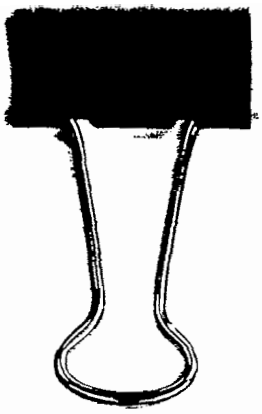

Interaction Notes
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A STATISTICAL THEORY OF ELECTROMAGNETIC FIELDS IN COMPLEX CAVITIES

T.H. LEHMAN, Consultant
Albuquerque, NM

Abstract- A statistical model for electromagnetic field variables in complex cavities is derived from deterministic expressions for the fields. The deterministic expression for the electric field in a cavity with perfectly conducting walls and an arbitrary shape is obtained by expanding the vector potential in a complete set of vector functions which satisfy the wave equation, the divergence condition, the boundary conditions and the orthogonality condition. These results are extended to obtain the steady state expression for the electric field in cavities containing infinitesimal dipole sources whose walls are good conductors. An expression for the eigenvectors is derived and this expression is used to evaluate volume averages of simple functions of the cartesian components of the eigenvectors for cavities whose shapes are sufficiently complex. It is assumed that the position vector in the cavity is a random variable distributed according to a uniform distribution and the equivalence of volume averages and expectation values is demonstrated. The steady state expressions for the electric field are used in conjunction with the calculus of probability theory to derive the statistical models. These models consist of a probability density function and both a spatial and a temporal correlation function. Statistical models for the amplitude of the components of the partial cavity fields, the magnitude of a component of the time averaged electric field, the square of a component of the time average field, the time averaged component of the power density and the total energy density are derived. Potential applications of the theory are discussed.

1.0 INTRODUCTION

Since the turn of the century, statistics has played an important role in understanding the nature of physics. More recently, statistics has played and is playing an increasingly significant role in the understanding and interpretation of large system behavior. This includes the areas of risk assessment, reliability, survivability and vulnerability assessment to name a few. However, in general, statistics has not played a very significant role in areas of technology concerned with the interactions of classical waves with large systems. For these technologies, the major emphasis has been in developing large deterministic numerical codes and large test facilities primarily for the purpose of simulating "worst case" events. As the environments and systems of concern become more complex and/or susceptible, this approach becomes less and less viable. It is not to be implied that these deterministic approaches are without merit. The implication is there exists a general class of problems too complex to address deterministically, i.e., the number of model variations is too large to even consider generating the statistical models using Monte Carlo techniques. One such problem, the characterization of electromagnetic fields in electrically large, complex cavities is addressed in this paper. This problem, of course, is not the only one of concern [1]. However, it potentially could provide a starting point for addressing the complete set of problems.

In principle, the instantaneous variations of classical EM fields in any cavity can be followed in detail, i.e., analyzed deterministically, no matter how intricate the boundary conditions. This is also true for particles in a cavity (box) whose motion is governed by Newtonian mechanics. However, no one attempts to determine the instantaneous position and momentum of each and/or every molecule in a gas. Experience has shown that many of the important questions associated with the motion of gas can be adequately addressed and answered using statistical techniques without a detailed knowledge of the motion of the individual molecules.

Statistics has many faces and it is important to recognize that all statistical problems are not equivalent. For example, randomly drawing colored balls from a box is not equivalent to the "random motion" experienced by molecules in a box. No fundamental laws of nature govern the distribution of the balls in the box, i.e., the distribution is not constrained by the laws of physics. The distribution of colors, for example, can be any pre-determined function. The distribution functions themselves can be visualized as being random and their number is, in principle, unbounded. On the other hand, the motion of the molecules must obey Newtonian mechanics, no matter how "randomized" this motion may be. This establishes constraints on the statistical models. They must be compatible with the laws of physics and the results of model predictions must not violate these laws. As a consequence, only a few

statistical models (defined to be joint probability density functions and associated parameters) are needed to characterize the properties of gases. In particular, for an ideal gas, the velocity distribution is always Maxwellian, independent of the shape of the box. In addition, the measurable parameters (temperature, pressure and volume) are related through the ideal gas law.

The above properties of the statistical models and parameters have helped to guarantee the success of statistical mechanics. The simple models for the idealized cases provide the foundation for the statistical analysis of more complex problems. If, for example, the statistical models for the idealized gas were dependent on the shape of the box (cavity), statistical mechanics would be a more complicated branch of physics. Many more problems would need to be addressed starting from first principals. It is these properties --simplicity, robustness and the connection with deterministic classical physics-- which must be preserved in the development of a viable statistical theory associated with electromagnetic fields. Experimental evidence strongly supports the existence of such a theory.

Statistical characterizations of electromagnetic fields and their interactions are not new. Statistics is used extensively in the theory of partial coherence, scattering from rough or random surfaces and the propagation of EM fields through random media to name a few. In addition, statistical techniques have and are being used to support the characterization of the radar cross section of large complex objects [2]. However, the effects of electromagnetic fields on the operation of electronic equipment in large systems are of primary interest in this paper.

Historically, applications of statistical techniques to large system interactions for the purpose of quantifying electromagnetic effects (EME) have not been very successful. Empirical statistics are typically used to construct the statistical models from very sparse data sets. Almost without exception, the results of these empirical efforts are inconclusive and it is usually assumed that the electromagnetic variables (power in watts, current in amps, voltage in volts) are distributed according to log-normal distributions. Consequently, predictions of the effects of electromagnetic environments on large systems often contain large errors, i.e., usually it is predicted that significant detrimental effects occur for nominal values of induced voltages and currents well below the operating levels of the system electronics. The source of this error can be attributed to the statistical models, i.e., the log-normal distributions. While these distributions seem to be adequate for characterizing the variable near the mean of the distribution, they are inadequate for characterizing rare or extreme events. Typically, the log-normal distributions predict with a high probability of occurrence, the existence of induced currents and voltages that are well above levels allowed by the physics of the interaction.

Most physical variables have upper or lower limits and sometimes both. It is for this reason that the use of truncated distributions is often recommended. However, this approach only serves to transform the problem from one of finding the correct distribution to one of finding the correct values for the "cut-offs". The solution to the second problem is often equally as difficult or more difficult than the solution to the first. In addition, this approach does not address the equally important problem, but usually more difficult one, of finding the correlation functions for the complete set of random variables. An alternative to empirical statistics is the derivation of the statistical models from physical models that characterize the process or interaction of interest. This method is sometimes referred to as "objective Bayesian statistics" and it is used exclusively in this paper.

The techniques of objective Bayesian statistics are not new and the method has often been suggested as a possible alternative to empirical statistics. Specifically, for applications of statistics to the interaction and coupling of electromagnetic fields to the electronics of complex systems, the use of the techniques of objective Bayesian statistics are proposed in [3] and [4]. In [3], a possible technique for the statistical analysis of load excitations on an unshielded N-wire random cable illuminated by an incident time-harmonic field using the reciprocity theorem in conjunction with a subset representation of a statistical ensemble is discussed. In [4], the continuous wave coupling to a complicated electrical system in terms of random small dipole interactions in the low frequency limit is analyzed. Both random coupling to the incident field and random interactions between dipoles are considered in this paper. In both of these papers, the starting point for the statistical analysis is a deterministic model of the relevant physical interactions.

In [5], questions regarding the analysis and use of less than perfect test data are addressed. Although no statistical models are explicitly developed in this paper, probability theoretic techniques are used in conjunction with physical considerations to address some of these questions. The issues raised in this paper are of great importance and are usually ignored. Since most large system data sets (both experimental and numerical) are imperfect, techniques are needed for extracting the information needed by the ultimate users, i.e. decisionmakers, designers, etc. Methods based on the techniques of Bayesian statistics may help resolve some of these issues.

The deterministic characterization of the EM cavity fields is an integral part of the formulation of the statistical model presented in this paper. The literature on the deterministic analysis of cavities is extensive and it varies from analytical characterizations of canonical shaped cavities to the numerical analysis of relatively simple cavity configurations (for electrically large cavities), i.e. cavities containing only a few important system elements and

features. The details of these deterministic characterizations are not discussed here. Only the general properties of rectangular cavities are pertinent to the development of the statistical EM model.

The history of the statistical analysis of EM cavity fields is not nearly as extensive nor as diverse as its deterministic counterpart. In 1963, it was shown [6] that the distribution of the components of the electric field vector $\boldsymbol{E(r,t)}$ in a rectangular cavity is gaussian (normal) for frequencies in the optical region of the EM spectrum. Quantum electrodynamics techniques are used in this derivation.

In 1968, a new technique [7] was suggested for making EM measurements in shielded enclosures at microwave frequencies. The shapes of these shielded, rectangular enclosures are made to be complex by the addition of an electrically large irregular shaped, metallic object (a stirrer) to the interior of the cavity. This test method is referred to as mode-stirred chamber testing or reverberation chamber testing [8]. Currently, this test method is used in EMI/EMC testing. The results [8] of measurements performed in these complex cavities indicate that variations in normalized average values are not significant from one frequency to another and from one chamber design to another over a wide range of test conditions and cavity shapes. These results suggest the existence of a model and/or theory that is valid for a large class of complex cavity shapes and frequencies.

A result [9] that significantly impacted the development of the statistical models for cavities was reported in 1988. In this paper, empirical statistics (goodness-of-fit tests) are used to demonstrate that for a wide class of complex cavities (Q, frequency, shape and volume), the power as measured by a uni-directional probe is distributed according to a chi-squared distribution with two degrees of freedom (an exponential distribution). A theoretical derivation for the observed distribution using a modal expansion of the cavity fields is also presented. The results of this derivation are inconclusive because the mode expansion is not complete and neither the complexity of the cavity nor the effects of a source on the distribution is incorporated into the derivation.

In 1991, a statistical model [10] for mode-stirred chambers was presented. This statistical model was derived by assuming that a large number of modes is excited in the cavity and that the mode amplitudes are random variables. Using the central limit theorem, it is argued that both the in-phase and the quadrature contributions to a component of the measured fields are normally distributed. It follows that the amplitude of a field component is distributed according to a Rayleigh distribution and the square of the field amplitude (power) is distributed according to an exponential distribution. These distributions were verified using chi-squared goodness-of-fit tests. While this simple model is adequate for

deriving the distributions for an over-moded cavity, it does not provide the complete statistics, i.e., the joint distribution functions including correlation coefficients. Also, it provides little insight into establishing the range of validity of the assumptions used in the derivation.

