Interaction Notes

Note 492

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The SEM Representation of Scattering From Perfectly Conducting Targets in Simple Lossy Media

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Abstract

This paper considers the general problem of the response of perfectly conducting targets in a lossy dielectric characterized by frequency-independent permittivity and conductivity, and free-space permeability. There are scaling relationships involving the propagation constant and wave impedance of the medium (as compared to free space) which can be used to relate the various parameters to the free-space case. Based on this one can use free-space parameters to find the singularity-expansion-method (SEM) representation of the target response in the lossy medium, including natural frequencies, natural modes, and coupling coefficients. There is also a branch-cut term introduced which is most significant for low frequencies (or late times).

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I. Introduction

An important problem in remote sensing is the detection, location, and identification of various objects buried in a lossy dielectric such as soil (or water, etc.) [27]. As indicated in fig. 1.1 let there be some finite-size scatterer (target) enclosed in a minimum circumstribing sphere of radius a. Let the medium surrounding the target be a uniform isotropic dielectric with constitutive parameters

$$\begin{aligned} \mu_o &\equiv \text{permeability (free space)} \\ \epsilon = \epsilon_r \epsilon_o &\equiv \text{permittivity} \\ \epsilon_r &\equiv \text{relative permittivity} \\ \sigma &\equiv \text{conductivity} \end{aligned} \tag{1.1}$$

This gives

$$\tilde{\gamma}(s) = \left[s\mu_o(\sigma + s\,\epsilon)\right]^{\frac{1}{2}} \equiv \text{propagation constant}$$

$$\tilde{Z}(s) = \left[\frac{s\mu_o}{\sigma + s\,\epsilon}\right]^{\frac{1}{2}} \equiv \text{wave impedance}$$

$$s \equiv \Omega + j\omega \equiv \text{complex frequency (two-sided-Laplace-transform variable)}$$

$$\sim \; \equiv \text{Laplace transform (two sided)}$$

$$\tilde{\gamma}(s)\,\tilde{Z}(s) = s\mu_o \tag{1.2}$$

$$\frac{\tilde{\gamma}(s)}{\tilde{Z}(s)} = \sigma + s\,\epsilon$$

For the present paper the constitutive parameters in (1.1) are taken as constant (frequency independent) which will simplify the problem somewhat. In the high-frequency limit we have

$$\tilde{Z}(s) = Z_\infty + O\left(s^{-1}\right) \text{ as } s \to \infty$$

$$Z_\infty \equiv \left[\frac{\mu_o}{\epsilon}\right]^{\frac{1}{2}} = \epsilon_r^{-\frac{1}{2}} Z_o \;,\; Z_o \equiv \left[\frac{\mu_o}{\epsilon_o}\right]^{\frac{1}{2}}$$

$$\tilde{\tilde{Z}}_t(\vec{r}_s,\vec{r}_s';s) = \vec{1}_S(\vec{r}_s)\cdot\tilde{\tilde{Z}}(\vec{r}_s,\vec{r}_s';s)\cdot\vec{1}_S(\vec{r}_s')$$

$$= s\mu_o\,\vec{1}_S(\vec{r}_s)\cdot\left\{\left[-2\zeta^{-3}-2\zeta^{-2}\right]e^{-\zeta}\vec{1}_R\,\vec{1}_R+\left[\zeta^{-3}+\zeta^{-2}+\zeta^{-1}\right]e^{-\zeta}\left[\vec{1}-\vec{1}_R\,\vec{1}_R\right]\right\}$$

$$R \equiv |\vec{r}_s-\vec{r}_s'|\ ,\ \vec{1}_R = \frac{\vec{r}_s-\vec{r}_s'}{|\vec{r}_s-\vec{r}_s'|}\ \text{for}\ \vec{r}_s\neq\vec{r}_s' \tag{1.5}$$

$$\vec{1} \equiv \vec{1}_x\,\vec{1}_x+\vec{1}_y\,\vec{1}_y+\vec{1}_z\,\vec{1}_z\ \text{(identity dyad)}$$

$$\zeta \equiv \tilde{\gamma}(s)R$$

Here the form is the same as in free space with $\epsilon_o \to \epsilon+\sigma/s$. This is well defined in the right half s plane and is extended into the left half s plane by analytic continuation, noting the branch cut on $s = \Omega$ for $-\sigma/\epsilon \le \Omega \le 0$.

Having the surface current density one can now find the scattered far electric field

$$\tilde{\vec{E}}_f(\vec{r},s) = \frac{-s\mu_o e^{-\gamma r}}{4\pi r}\left\langle\vec{1}_r e^{\tilde{\gamma}(s)\vec{1}_r\cdot\vec{r}_s};\tilde{\vec{J}}_s(\vec{r}_s,s)\right\rangle$$

$$= -E_o\tilde{f}(s)\frac{s\mu_o e^{-\gamma r}}{4\pi r}\left\langle\vec{1}_r e^{\tilde{\gamma}(s)\vec{1}_r\cdot\vec{r}_s};\tilde{\tilde{Z}}_t^{-1}(\vec{r}_s,\vec{r}_s';s);\vec{1}_1 e^{-\tilde{\gamma}(s)\vec{1}_1\cdot\vec{r}_s'}\right\rangle \tag{1.6}$$

$$\vec{1}_r \equiv \vec{1}-\vec{1}_r\,\vec{1}_r\ ,\ \vec{1}_1 \equiv \vec{1}-\vec{1}_1\,\vec{1}_1$$

The far scattered magnetic field is just

$$\tilde{\vec{H}}_f(\vec{r},s) = \tilde{Z}^{-1}(s)\ \vec{1}_r\times\tilde{\vec{E}}(\vec{r},s) \tag{1.7}$$

The far scattering is characterized by a dyadic scattering operator [12, 13]

$$\tilde{\vec{E}}_f(\vec{r},s) = \frac{e^{-\gamma r}}{4\pi r}\,\tilde{\vec{\Lambda}}\left(\vec{1}_r,\vec{1}_1;s\right)\cdot\tilde{\vec{E}}^{(inc)}(0,s)$$

$$\begin{aligned}\tilde{\vec{\Lambda}}\left(\vec{1}_r,\vec{1}_1;s\right) &= -s\mu_o\left\langle \vec{1}_r e^{\tilde{\gamma}(s)\vec{1}_r\cdot\vec{r}_s};\ \tilde{\vec{Z}}_t^{-1}(\vec{r}_s,\vec{r}_s{}';s);\ \vec{1}_1 e^{-\tilde{\gamma}(s)\vec{1}_1\cdot\vec{r}_s{}'}\right\rangle \\ &= \tilde{\vec{\Lambda}}^T\left(-\vec{1}_1,-\vec{1}_r;s\right)\quad\text{(reciprocity)} \\ &\equiv \text{dyadic scattering operator}\end{aligned} \tag{1.8}$$

For comparison to our lossy-medium scattering problem consider the same target in free space, for which case we use a superscript 0 to distinguish the appropriate parameters. For this we have

$$\tilde{\gamma}^{(0)}(s) = \frac{s}{c}\quad,\quad \tilde{Z}^{(0)}(s) = Z_o \tag{1.9}$$

The comparison problem is related to the lossy-medium problem by the substitution

$$\tilde{\gamma}^{(0)}(s)\to\tilde{\gamma}(s)\quad,\quad \tilde{Z}^{(0)}(s)\to\tilde{Z}(s) \tag{1.10}$$

With this in mind the solution of the two problems is the same, with due care of the branch cut in the lossy-medium problem. Note that the impedance kernel in (1.5) can be written as

$$\tilde{\vec{Z}}_t(\vec{r}_s,\vec{r}_s{}';s) = \tilde{Z}(s)\,\tilde{\gamma}(s)\,\vec{1}_S(\vec{r}_s)\cdot\left\{\left[-2\zeta^{-3}-2\zeta^{-2}\right]e^{-\zeta}\,\vec{1}_R\,\vec{1}_R + \left[\zeta^{-3}+\zeta^{-2}+\zeta^{-1}\right]e^{-\zeta}\left[\vec{1}-\vec{1}_R\,\vec{1}_R\right]\right\} \tag{1.11}$$

which shows not only the scaling by $\tilde{\gamma}(s)$ (with $\zeta = \tilde{\gamma}(s)\,R$), but also a coefficient in front as $\tilde{Z}(s)$. So surface current density, magnetic field, and admittances have this additional scaling factor of $\tilde{Z}^{-1}(s)$ after connecting the two problems via the propagation constant.

