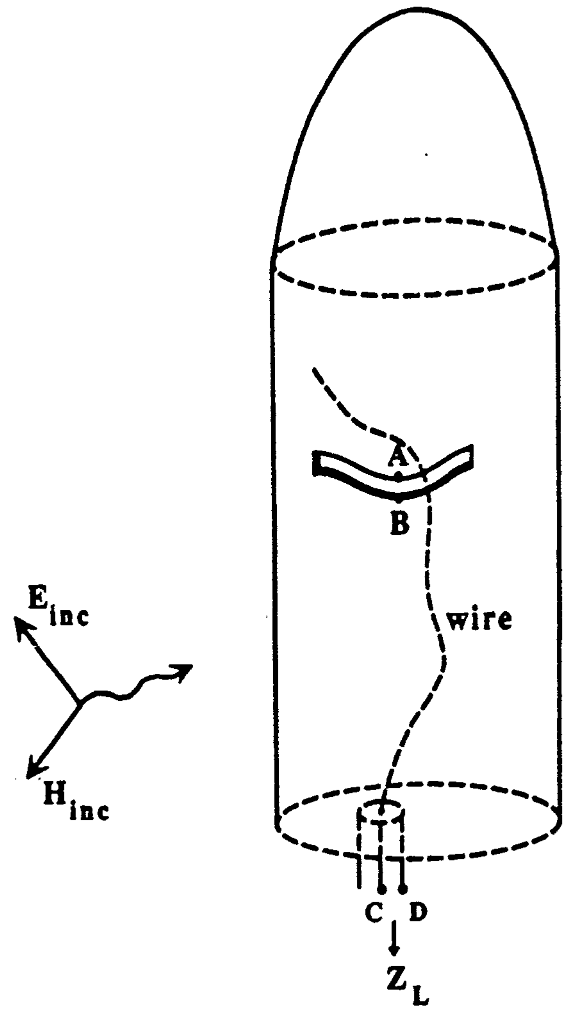

Figure 1. Coupling to a wire inside a slotted cavity.

energy loss, with one to radiation to infinity and the other to the resistive load of the wire. Then one may say, with the aid of (7a) and (7b),

$$\begin{aligned} \text{Energy Coupled to Wire Load} &< \frac{V_\delta^2}{2L}, \quad \text{for impulse} \\ &< \frac{1}{2} C V_u^2, \quad \text{for step-function pulse} \end{aligned} \tag{8}$$

The inductance L and capacitance C are the combination of the external and internal inductances and capacitances across A, B.

In the Appendix we will work out all the details of (7a) and 7b) for an L, R, C circuit, where the elements are frequency-independent, lumped circuit parameters.

In this section we consider the application of (3) to thin wires. In particular, we specialize the wires to straight thin wires and consider a delta-function incident wave.

Let us first consider a straight thin perfectly conducting wire of length ℓ. Using the method of natural modes we may write for the induced current $\tilde{I}(z,s)$ [6]

$$\tilde{I}(z,s) \doteq \frac{4\pi c}{Z_0 \Omega \ell} \sum_n \frac{1}{s - s_n} \left[\int_0^{\ell} \sin(n\pi z'/\ell)\, \tilde{E}_{inc}(z',s)\,dz' \right] \sin(n\pi z/\ell) \tag{9}$$

where $n = \pm 1, \pm 2, \cdots$, $\Omega = 2\ell n\,(\ell/a)$, $Z_0 = 120\pi$ ohms, c = free-space speed of light, and the natural frequency s_n is given by

$$s_n = i\, n\pi c/\ell - \frac{c}{\Omega \ell}\left[\mathrm{Cin}(2|n|\pi) + i\,\mathrm{Si}(2n\pi)\right] + 0\left(\Omega^{-2}\right)$$

with Si (x) being the sine integral and

$$\mathrm{Cin}\,(x) = \int_0^x \frac{1 - \cos t}{t}\,dt$$

We now proceed to evalutate the forward scattering amplitude A_1' for $\omega \to \infty$. Recalling that the scattering amplitude **A** is given by