In this paper, a statistical model for EM fields in complex cavities is derived. The derivation is presented in two parts. The first part results in deterministic expressions for the fields and the volume averages of field variables in arbitrarily shaped and electrically large complex cavities. The details are presented in Section 2.0. The second part results in a statistical model based on the deterministic expressions and is presented in Section 3.0.

The deterministic derivation begins by treating cavities in the usual fashion. The steady state fields for a cavity with finite Q and an infinitesimal dipole source are expanded in a complete set of orthogonal vector functions (eigenvectors). Next, a generalized expression for the eigenvectors of an arbitrarily shaped cavity is derived. This generalized expression for the eigenvectors is used to obtain volume averages of simple functions of the eigenvector components in cartesian coordinates. A definition of a complex cavity is used to perform this volume averaging.

The concept of a complex cavity is central to the evaluation of the volume averages. The irregularity provided by the walls of a complex cavity is needed to insure that the coherent contribution to the field intensities in the cavity are small. In this formulation it is assumed that this contribution to the volume averages is vanishingly small and as a consequence, only the contribution from the incoherent part of the fields is retained. In the limit of randomly shaped cavity walls (with respect to wavelength dimensions), the cavity fields can be visualized as resulting from the superposition of a large number of photons. In this case, the location of each photon in the cavity as well as its propagation direction are random.

The derivation of the statistical model is straight forward. It is shown that the volume averages are equivalent to expectation values of random variables when the position vector in the cavity is assumed to be a random variable with a uniform density function. The moment theorem of probability theory is used to show that the cartesian components of the eigenvectors for a complex cavity can be treated as random variables distributed according to a normal distribution. Using the calculus of probability theory, the deterministic expressions for the fields in the cavity are used to derive the statistical model.

Finally, applications of the statistical models are considered. In particular, the applicability of the model to the analysis of reverberation chamber testing and to RF system survivability/vulnerability assessment is discussed. A brief description of a few elementary concepts of probability theory is included as an appendix.

2.0 DETERMINISTIC CHARACTERIZATION OF FIELDS IN COMPLEX CAVITIES

Some of the cavity development presented in this formulation is not new and can be found elsewhere. It is included for completeness and closely follows the development given in [11]. An ideal, arbitrarily shaped cavity (perfectly conducting walls) is considered first. The usual assumption that the electric field $E(r,t)$ and the vector potential $A(r,t)$ can be expanded in a complete set of vector functions is made. The general properties of these vector functions are established. Next, the results of the ideal cavity are used to find expressions for the steady state electric field in a real cavity containing an infinitesimal dipole source. These expressions are in terms of the source parameters , the Q of the cavity, the number of modes excited in the cavity and the vector functions (eigenvectors) of the ideal cavity. Finally, approximate analytical expressions for the eigenvectors of an arbitrarily shaped cavity are derived. It would require considerable effort to evaluate these expressions numerically. However, it is shown that they are adequate for obtaining the volume averages of field variables when the cavity shape is complex and the cavity dimensions are large compared to the wavelength.

2.1 The Ideal Cavity

Consider the source free cavity of Figure 1. It is represented as a finite volume V bounded by an arbitrarily shaped, perfectly conducting surface S. The only restriction is that the surface is closed, although certain non-simply connected, pathological shapes must be excluded from consideration [9].

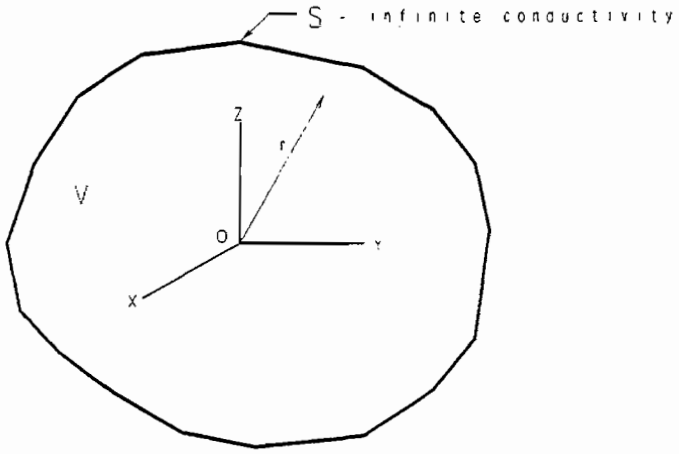

Figure 1. An ideal cavity with arbitrary shape.

It is assumed that the electric field vector and the vector potential in the cavity can be expanded in a complete set of vector functions $u_k(r)$ where r is the position vector in the cavity or

$$E(r,t) = \sum_k u_k(r)E_k(t) \tag{1}$$

and

$$A(r,t) = \sum_k u_k(r)A_k(t). \tag{2}$$

From Maxwell's equations, it follows that the cartesian components (cartesian coordinates are used exclusively) of the vector functions $u_k(r)$ satisfy the second-order wave equation

$$\nabla^2 u_{ks} + \frac{\omega_k^2}{c^2} u_{ks} = 0, \qquad s=x,y,z \tag{3}$$

and satisfy the divergence condition

$$\nabla \cdot u_k = 0 \tag{4}$$

where ω_k is an eigenfrequency (resonant frequency) of the cavity and c is the speed of light. Since the tangential components of the electric field and the vector potential vanish at the surface of a perfect conductor, the boundary condition for the vector functions is

$$n \times u_k(r)|_{r\in S} = 0 \tag{5}$$

where n is a unit vector normal to the surface of the walls of the cavity.

Using (3-5) and with the help of Green's theorem [11], it can be shown that any two of these functions $u_k(r)$ and $u_{k'}(r)$ corresponding to two different eigenfrequencies ω_k and $\omega_{k'}$ are orthogonal to each other or

$$\int_V u(r)_k \cdot u_{k'}(r)dV = \delta_{kk'} \tag{6}$$

and $u_k(r)$ is called an eigenvector. Also, it can be shown [12] that for an arbitrarily shaped cavity with perfectly conducting wall and dimensions large compared to a wavelength, the mode density $dM/d\omega$ is

$$\frac{dM}{d\omega}d\omega = \frac{V\omega^2}{\pi^2 c^3}d\omega \tag{7}$$

where dM is the number of allowed modes in the angular frequency interval ω to $\omega+d\omega$.

The above properties of an ideal cavity are next used to develop the expression for the field in a cavity having a finite Q and containing a source (a real cavity).

2.2 The Real Cavity

In the preceding formulation, the cavity walls are assumed to be perfectly conducting. Sources are excluded because in the absence of damping, the mode amplitudes grow without bound. If sources are included then the effects of dissipative forces must also be included. In this development only linear ohmic losses (Joule heating) are considered.

In a strict sense, the formulation for cavities with perfectly conducting walls is not applicable to cavities containing lossy materials (real cavities). A self-consistent solution for the response of real cavities requires the total fields to be expanded in a complete set of natural modes where each natural mode has a unique natural frequency. The natural modes correspond to field distributions which are supported in the cavity in the absence of sources. In this formulation, both the natural modes and the natural frequencies are complex variables. The real part of the complex frequency insures that the undriven contributions to the fields vanish after the source has been turned off. A self-consistent solution of real cavity responses is, in general, very difficult and will not be attempted. Instead, an approximate, phenomenological model is used.

To extend the above results of the ideal cavity to real cavities, the walls of the real cavities are assumed to be good conductors. For this case, the eigenvectors of the ideal cavity will be approximately equal to the natural modes of the ideal cavity and the expansions (1) and (2) for the electric field and vector potential are approximately valid. (In structural dynamics applications, this approximation provides satisfactory results except for those cases where the structure exhibits heavy nonproportional damping.) With this approximation, the functions defined by the terminology "eigenvector" and "natural mode" are synonymous. To avoid confusion, the term eigenvector will be used throughout the remainder of this paper to denote both. It should be noted that ω_k is not the complex part of the natural frequency. ω_k is the free oscillation frequency (resonant frequency) of the ideal cavity and it is used in this context in the remainder of this paper.

With the above approximation, loss must be treated as a phenomenological effect. A modal damping coefficients γ_k is incorporated into the equations of motion for the modal amplitudes to account for the loss.

Ignoring for the moment the effects of the loss mechanism and substituting (2) into (4) yields

$$\nabla \cdot A(r,t) = 0 \tag{8}$$

everywhere in the cavity. This condition characterizes the Coulomb gauge. In this gauge, the vector and scalar potentials satisfy the following equations [13]:

$$\left(\nabla^2 - \frac{1}{c^2}\right)A = -\mu_0 j + \frac{1}{c}\nabla\left(\frac{\partial \phi}{\partial t}\right) \tag{9}$$

and

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0} \tag{10}$$

where $j(r,t)$ and $\rho(r,t)$ are the current and charge densities produced by electromotive forces external to the cavity. Substituting (2) into (9) and using (3) yields

$$\sum_k \left[\frac{\partial^2 A_k(t)}{\partial t^2} + \omega_k^2 A_k(t)\right] u_k = \frac{1}{\epsilon_0} j - c\nabla\left(\frac{\partial \phi}{\partial t}\right). \tag{11}$$

Taking the innerproduct of (11) with u_ν, integrating over the volume of the cavity and using the orthogonality condition (6), the differential equation for $A_k(t)$ follows

$$\frac{d^2 A_k(t)}{dt^2} + \omega_k^2 A_k(t) = F_k(t) - G_k(t) \tag{12}$$

where

$$F_k(t) = \frac{1}{\epsilon_0}\int_V j(r,t) \cdot u_k(r) dV \tag{13}$$

and

$$G_k(t) = c\int_V \nabla\left(\frac{\partial \phi}{\partial t}\right) \cdot u_k(r) dV. \tag{14}$$

$G_k(t)$ can be shown to vanish with the help of Gauss's theorem and the boundary condition (5). For a linear ohmic loss mechanism, the damping term is proportional to the time

derivative of $A_k(t)$ and including it into (12) yields the equation of motion for a driven oscillator with damping or

$$\frac{d^2A_k(t)}{dt^2} + \gamma_k\frac{dA_k(t)}{dt} + \omega_k^2A_k(t) = F_k(t) \tag{15}$$

where γ_k, as before, is defined to be the modal damping coefficient for the kth mode.