II. Scaling Complex Frequencies: Free Space to Simple Lossy Medium

As discussed in [7, 25] there is a not-too-complicated scaling of frequencies from free space to our simple lossy medium by equating the two propagation constants as

$$\tilde{\gamma}^{(0)}\left(s^{(0)}\right) = \frac{s^{(0)}}{c} = \left[s\mu_o(\sigma + s\in)\right]^{\frac{1}{2}} = \tilde{\gamma}(s) \tag{2.1}$$

In particular, for natural frequencies s_α, defined by

$$\left\langle \tilde{\tilde{Z}}_t(\vec{r}_s,\vec{r}_s';s_\alpha); \vec{j}s_\alpha(\vec{r}'s)\right\rangle = \vec{0} \tag{2.2}$$

these are related to the free-space natural frequencies by

$$\gamma_\alpha^{(0)} \equiv \frac{s_\alpha^{(0)}}{c} = \left[s_\alpha\mu_o(\sigma + s_\alpha \in)\right]^{\frac{1}{2}} \equiv \gamma_\alpha$$

$$s_\alpha^2 + \frac{\sigma}{\in}s_\alpha - \frac{1}{\in_r}s_\alpha^{(0)^2} = 0 \tag{2.3}$$

$$s_\alpha = -\frac{\sigma}{2\in} \pm \left[\left[\frac{\sigma}{2\in}\right]^2 + \frac{1}{\in_r}s_\alpha^{(0)^2}\right]^{\frac{1}{2}}$$

Noting in (2.1) that s in the right half plane corresponds to $s^{(0)}$ in the right half plane, then the plus sign in (2.3) is the proper choice giving

$$s_\alpha = -\frac{\sigma}{2\in} + \left[\left[\frac{\sigma}{2\in}\right]^2 + \frac{1}{\in_r}s_\alpha^{(0)^2}\right]^{\frac{1}{2}} \tag{2.4}$$

For $s_\alpha^{(0)}$ in the second quadrant we have

$$\frac{\pi}{2} < \arg\left(\frac{1}{\in_r^{\frac{1}{2}}}s_\alpha^{(0)}\right) < \pi \qquad \text{(second quadrant)}$$

$$\pi < \arg\left(\frac{1}{\in_r}s_\alpha^{(0)^2}\right) < 2\pi \qquad \text{(third and fourth quadrants)}$$

$$\pi < \arg\left(\left[\frac{\sigma}{2\in}\right]^2 + \frac{1}{\in_r} s_\alpha^{(0)^2}\right) < 2\pi \qquad \text{(third and fourth quadrants)}$$

$$\frac{\pi}{2} < \arg\left(\left[\left[\frac{\sigma}{2\in}\right]^2 + \frac{1}{\in_r} s_\alpha^{(0)^2}\right]^{\frac{1}{2}}\right) < \pi \qquad \text{(second quadrant)} \qquad (2.5)$$

$$\frac{\pi}{2} < \arg(s_\alpha) = \arg\left(-\frac{\sigma}{2\in} \pm \left[\left[\frac{\sigma}{2\in}\right]^2 + \frac{1}{\in_r} s_\alpha^{(0)^2}\right]^{\frac{1}{2}}\right) < \pi \qquad \text{(second quadrant)}$$

Similarly for $s_\alpha^{(0)}$ in the third quadrant s_α lies in the third quadrant, for which it is most convenient to consider the arg function as negative angles. For the origin in the s plane ($s_\alpha^{(0)} = 0$ being appropriate to certain input admittances) we have

$$s_\alpha^{(0)} = 0$$

$$s_\alpha^{(0)} = 0\,, \quad -\frac{\sigma}{\in} \qquad (2.6)$$

these two points being branch points. For the negative real axis we have

$$s_\alpha^{(0)} = \Omega_\alpha^{(0)} < 0$$

$$s_\alpha = \Omega_\alpha = -\frac{\sigma}{2\in} - \left[\left[\frac{\sigma}{2\in}\right]^2 + \frac{1}{\in_r} s_\alpha^{(0)^2}\right]^{\frac{1}{2}} < -\frac{\sigma}{\in} \qquad (2.7)$$

with positive square root, thereby placing the s_α to the left of the branch cut. Note also that (2.2) implies

$$\vec{j}_{s_\alpha}(\vec{r}_s) = \vec{j}_{s_\alpha}^{(0)}(\vec{r}_s) \qquad (2.8)$$

(with the arbitrary scalar coefficient taken as unity) i.e., that the natural modes are unchanged in the scaling, noting the shift of the corresponding natural frequencies.

For targets with natural frequencies much larger in magnitude than $\sigma / \in$ (the relaxation frequency of the medium) we have

$$\left|s_\alpha\right| >> \frac{\sigma}{\in} \;,\; \left|s_\alpha^{(0)}\right| >> \in_r^{\frac{1}{2}} \frac{\sigma}{\in}$$

$$s_\alpha = -\frac{\sigma}{2\in} + \in_r^{-\frac{1}{2}} s_\alpha^{(0)} \left[1 + \in_r s_\alpha^{(0)^{-2}} \left[\frac{\sigma}{2\in}\right]^2\right]^{\frac{1}{2}}$$

$$= -\frac{\sigma}{2\in} + \in_r^{-\frac{1}{2}} s_\alpha^{(0)} \left[1 + \frac{\in_r}{2} s_\alpha^{(0)^{-2}} \left[\frac{\sigma}{2\in}\right]^2 + O\left(s_\alpha^{(0)^{-3}}\right)\right] \tag{2.9}$$

$$= \in_r^{-\frac{1}{2}} s_\alpha^{(0)} - \frac{\sigma}{2\in} + O\left(s_\alpha^{(0)^{-1}}\right) \text{ as } s_\alpha^{(0)} \to \infty$$

This is a rather simple transformation consisting of a dilation $\left(\in_r^{-\frac{1}{2}}\right)$ and a translation $(-\sigma/(2\in))$ in the complex plane. For high natural frequencies the pattern is then approximately the same in $s^{(0)}$ and s planes. This may help in recognizing the pole pattern of a target in the lossy medium, given the pole pattern in free space. As an added benefit one can obtain an estimate of $\in_r$ (and hence $\in$) and σ (from $-\sigma/(2\in)$), at least as effective parameters appropriate to the frequency region of the natural frequencies of interest. Note that while the foregoing discussion is in terms of natural frequencies it applies to all s (not on the branch cut) and their relation to $s^{(0)}$.

III. Branch Cut

In the SEM representation one class of terms of interest is branch cuts [8, 30, 32]. The present problem has one such cut. Consider this branch cut in more detail. As in fig. 3.1A, there is a positive (counterclockwise) contour C_b surrounding the branch cut in the s plane. This corresponds (maps one to one) with the contour C_o in the $s^{(0)}$ plane. Note the four points s_1 through s_4 on the branch cut where s_2 is taken just above (second quadrant) and s_4 just below (third quadrant) the branch cut. These correspond to $s_1^{(0)}$ through $s_4^{(0)}$, respectively, with $s_1^{(0)}$ just to the right and $s_3^{(0)}$ just to the left of the origin in the $s^{(0)}$ plane.

Let Ω' be a parameter with

$$-\frac{\sigma}{\epsilon} \le \Omega' \le 0 \tag{3.1}$$

giving the position along the branch cut. For positions above the branch (second quadrant) one can use Ω'_+, and similarly Ω'_- for those below. In the $s^{(0)}$ plane points on the branch are mapped into points given by

$$\begin{aligned} s^{(0)} &= j\omega^{(0)'} \\ \omega_4^{(0)} &= -\omega_2^{(0)} \le \omega^{(0)'} \le \omega_2^{(0)} \end{aligned} \tag{3.2}$$

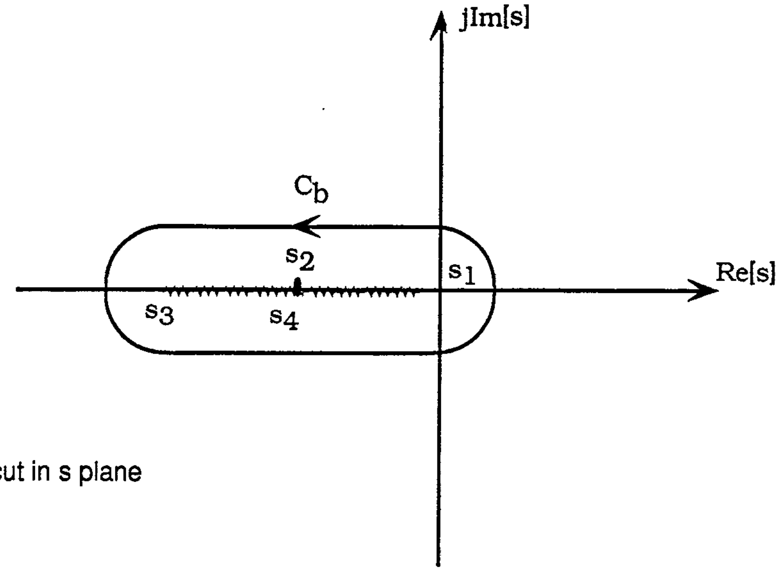
Note that $\omega^{(0)'}$ also corresponds to two points in general to the right and left of the $j\omega^{(0)}$ axis. From (2.1) consider the mapping of Ω'_+ as

$$\omega^{(0)'^2} = -\Omega'_+ \left[\frac{\sigma}{\epsilon_o} + \epsilon_r \, \Omega'_+\right] \tag{3.3}$$