$$\mathbf{A} = -i\omega\mu\, \mathbf{1}_r \times \left[\mathbf{1}_r \times \int_0^{\ell} e^{-ik\mathbf{1}_r \cdot \mathbf{r}'} \mathbf{I}(\mathbf{r}')\,dz' \right] \tag{10}$$

we obtain, by substituting (9) in (10), setting $s = -i\omega$ and noting that $\mathbf{1}_r \cdot \mathbf{r}' = 0$ in the forward direction,

$$\left(A_1'\right)_{\omega \to \infty} = -\frac{4\pi c\mu}{Z_0 \Omega \ell}\, E_0\, \frac{4\ell^2}{\pi^2} \sum_n \frac{1}{n^2}, \qquad n = \pm 1, \pm 3, \cdots$$

$$= -\frac{4\pi\ell}{\Omega} E_0 \tag{11}$$

where E_o is the amplitude of the incident electric field parallel to the axis of the wire. The integral of σ_t over all frequencies is then given by

$$\int_o^\infty \sigma_t(\omega)\,d\omega = \frac{\pi^2 c\ell}{\ell n(\ell / a)} \tag{12}$$

Let us now take another route to evaluate the integral of σ_t over all frequencies, first calculating the energy spectrum via the EMF method and then evaluating the scattered energy by integrating over all frequencies. Denoting the energy spectrum by P_ω we then have

$$P_\omega = \mathrm{Re}\int \tilde{I}(z)\,\tilde{E}^*_{inc}(z)dz$$

$$= \frac{4\pi c}{Z_o\Omega\ell} E_o^2 \sum_{n=1,3,5\cdots} \mathrm{Re}\left(\frac{1}{s-s_n} + \frac{1}{s-s_{-n}}\right)\cdot \frac{4\ell^2}{\pi^2}\cdot\frac{1}{n^2}$$

Since σ_t is defined to be $P_\omega / \left(E_o^2 / Z_o\right)$, we have

$$\int_o^\infty \sigma_t(\omega)\,d\omega = \frac{Z_o}{E_o^2}\int_o^\infty P_\omega\,d\omega$$

$$= \frac{16c\ell}{\Omega\pi}\sum_{n=1,3,5\cdots}\frac{1}{n^2}\int_o^\infty \mathrm{Re}\left(\frac{1}{s-s_n} + \frac{1}{s-s_{-n}}\right)d\omega \tag{13}$$

where

$$\mathrm{Re}\left(\frac{1}{s-s_n} + \frac{1}{s-s_{-n}}\right) = \frac{4\alpha_n\,\omega(\omega-\alpha_n)}{\left(\omega^2-\omega_n^2+\alpha_n^2\right)^2 + 4\alpha_n^2\omega^2}$$

$$\rightarrow \frac{4\alpha_n\omega^2}{\left(\omega^2-\omega_n^2\right)^2 + 4\alpha_n^2\omega^2}, \text{ for small } \alpha_n$$

Noting that [7]

$$\int_o^\infty \frac{4\alpha_n\omega^2}{\left(\omega^2-\omega_n^2\right)^2 + 4\alpha_n^2\,\omega^2}\,d\omega = \pi$$

we finally obtain from (13)

$$\int_0^\infty \sigma_t(\omega)\, d\omega = \frac{16c\ell}{\Omega} \sum_{n=1,3,5\cdots} \frac{1}{n^2} = \frac{\pi^2 c\ell}{\ell n(\ell / a)} \tag{14}$$

which is identical to (12). We have just simply verified the correctness of (3).

One may make use of (14) to calculate the total scattered energy for an impulsive incident plane wave. Let E_o be the amplitude of the impulse. Then

$$\text{Total Scattered Energy} = 2 \cdot \frac{E_o^2}{Z_o} \cdot \int_0^\infty \sigma_t(\omega)\, df$$

$$= \frac{E_o^2}{Z_o} \cdot \frac{1}{\pi} \cdot \frac{\pi^2 c\ell}{\ell n(\ell / a)} = \frac{(E_o \ell)^2}{2L} \tag{15}$$

where $L = \mu_o \Omega\, \ell/(4\pi)$, the inductance of the wire. It is interesting to note the similarity between (15) and (7a).