Now consider the cavity of Figure 2. It contains a sinusoidal dipole source located at the position $\boldsymbol{r}_0$ directed along the x-axis (the direction is arbitrary). It is assigned an amplitude I_0 and an infinitesimal length $\Delta\ell$ A mathematical representation of the current density for this source is

$$\boldsymbol{j}(\boldsymbol{r},t) = \boldsymbol{e}_x\Delta\ell I_0\delta(\boldsymbol{r}-\boldsymbol{r}_0)\sin\omega t \tag{16}$$

where $\delta(\boldsymbol{r}-\boldsymbol{r}_0)$ is the three dimensional Dirac delta function and $\boldsymbol{e}_x$ is a unit vector in the x-direction.

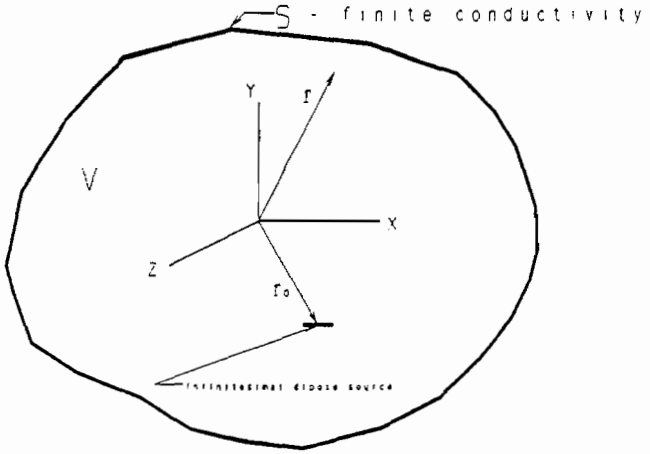

Figure 2. A real cavity with a source and arbitrary shape.

Substituting (16) into (13), the steady state solution of (15) is given in [14] as

$$A_k(t) = \frac{\Delta\ell I_0}{\epsilon_0}\alpha_k u_{kx}(r_0)\sin(\omega t - \phi_k) \tag{17}$$

where $u_{kx}(r_0)$ is the x-component of the eigenvector evaluated at the location of the source,

$$\alpha_k = \frac{1}{\sqrt{\gamma_k^2\omega^2 + (\omega^2 - \omega_k^2)^2}} \tag{18}$$

and

$$\tan\phi_k = \frac{\gamma_k\omega}{(\omega_k^2 - \omega^2)}. \tag{19}$$

Defining the cavity Q to be the number of cycles required for the wave amplitude to be reduced by a factor of e^{-1} and using the natural frequency $s_k = -\gamma_k/2 \pm j(\omega_k^2+\gamma_k^2/4)^{1/2}$ of the damped oscillator defined by (15), it follows that

$$\gamma_k^2 = \frac{4\omega_k^2}{1 + 4\pi^2 Q_k^2} \tag{20}$$

where Q_k is the Q of the kth mode. For reasonable values of Q, $\gamma_k \approx \omega_k/\pi Q_k$, the expressions for α_k and ϕ_k become

$$\alpha_k = \left[\frac{\omega^2\omega_k^2}{\pi^2 Q_k^2} + \left(\omega^2 - \omega_k^2\right)^2\right]^{-1/2} \tag{21}$$

and

$$\tan\phi_k = \frac{\omega_k\omega}{\pi Q_k(\omega_k^2 - \omega^2)}. \tag{22}$$

In the Coulomb gauge, the propagating electric field is given by $E(r,t) = -\partial A(r,t)/\partial t$ and the expression for the electric field is

$$E(r,t) = -\frac{\Delta\ell I_0\omega}{\epsilon_0}\sum_k \alpha_k u_{kx}(r_0)\, u_k(r,t)\cos(\omega t - \phi_k). \tag{23}$$

In principle, the above expression for the electric field in the cavity can be evaluated deterministically once all of the eigenvectors for the cavity are known.

2.3 A Generalized Expression for the Eigenvectors of an Arbitrarily Shaped Cavity

Analytical solutions for the eigenvectors exist for only a few cavities with canonical shapes. The only other alternative is to use a numerical technique. As was implied earlier, the numerical solutions are not satisfactory. Therefore, some other approach is required.

To derive the generalized expressions for the eigenvectors, solutions to the vector Helmholtz equation for cavities with canonical shapes are examined. The properties of these solutions are used to assume a form for the eigenvectors for cavities with arbitrary shapes. Using this assumed form, a method for constructing the generalized expressions is developed. Arguments based on numerical solution techniques are used to establish the properties of the parameters in the assumed form of the solution.

2.3.1 Assumed Form of the Solution

It is well-known [14] that the eigenvectors for a rectangular cavity can be expressed as the sum of eight plane waves or as will be used in this derivation the sum of four standing waves (this keeps the formulation real) so that

$$\boldsymbol{u}_k(\boldsymbol{r}) = \sum_{i=1}^{4} \boldsymbol{U}_i \cos \boldsymbol{K}_i \cdot \boldsymbol{r} \tag{24}$$

where $|\boldsymbol{K}_i| = k$ for all i. Here, k is the cavity wave number $(k = \omega_k/c)$ and is specified by three integers, l,m,p. Using the plane wave expansions for the cylindrical Bessel functions, similar expressions follow for the eigenvectors of a right cylindrical cavity except that the resulting expressions are in terms of a sum and an integral. This suggests that the eigenvectors for any cavity can be expressed as the sum of a large number N of standing waves or

$$\boldsymbol{u}_k(\boldsymbol{r}) = \sum_{n=1}^{4N} \boldsymbol{U}_n \cos(\boldsymbol{K}_n \cdot \boldsymbol{r} + \Psi_n) \tag{25}$$

where $|\boldsymbol{K}_n| = k$ for all n as was the case for the rectangular cavity, Ψ_n is a phase factor and N is one for a rectangular cavity but in general, is a very large number for most cavity shapes (infinite). For a rectangular cavity whose origin is one of the corners of the cavity, the phase angle is zero and the wave vectors $\boldsymbol{K}_n$ are functions of the dimensions L_x, L_y, L_z of the cavity. For a cavity with arbitrary shape, it is expected that the wave vectors $\boldsymbol{K}_n$ will, in some sense, reflect all the dimensions of the cavity.

It should be noted that the cosine functions of (25) do not form a complete set and that each of them do not satisfy the boundary conditions except for the case of a rectangular

boundary. In order for (25) to be a valid expression for the eigenvectors, it must be shown under what conditions (25) satisfies (3)-(6).

It is easy to show that (25) satisfies the wave equation (3) without any restrictions. Imposing the divergence condition (4) on (25) results in

$$\sum_{n=1}^{4N} \boldsymbol{K}_n \cdot \boldsymbol{U}_n \cos(\boldsymbol{K}_n \cdot \boldsymbol{r} + \Psi_n) = 0. \tag{26}$$

The divergence condition is satisfied if the amplitude vectors $\boldsymbol{U}_n$ are chosen such that

$$\boldsymbol{K}_n \cdot \boldsymbol{U}_n = 0 \tag{27}$$

for all n. However, there exists another set of vectors $\boldsymbol{W}_n$ such that

$$\boldsymbol{W}_n \cdot \boldsymbol{U}_n = 0 \qquad \text{and} \qquad \boldsymbol{W}_n \cdot \boldsymbol{K}_n = 0 \tag{28}$$

for all n. Therefore, if $\boldsymbol{u}_k$ is an eigenvector, then

$$\boldsymbol{w}_n(\boldsymbol{r}) = \sum_{n=1}^{4N} \boldsymbol{W}_n \cos(\boldsymbol{K}_n \cdot \boldsymbol{r} + \Psi_n) \tag{29}$$

is also an eigenvector. These two eigenvectors, in some sense, must correspond to the two possible polarizations for transverse waves. To complete this demonstration, it is necessary to show that the eigenvectors satisfy the boundary conditions and the orthogonality condition. With the eigenvectors in the above general form, this demonstration is not possible.

2.3.2 Solution by Construction

While it can be argued that the vector coefficients and the phases can be chosen so that (25) and (29) satisfy both the boundary conditions (5) and the orthogonality condition (6), the solutions (26) and (29) are too general to be of much value. To put the above expressions in a more useable form, the following prescription for constructing solutions for the eigenvectors of arbitrarily shaped cavities is introduced. Using these constructed solutions, it is shown that the amplitude vectors and the phases are, in principle, known. The prescription for construction is as follows:

(a) It is assumed that all the wave numbers k for the eigenvalues ω_k are known. These could have been determined by using some numerical technique such as the finite element method. This results in a discrete set of values for k or $k = k_1, k_2, k_3, \ldots$. In the following, the subscript on k is dropped and it must be remembered that k is not a continuous variable.

(b) The cavity volume is discretized into a large number N of identical cubic cells. Each cell

has a volume $\Delta V=(\Delta x)^3$ such that $k\Delta x \ll 1$ and $V \approx N\Delta V$.

(c) Each small cube in the discretized volume of the cavity is used to construct a much larger rectangular cavity, i.e., a cavity with a volume on the order of the original cavity. Since these large rectangular cavities do not really exist, they are referred to as *virtual* cavities.