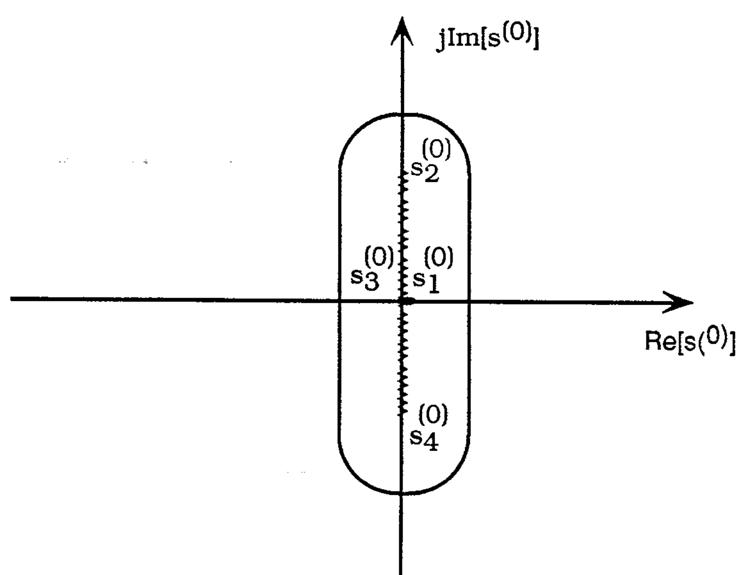
The maximum $\omega^{(0)'}$ (i.e. $\omega_2^{(0)}$) is found by setting the derivative with respect to Ω'_+ to zero, giving

$$\Omega_2 = -\frac{\sigma}{2\,\epsilon} \; , \; \omega_2^{(0)} = \epsilon_r^{-\frac{1}{2}} \frac{\sigma}{2\,\epsilon_o} = \epsilon_r^{\frac{1}{2}} \frac{\sigma}{2\,\epsilon} \tag{3.4}$$

Considering the symmetry we then have the four points around the contour as



A. Branch cut in s plane



B. Corresponding contour in $s^{(0)}$ plane

Fig. 3.1. Branch Contribution

$$
\begin{aligned}
s_1^{(0)} &= 0 \qquad\qquad\quad , \quad s_1 = 0 \\
s_2^{(0)} &= j\,\epsilon_r^{\frac{1}{2}}\,\frac{\sigma}{2\,\epsilon} \quad , \quad s_2 = -\frac{\sigma}{2\,\epsilon} \\
s_3^{(0)} &= 0 \qquad\qquad\quad , \quad s_3 = -\frac{\sigma}{\epsilon} \\
s_4^{(0)} &= -j\,\epsilon_r^{\frac{1}{2}}\,\frac{\sigma}{2\,\epsilon} \quad , \quad s_4 = -\frac{\sigma}{2\,\epsilon}
\end{aligned}
\tag{3.5}
$$

noting that these points are approached from the directions indicated previously.

Suppose that we have $\tilde{F}(s)$, a function of s with this branch cut. Let this be conjugate symmetric, i.e.

$$
\tilde{F}\left(s^*\right) = \tilde{F}^*(s) \tag{3.6}
$$

corresponding to the Laplace transform of a real-valued time function. This leads to a discontinuity across the branch cut. As discussed in [32] the branch cut then gives a contribution

$$
\begin{aligned}
\tilde{F}_b(s) &= \frac{1}{2\pi j}\oint_{C_b} \frac{\tilde{F}(s')}{s-s'}\,ds' \\
F_b(t) &= \frac{1}{2\pi j}\oint_{C_b} \tilde{F}(s)\;e^{st}\;u(t)\,ds'
\end{aligned}
\tag{3.7}
$$

Utilizing the conjugate symmetry we have

$$
\begin{aligned}
\mathrm{Re}\left[\tilde{F}(\Omega'_+)\right] &= \mathrm{Re}\left[\tilde{F}(\Omega'_-)\right] \\
\mathrm{Im}\left[\tilde{F}(\Omega'_+)\right] &= -\,\mathrm{Im}\left[\tilde{F}(\Omega'_-)\right]
\end{aligned}
\tag{3.8}
$$

so that only the imaginary part contributes giving

$$\tilde{F}_b(s) = \frac{1}{\pi}\int_0^{-\frac{\sigma}{\epsilon}} \frac{\mathrm{Im}\left[\tilde{F}(\Omega'_+)\right]}{s-\Omega'_+}\, d\Omega'_+$$

$$F_b(t) = \frac{1}{\pi}\int_0^{-\frac{\sigma}{\epsilon}} \mathrm{Im}\left[\tilde{F}(\Omega'_+)\right] e^{\Omega'_+ t}\, u(t)\; d\Omega'_+ \tag{3.9}$$

Note that the inclusion of u(t) in (3.7) and (3.9) is somewhat arbitrary, due to the left-right decomposition in the s plane as discussed in [32]. By a time shift this unit step can be turned on at various times as convenient.

Now let us transform these integrals from the s to the $s^{(0)}$ plane. From (2.1) we have

$$s^{(0)2} = s\left[\frac{\sigma}{\epsilon_o} + \epsilon_r\, s\right]$$

$$2s^{(0)}\; ds^{(0)} = \left[\frac{\sigma}{\epsilon_o} + 2\,\epsilon_r\, s\right] ds$$

$$s = -\frac{\sigma}{2\,\epsilon} + \left[\left[\frac{\sigma}{2\,\epsilon}\right]^2 + \frac{1}{\epsilon_r}\, s^{(0)2}\right]^{\frac{1}{2}}$$

$$ds = \left[\left[\frac{\sigma}{2\,\epsilon}\right]^2 + \frac{1}{\epsilon_r}\, s^{(0)2}\right]^{-\frac{1}{2}} \frac{s^{(0)}}{\epsilon_r}\, ds^{(0)} \tag{3.10}$$

with the square root as discussed previously. Next assume that $\tilde{F}(s)$ transforms in the "electric" sense as

$$\tilde{F}(s) = \tilde{F}^{(0)}\left(s^{(0)}\right)$$

$$\tilde{\gamma}(s) = \tilde{\gamma}^{(0)}\left(s^{(0)}\right) \tag{3.11}$$

Then (3.7) can be written as

$$\tilde{F}_b(s) = \frac{1}{2\pi j}\oint_{C_o} \frac{s^{(0)'}}{\epsilon_r}\left[\left[\frac{\sigma}{2\epsilon}\right]^2 + \frac{s^{(0)'^2}}{\epsilon_r}\right]^{-\frac{1}{2}} \left\{s + \frac{\sigma}{2\epsilon} - \left[\left[\frac{\sigma}{2\epsilon}\right]^2 + \frac{s^{(0)'^2}}{\epsilon_r}\right]^{\frac{1}{2}}\right\}^{-1} \tilde{F}^{(0)}\left(s^{(0)'}\right) ds^{(0)'}$$

$$F_b(t) = \frac{1}{2\pi j}\oint_{C_o} \frac{s^{(0)'}}{\epsilon_r}\left[\left[\frac{\sigma}{2\epsilon}\right]^2 + \frac{s^{(0)'^2}}{\epsilon_r}\right]^{-\frac{1}{2}} \tilde{F}^{(0)}\left(s^{(0)'}\right) e^{\left[-\frac{\sigma}{2\epsilon} + \left[\left[\frac{\sigma}{2\epsilon}\right]^2 + \frac{s^{(0)'^2}}{\epsilon_r}\right]^{\frac{1}{2}}\right]t} u(t)\, ds^{(0)'} \qquad (3.12)$$

In the $s^{(0)}$ plane $\tilde{F}^{(0)}$ may be analytic inside and on C_0. However, there is a branch cut described by (3.2) associated with the other terms coming from the transformation.

Using variables on the branch cut

$$\omega^{(0)'^2} = -\Omega'_+\left[\frac{\sigma}{\epsilon_o} + \epsilon_r\, \Omega'_+\right]$$

$$2\omega^{(0)'}\, d\omega^{(0)'} = -\left[\frac{\sigma}{\epsilon_o} + 2\,\epsilon_r\, \Omega'_+\right] d\Omega'_+$$

$$\Omega'_+ = \begin{cases} -\dfrac{\sigma}{2\epsilon} + \left[\left[\dfrac{\sigma}{2\epsilon}\right]^2 - \dfrac{1}{\epsilon_r}\,\omega^{(0)'^2}\right]^{\frac{1}{2}} & \text{for } -\dfrac{\sigma}{2\epsilon} \le \Omega'_+ \le 0 \\ -\dfrac{\sigma}{2\epsilon} - \left[\left[\dfrac{\sigma}{2\epsilon}\right]^2 - \dfrac{1}{\epsilon_r}\,\omega^{(0)'^2}\right]^{\frac{1}{2}} & \text{for } -\dfrac{\sigma}{\epsilon} \le \Omega'_+ \le -\dfrac{\sigma}{2\epsilon} \end{cases}$$

$$d\,\Omega'_+ = \begin{cases} -\left[\left[\dfrac{\sigma}{2\epsilon}\right]^2 - \dfrac{1}{\epsilon_r}\,\omega^{(0)'^2}\right]^{-\frac{1}{2}} \dfrac{\omega^{(0)'}}{\epsilon_r}\, d\omega^{(0)'} & \text{for } -\dfrac{\sigma}{2\epsilon} \le \Omega'_+ \le 0 \\ +\left[\left[\dfrac{\sigma}{2\epsilon}\right]^2 - \dfrac{1}{\epsilon_r}\,\omega^{(0)'^2}\right]^{-\frac{1}{2}} \dfrac{\omega^{(0)'}}{\epsilon_r}\, d\omega^{(0)'} & \text{for } -\dfrac{\sigma}{\epsilon} \le \Omega'_+ \le -\dfrac{\sigma}{2\epsilon} \end{cases} \qquad (3.13)$$