Up to now our considerations have been restricted to a perfectly conducting wire without any impedance loading. Suppose the wire acts as a receiving antenna with a load Z_L across a gap at, say, $z = z_o$. We ask, what is the total extinction cross section for this case? To answer this question we will seek a relation between the forward scattering amplitudes of the receiving problem (with Z_L) and the parasitic problem (with $Z_L = 0$).

Let

$\mathbf{E}_r$ = total field in the receiving problem with Z_L

$\mathbf{E}_t$ = total field in the transmitting problem with $\mathbf{E}_t = -V_t\, \mathbf{1}_{tan}\, \delta\, (z-z_o)$ on antenna surface

$\mathbf{E}_p$ = total field in the parasitic problem with $Z_L = 0$.

By superposition we then have

$$\mathbf{E}_r = \mathbf{E}_p + \alpha\, \mathbf{E}_t \tag{16}$$

everywhere outside and on the antenna surface. To determine α we make use of the boundary conditions on the antenna surface, namely,

$$\int \mathbf{E}_r \cdot dz = V_r, \qquad \int \mathbf{E}_t \cdot dz = -V_t, \qquad \mathbf{E}_{p,tan} = 0 \tag{17}$$

and we get

$$\alpha = -\frac{V_r}{V_t} = -\frac{1}{V_t}\,\frac{I_p Z_T Z_L}{Z_T + Z_L} \tag{18}$$

where I_p is the short-circuit current of the parasitic problem at $z = z_o$ and Z_T is the antenna driving-point impedance.

In the far zone, $r \to \infty$, we write

$$\begin{aligned} \mathbf{E}_r &\to \mathbf{E}_{inc} + \mathbf{A}_r \frac{e^{ikr}}{4\pi r} \\ \mathbf{E}_r &\to \mathbf{E}_{inc} + \mathbf{A}_p \frac{e^{ikr}}{4\pi r} \\ \mathbf{E}_t &\to \mathbf{A}_t \frac{e^{ikr}}{4\pi r} \end{aligned} \tag{19}$$

from which we have

$$\mathbf{A}_r = \mathbf{A}_p + \alpha\, \mathbf{A}_t \tag{20}$$

We now invoke the reciprocity theorem to say

$$V_t\, I_p = -\frac{i}{k Z_o}\, \mathbf{E}_{inc} \cdot \mathbf{A}_t \tag{21}$$

With (18) and (21) we obtain from (20)

$$\mathbf{E}_{inc} \cdot \mathbf{A}_r = \mathbf{E}_{inc} \cdot \mathbf{A}_p - i k Z_o \frac{Z_L Z_T}{Z_L + Z_T}\, I_p^2 \tag{22}$$

which is what we set out to seek.

Let us carefully examine the second term on the right hand side of (22). At low frequencies, $I_p \sim \omega$. Thus, the second term goes as ω^3 unless both Z_L and Z_T behave like a capacitance, in which case it goes as ω^2. At high frequencies, $I_p \sim \omega^{-1}$ and thus the term goes as ω^{-1} unless both Z_T and Z_L behave like an inductance, in which case it reduces to a constant. To sum up, we have from (22)

$$\left(\mathbf{1}_1 \cdot \mathbf{A}_r\right)_{\omega\to\infty} = \left(\mathbf{1}_1 \cdot \mathbf{A}_p\right)_{\omega\to\infty} \tag{23a}$$

unless both Z_T and Z_L behave like an inductance as $\omega \to \infty$, and

$$\left(\frac{1}{\omega^2}\ \mathbf{1}_1 \cdot \mathbf{A}_r\right)_{\omega\to 0} = \left(\frac{1}{\omega^2}\mathbf{1}_1 \cdot \mathbf{A}_p\right)_{\omega\to 0} \tag{23b}$$

unless both Z_T and Z_L behave like a capacitance as $\omega \to 0$. Equations (23a) and (23b) mean that for impulsive or step-function incident waves, the total energy absorbed plus the total energy scattered is equal to the total energy scattered by the same antenna without impedance loading. Perhaps, the no-load case sets the upper limit for the loading case at least for these two types of waveforms.