(d) With reference to Figure 3, the *n*th *virtual* cavity is constructed as follows: First, each of the six surface elements of the *n*th cubic cell is projected onto the walls of the real cavity. This projection defines the six surface elements located on the walls of the real cavity as shown in Figure 3. Remember that the cavity volume has been discretized and the cavity walls are no longer continuous but are comprised of a large number of small square plates. Each plate has a surface area $\Delta S = (\Delta x)^2$. Two of the projected surface elements lie in x-y plane, two in the x-z plane and two in the y-z plane. Now all six of the small surface elements are projected to $\pm\infty$ in both directions. The intersection of this set of six planes forms the large rectangular *virtual* cavity shown Figure 3. The dimensions of this *virtual* cavity are X_n, Y_n, Z_n and the position vector to any one of the corners of the *virtual* cavity with respect to the origin O of the coordinate system is denoted by $\boldsymbol{r}_n$. The set of *virtual* cavity parameters are all deterministic (since in principal they are known by construction) but they are not unique since the orientation of the coordinate system is arbitrary. Note that the set of *N virtual* cavities defines all of the dimensions of the real cavity within the accuracy of the

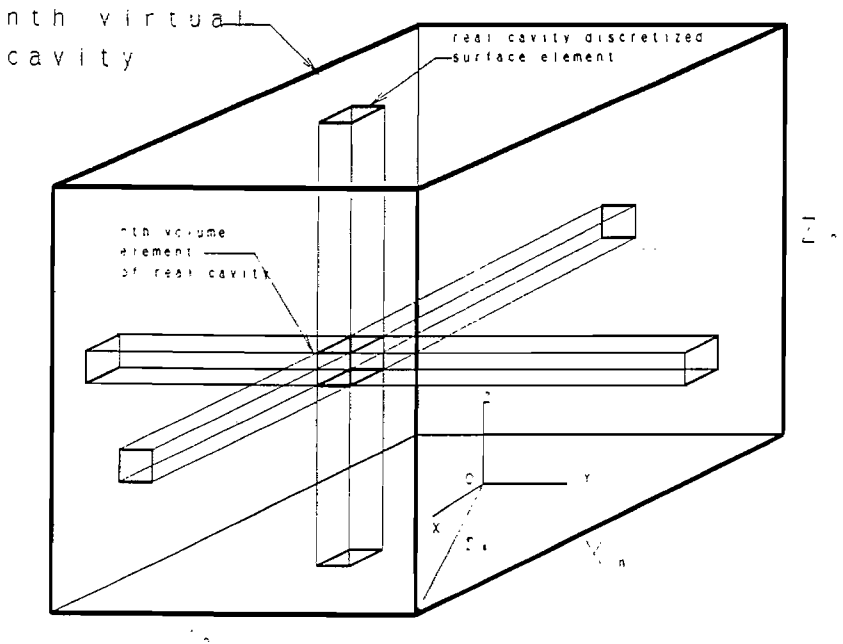

Figure 3. The *n*th *virtual* cavity construct.

spatial discretization scheme.

(e) Now it is assumed that the walls of each of the *virtual* cavities are perfectly conducting. Then the set of all eigenvectors, $\boldsymbol{u}_{\kappa(n)}(\boldsymbol{r})$ and $\boldsymbol{w}_{\kappa(n)}(\boldsymbol{r})$ for the nth *virtual* cavity are

$$\boldsymbol{u}_{\kappa(n)}(\boldsymbol{r}) = \sum_{i=1}^{4} \boldsymbol{U}_{n,i} \cos[\mathbf{K}_{n,i} \cdot (\boldsymbol{r} - \boldsymbol{r}_n)] \tag{30}$$

and

$$\boldsymbol{w}_{\kappa(n)}(\boldsymbol{r}) = \sum_{i=1}^{4} \boldsymbol{W}_{n,i} \cos[\mathbf{K}_{n,i} \cdot (\boldsymbol{r} - \boldsymbol{r}_n)] \tag{31}$$

where

$$\kappa_x(n) = \frac{\pi l}{X_n}, \qquad \kappa_y(n) = \frac{\pi m}{Y_n}, \qquad \kappa_z(n) = \frac{\pi p}{Z_n}, \tag{32}$$

l,m,p are positive integers, X_n, Y_n, Z_n are the dimensions of the nth *virtual* cavity as before and $\boldsymbol{r}_n$ is the position vector to one of the corners of the nth *virtual* cavity (Figure 3). For this formulation, the four wave vectors are chosen so that they always lie in the upper half-plane of k-space, i.e., their z-component is always positive or

$$\begin{aligned} \mathbf{K}_{n,1} &= (\kappa_x(n), \kappa_y(n), \kappa_z(n)) \\ \mathbf{K}_{n,2} &= (\kappa_x(n), -\kappa_y(n), \kappa_z(n)) \\ \mathbf{K}_{n,3} &= (-\kappa_x(n), \kappa_y(n), \kappa_z(n)) \\ \mathbf{K}_{n,4} &= (-\kappa_x(n), -\kappa_y(n), \kappa_z(n)). \end{aligned} \tag{33}$$

The allowed frequencies are

$$\frac{\omega_\kappa^2}{c^2} = \kappa^2(n) = \pi^2 \left(\frac{l^2}{X_n^2} + \frac{m^2}{Y_n^2} + \frac{p^2}{Z_n^2} \right) \tag{34}$$

and it follows from the divergence condition (4) that

$$\mathbf{K}_{n,i} \cdot \boldsymbol{U}_{n,i} = 0, \qquad \mathbf{K}_{n,i} \cdot \boldsymbol{W}_{n,i} = 0, \qquad \boldsymbol{U}_{n,i} \cdot \boldsymbol{W}_{n,i} = 0 \tag{35}$$

for all n and i.

(f) The final step in the prescription for deriving a generalized expression for the

eigenvectors of an arbitrarily shaped cavity is to chose one eigenvector from the infinite set of eigenvectors for each *virtual* cavity. The criteria for selecting this virtual cavity eigenvector is to simply choose the one whose eigenvalue is closest in magnitude to the eigenvalue k of interest for the real cavity or

$$\kappa^2(n;l_k,m_k,p_k) \approx k^2 \tag{36}$$

where l_k,m_k,p_k are the values of the three quantum numbers for the nth *virtual* cavity which provide the best approximation. The real cavity eigenvector for the eigenvalue k is written as the superposition of all the selected *virtual* cavity eigenvectors or

$$\begin{aligned} u_k(\boldsymbol{r}) &= \sum_{n=1}^{N} u_{\kappa(n)}(\boldsymbol{r})\big|_{\kappa^2(n)\approx k^2} \\ &= \sum_{n=1}^{N}\sum_{i=1}^{4} U_{n,i}\cos[\mathbf{K}_{n,i}\cdot(\boldsymbol{r}-\boldsymbol{r}_n)]\big|_{\mathbf{K}_{n,i}^2=k^2}. \end{aligned} \tag{37}$$

where it is assumed that the *virtual* cavity eigenvectors are valid throughout the volume of the real cavity. For convenience the i subscript is assumed and the greek notation is dropped. The expressions for the eigenvectors become

$$u_k(\boldsymbol{r}) = \sum_{n=1}^{N} U_n\cos[\boldsymbol{K}_n\cdot(\boldsymbol{r}-\boldsymbol{r}_n)] \tag{38}$$

and

$$w_k(\boldsymbol{r}) = \sum_{n=1}^{N} W_n\cos[\boldsymbol{K}_n\cdot(\boldsymbol{r}-\boldsymbol{r}_n)] \tag{39}$$

where the magnitude of the wave vectors are equal or

$$k^2 = \boldsymbol{K}_n^2 = \boldsymbol{K}_m^2 \tag{40}$$

for all n and m, but, in general, their directions are not the same or

$$\boldsymbol{K}_n \neq \boldsymbol{K}_m \tag{41}$$

for $n \neq m$. It should be noted that the set of wave vectors $\boldsymbol{K}_n$ contains all of the dimensions of the cavity.

Since the eigenvalues k, the dimensions of the virtual cavities and their location in space are all known, the only parameters yet to be determined are the vector amplitude coefficients U_n and W_n. These must be determined from the boundary conditions and the orthogonality condition.

2.3.3 Boundary Condition Considerations

The three vectors K_n, U_n and W_n form an orthogonal set. It is convenient to express their components in terms of the three Eulerian angles θ_n, ϕ_n and ψ_n [13]. In terms of these angles, the components of K_n are

$$\begin{aligned} K_{nx} &= k\sin\theta_n\sin\phi_n \\ K_{ny} &= -k\sin\theta_n\cos\phi_n \\ K_{nz} &= k\cos\theta_n \end{aligned} \tag{42}$$

the components of U_n are

$$\begin{aligned} U_{nx} &= U_n(\cos\psi_n\cos\phi_n - \sin\psi_n\sin\phi_n\cos\theta_n) \\ U_{ny} &= U_n(\cos\psi_n\sin\phi_n - \sin\psi_n\cos\phi_n\cos\theta_n) \\ U_{nz} &= U_n\sin\psi_n\sin\theta_n \end{aligned} \tag{43}$$

and the components of W_n are

$$\begin{aligned} W_{nx} &= W_n(-\sin\psi_n\cos\phi_n - \cos\psi_n\sin\phi_n\cos\theta_n) \\ W_{ny} &= W_n(-\sin\psi_n\sin\phi_n + \cos\psi_n\cos\phi_n\cos\theta_n) \\ W_{nz} &= W_n\cos\psi_n\sin\theta_n. \end{aligned} \tag{44}$$

Both ϕ_n and θ_n are known, since all of the wave vectors K_n are known by construction and the eigenvalue analysis or

$$\phi_n = \tan^{-1}\left[\frac{K_{nx}}{-K_{ny}}\right] \tag{45}$$

and

$$\theta_n = \cos^{-1}\left[\frac{K_{nz}}{k}\right]. \tag{46}$$

This leaves the values of U_n, W_n and ψ_n to be fixed by the boundary conditions and the

orthogonality condition. However, only N unknowns can be fixed by the boundary conditions and only one can be fixed by the orthogonality condition. Therefore, U_n and W_n are chosen to be the arbitrary unknowns. Their values are chosen to be $U_n = U$ and $W_n = W$ for all n. Since both U and W are fixed by the same normalization condition, it follows that $W = U$.