the integrals in (3.9) take the form

$$\tilde{F}_b(s) = -\frac{1}{\pi}\int_0^{\omega_2^{(0)}} \frac{\omega^{(0)'}}{\epsilon_r}\left[\left[\frac{\sigma}{2\epsilon}\right]^2 - \frac{\omega^{(0)'^2}}{\epsilon_r}\right]^{-\frac{1}{2}}\left\{\left[s+\frac{\sigma}{2\epsilon}-\left[\left[\frac{\sigma}{2\epsilon}\right]^2-\frac{\omega^{(0)'^2}}{\epsilon_r}\right]^{\frac{1}{2}}\right]^{-1}\right.$$

$$\left.+\left[s+\frac{\sigma}{2\epsilon}+\left[\left[\frac{\sigma}{2\epsilon}\right]^2-\frac{\omega^{(0)'^2}}{\epsilon_r}\right]^{\frac{1}{2}}\right]^{-1}\right\} \operatorname{Im}\left[\tilde{F}^{(0)}\left(j\omega^{(0)'}\right)\right] d\omega^{(0)'}$$

$$F_b(t) = -\frac{1}{\pi}\int_0^{\omega_2^{(0)}} \frac{\omega^{(0)'}}{\epsilon_r}\left[\left[\frac{\sigma}{2\epsilon}\right]^2 - \frac{\omega^{(0)'^2}}{\epsilon_r}\right]^{-\frac{1}{2}}\left\{e^{\left[-\frac{\sigma}{2\epsilon}+\left[\left[\frac{\sigma}{2\epsilon}\right]^2-\frac{\omega^{(0)'^2}}{\epsilon_r}\right]^{\frac{1}{2}}\right]t}\right.$$

$$\left.+\, e^{\left[-\frac{\sigma}{2\epsilon}-\left[\left[\frac{\sigma}{2\epsilon}\right]^2-\frac{\omega^{(0)'^2}}{\epsilon_r}\right]^{\frac{1}{2}}\right]t}\right\} \operatorname{Im}\left[\tilde{F}^{(0)}\left(j\omega^{(0)'}\right)\right] u(t)\, d\omega^{(0)'} \qquad (3.14)$$

Note the two terms corresponding to integration both up and down the $j\omega^{(0)'}$ axis given by the range of Ω'_+. So the branch contribution can be expressed in both frequency and time domains by integrals over the branch cut in either s or $s^{(0)}$ planes, whichever is more convenient.

There are also magnetic parameters proportional to

$$
\begin{aligned}
\tilde{Z}^{-1}(s)\ \tilde{F}(s) &= \left[\frac{\sigma + s \in}{s\mu_o}\right]^{\frac{1}{2}} \tilde{F}(s) \\
&= \frac{\tilde{\gamma}(s)}{s\mu_o}\ \tilde{F}(s) = \frac{\tilde{\gamma}^{(0)}\left(s^{(0)}\right)}{s\left(s^{(0)}\right)\mu_o}\ \tilde{F}^{(0)}\ \left(s^{(0)}\right)
\end{aligned}
\tag{3.15}
$$

with $s\left(s^{(0)}\right)$ determined as discussed previously. This type of scaling can be substituted for $\tilde{F}(s)$ scaling in (3.11) in the above results ((3.12) and (3.14)).

IV. Scaling of Eigenmode Parameters

Extending on the formulation in Section I, one can diagonalize the symmetric kernel of the integral equation via

$$\left\langle \tilde{\bar{\bar{Z}}}_t(\bar{r}_s, \bar{r}_s';s); \tilde{\bar{\bar{j}}}_{s_\beta}(\bar{r}_s', s)\right\rangle = \tilde{Z}_\beta(s)\; \tilde{\bar{\bar{j}}}_{s_\beta}(\bar{r}_s, s)$$

$$\tilde{\bar{\bar{Z}}}_t^{\upsilon}(\bar{r}_s, \bar{r}_s';s) = \sum_\beta \tilde{Z}_\beta^{\upsilon}(s)\; \tilde{\bar{\bar{j}}}_{s_\beta}(\bar{r}_s, s)\; \tilde{\bar{\bar{j}}}_{s_\beta}(\bar{r}_s', s)$$

$$\left\langle \tilde{\bar{\bar{j}}}_{s_{\beta_1}}(\bar{r}_s, s)\, ;\, \tilde{\bar{\bar{j}}}_{s_{\beta_2}}(\bar{r}_s, s)\right\rangle = 1_{\beta_1,\beta_2} \;\; \text{(orthonormal)} \tag{4.1}$$

$$\upsilon = \begin{cases} 1 \Rightarrow \text{kernel} \\ 0 \Rightarrow \text{identity} \\ 1 \Rightarrow \text{inverse kernel} \end{cases}$$

This is the eigenmode expansion method (EEM) for representing the scattering [8, 10, 11, 13, 14, 32, 35]. The far scattering in (1.6) takes the form

$$\tilde{\bar{E}}_f(\bar{r}, s) = \frac{e^{-\tilde{\gamma}(s)r}}{4\pi r}\, \tilde{\bar{\bar{\Lambda}}}\left(\bar{1}_r, \bar{1}_1; s\right) \cdot \tilde{\bar{E}}^{(inc)}(0, s)$$

$$\tilde{\bar{\bar{\Lambda}}}\left(\bar{1}_r, \bar{1}_1; s\right) = -s\mu_o \left\langle \bar{1}_r e^{\tilde{\gamma}(s)\,\bar{1}_r \cdot \bar{r}_s}\, ;\, \tilde{\bar{\bar{Z}}}_t^{-1}(\bar{r}_s, \bar{r}_s';s)\, ;\, \bar{1}_1\, e^{-\tilde{\gamma}(s)\bar{1}_1 \cdot \bar{r}_s'} \right\rangle$$

$$= -s\mu_o \sum_\beta \tilde{Z}_\beta^{-1}(s)\, \tilde{\bar{C}}_{r_\beta}\left(\bar{1}_r, s\right)\; \tilde{\bar{C}}_\beta\left(\bar{1}_1, s\right) = -\tilde{Z}(s)\, \tilde{\gamma}(s) \sum_\beta \tilde{Z}_\beta^{-1}(s)\, \tilde{\bar{C}}_{r_\beta}\left(\bar{1}_r, s\right)\; \tilde{\bar{C}}_\beta\left(\bar{1}_1, s\right)$$

$$\tilde{\bar{C}}_{r_\beta}\left(\bar{1}_r, s\right) = \left\langle \bar{1}_r e^{\gamma \bar{1}_r \cdot \bar{r}_s'}\, ;\, \tilde{\bar{\bar{j}}}_{s_\beta}(\bar{r}_s', s)\right\rangle \tag{4.2}$$

$$\tilde{\bar{C}}_\beta\left(\bar{1}_1, s\right) = \tilde{\bar{C}}_{r_\beta}\left(-\bar{1}_1, s\right) = \left\langle \bar{1}_1 e^{-\gamma \bar{1}_1 \cdot \bar{r}_s'}\, ;\, \tilde{\bar{\bar{j}}}_{s_\beta}(\bar{r}_s', s)\right\rangle$$

where for backscattering this simplifies to

$$\bar{1}_r = -\bar{1}_1$$

$$\tilde{\bar{\bar{\Lambda}}}(\bar{1}_1, s) \equiv \tilde{\bar{\bar{\Lambda}}}(-\bar{1}_1, \bar{1}_1; s) = -\tilde{Z}(s)\,\tilde{\gamma}(s) \sum_{\beta} \tilde{Z}_{\beta}^{-1}(s)\, \tilde{\bar{C}}_{\beta}(\bar{1}_1, s)\, \tilde{\bar{C}}_{\beta}(\bar{1}_1, s) \tag{4.3}$$

which exhibits the symmetry (reciprocity) in the backscattering dyadic.

As discussed previously, there is a scaling of parameters from the free-space parameters (superscript zero). Frequencies scale as

$$s^{(0)} = \left[s \left[\frac{\sigma}{\epsilon_o} + \epsilon_r\, s \right] \right]^{\frac{1}{2}}$$

$$s = -\frac{\sigma}{2\epsilon} + \left[\left[\frac{\sigma}{2\epsilon} \right]^2 + \frac{1}{\epsilon_r} s^{(0)^2} \right]^{\frac{1}{2}} \tag{4.3}$$

Noting that the free-space impedance kernel scales as

$$Z_0^{-1}\, \tilde{\bar{\bar{Z}}}_t^{(0)}\left(\bar{r}_s, \bar{r}_s'; s^{(0)}\right) = \tilde{Z}^{-1}(s)\, \tilde{\bar{\bar{Z}}}_t(\bar{r}_s, \bar{r}_s'; s) \tag{4.4}$$

we have from the EEM representation

$$\tilde{\bar{\bar{Z}}}_t^{(0)^{v}}\left(\bar{r}_s, \bar{r}_s'; s^{(0)}\right) = \sum_{\beta} \tilde{Z}_{\beta}^{(0)}\left(s^{(0)}\right) \tilde{\bar{j}}_{s_\beta}^{(0)}\left(\bar{r}_s, s^{(0)}\right) \tilde{\bar{j}}_{s_\beta}^{(0)}\left(\bar{r}_s', s^{(0)}\right) \tag{4.5}$$

the scaling relationships

$$\tilde{\bar{j}}_{s_\beta}(\bar{r}_s, s) = \tilde{\bar{j}}_{s_\beta}^{(0)}\left(\bar{r}_s, s^{(0)}\right)$$

$$\frac{1}{\tilde{Z}(s)}\, \tilde{Z}_{\beta}(s) = \frac{1}{Z_o}\, \tilde{Z}_{\beta}^{(0)}\left(s^{(0)}\right) \tag{4.6}$$

for the basic EEM parameters.