In this section we will explore ways to calculate the foward scattering amplitude $A_1^{'}$ introduced in (1). The scatterer is of general shape and perfectly conducting. In terms of the surface currents **K**, **A** is given by

$$\mathbf{A} = -i\omega\mu\, \mathbf{1}_r \times \left[\mathbf{1}_r \times \int_S e^{-ik\mathbf{1}_r \cdot \mathbf{r}'} \mathbf{K}(\mathbf{r}') dS' \right] \qquad (24)$$

which reduces to (10) for the case of a thin wire. Let $\mathbf{1}_1$ be the unit vector parallel to the electric field of the incident plane wave. Then

$$A_1^{'} = \mathrm{Re}(\mathbf{1}_1 \cdot \mathbf{A}) = -\omega\mu \left[\mathrm{Im} \int_S \mathbf{1}_1 \cdot \mathbf{K}\, e^{-ik\mathbf{1}_r \cdot \mathbf{r}'} dS' \right] \qquad (25)$$

Let us first examine the low-frequency case where we know what (25) should give. Expanding the exponential in small argument and keeping only the first two terms we get

$$\int_S \mathbf{1}_1 \cdot \mathbf{K}\, e^{-ik\mathbf{1}_r \cdot \mathbf{r}'} dS' = \int_S \mathbf{1}_1 \cdot \mathbf{K}\, dS' - ik \int_S (\mathbf{1}_1 \cdot \mathbf{K})(\mathbf{1}_r \cdot \mathbf{r}')\, dS'$$

The first term is just the dipole term, namely,

$$\int_S \mathbf{1}_1 \cdot \mathbf{K}\, dS' = -i\omega\, \mathbf{1}_1 \cdot \mathbf{p} \qquad (26)$$

To work out the second term we let, without loss of generality, $\mathbf{1}_1$ in the x-direction and $\mathbf{1}_r$ in th z-direction. Then expanding

$$(\mathbf{1}_1 \cdot \mathbf{K})(\mathbf{1}_r \cdot \mathbf{r}') = z'K_x$$

$$= \frac{1}{2}(z'K_x - x'K_z) + \frac{1}{2}(z'K_x + x'K_z)$$

we have

$$\int (\mathbf{1}_1 \cdot \mathbf{K})(\mathbf{1}_r \cdot \mathbf{r}')dS' = m_y - \frac{1}{2} i\omega Q_{xz} \tag{27}$$

where m_y is the magnetic dipole moment in the y-direction and Q_{xz} is the xz-component of the quadrupole moment **Q**. Substitution of (26) and (27) in (25) gives

$$A_1^{'} = \omega^2\mu \, \mathbf{1}_1 \cdot \mathbf{p} + \frac{\omega^2\mu}{c} \mathbf{1}_2 \cdot \mathbf{m} + 0\,(\omega^3) \tag{28}$$

in agreement of previous results [1].

To evaluate the forward scattering amplitude at the high-frequency limit we will start with the expression for the scattered field that makes no approximations on the phase function, namely,

$$\mathbf{1}_1 \cdot \mathbf{E}_{sc} = \frac{i\omega\mu}{4\pi r} \int K_1 \;\; e^{ik|\mathbf{r}-\mathbf{r}'|} dS'$$

$$= \frac{i\omega\mu}{4\pi r} \int K_{10} \;\; e^{ik(z'+|\mathbf{r}-\mathbf{r}'|)} dS' \tag{29}$$

In the latter expression of (29) the propagation factor of the incident wave has been factored out (see Fig.2). The phase function f(x',y') for observation points on the z-axis, i.e., along the direction of propagation, is given by