For all cavities of interest, the number N of cavity volume elements ΔV will be much greater than the number N' of cavity surface elements ΔS. If point matching at the center of each surface element is used to solve for the unknowns, there will be two equations for each surface elements. However, the number $N'' = N\text{-}2N'$ will still be very large. This implies that for a unique solution to exist, N'' equations of constraint are required. These equations of constraint result from the requirement that the eigenvectors $\boldsymbol{u}_n(\boldsymbol{r})$ must be invariant under rotations about the origin, i.e., the orientation of the cavity coordinate system is arbitrary. Since the cavity space is discrete, there will not be an infinite number of allowed rotations. For a sphere, one can show the number of allowed rotations is on the order of N.

The exact values of the ψ_n are not needed for the derivation of the statistical model. However, they are unique and can be determined by forcing the eigenvectors expressions (38) and (39) to satisfy the boundary condition (5) at the center of each of the surface elements. The value of U is yet to be determined and it must be shown under what conditions the expressions for the eigenvectors can be made to satisfy the orthogonality condition.

2.3.4 Orthogonality Condition Considerations -The Complex Cavity Definition

It is not possible to evaluate the coefficient U using (38) and (39) nor is it possible to show, in general, that the two eigenvectors satisfy the orthogonality condition (6), except for rectangular cavities or cavities with special properties. This is best illustrated by first considering the evaluation of U. U is evaluated by using the orthogonality condition (6) for the case when $k' = k$. Substituting (38) into (6) and with the help of the identity

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)], \tag{47}$$

the orthogonality condition (6) for $k' = k$ becomes

$$1 = \frac{NVU^2}{2}$$

$$+ \frac{1}{2}\sum_{n\neq m}^{N}\sum_{m\neq n}^{N} \boldsymbol{U}_n\cdot\boldsymbol{U}_m\int_V \cos[\boldsymbol{K}_n\cdot(\boldsymbol{r}-\boldsymbol{r}_n) - \boldsymbol{K}_m\cdot(\boldsymbol{r}-\boldsymbol{r}_m)]dV \qquad (48)$$

$$+ \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N} \boldsymbol{U}_n\cdot\boldsymbol{U}_m\int_V \cos[\boldsymbol{K}_n\cdot(\boldsymbol{r}-\boldsymbol{r}_n) + \boldsymbol{K}_m\cdot(\boldsymbol{r}-\boldsymbol{r}_m)]dV$$

where $U_n=U$ has been used. The first term on the right comes from the difference in the cosine argument in the identity (47) which vanishes when $m = n$. The second term on the right in (48) comes from the same cosine argument. Since the wave vectors were restricted to be in the upper half-plane of k-space, none of the cosine arguments vanish when $m \neq n$. The third term of (48) comes from the sum of the cosine argument in the identity (47). Again, because of the restriction on the wave vectors, none of these arguments vanish for any values of n and m.

The integrals of the second and third terms in the right half of (48) are not, in general, integrable; the only notable exception is for the case of a rectangular cavity. However, some important observations can be made regarding these two integrals. If the shape of the cavity is regular, i.e., relatively smooth compared to a wavelength, then the contribution of these two integrals will be large. Their phase factors will be slowly varying functions because the position vectors to the corners of the *virtual* cavities will all be highly correlated. As a result, a large in-phase contribution is expected. It would also be expected that the values of these integrals would be a function of the shape of the cavity. On the other hand, if the shape of the cavity is electrically rough (very irregular compared to a wavelength), then the contributions from these two integrals will be small. This follows, since the magnitudes of the coefficients of the eigenvectors are the same so that one cosine term does not dominate. In the limit of a random shape (again compared to a wavelength), it would be expected that the contribution of these integrals would vanish.

The integrals of the form given in (48) are used to define cavities with complex shapes. Specifically, a complex cavity is defined to be a cavity for which integrals of the form contained in (48) are vanishingly small or

$$\sum_{n=1}^{N} \int_V \cos[\boldsymbol{K}_n \cdot (r - r_n)] dV \approx 0. \tag{49}$$

There is no way to *a priori* test to determine if a cavity is complex. The only way to proceed is to assume cavity complexity based on experience with other cavities and then use experimental data to justify the assumption. Any other approach must address the details of each cavity separately.

With the above definition for a complex cavity, it is now possible to evaluate the normalization coefficient U and to demonstrate the orthogonality of the eigenvectors. To evaluate the normalization coefficient, the second and third terms on the right hand side are assumed to be vanish and the value for U follows immediately or

$$U^2 = \frac{2}{NV}. \tag{50}$$

For the case where $k' \neq k$, the orthogonality condition (6) becomes

$$\begin{aligned} 0 = & \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N} \boldsymbol{U}_n \cdot \boldsymbol{U}'_n \int_V \cos[\boldsymbol{K}_n \cdot (r - r_n) - \boldsymbol{K}'_m \cdot (r - r_m)] dV \\ & + \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N} \boldsymbol{U}_n \cdot \boldsymbol{U}'_m \int_V \cos[\boldsymbol{K}_n \cdot (r - r_n) + \boldsymbol{K}'_m \cdot (r - r_m)] dV. \end{aligned} \tag{51}$$

Since $\boldsymbol{K}_n \neq \boldsymbol{K}'_m$ for all n and m, the two terms on the right hand side of (51) are always zero for cavities with complex shapes and orthogonality is established. Finally, it must be demonstrated that

$$\int_V \boldsymbol{u}_k \cdot \boldsymbol{w}_{k'} dV = 0 \tag{52}$$

for all k and k'. If $k = k'$, then

$$\begin{aligned} \int_V \boldsymbol{u}_k \cdot \boldsymbol{w}_{k'} dV = & \frac{V}{2}\sum_{n=1}^{N} \boldsymbol{U}_n \cdot \boldsymbol{W}_n \\ & + \text{two non-contributing terms.} \end{aligned} \tag{53}$$

Since $U_n \cdot W_n = 0$ for all n, (52) is satisfied. Using similar arguments, it can be shown that (52) is satisfied for the case when $k \neq k'$.

This completes the development of the generalized expression for the eigenvectors. It has been shown the expressions (38) and (39) for the eigenvectors satisfy the wave equation (3) and the divergence condition (4) without any restrictions. These expressions can be made to uniquely satisfy the boundary condition (5) by enforcing the boundary condition at the center of each small surface element. These results are independent of the shape of the cavity. However, the orthogonality condition is more restrictive. While it can be shown that orthogonality condition (6) holds for the trivial case, i.e. a rectangular cavity, it has not been demonstrated that this is true for any arbitrarily shaped cavity. The additional requirement that the shape of the cavity is complex must be imposed before it can be demonstrated that the orthogonality condition is satisfied. Next, the expressions (38) and (39) for the eigenvectors along with the definition of a complex cavity are used to find the volume averages of the components of the eigenvectors.

2.4 Volume Averages for Complex Cavities

The generalized expressions for the eigenvectors as developed in the previous section are of little use for obtaining exact values of quantities everywhere in the volume of the cavity. However, as will be shown, they are adequate for finding the volume averages of quantities of interest when the cavity is complex and electrically large.

The volume average $\langle H \rangle$ of a scalar quantity $H(\mathbf{r})$ is defined by the integral

$$\langle H \rangle = \frac{1}{V}\int_V H(\mathbf{r})dV. \tag{54}$$

The quantities of interest are the volume averages of the components of the eigenvectors and the volume averages of all multiplicative combinations of these components. The moments of the eigenvectors themselves are not considered. It is not clear if vector moments have any physical or mathematical meaning.

Since it was assumed that the orientation of the cavity coordinate system is arbitrary, the volume averages for one component must equal the volume averages for the other two components or

$$\langle u_{kx}^m \rangle = \langle u_{ky}^m \rangle = \langle u_{kz}^m \rangle \tag{55}$$

and

$$\langle u_{kx}^m u_{ky}^l \rangle = \langle u_{kx}^m u_{kz}^l \rangle = \langle u_{ky}^m u_{kz}^l \rangle \tag{56}$$

where m and l are positive integers. This is called the isotropy condition. Also, the volume averages for the components of the eigenvector w_k must be equal to the averages for u_k. The x-component of u_k will be considered first. Then, the volume averages for the product of two different components are evaluated.

2.4.1 **Volume Averages of a Single Component**

The volume averages of all the powers of a single component of the eigenvectors are required for the development of the statistical model. In particular, the volume averages are defined by

$$< u_{kx}^m > = \frac{1}{V}\int_V u_{kx}^m(r)\,dV \tag{57}$$

for all positive, non-zero m are evaluated.

(1) $m = 1$: Using (38) and (57) , the volume average for $m = 1$ is

$$< u_{kx} > = \frac{1}{V}\sum_{n=1}^{N} U_{nx}\int_V \cos[K_n\cdot(r - r_n)]dV. \tag{58}$$

From the definition (49) of a complex cavity, it follows that

$$< u_{kx} > = 0. \tag{59}$$

(2) $m = 2$: The volume average for this case is

$$< u_{kx}^2 > = \frac{1}{3V}. \tag{60}$$

This result follows from the orthogonality condition (6) and the isotropy condition (55).