Carrying on the development the far scattering parameters scale as

$$e^{-\tilde{\gamma}(s)r} = e^{-\tilde{\gamma}^{(0)}\left(s^{(0)}\right)r}$$

$$\tilde{\bar{C}}_{\beta}\left(\bar{1}_1, s\right) = \tilde{\bar{C}}_{\beta}^{(0)}\left(\bar{1}_1, s^{(0)}\right)$$

$$\tilde{\bar{C}}_{r_\beta}\left(\bar{1}_r, s\right) = \tilde{\bar{C}}_{r_\beta}^{(0)}\left(\bar{1}_r, s^{(0)}\right) \tag{4.7}$$

$$\tilde{\bar{\Lambda}}\left(\bar{1}_r, \bar{1}_1; s\right) = \tilde{\bar{\Lambda}}^{(0)}\left(\bar{1}_r, \bar{1}_1; s^{(0)}\right)$$

The natural frequencies s_α are just the zeros of $\tilde{Z}_\beta$ for which

$$s_\alpha = s_{\beta,\beta'}$$

$$\tilde{Z}_\beta\left(s_{\beta,\beta'}\right) = 0 \quad (\beta'\text{th root of } \beta\text{th eigenimpedance}) \tag{4.8}$$

These roots of course scale from the roots of the free-space eigenimpedancces $\tilde{Z}_{\beta}^{(0)}\left(s^{(0)}\right)$ via (4.3) and (4.6). For the branch cut as in Section III, one can evaluate the integrals in either s or $s^{(0)}$ plane, where the various integrands are now represented as sums over β. Note that the branch cut in the s plane applies to various eigenterms (eigenmodes, eigenimpedances, etc.) unless certain terms are frequency independent (e.g. the eigenmodes of the sphere [13]), in which case they are frequency independent in both planes.

V. Propagation of Incident and Scattered Fields

As indicated in fig. 5.1 our complete scattering or radar problem consists of the production of an incident wave from some transmitting antenna, propagation of some distance d_{inc} to the target, and scattering to some receiving antenna a distance d_{sc} away. This involves the propagation of the wave a distance

$$d = d_{inc} + d_{sc} \tag{5.1}$$

from the transmitter, via the target, to the receiver. One can regard d as some minimum propagation distance, or some average distance such as via the coordinate origin in fig. 1.1.

The antennas have various frequency- and angular-dependent characteristics as does the target, complicating the analysis considerably. For present purposes consider a plane wave propagating a distance d in the conducting dielectric to estimate the effect of the lossy medium on the pulse propagation. For simplicity let this propagate in the z direction as

$$\begin{aligned} \tilde{\vec{E}}(\vec{r},s) &= \vec{1}_p\, E_o\, \tilde{g}(s)\, e^{-\tilde{\gamma}(s)z} \\ \tilde{\vec{H}}(\vec{r},s) &= \vec{1}_z \times \vec{1}_p\, \frac{E_o}{\tilde{Z}(s)}\, \tilde{g}(s)\, e^{-\tilde{\gamma}(s)z} \end{aligned} \tag{5.2}$$

where $\tilde{g}(s)$ characterizes the waveform of the electric field on the z = 0 plane. For convenience let us take this exciting waveform as a delta function, i.e.

$$\tilde{g}(s) = 1 \quad , \quad g(t) = \delta(t) \tag{5.3}$$

Then define our normalized fields as

$$\begin{aligned} \tilde{e}_o(\vec{r},s) &= e^{-\tilde{\gamma}(s)z} \\ \tilde{h}_o(\vec{r},s) &= \tilde{Z}^{-1}(s)\, e^{-\tilde{\gamma}(s)z} = \frac{\tilde{\gamma}(s)}{s\mu_o}\, e^{-\tilde{\gamma}(s)z} \end{aligned} \tag{5.4}$$

Note that one can just as easily normalize to the magnetic-field waveform at z = 0.

Using (1.3) the normalized fields can be approximated for high frequencies as

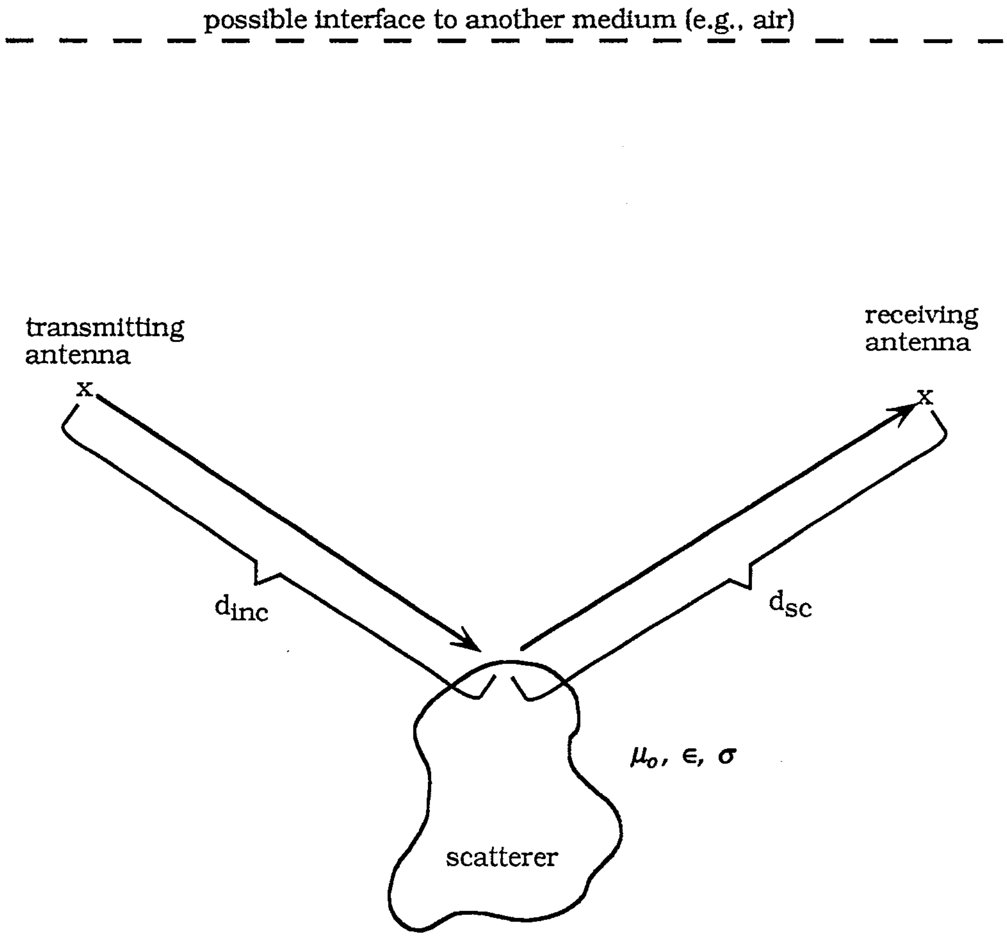


Fig. 5.1. Propagation in Lossy Medium

$$\tilde{e}_o(\bar{r},s) = e^{-\frac{s}{v}z} \; e^{-\frac{\sigma Z_\infty z}{2}} \left[1+O\left(s^{-1}\right)\right] \qquad \text{as} \qquad s \to \infty$$

$$\tilde{h}_o(\bar{r},s) = Z_\infty^{-1} \; e^{-\frac{s}{v}z} \; e^{-\frac{\sigma Z_\infty z}{2}} \left[1+O\left(s^{-1}\right)\right] \qquad \text{as} \qquad s \to \infty \qquad (5.5)$$

For low frequencies these take a different form as

$$\tilde{e}_o(\bar{r},s) = e^{-\sqrt{s\mu_o\sigma}\, z} \left[1+O(s)\right] \qquad \text{as} \qquad s \to 0$$

$$\tilde{h}_o(\bar{r},s) = \left[\frac{\sigma}{s\mu_o}\right]^{\frac{1}{2}} e^{-\sqrt{s\mu_o\sigma}\, z} \left[1 + O(s)\right] \qquad \text{as} \qquad s \to 0 \qquad (5.6)$$

As one can readily see the high- and low-frequency behaviours are quite different.