$$f(x',y') = z' + \sqrt{(z-z')^2 + x'^2 + y'^2} \tag{30}$$

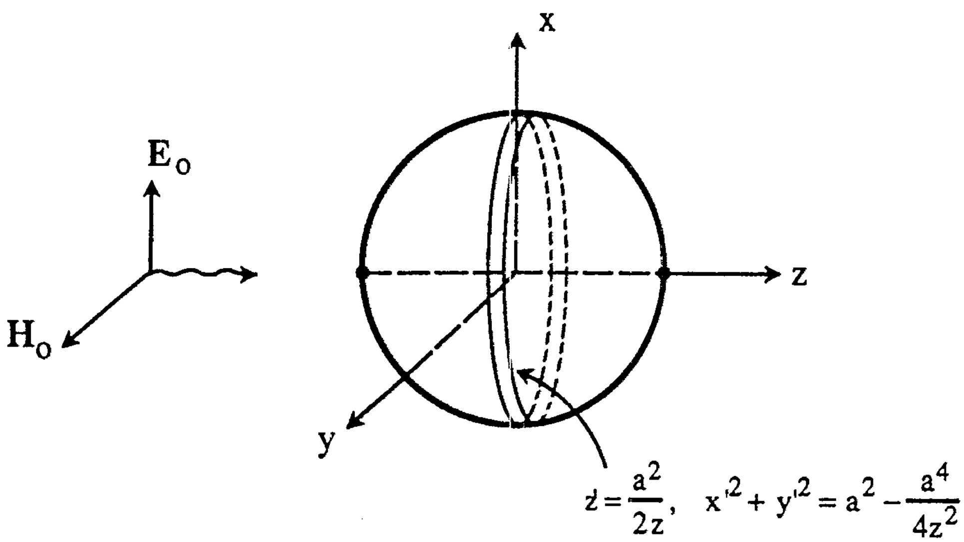

Figure 2. Stationary points at $x' = 0$, $y' = 0$, $z' = \pm a$ and on circle: $x'^2 + y'^2 = a^2 - a^4/(4z^2)$, $z > 0$.

In what follows we will work out the stationary-phase evaluation of (29) for a sphere of radius a. That is, we will evaluate the integral

$$I = \int K_{10} \; e^{ikf} dS' \qquad \text{for } k \to \infty \tag{31}$$

where

$$f(x',y') = z' + \sqrt{z^2 - 2zz' + a^2} \tag{32}$$

Differentiating f with respect to x' one gets

$$\frac{\partial f}{\partial x'} = \frac{x'}{z'} \left[\frac{z}{\sqrt{z^2 - 2zz' + a^2}} - 1 \right] \tag{33}$$

$$\frac{\partial^2 f}{\partial x'^2} = \left[\frac{1}{z'} + \frac{x'^2}{z'^3} \right] \left[\frac{z}{\sqrt{z^2 - 2zz' + a^2}} - 1 \right] - \left(\frac{x'}{z'} \right)^2 \frac{z^2}{\left(z^2 - 2zz' + a^2\right)^{3/2}} \tag{34}$$

The expressions for the derivatives with respect to y' can be obtained from (33) and (34) with x' replaced by y'. Setting

$$\frac{\partial f}{\partial x'} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y'} = 0$$

one obtains the following stationary points:

$$x' = 0, \; y' = 0, \; z' = \pm a \tag{35a}$$

and

$$z' = a^2/(2z), \qquad x'^2 + y'^2 = a^2 - a^4/(4z^2) \qquad (z > 0) \tag{35b}$$

Since the surface current K_{10} at $x' = 0$, $y' = 0$, $z = a$ is zero when $k \to \infty$ we will ignore this stationary point in the following discussions.

At the stationary point $x' = 0$, $y' = 0$, $z' = -a$ (Fig.2), the phase function (32) takes the form

$$f = -a + |z+a| - \frac{1}{2}\cdot\left(\frac{z}{|z+a|}\right)x'^2 - \frac{1}{2}\cdot\frac{1}{a}\left(\frac{z}{|z+a|}-1\right)y'^2$$

$$= z + \frac{1}{2}\frac{x'^2}{z+a} + \frac{1}{2}\frac{y'^2}{z+a} \qquad z > 0 \text{ (forward scattering)} \qquad (36a)$$

$$= -z + \frac{x'^2}{a} + \frac{y'^2}{a} \qquad z \to -\infty \text{ (backscattering)} \qquad (36b)$$

The latter expression (36b) leads to πa^2 for the backscatter cross section of a sphere at the high-frequency limit, as it should.