(3) $m = 3$: The volume average for this case follows by substituting (38) into (57) and applying the cosine identity (47) twice. This yields for the volume average

$$< u_{kx}^3 > = \frac{1}{4V}\sum_{l,n,p}^{N} U_{lx}U_{nx}U_{px}\sum_{i=1}^{4}\int_V \cos\Psi_i\,dV \tag{61}$$

where the arguments of the cosine are

$$\begin{aligned}
\Psi_1 &= \boldsymbol{K}_l \cdot \boldsymbol{R}_l + \boldsymbol{K}_n \cdot \boldsymbol{R}_n + \boldsymbol{K}_p \cdot \boldsymbol{R}_p \\
\Psi_2 &= \boldsymbol{K}_l \cdot \boldsymbol{R}_l - \boldsymbol{K}_n \cdot \boldsymbol{R}_n + \boldsymbol{K}_p \cdot \boldsymbol{R}_p \\
\Psi_3 &= \boldsymbol{K}_l \cdot \boldsymbol{R}_l + \boldsymbol{K}_n \cdot \boldsymbol{R}_n - \boldsymbol{K}_p \cdot \boldsymbol{R}_p \\
\Psi_4 &= \boldsymbol{K}_l \cdot \boldsymbol{R}_l - \boldsymbol{K}_n \cdot \boldsymbol{R}_n - \boldsymbol{K}_p \cdot \boldsymbol{R}_p
\end{aligned} \tag{62}$$

and

$$\boldsymbol{R}_n = \boldsymbol{r} - \boldsymbol{r}_n. \tag{63}$$

Since none of the Ψ_i can vanish, it follows from the definition of a complex cavity (49) that

$$< u_{kx}^3 > = 0. \tag{64}$$

(4) $m = 4$: The volume average for the case $m = 4$ follows by substituting (38) into (57) and applying the cosine identity (47) three times. This yields for the volume average

$$< u_{kx}^4 > = \frac{1}{8V} \sum_{l,n,p,q}^{N} U_{lx} U_{nx} U_{px} U_{qx} \sum_{i=1}^{8} \int_V \cos \Psi_i \, dV. \tag{65}$$

The arguments of the cosine in the above expression are

$$\Psi_i = \varepsilon_{1,i} \boldsymbol{K}_l \cdot \boldsymbol{R}_l + \varepsilon_{2,i} \boldsymbol{K}_n \cdot \boldsymbol{R}_n + \varepsilon_{3,i} \boldsymbol{K}_p \cdot \boldsymbol{R}_p + \varepsilon_{4,i} \boldsymbol{K}_q \cdot \boldsymbol{R}_q \tag{66}$$

where $\varepsilon_{1,i}$ equals +1 for all i; $\varepsilon_{2,i}$ equals +1 for i = 1,2,3,4 and equals -1 for i = 5,6,7,8; $\varepsilon_{3,i}$ equals +1 for even i and -1 for odd i; and $\varepsilon_{4,i}$ equals +1 for 1,4,5,8 and -1 for i = 2,3,6,7. Since Ψ_i can not be zero for i = 1,3,4,5,6, it follows from the complex cavity definition that these terms do not contribute to (65). For $i = 2$, the argument of the cosine vanishes when $l = p$ and $n = q$ and when $l = q$ and $n = p$. It follows that

$$\int_V \cos \Psi_2 \, dV = V(\delta_{lp}\delta_{nq} + \delta_{lq}\delta_{np}). \tag{67}$$

Using similar arguments for i = 7 and i = 8 it can be shown that

$$\int_V \cos \Psi_7 \, dV = \int_V \cos \Psi_8 \, dV = V(\delta_{ln}\delta_{pq} + \delta_{lq}\delta_{np}). \tag{68}$$

Substituting (67) and (68) into (65) yields

$$< u_{kx}^4 > = \frac{3}{4} \sum_{n=1}^{N} \sum_{l=1}^{N} U_{nx}^2 U_{lx}^2 . \tag{69}$$

Defining the parameter S as

$$S = \sum_{n=1}^{N} U_{nx}^2 , \tag{70}$$

(69) reduces to

$$< u_{kx}^4 > = \frac{3}{4} S^2 . \tag{71}$$

The quantity S is evaluated by writing U_{nx} in the form

$$U_{nx} = U \Phi_{nx} \tag{72}$$

where Φ_{nx} is the direction cosine between the vector U_n and the x-axis of the cavity's coordinate system. S now becomes

$$S = U^2 \sum_{n=1}^{N} \Phi_{nx}^2 = NU^2 \left(\frac{1}{N} \sum \Phi_{nx}^2 \right) = NU^2 [\Phi_x^2]_{ave} . \tag{73}$$

In spherical polar coordinates, the average value of the direction cosine squared is given by

$$[\Phi_x^2]_{ave} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \sin^3 \theta \cos^2 \phi \, d\theta \, d\phi = \frac{1}{3} . \tag{74}$$

Using $U^2 = (2/NV)$, S reduces to

$$S = \frac{2}{3V} \tag{75}$$

and (69) becomes

$$< u_{kx}^4 > = \frac{1}{3V^2} \tag{76}$$

or

$$< u_{kx}^4 > \; = \; 1 \cdot 3 \left(\frac{1}{3V} \right)^2 . \tag{77}$$

The rationale for writing (72) in this form will become immediately obvious.

(5) $m = 5$: For this case, the approach is the same as was used for $m=3$. It follows that if m is odd, there is no combination of the indices l,n,p,q,r for which the cosine argument will vanish. Therefore, from the definition of a complex cavity

$$< u_{kx}^5 > \; = \; 0. \tag{78}$$

(6) $m = 6$: For this case, the volume average is

$$< u_{kx}^6 > \; = \; \frac{1}{32V} \sum_{l=1}^{N} \cdots \sum_{s=1}^{N} U_{lx}^2 \cdots U_{sx}^2 \sum_{i=1}^{32} \int_V \cos \Psi_i \, dV. \tag{79}$$

The 32 angles in the above equation are similar to the ones given in (66) for $m=4$ except in this case there are 6 indices: l,n,p,q,r,s. The thirty two angles result from applying the cosine identity (47) five times, i.e. number of angles equal $2^{m-1}=32$. Of the thirty two Ψ_i, only 10 contribute. To show this, consider the following argument: For any of the Ψ_i 's to vanish, each Ψ_i must contain an equal number of plus and minus signs similar to the discussion associated with (66). For this case the number of plus and minus signs is $m/2=3$. There are $m!=6!$ ways to order the six $\boldsymbol{K}_l \cdot \boldsymbol{R}_l$ terms contained in each Ψ_i. Many of these $m!$ ways do not give independent distinguishable arrangements into groups of $m/2$ plus signs and $m/2$ negative signs. Interchanges of the $m/2$ plus signs purely among themselves does not give anything new and there are $(m/2)!$ such interchanges. Similarly the $(m/2)!$ interchanges of the $m/2$ negative signs does not give new arrangements. Thus the total number of independent arrangements of the six $\boldsymbol{K}_l \cdot \boldsymbol{R}_l$ terms contained in a Ψ_i is $m![(m/2)!]^{-2}=20$. However, the first term in the cosine expansion is always positive; therefore, the number of non-vanishing angles is reduced by a factor of 2 or the number of contributing terms is given by $m!\{2[(m/2)!]^2\}^{-1} = 10$. Each contributing angle results in 6 identical terms $(m/2)!$ where again m is the number of indices. Therefore, (79) reduces to

$$< u_{kx}^6 > \; = \; \frac{60}{32} \sum_{l=1}^{N} \sum_{n=1}^{N} \sum_{p=1}^{N} U_{lx}^2 U_{nx}^2 U_{px}^2 \tag{80}$$

or in terms of S as defined by (70)

$$< u_{kx}^6 > = \frac{15}{8} S^3. \tag{81}$$

Using the value of S given by (75), then

$$< u_{kx}^6 > = \frac{5}{9V^3} \tag{82}$$

or

$$< u_{kx}^6 > = 1 \cdot 3 \cdot 5 \left(\frac{1}{3V} \right)^3. \tag{83}$$

(7) $m = 7$: Since m is odd, there are no contributing terms and

$$< u_{kx}^7 > = 0. \tag{84}$$

(8) $m = 8$: For $m = 8$, the volume average is

$$< u_{kx}^8 > = \frac{1}{132V} \sum_{l=1}^{N} \cdots \sum_{v=1}^{N} U_{lx}^2 \cdots U_{vx}^2 \sum_{i=1}^{132} \int_V \cos \Psi_i \, dV. \tag{85}$$

Of the $2^{m-1}=132$ angles, only $m!/\{2[(m/2)!]^2\}^{-1}=35$ have combinations of the indices for which they vanish. Here the same argument used for the case $m=6$ is applicable. Each of these vanishing angles contributes $(m/2)!=24$ identical terms. As a result (85) reduces to

$$< u_{kx}^8 > = \frac{840}{132} \sum_{l=1}^{N} \sum_{n=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} U_{lx}^2 U_{nx}^2 U_{px}^2 U_{qx}^2 \tag{86}$$

or in terms of S

$$< u_{kx}^8 > = \frac{105}{16} S^4. \tag{87}$$

Using (75) for S results in

$$< u_{kx}^8 > = \frac{35}{27V^4} \tag{88}$$

or

$$< u_{kx}^8 > = 1 \cdot 3 \cdot 5 \cdot 7 \left(\frac{1}{3V} \right)^4. \tag{89}$$

(9) ***arbitrary m***: By induction, it follows that the volume average (57) for an arbitrary value of m can be written as

$$< u_{kx}^m > = \begin{cases} 1 \cdot 3 \cdot 5 \cdot 7 \cdots (m-1) \left(\frac{1}{3V} \right)^m & \text{for } m \text{ even} \\ 0 & \text{for } m \text{ odd.} \end{cases} \tag{90}$$

From (39) and the isotropy condition (55), it follows that

$$< u_{kx}^m > = < u_{ky}^m > = < u_{kz}^m > = < w_{kx}^m > = < w_{ky}^m > = < w_{kz}^m >. \tag{91}$$

2.4.2 Volume Averages of Products of Different Eigenvector Components

In order to develop the statistical model, the volume averages of the product of two different components of the eigenvectors are required. Two cases are of interest. For the first case, two components having the same wave numbers $k = k'$ are considered. For the second case, the component wave numbers are not equal $k \neq k'$.