In time domain the normalized fields can be used for convolution with $\tilde{g}(s)$, some assumed exciting waveform. The time-domain form of (5.4) is a long-studied problem [1, 2, 3, 5, 6, 15-19, 23, 24]. Define some appropriate parameters as

$$t_o \equiv \frac{z}{v} = z\sqrt{\mu_o \in} \quad , \; t_1 \equiv \frac{2\in}{\sigma} \quad , \; z_o \equiv \frac{2}{\sigma Z_\infty}$$

$$\frac{t_o}{t_1} = \frac{z}{z_o} \qquad (5.7)$$

Noting that $z \geq 0$ we have the well-known result for the electric field

$$e_o(\bar{r},t) = e^{-\frac{t}{t_1}} \left\{ \delta(t-t_o) + \frac{t_o}{t_1}\left[t^2 - t_o^2\right]^{-\frac{1}{2}} I_1\left(t_1^{-1}\left[t^2-t_o^2\right]^{\frac{1}{2}}\right) u(t-t_o) \right\}$$

$$= e^{-\frac{z}{z_o}} \delta(t-t_o) + e^{-\frac{t}{t_1}} \frac{t_o}{t_1} \left[t^2 - t_o^2\right]^{-\frac{1}{2}} I_1\left(t_1^{-1}\left[t^2-t_o^2\right]^{\frac{1}{2}}\right) u(t-t_o) \qquad (5.8)$$

$$= e^{-\frac{z}{z_o}} \left\{ \delta(t-t_o) + \frac{z}{z_0} e^{-\frac{t-t_o}{t_1}} \left[t^2 - t_o^2\right]^{-\frac{1}{2}} I_1\left(t_1^{-1}\left[t^2-t_o^2\right]^{\frac{1}{2}}\right) u(t-t_o) \right\}$$

The magnetic field has a more complicated expression which is found in [6] and is given by

$$h_o(\bar{r},t) = Z_\infty^{-1}\Bigg\{ e^{-\frac{t}{t_1}}\,\delta(t-t_o) + \left[t_1^{-1} + t_o\,t_1^{-2}\right] e^{-\frac{t_o}{t_1}}\,u(t-t_o)$$

$$+\,u(t-t_o)\,t_1^{-2}\int_{t_o}^{t} e^{-\frac{t'}{t_1}}\left[-\frac{I_1\left(t_1^{-1}\left[t'^2-t_o^2\right]^{\frac{1}{2}}\right)}{t_1^{-1}\left[t'^2-t_o^2\right]^{\frac{1}{2}}} + \frac{1}{4}\left(\frac{t_o}{t_1}\right)^2\left[\frac{I_o\left(t_1^{-1}\left[t'^2-t_o^2\right]^{\frac{1}{2}}\right)}{t_1^{-2}\left[t'^2-t_o^2\right]}\right.\right.$$

$$\left.\left.-2\,\frac{I_1\left(t_1^{-1}\left[t'^2-t_o^2\right]^{\frac{1}{2}}\right)}{t_1^{-3}\left[t'^2-t_o^2\right]^{\frac{3}{2}}}\right]\right]dt'\Bigg\}$$

$$= Z_\infty^{-1}\Bigg\{ e^{-\frac{z}{z_o}}\left[\delta(t-t_o) + t_1^{-1}\left[1+\frac{z}{z_o}\right]u(t-t_o)\right]$$

$$+\,u(t-t_o)\,t_1^{-2}\int_{t_o}^{t} e^{-\frac{t'}{t_1}}\left[-\frac{I_1\left(t_1^{-1}\left[t'^2-t_o^2\right]^{\frac{1}{2}}\right)}{t_1^{-1}\left[t'^2-t_o^2\right]^{\frac{1}{2}}} + \frac{1}{4}\left(\frac{z}{z_o}\right)^2\frac{I_2\left(t_1^{-1}\left[t'^2-t_o^2\right]^{\frac{1}{2}}\right)}{t_1^{-2}\left[t'^2-t_o^2\right]}\right]dt'\Bigg\} \qquad (5.9)$$

where we have made use of a recurrence relation for the modified Bessel functions [29]. One can also start with the magnetic field specified at z = 0, giving (5.7) for the magnetic field and a somewhat simpler expression (than (7.9)) for the electric field. However, the above will do for present purposes.

An interesting time regime is for retarded time $t - t_o << t_1$. The normalized electric field behaves like

$$e_o(\bar{r},t) = e^{-\frac{z}{z_o}}\left\{\delta(t-t_o)+\frac{z}{z_o}\,e^{-\frac{t-t_o}{t_1}}\,\frac{1+O\left(t_1^{-2}\left[t^2-t_o^2\right]\right)}{2t_1}\,u(t-t_o)\right\}$$

$$= e^{-\frac{z}{z_o}}\left\{\delta(t-t_o)+\frac{z}{z_o}\,\frac{1}{2t_1}\,u(t-t_o)+O(t-t_o)\right\} \text{ as } t-t_o\to 0 \tag{5.10}$$

Convolving with the excitation function gives (assuming g(t) = 0 for t < 0)

$$\tilde{\bar{E}}(\bar{r},t) = \bar{1}_p\,E_o\,g(t)\circ e_o(\bar{r},t)$$

$$= \bar{1}_p E_o\,e^{-\frac{z}{z_o}}\left\{g(t-t_o)+\frac{z}{z_o}\,\frac{1}{2t_1}\int_{t_o}^{t} g(t'-t_o)\,dt'\right. \tag{5.11}$$

$$\left.+\,O\left(\int_{t_o}^{t}\int_{t_o}^{t'} g(t''-t_o)\,dt''\,dt'\right\} \text{ as } t-t_o\to 0$$

Comparing the second to the first term one can estimate conditions for which the first term is an adequate approximation. In particular if for some t of interest we define

$$g_o \equiv \sup_{t_o<t'<t} g(t'-t_o) \tag{5.12}$$

then we have a relative error as

$$\left|\frac{1}{g_o}\,\frac{z}{z_o}\,\frac{1}{2t_1}\int_{t_o}^{t} g(t'-t_o)\,dt'\right| \le \frac{z}{z_o}\,\frac{t-t_o}{2t_1} \tag{5.13}$$

If this is to be small compared to unity we have

$$t-t_o \ll 2t_1\,\frac{z_o}{z} \tag{5.14}$$

which includes both the relaxation time t_1 and depth in number of e-folds. This can be used to bound what might be termed early time, at least for plane-wave propagation.

Considering the magnetic field for early retarded times we have the normalized form

$$h_o(\bar{r},t) = Z_\infty^{-1}\ e^{-\frac{z}{z_o}}\left\{\delta(t-t_o)+t_1^{-1}\left[1+\frac{z}{z_o}\right]\ u(t-t_o)\right.$$

$$\left. + u(t-t_o)\,t_1^{-2}\int_{t_o}^{t} e^{-\frac{t'-t_o}{t_1}}\left[-\frac{1}{2}+\frac{1}{32}\left(\frac{z}{z_o}\right)^2 + O(t'-t_o)\right]dt'\right\} \tag{5.15}$$

$$= Z_\infty^{-1}\ e^{-\frac{z}{z_o}}\left\{\delta(t-t_o)+t_1^{-1}\left[1+\frac{z}{z_o}\right]u(t-t_o)+O(t-t_o)\right\}\ \text{as}\ t-t_o\to 0$$

and

$$\bar{H}(\bar{r},t) = \bar{1}_z\times\bar{1}_p\,\frac{E_o}{Z_\infty}\,e^{-\frac{z}{z_o}}\left\{g(t-t_o)+t_1^{-1}\left[1+\frac{z}{z_o}\right]\int_{t_o}^{t} g(t'-t_o)\,dt'\right.$$

$$\left. + O(\int_{t_o}^{t}\int_{t_o}^{t'} g(t''-t_o)\,dt''\,dt'\right\}\ \text{as}\ t-t_o\to 0 \tag{5.16}$$

This is similar to the previous result for the electric field. Now we have an estimate of the relative error as

$$\left|\frac{1}{g_o}\left[1+\frac{z}{z_o}\right]\frac{1}{t_1}\int_{t_o}^{t} g(t'-t_o)\,dt'\right| \le \left[1+\frac{z}{z_o}\right]\ \frac{t-t_o}{t_1} \tag{5.17}$$

which when compared to (5.13) shows a slightly larger error for a given $t - t_o$.

Note that by judicious choice of g(t) these "errors" can be made less. Such is the case if $g(t - t_o)$ is somewhat oscillatory, or is limited to times short compared to $t - t_o$, thereby reducing its time integral in (5.13) and (5.17). One might even tailor g(t) in such a way as to partially compensate for the propagation distortion at some z of interest.

VI. Scaling of Pole Parameters

As observed in Section II the natural frequencies scale from free space to the simple lossy medium via the propagation constants of the two media. The natural modes (for currents on the perfectly conducting scatterer) are unchanged in the transformation. This leaves the coupling coefficients to consider.