The evaluation of the integral in (29) with (36a) gives

$$\mathbf{1}_1 \cdot \mathbf{E}_{sc} = -E_0\, e^{ikz}, \qquad z > 0 \qquad (37a)$$

where $K_{10} = 2H_o = 2E_o/Z_o$ has been taken. Thus in the forward scattering direction, the scattered field is just the negative of the incident field from the contribution of the stationary point $x'=0$, $y' = 0$, $z = -a$. To be more accurate, (37a) should be replaced by

$$\mathbf{1}_1 \cdot \mathbf{E}_{sc} = -\left(1+\frac{a}{z}\right)E_0\, e^{ikz}, \quad z > 0 \qquad (37b)$$

The stationary points given by (35b) lie on the circle formed by the spherical surface and the plane $z' = a^2/(2z)$ perpendicular to the direction of incidence (Fig. 2). To evaluate the contributions of these stationary points to the integral in (29) we revert to the spherical coordinates. The stationary points defined by (35b) are now described by

$$\cos\theta_o' = a/(2z), \qquad 0 \le \phi' < 2\pi \tag{38}$$

and the phase function (32) is now expressed as

$$f(\theta',\phi') = f(\theta_o',\phi') + \frac{1}{2}\left.\frac{\partial^2 f}{\partial\theta'^2}\right|_{\theta_o'} (\theta'-\theta_o)^2 + \cdots$$

$$\approx z - \frac{a^2}{2z}(\theta'-\theta_o)^2 + \cdots \tag{39}$$

With (39) a stationary-phase evaluation of (29) gives, for $k \to \infty$ and in the forward scattering direction,

$$\mathbf{1}_1 \cdot \mathbf{E}_{sc} = \frac{1+i}{4\pi} \frac{e^{ikz}}{\sqrt{z}} \sqrt{\pi k a^2} \int_0^{2\pi} Z_o\, K_{10}\left(\theta_o',\phi'\right) d\phi' \tag{40}$$

From the discussions of [9] it is possible that $K_{10}(\theta_o', \phi')$ goes to zero faster than $k^{-1/2}$. Thus the stationary points on the circle defined by (35b) would not contribute to the forward scattering at the optical limit.

From the second term of (37b) and setting $E_o = 1$ we find $A_1' = -4\pi a$ for the real part of the normalized forward scattering amplitude at the optical limit. Finally,

$$\int_0^\infty \left[\sigma_t(\omega) - 2\pi a^2\right] d\omega = 2c\pi^2 a \tag{41}$$

for the case of a perfectly conducting sphere.

APPENDIX

In this Appendix we will work out the energy loss to the resistor R of a simple L, R, C circuit using (7a) and (7b) for an impulsive as well as a step-function voltage source. The circuit elements L, R, C are constant parameters, although (7a) and (7b) work even for frequency-dependent parameters. The circuit is shown in Fig. A.

First, let $v(t) = V_u\, u(t)$, i.e., the voltage source is a battery and the switch S is closed at $t = 0$. Using Equations (8), (9) and (10) in Symthe [8] for q(t), the charge in C at time t, we find i from $\dot{q}$ and calculate the total energy loss to R by direct integration to get

$$R \int_0^\infty i^2\, dt \equiv \frac{1}{2} C V_u^2 \tag{A·1}$$

which is predicted by (7b). We now want to use, instead, the left-hand side of (7b) to calculate the energy loss. The conductance G is given by

$$G(\omega) = \text{Re } Y(\omega) = \frac{R}{L^2} \frac{\omega^2}{\left(\omega^2 - \omega_o^2\right)^2 + R^2\omega^2 / L^2} \tag{A·2}$$

where $\omega_o = (LC)^{-1/2}$. We find

$$2\int_0^\infty V_u^2\, \frac{G(\omega)}{\omega^2}\, df = \frac{1}{\pi}\, V_u^2\, \frac{R}{L^2} \int_0^\infty \frac{d\omega}{\left(\omega^2 - \omega_o^2\right)^2 + R^2\omega^2 / L^2}$$

$$= \frac{1}{\pi}\, V_u^2\, \frac{R}{L^2} \cdot \frac{CL^2}{R} \int_0^\infty \frac{dx}{\left(a x^2 - 1\right)^2 + x^2}$$

$$= \frac{1}{2} C V_u^2 \tag{A·3}$$

since the integral has the value $\pi/2$ for $a \geq 0$. Again, (7b) is confirmed.