Case 1 - $k = k'$: Only two products need be considered. The rest can be inferred using the isotropy condition (56). The first volume average of concern $<u_{kx}u_{ky}>$ can be shown to be

$$< u_{kx}u_{ky} > = \frac{1}{2} \sum_{n=1}^{N} U_{nx}U_{ny} \tag{92}$$

by using the previously established procedures. The sum in (92) is easily evaluated by writing the components of the U_n in terms of their direction cosines or

$$\sum_{n=1}^{N} U_{nx}U_{ny} = NU^2 \left(\frac{1}{N} \sum_{n=1}^{N} \Phi_{nx}\Phi_{ny} \right) = NU^2 [\, \Phi_x \Phi_y]_{ave}. \tag{93}$$

The average value of the product of the direction cosine in spherical polar coordinates is

$$[\Phi_x \Phi_y]_{ave} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \sin^3\theta \sin\phi \cos\phi \, d\theta \, d\phi = 0. \tag{94}$$

Therefore, it follows that

$$< u_{kx} u_{ky} > \; = \; < u_{kx} u_{kz} > \; = \; < u_{ky} u_{kz} > \; = \; 0. \tag{95}$$

It also follows immediately from the orthogonality condition (6) and the isotropy condition (57) that

$$< u_{kx} w_{kx} > \; = \; < u_{ky} w_{ky} > \; = \; < u_{kz} w_{kz} > \; = \; 0. \tag{96}$$

For volume averages of the form $<u_{kx} w_{ky}>$, it can be shown that

$$< u_{kx} w_{ky} > \; = \; \frac{1}{2} \sum_{n=1}^{N} U_{nx} W_{ny} \, . \tag{97}$$

This expression can be evaluated by writing both W_{ny} and U_{nx} in terms of their direction cosines in spherical polar coordinates. Then

$$\begin{aligned} U_{nx} \; &= \; U \, \Phi_{nx} \\ &= \; U \sin\theta_n \cos\phi_n \end{aligned} \tag{98}$$

and

$$\begin{aligned} W_{ny} \; &= \; U \, \Phi'_{ny} \\ &= \; U \sin\theta'_n \sin\phi'_n \, . \end{aligned} \tag{99}$$

Since $U_n \cdot W_n = 0$ for all n, it follows that

$$\sin\theta_n \sin\theta'_n \cos(\phi_n - \phi'_n) \; + \; \cos\theta_n \cos\theta'_n \; = \; 0. \tag{100}$$

This expression is satisfied if (there are a number of other conditions, but they all yield the same result)

$$\begin{aligned} \phi_n - \phi'_n \; &= \; 0 \\ \theta_n - \theta'_n \; &= \; \frac{\pi}{2}. \end{aligned} \tag{101}$$

Using (98), (99) and (101), then the average value of the product of directions cosines for the two components becomes

$$[\Phi_{nx} \Phi'_{ny}]_{ave} \; = \; \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \sin^2\theta \cos\theta \sin\phi \cos\phi \, d\theta \, d\phi \; = \; 0 \tag{102}$$

which upon substitution into (97) yields

$$< u_{kx} w_{ky} > = 0. \tag{103}$$

Similar arguments and/or the isotropy condition (57) are used to evaluate the other two cross product terms. It follows that

$$< u_{kx} w_{ky} > = < u_{kx} w_{kz} > = < u_{ky} w_{kz} > = 0. \tag{104}$$

Case 2 - $k \neq k'$. For this case, it follows directly from the complex cavity definition that

$$< u_{ks} u_{k's} > = < w_{ks} w_{k's} > = < u_{ks} w_{k's} > = 0 \qquad s = x,y,z. \tag{105}$$

2.5 Summary of Deterministic Characterization

Before the results of the above development are summarized, it is necessary to express the electric field in the cavity in terms of the eigenvectors u_k and w_k, given by (38) and (39). A natural choice is to write the total electric field as the superposition of the contribution due to the eigenvectors u_k and the contribution due to the eigenvectors w_k. The electric field would then be given by

$$E_1(r,t) = E_u(r,t) + E_w(r,t) \tag{106}$$

where E_u and E_w are the partial fields as determined by (23) using the eigenvectors u_k and w_k, respectively. If

$$E_u \cdot E_w = 0 \tag{107}$$

for all $r \in V$, then (106) would be the only solution. However, (106) is not the only solution since, in general

$$u_k \cdot w_k \neq 0. \tag{108}$$

Therefore, there exists another linearly independent solution for the electric field of the form

$$E_2 = E_u - E_w. \tag{109}$$

This follows since the amplitudes of the partial fields E_u and E_w are equal.

Using the two linearly independent solutions, the deterministic characterization of the fields in an electrically large, complex and lossy cavity can be summarized as follows:
(1) The total electric field $\in(r,t)$ is expressed as a doublet in polarization space (in the notation used in [1], it is a supervector) or

$$\in (r,t) = \begin{pmatrix} E_1(r,t) \\ E_2(r,t) \end{pmatrix}. \tag{110}$$

This notation is convenient for developing the statistical model. In this space the total electric field is not a 6x1 vector with 6 Euclidean space scalar components, but is a 2x1 vector with 2 Euclidean space vector components. The matrix operations are independent of the Euclidean space vector operations and it is usually less confusing to perform the matrix operations first. Using (106) and (109), the total electric field in terms of the partial fields E_u and E_w is

$$\in (r,t) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \odot \begin{pmatrix} E_u(r,t) \\ E_w(r,t) \end{pmatrix} \tag{111}$$

where $2^{-1/2}$ is a normalization factor and the symbol $\odot$ denotes the generalized dot product. (2) For a cavity with a modal loss factor Q_k containing an infinitesimal sinusoidal dipole source with angular frequency ω, length $\Delta\ell$ and amplitude I_0 directed along the x-axis and located at the position r_0, the partial fields E_u and E_w are

$$E_u(r,t) = -\frac{\Delta\ell I_0 \omega}{\epsilon_0} \sum_k \alpha_k u_{kx}(r_0) u_k(r) \cos(\omega t - \phi_k) \tag{112}$$

and

$$E_w(r,t) = -\frac{\Delta\ell I_0 \omega}{\epsilon_0} \sum_k \alpha_k w_{kx}(r_0) w_k(r) \cos(\omega t - \phi_k) \tag{113}$$

where the constants α_k and ϕ_k are

$$\alpha_k = \left[\frac{\omega^2 \omega_k^2}{\pi^2 Q_k^2} + \left(\omega^2 - \omega_k^2\right)^2 \right]^{-1/2} \tag{114}$$

and

$$\tan\phi_k = \frac{\omega \omega_k}{\pi Q_k (\omega_k^2 - \omega^2)}. \tag{115}$$

(3) The eigenvectors u_k and w_k are

$$u_k(r) = \sum_{n=1}^{N} U_n \cos[K_n \cdot (r - r_0)] \tag{116}$$

and

$$w_k(r) = \sum_{n=1}^{N} W_n \cos[K_n \cdot (r - r_0)] \tag{117}$$

where

$$K_n \cdot U_n = K_n \cdot W_n = U_n \cdot W_n = 0 \tag{118}$$

and

$$K_n^2 = k^2 \quad \& \quad U_n^2 = W_n^2 = U^2 = \frac{2}{NV} \tag{119}$$

for all n. N is the number of *virtual* cavities.

(4) All the volume averages of single components of an eigenvector for complex cavities are

$$< u_{ks}^m > = < w_{ks}^m > = \begin{cases} 1 \cdot 3 \cdot 5 \cdots (m-1) \left(\frac{1}{3V}\right)^m & \text{for } m \text{ even} \\ 0 & \text{for } m \text{ odd} \end{cases} \tag{120}$$

for $s = x,y,x$ and m a positive non-zero integer.

(5) The volume averages of products of the different components of the eigenvectors are all zero or

$$< u_{ks} u_{ks'} > = < w_{ks} w_{ks'} > = 0 \tag{121}$$

for $s \neq s'$,

$$< u_{ks} w_{ks'} > = 0 \tag{122}$$

for all s and s' and

$$< u_{ks} u_{k's'} > = < w_{ks} w_{k's'} > = < u_{ks} w_{k's'} > = 0 \tag{123}$$

for all s and s' when $k' \neq k$.

(6) The number of modes $dM/d\omega$ in the interval $d\omega$ is

$$\frac{dM}{d\omega}d\omega = \frac{V\omega^2}{\pi^2 c^3}d\omega. \tag{124}$$

Next, equations (110)-(124) are used to derive the statistical model for EM fields in complex cavities.

3.0 THE STATISTICAL MODEL

The derivation of the statistical model for EM fields in complex cavities is presented in two parts. In the first part, equations (116)-(123) are used to develop a statistical model for the eigenvectors. In the second part, the eigenvector statistical model and equations (110)-(115) and (124) are used to derive the statistical model for the fields in complex cavities.

Before either model can be derived, it is necessary to establish a correspondence between probability theory and the deterministic cavity parameters developed above. This correspondence is addressed next.

3.1 Random Variable Considerations

Consider the cavity of Figure 2. As stated in the introduction, the fields everywhere in the cavity are known (the problem is deterministic). With reference to this cavity, the following experiment is performed: A large number Ξ of position vectors $\boldsymbol{r}_\xi = \boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_\Xi$ are randomly selected with equal weight, i.e., every position in the cavity has an equal chance of being selected. At each of these random positions, a field variable is measured. This field variable could correspond to a component of the electric field or a component of an eigenvector.

For this experiment, the position vector $\boldsymbol{r}$ in the cavity can be treated as a random variable (r.v.). Since any position is equally likely, $\boldsymbol{r}$ is distributed according to a uniform distribution and its probability density function $f_r(\boldsymbol{r})$ is

$$f_r(\boldsymbol{r}) = \begin{cases} \dfrac{1}{V} & \boldsymbol{r} \in V \\ 0 & \text{elsewhere.} \end{cases} \tag{125}$$

If $\boldsymbol{r}$ is a r.v., the field variables are also r.v. since they are functions of $\boldsymbol{r}$ or for example

$$u_{kx} = u_{kx}(\boldsymbol{r}) \tag{126}$$

is a random variable since it is a function of $\boldsymbol{r}$. Using (A-12) and (A-13) of the appendix, the expectation value $e\{u_{kx}(\boldsymbol{r})\}$ of the r.v. $u_{kx}(\boldsymbol{r})$ can be determined if both the functional form of $u_{kx}(\boldsymbol{r})$ and $f_r(\boldsymbol{r})$ are known or

$$e\{u_{kx}(\boldsymbol{r})\} = \int u_{kx}(\boldsymbol{r}) f_r(\boldsymbol{r})\, dV \tag{127}$$

where the volume integral extends over all space. Substituting (125) into (127) yields

$$e\{u_{k\alpha}(r)\} = \frac{1}{V}\int_V u_{k\alpha}(r)\,dV. \tag{128}$$

The right hand side of (128) is equal to the right hand side of (54), the definition of the volume averages used to obtain the results given by (120)-(123). Therefore, it follows that

$$e\{u_{k\alpha}(r)\} = < u_{k\alpha}(r) > \tag{129}$$

or, in general, the expectation value $e\{g(r)\}$ of any function $g(r)$ is equal to its volume average $<g(r)>$ or

$$e\{g(r)\} = < g(r) >. \tag{130}$$

With the above equivalence relationship, it is now possible to develop the statistical model for the eigenvectors of a complex cavity. In the remainder of this paper, the brackets < > will be used to denote both average values and expectation values.