As discussed in [10, 11, 13, 26, 35] the surface currents on a perfectly conducting scatterer in free space are (assuming first-order poles)

$$\tilde{\vec{J}}_s^{(0)}\left(\vec{r}_s, s^{(0)}\right) = E_o \sum_\alpha \tilde{f}\left(s_\alpha^{(0)}\right) \eta_\alpha^{(0)}\left(\vec{1}_1, \vec{1}_p\right) \vec{j}_{s_\alpha}(\vec{r}_s)\left[s^{(0)} - s_\alpha^{(0)}\right]^{-1} e^{-\left(s^{(0)} - s_\alpha^{(0)}\right) t_i}$$

$$+ \text{ singularities of } \tilde{f}\left(s^{(0)}\right)$$

$$+ \text{ possible entire function}$$

$$\eta_\alpha\left(\vec{1}_1, \vec{1}_p\right) = \frac{\vec{1}_p \cdot \left\langle e^{-\gamma_\alpha^{(0)} \vec{1}_1 \cdot \vec{r}_s'}, \vec{j}_{s_\alpha}(\vec{r}\,')\right\rangle}{\left\langle \vec{j}_{s_\alpha}(\vec{r}_s); \frac{\partial}{\partial s^{(0)}} \tilde{\vec{Z}}_t^{(0)}(\vec{r}_s, \vec{r}_s'; s)\Big|_{s^{(0)} = s_\alpha^{(0)}}; \vec{j}_{s_\alpha}(\vec{r}_s')\right\rangle} \quad (6.1)$$

$$\equiv \text{ coupling coefficient}$$

$$t_i \equiv \text{ initial time or turn-on time (chosen for convenience)}$$

In our simple lossy medium this takes the form

$$\tilde{\vec{J}}_s(\vec{r}_s,s) = E_o \sum_{\alpha} \tilde{f}(s_\alpha)\,\eta_\alpha\left(\vec{1}_1,\vec{1}_p\right)\,\vec{j}_s(\vec{r}_s)[s-s_\alpha]^{-1} e^{-(s-s_\alpha)\,t_i}$$

+ singularities of $\tilde{f}(s)$

+ branch contribution

+ possible entire function

$$\eta_\alpha\left(\vec{1}_1,\vec{1}_p\right) = \frac{\vec{1}_p \cdot \left\langle e^{-\gamma_\alpha \vec{1}_1\cdot\vec{r}_s'}\,,\ \vec{j}_{s_\alpha}(\vec{r}_s')\right\rangle}{\left\langle \vec{j}_{s_\alpha}(\vec{r}_s); \frac{\partial}{\partial s}\tilde{\vec{Z}}_t(\vec{r}_s,\vec{r}_s';s)\Big|_{s=s_\alpha}; \vec{j}_{s_\alpha}(\vec{r}_s')\right\rangle} \tag{6.2}$$

To scale this note that

$$\begin{aligned}\frac{\partial}{\partial s}\tilde{\vec{Z}}_t(\vec{r}_s,\vec{r}_s';s) &= \frac{\partial}{\partial s}\left[\frac{\tilde{Z}(s)}{\tilde{Z}_o}\,\tilde{\vec{Z}}_t^{(0)}\left(\vec{r}_s,\vec{r}_s';s^{(0)}\right)\right] \\ &= \tilde{\vec{Z}}_t^{(0)}\left(\vec{r}_s,\vec{r}_s';s^{(0)}\right)\frac{\partial}{\partial s}\left[\frac{\tilde{Z}(s)}{Z_o}\right] + \frac{\tilde{Z}(s)}{Z_o}\,\frac{\partial}{\partial s}\,\tilde{\vec{Z}}_t^{(0)}\left(\vec{r}_s,\vec{r}_s';s^{(0)}\right)\end{aligned} \tag{6.3}$$

When operating on the natural mode the first term gives zero, so we have

$$\begin{aligned}&\left\langle \vec{j}_{s_\alpha}(\vec{r}_s); \frac{\partial}{\partial s}\tilde{\vec{Z}}_t(\vec{r}_s,\vec{r}_s';s)\Big|_{s=s_\alpha}; \vec{j}_{s_\alpha}(\vec{r}_s')\right\rangle \\ &= \frac{\tilde{Z}(s_\alpha)}{Z_o}\,\frac{ds^{(0)}}{ds}\Big|_{s=s_\alpha}\left\langle \vec{j}_{s_\alpha}(\vec{r}_s); \frac{\partial}{\partial s^{(0)}}\tilde{\vec{Z}}_t^{(0)}(\vec{r}_s,\vec{r}_s';s)\Big|_{s^{(0)}=s_\alpha^{(0)}}; \vec{j}_{s_\alpha}(\vec{r}_s')\right\rangle\end{aligned} \tag{6.4}$$

Next we have

$$\frac{ds^{(0)}}{ds} = c\frac{d}{ds}\tilde{\gamma}(s) = c\frac{d}{ds}\left[s\mu_o(\sigma + s\in)\right]^{\frac{1}{2}}$$

$$= \frac{c}{2\tilde{\gamma}(s)}\left[\mu_o\sigma + 2s\mu_o\in\right] = \frac{c}{2\tilde{\gamma}^{(0)}\left(s^{(0)}\right)}\left[\mu_o\sigma + 2s\mu_o\in\right]$$

$$= \frac{c^2}{2s^{(0)}}\left[\mu_o\sigma + 2s\mu_o\in\right] = \frac{1}{2s^{(0)}}\left[\frac{\sigma}{\in_o} + 2\in_r s\right] \tag{6.5}$$

$$\frac{\tilde{Z}(s)}{Z_o}\frac{ds^{(0)}}{ds} = \frac{s\mu_o}{Z_o\tilde{\gamma}(s)}\frac{ds^{(0)}}{ds} = \frac{s}{c\tilde{\gamma}(s)}\frac{ds^{(0)}}{ds}$$

$$= \frac{s}{2\tilde{\gamma}(s)}\left[\mu_o\sigma + 2s\mu_o\in\right] = \frac{\sigma + 2s\in}{2\sigma + 2s\in}$$

$$= \frac{1 + st_1}{2 + st_1}$$

which is a meromorphic function of s. Using (6.4) then the coupling coefficients in the lossy medium scale as

$$\eta_\alpha\left(\vec{1}_1, \vec{1}_p\right) = \frac{2 + s_\alpha t_1}{1 + s_\alpha t_1}\,\eta_\alpha^{(0)}\left(\vec{1}_1, \vec{1}_p\right) \tag{6.6}$$

Note that if the conductivity goes to zero (giving a lossless and dispersionless) dielectric medium then

$$\eta_\alpha\left(\vec{1}_1, \vec{1}_p\right)\Big|_{\sigma=0} = \eta_\alpha^{(0)}\left(\vec{1}_1, \vec{1}_p\right)$$

$$s_\alpha\Big|_{\sigma=0} = \in_r^{-\frac{1}{2}}\, s_\alpha^{(0)} \tag{6.7}$$

which, along with unchanged natural modes, is rather simple.

For far scattering we have in class 1 form [13]

$$\tilde{\tilde{\Lambda}}(\vec{1}_r,\vec{1}_1;s) = \sum_{\alpha} \frac{e^{-(s-s_\alpha)t_i}}{s-s_\alpha} W_\alpha \;\; \tilde{C}_{r_\alpha}(\vec{1}_r)\; \tilde{C}_\alpha(\vec{1}_1)$$

+ possible entire function

+ possible branch contribution

$$\tilde{C}_{r_\alpha}(\vec{1}_r) = \left\langle \vec{1}_r e^{\gamma_\alpha \vec{1}_r\cdot\vec{r}_s'} ; \vec{j}_{s_\alpha}(\vec{r}_s') \right\rangle$$

$$\tilde{C}_\alpha(\vec{1}_1) = \left\langle \vec{1}_1 e^{-\gamma_\alpha \vec{1}_1\cdot\vec{r}_s'} ; \vec{j}_{s_\alpha}(\vec{r}_s') \right\rangle \tag{6.8}$$

$$W_\alpha = -s_\alpha \mu_o \left\langle \vec{j}_{s_\alpha}(\vec{r}_s) ; \frac{\partial}{\partial s} \tilde{\tilde{Z}}_t(\vec{r}_s,\vec{r}_s';s)\Big|_{s=s_\alpha} ; \vec{j}_{s_\alpha}(\vec{r}_s') \right\rangle^{-1}$$

Comparing to free space conditions

$$\tilde{\tilde{\Lambda}}^{(0)}(\vec{1}_r,\vec{1}_1;s) = \sum_{\alpha} \frac{e^{-\left(s^{(0)}-s_\alpha^{(0)}\right)t_i}}{s-s_\alpha} W_\alpha^{(0)} \;\; \tilde{C}_{r_\alpha}^{(0)}(\vec{1}_r)\; \tilde{C}_\alpha^{(0)}(\vec{1}_1) \tag{6.9}$$

we have (using (6.4))

$$\tilde{C}_{r_\alpha}(\vec{1}_r) = \tilde{C}_{r_\alpha}^{(0)}(\vec{1}_r) \quad , \quad \tilde{C}_\alpha(\vec{1}_1) = C_\alpha^{(0)}(\vec{1}_1)$$

$$\frac{W_\alpha}{W_\alpha^{(0)}} = \frac{s_\alpha}{s_\alpha^{(0)}} \frac{Z_o}{\tilde{Z}(s_\alpha)} \frac{ds}{ds^{(0)}}\Big|_{s=s_\alpha} = \frac{\gamma_\alpha}{\gamma_\alpha^{(0)}} \frac{ds}{ds^{(0)}}\Big|_{s=s_\alpha} \tag{6.10}$$

$$= \frac{ds}{ds^{(0)}}\Big|_{s=s_\alpha} = \frac{s_\alpha^{(0)}}{t_1^{-1} + \epsilon_r \, s_\alpha}$$

as the scaling relations for the pole parameters for far scattering.