Let us now take an impulsive voltage source, $v(t) = V_\delta\, \delta(t)$. The left-hand side of (7a) is

$$2\int_0^\infty V_\delta^2\, G(\omega)\,df = \frac{1}{\pi} V_\delta^2 \int_0^\infty G(\omega)\,d\omega$$

$$= \frac{1}{\pi} V_\delta^2 \frac{R}{L^2} \int_0^\infty \frac{\omega^2 d\omega}{\left(\omega^2 - \omega_o^2\right)^2 + R^2\omega^2 / L^2}$$

$$= \frac{V_\delta^2}{2L} \qquad \text{(A·4)}$$

since the integral has the value $\pi/(2R/L)$ [7]. Equation (7a) is thus verified for the simple L, R, C circuit shown in Fig. A.

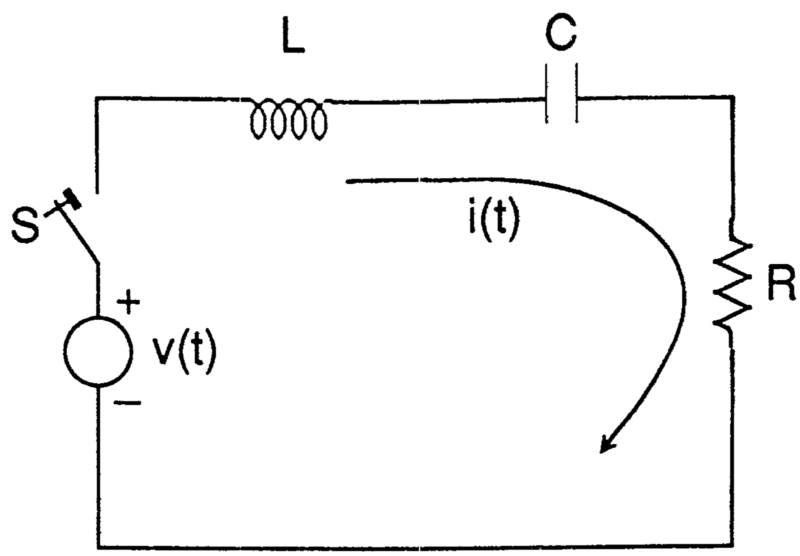

Figure A. Simple L, R, C circuit.

REFERENCES

[1] K.S.H. Lee and F.C. Yang, "Bounding Calculations in RF Coupling," DC-FR-9007-1, Kaman Sciences Corporation, Santa Monica, California, September 1989.

[2] L.K. Warne and K.C. Chen, "A Bound on EMP Coupling, "Interaction Note 466, June 1988; also IEEE Trans. Electromagnetic Compactibility, EMC Vol. 32, No. 3, pp. 217-221, August 1990.

[3] C.E. Baum, "Scattering of Transient Plane Waves, Air Force Weapons Laboratory (Phillips Laboratory) Interaction Note Series, Note 473, August 17, 1988.

[4] E.A. Guillemin, Synthesis of Passive Networks, John Wiley & Sons, Inc., New York, 1959.

[5] K.S.H. Lee and F.C. Yang, "Maximum RF Pickup by Wires Inside a Slotted Cavity," Submitted to SANDIA, October 25, 1990.

[6] K.S.H. Lee, et al, "Transient Response of a Thin Wire to a Moving Charged Particle," IEEE Trans. Antennas and Propagation, Vol. AP-25, No. 5, pp. 671-675, September 1977.

[7] I.S. Gradshteyn and I.M Ryzhik, Table of Integrals, Series and Products, Academic Press, New York, 1965, p. 296.

[8] W.R. Symthe, Static and Dynamic Electricity, McGraw-Hill, 1950, p.330.

[9] J.A. Cullen, "Surface Currents Induced by Short-Wavelength Radiation," Phys. Rev. Vol. 109, No. 6, pp. 1863-1867, March 1958.