VII. Dipole Scattering

Even with the target resonance's all at high frequencies (i.e. $|s_\alpha| >> \sigma / \epsilon$) there is still a scattering contribution near s = 0 due to the branch cut in Section III. In free space s = 0 is not a singularity since the far scattering is proportional to s^2 and is characterized by induced electric and magnetic dipoles. From [4, 9] we have

$$\tilde{\vec{E}}_f^{(0)}\left(\vec{r},s^{(0)}\right) = \frac{e^{-\gamma r}}{4\pi r}\ \mu_o s^{(0)^2}\ \vec{1}_r \cdot \left[-\tilde{\vec{p}}\left(s^{(0)}\right) + \frac{1}{c}\ \vec{1}_r \times \tilde{\vec{m}}\left(s^{(0)}\right)\right] \text{ as } s^{(0)} \to 0 \tag{7.1}$$

where the dipole moments are related to the incident field by polarizability dyads as

$$\begin{aligned} \tilde{\vec{p}}\left(s^{(0)}\right) &= \epsilon_o\ \tilde{P} \cdot \tilde{\vec{E}}^{(inc)}\left(\vec{0},s^{(0)}\right) = E_o\ \tilde{f}\left(s^{(0)}\right)\ \tilde{P} \cdot \vec{1}_p \\ \tilde{\vec{m}}\left(s^{(0)}\right) &= \tilde{M} \cdot \tilde{\vec{H}}^{(inc)}\left(\vec{0},s^{(0)}\right) = \frac{E_o}{Z_o}\ \tilde{f}\left(s^{(0)}\right) \tilde{M} \cdot \left[\vec{1}_1 \times \vec{1}_p\right] \end{aligned} \tag{7.2}$$

These polarizabilities for perfectly conducting objects are frequency independent and are known for various canonical shapes [33]. The scattering dyadic then becomes

$$\tilde{\tilde{\Lambda}}\left(\vec{1}_r,\vec{1}_1;s^{(0)}\right) = \tilde{\gamma}^{(0)^2}\left(s^{(0)}\right)\left[-\vec{1}_r \cdot \tilde{P} \cdot \vec{1}_1 + \vec{1}_r \times \tilde{M} \times \vec{1}_1\right] \text{ as } s^{(0)} \to 0 \tag{7.3}$$

In the conducting dielectric medium we have the scattering dyadic

$$\begin{aligned} \tilde{\tilde{\Lambda}}\left(\vec{1}_r,\vec{1}_1;s\right) &= \tilde{\gamma}^2(s)\left[-\vec{1}_r \cdot \tilde{P} \cdot \vec{1}_1 + \vec{1}_r \times \tilde{M} \times \vec{1}_1\right] \text{ as } s \to 0 \\ &= s\mu_o(\sigma + s\,\epsilon)\left[-\vec{1}_r \cdot \tilde{P} \cdot \vec{1}_1 + \vec{1}_r \times \tilde{M} \times \vec{1}_1\right] \text{ as } s \to 0 \end{aligned} \tag{7.4}$$

this being consistent with the general scaling in (4.7). Note that due to being proportional to $\tilde{\gamma}^2(s)$ there is no branch cut in this expression. In fact, this is an entire function (singularity at ∞). In (3.9) the branch contribution is zero since the frequency function in (7.4) is real for negative real $s = \Omega$. Also note, however, that (7.1) only represents the leading term in dipole scattering. The second term is proportional to $\tilde{\gamma}(s)\ r^{-2}$ in the lossy medium; this near-field term does have a branch cut. The third term is proportional to r^{-3} and again has no branch cut.

While (7.4) represents the low-frequency far scattering, this is still just the leading term. Odd powers in the expansion around $\tilde{\gamma}(s) = 0$ contain a branch contribution. However, suppose that the natural frequencies s_α are all large in the sense that

$$\left| s_\alpha \right| >> \frac{\sigma}{\epsilon} \text{ for all } \alpha \tag{7.5}$$

Then the form in (7.4) will approximate the far scattering over the range of s from 0 to $-\sigma / \epsilon$ in (3.9), thereby giving a negligible branch contribution to the scattering dyadic. So for the case that the first natural frequency is sufficiently large the SEM representation of the scattering dyadic is simplified.

As discussed in Section V there is a branch-cut contribution in the incident wave due to the term $e^{-\tilde{\gamma}(s)z}$. So the low-frequency scattered field includes a branch contribution. If d as in (5.1) represents the distance for propagation of both the incident and scattered field (from transmitter via target to receiver) then one needs to consider the form of

$$\tilde{\vec{E}}_f(\vec{r},s) = \frac{e^{-\tilde{\gamma}(s)d}}{4\pi r} \tilde{\vec{\Lambda}}\left(\vec{1}_r, \vec{1}_1; s\right) \cdot \vec{1}_p \; E_o \; \tilde{g}(s) \tag{7.6}$$

where $\tilde{g}(s)$ represents the incident waveform leaving the transmitter. Note (7.6) is written for plane-wave incidence and is modified somewhat for real transmitting antennas.

So the low-frequency form of the scattering goes like

$$\tilde{\vec{\Lambda}}\left(\vec{1}_r, \vec{1}_1; s\right) \frac{e^{-\tilde{\gamma}(s)d}}{4\pi r} = \frac{\tilde{\gamma}^2(s)}{4\pi r} \, e^{-\tilde{\gamma}(s)d}\left[-\vec{1}_r \cdot \vec{P} \cdot \vec{1}_1 + \vec{1}_r \times \vec{M} \times \vec{1}_1\right] \tag{7.7}$$

in the far-field approximation. However, as $s \to 0$ near fields begin to dominate, eventually proportional to $e^{-\tilde{\gamma}(s)d} \; r^{-3}$.

One can evaluate the time-domain form of (7.7), but at early retarded time this contains first and second derivatives with respect to time of the delta function $\delta(t - t_d)$. (This comes from the entire-function part.) Of course, at the corresponding high frequencies the target does not behave as a dipole, making this early-time portion inappropriate. By taking the results for $e^{-\tilde{\gamma}(s)d}$ in (5.4) and in time domain in (5.8) (with $t_o = t_d$) one has the result by twice differentiating with respect to d.

The radiation from an elementary electric dipole in such a lossy medium has been treated in the literature [20, 21, 22, 28, 31]. The results for a magnetic dipole are dual to those for the electric dipole and thus have the same functional form. As indicated in (7.7) the two kinds of dipole fit together well. Here the dipole moments are induced by the incident field, making but a simple change in the formulas to allow for the propagation of incident as well as scattered fields.

Note that a dipole approximation for the scatterer is valid only for low frequencies, and that the scattering is small (proportional to $\tilde{\gamma}^2(s)$) in this region. Observe that the branch integral in (3.9) is from 0 to $-\sigma / \in$, and that for small conductivity the branch contribution is small. Furthermore for the case that the resonant frequencies have $\left| s_\alpha \right| >> \sigma / \in$ there is a significant separation of the important frequencies in the branch contribution away from the resonances. Under conditions where the target is not too far away from the transmitting and receiving antennas in terms of e-folding distance $z_o = 2/(\sigma Z_\infty)$, the relative significance of the branch will be small. These considerations define what has been termed a high-frequency window [34]. The lower frequencies can also be exploited, but using the diffusion approximation for which the dipole near fields are important [22, 23, 31].

VIII. Concluding Remarks

As discussed in [13] the far-scattering problem also has an entire-function contribution which is relevant for early times. Such early times being related to the transit time across the scatterer, then in the high-frequency approximation as in (2.9) this entire-function part can also be scaled. Basically the time for which this applies is scaled as c/v. A more general treatment would use the asymptotic treatment in [13] together with the general definition of the different SEM terms in terms of contour integrals as in [32]. As in the free-space case, the currents on the perfectly conducting scatterer can be expressed without an entire-function contribution. As discussed in Section V, however, the incident and scattered fields now include a branch-cut term which can considerably distort the propagated waveform.

Now then, we have a general procedure for the SEM representation of scattering from perfectly conducting objects in simple lossy media by a scaling from the free-space case. An important restriction concerns the assumption of a perfectly conducting scatterer, so that the constitutive parameters of the target do not enter into consideration. More general targets (dielectric, lossy, magnetic) need a more general treatment allowing for the fields inside the target (as, for example, resonances inside a dielectric target).

Another restriction concerns the lossy medium which has been assumed uniform and isotropic. Real earth can have layers. If the medium in the immediate vicinity of the target is uniform and isotropic, then the scaling of the natural frequencies as in Section II using the local parameters is an appropriate approximation. However, the propagation to and from the target (and hence the pole residues or coupling coefficients) can be significantly perturbed by the transmission of the incident and scattered waves through the various layers, including reflections at the interfaces.

While the present scaling results are quite general, the case of low medium conductivity is particularly significant for detection, location, and identification of targets. Then the natural frequencies have $\left| s_\alpha \right| >> \sigma / \epsilon$ and the scaling simplifies as in (2.9) which is a dilation and a shift in the complex frequency plane. Furthermore, as in Section V, the incident and scattered fields propagate at such frequencies in an approximately dispersionless manner with an attenuation given by $e^{-\frac{\sigma Z_\infty z}{2}}$ which can be tolerable for distances restricted such that $\frac{\sigma Z_\infty z}{2}$ is of order one or less. This also reduces the significance of the late-time distortion of the incident and scattered fields by the medium.

By suitable selection of the waveform launched by the transmitting antenna, the late-time distortion due to the medium can be compensated to some degree. This can also be allowed for in the processing of the scattered fields in the target-identification algorithms [26].

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