3.2 The Statistical Model for the Eigenvectors

For the case of a single r.v., the statistical model is usually defined as the probability density function (pdf) or the cumulative distribution function (cdf) of the r.v. For this problem, this definition is not appropriate since there are essentially an infinite number of r.v.'s of interest, namely the six components of the two eigenvectors for all the cavity eigenvalues k evaluated at the center of each of the N cells of the discretized space of the cavity. For the case of more than one r.v., the statistical model is usually more complicated. To illustrate consider the two random variables X and Y with pdf's $f_X(x)$ and $f_Y(y)$, respectively. The statistical model for this case is called the joint density function and is denoted by

$$f_{X,Y}(x,y). \tag{131}$$

Joint density functions are generally very difficult to determine or derive unless X and Y are distributed according to a normal distribution and/or are independent r.v.'s. For example, if X and Y are independent r.v.'s, then the joint density function is given by

$$f_{X,Y}(x,y) = f_X(x)f_Y(y). \tag{132}$$

No attempt will be made to write the complete joint density function for all the r.v.'s contained in this formulation. Instead, the complete statistics will be established by finding the pdf's for each r.v. and by establishing the statistical relationships (correlation, orthogonality and dependency) between the r.v.

3.2.1 The pdf's for the Components of the Eigenvectors

The pdf's for the components of the eigenvectors can be derived from their average values (120), the moment theorem (see Section A.5 of the Appendix) and the moments (A-32) of a normal distribution with zero mean and a standard deviation of β. From these considerations, it follows immediately that the pdf for $u_{ks}(\mathbf{r})$ is

$$f_{u_{ks}}(u_{ks}) = \frac{1}{\sqrt{2\pi}\beta} e^{-\frac{u_{ks}^2}{2\beta^2}} \tag{133}$$

and the pdf for $w_{ks}(\mathbf{r})$ is

$$f_{w_{ks}}(w_{ks}) = \frac{1}{\sqrt{2\pi}\beta} e^{-\frac{w_{ks}^2}{2\beta^2}} \tag{134}$$

for all k where $s = x,y,z$ and

$$\beta = \sqrt{\frac{1}{3V}}. \tag{135}$$

Thus, the complete set of r.v. formed by all the components of the eigenvectors (with k constrained by the definition of an electrically large, complex cavity) are all identical r.v. distributed according to a normal distribution with zero mean and a standard deviation β. It is important to note that these distributions are independent of the values k of the eigenvectors and the shape of the cavity (consistent with the definition of a complex cavity). To complete the statistics for the components of the eigenvectors, only the relationships between these r.v. are required. These relationships are established next.

3.2.2 Statistical Relationships Between the Eigenvector Components

The statistical relationships between the components of the eigenvectors are established by using the definitions presented in Sections A.7 and A.12 of the Appendix. Two general types of relationships are of interest. The first relationships establish the dependency between components with different spatial subscripts, eigenvector type and/or eigenvalue. It is shown that these r.v. are statistically independent. The second relationships establish the dependency between similar eigenvector components at different locations within the cavity. It is shown that these r.v. are correlated and dependent.

3.2.2.1 **Statistical Independent Random Variables**

Using (121)-(123), the equivalence of volume averages/expectation values and the definition of an orthogonal r.v. (A-25) of the Appendix, it follows that the sets of r.v.

$$\begin{aligned} &\{u_{ks}(r),u_{k's'}(r): s\neq s'\} \\ &\{w_{ks}(r),w_{k's'}(r): s\neq s'\} \\ &\{u_{ks}(r),w_{k's'}(r)\} \end{aligned} \tag{136}$$

are all statistically orthogonal for $s,s' = x,y,z$ and $k,k' = k_1,k_2,.....$ The set of r.v. $u_{ks}(r)$ and $w_{ks}(r)$ all have zero mean, therefore, the sets of r.v. are also uncorrelated (see the definition of uncorrelated r.v's (A-24)). Since the r.v. are all normally distributed, the theorem presented in Section A.13 of the appendix applies and it follows that the sets of r.v.'s given by (136) are also all independent. This means that the joint density function for the two r.v., say $u_{kx}(r)$ and $w_{ky}(r)$, is

$$f_{u_{kx},w_{ky}}(u,w) = f_{u_{kx}}(u)f_{w_{ky}}(w). \tag{137}$$

Similar forms for the joint density functions follow for any other combinations of the r.v. contained in the set defined by (136).

3.2.2.2 **Dependent Random Variables**

The eigenvectors $u_k(r)$ and $w_k(r)$ of (116) and (117) are continuous functions of r. Therefore, it follows that any two of these r.v.'s evaluated at two different positions, say $u_{kx}(r_1)$ and $u_{kx}(r_2)$ will not be independent. Their correlation coefficient can be determined by using the definition of the correlation coefficient κ [(A-43) of the Appendix] and the techniques previously used for performing volume averages. From the Appendix, the correlation coefficient of two similar components of an eigenvector is

$$\kappa[u_{ks}(r_1),u_{ks}(r_2)] = \frac{< u_{ks}(r_1)u_{ks}(r_2) >}{\beta^2} \tag{138}$$

where β is given by (135) and

$$< u_{ks}(r_1) > = < u_{ks}(r_2) > = 0. \tag{139}$$

The expectation value of (138) is evaluated using the same techniques that were previously used. Applying the isotropy condition (57) yields

$$< u_{ks}(r_1)u_{ks}(r_2) > \; = \; \frac{1}{3} < u_k(r_1) \cdot u_k(r_2) >. \tag{140}$$

Substituting (116) into (139) and with the help of the identity (47) and the complex cavity definition (49), the following expression for the correlation coefficient is obtained

$$\kappa[u_{ks}(r_1),u_{ks}(r_2)] \; = \; \frac{1}{N}\sum_{n=1}^{N} \cos[K_n \cdot (r_1 \; - \; r_2)]. \tag{141}$$

The quantity on the right hand side of (141) is just an average value which is easily evaluated using spherical polar coordinates. Since the wave vectors K_n are restricted to the upper half-plane of k-space, the integration is only over 2π steradian and the polar angle theta ranges from 0 to $\pi/2$. The correlation coefficient becomes

$$\kappa[u_{ks}(r_1),u_{ks}(r_2)] \; = \; \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi/2} \sin\theta \cos(k|r_1 - r_2|\cos\theta)\, d\theta\, d\phi. \tag{142}$$

Performing the integration yields

$$\kappa[u_{ks}(r_1),u_{ks}(r_2)] \; = \; \frac{\sin k|r_1 - r_2|}{k|r_1 - r_2|}. \tag{143}$$

Using this value for the correlation coefficient, it is now possible to write the joint density function for the two r.v. $x = u_{ks}(r_1)$ and $y = u_{ks}(r_2)$ or

$$f(x,y) \; = \; \frac{1}{2\pi\beta^2\sqrt{1-\kappa^2}} \exp\left[-\frac{1}{2\beta^2(1-\kappa^2)}(x^2 \; - \; 2\kappa xy \; + \; y^2)\right] \tag{144}$$

where κ is defined by (143). In the limit as $k\,|r_1 - r_2| \to \infty$, the correlation coefficient approaches zero and the two r.v. $u_{ks}(r_1)$ and $u_{ks}(r_2)$ are uncorrelated and independent. It is important to note that this statistic is dependent on the wave number of the eigenvector. The statistical model for the eigenvectors are used next to derive the statistical model for the fields in the cavity.

3.3 The Statistical Model for the Cavity Fields

The statistical model for the cavity fields follows from the deterministic expression for the fields (111)-(115), the mode density (124) and the statistical model for the components of the eigenvectors. Before the statistical model is derived, it is necessary to estimate the

number of modes excited in a cavity with an average damping factor Q and introduce some simplifying assumptions.

If a cavity is driven by a source with angular frequency ω, then the bandwidth in which the modes will be excited is defined by

$$\omega\left(1 - \frac{1}{2Q}\right) \leq \omega \leq \omega\left(1 + \frac{1}{2Q}\right). \tag{145}$$

To first order in inverse powers of Q, the number of modes M excited in the cavity is obtained by integrating (124) over the bandwidth or

$$M = \frac{V\omega^3}{\pi^2 c^3 Q}. \tag{146}$$

To simplify the expressions for the fields, the assumption is made that in the bandwidth defined by (145), each of the modal Q_k's is approximately equal to the average Q of the cavity or

$$Q_k \approx Q. \tag{147}$$

Also, it is assumed that in the bandwidth, the modal angular frequency ω_k is approximately equal to the excitation frequency ω or

$$\omega_k \approx \omega. \tag{148}$$

Using the approximations (147) and (148), the coefficients α_k and ϕ_k reduce to the simple expressions

$$\alpha_k \approx \frac{\pi Q}{\omega^2} \tag{149}$$

and

$$\phi_k \approx \pm\frac{\pi}{2}. \tag{150}$$

The sign in (150) is unimportant and the positive sign will be used in the following. These simplifying assumptions and expressions are incorporated into the derivation of the statistical model for the fields in a cavity. The pdf's for the fields are derived first. This is followed by a derivation of the statistical relationships needed to complete the statistics. Finally, the

statistics of the energy density in the cavity are developed.

3.3.1 The pdf's for the Field Variables

To derive the statistics for the total fields (111), the statistics of the partial fields E_u and E_w are developed first. These results are then applied to (111).

3.3.1.1 The Statistics of the Partial Fields

The field expressions for the partial fields E_u and E_w are identical upon the interchange of u and w. Thus, only the expression for E_u will be addressed and the subscript u is dropped. Substituting (149) and (150) into (112) yields for the three cartesian components of the partial field

$$\begin{aligned} E_x(r,t) &= a_0 \sum_{k'=k(1-M/2)}^{k(1+M/2)} u_{k'x}(r_0)u_{k'x}(r)\sin\omega t \\ E_y(r,t) &= a_0 \sum_{k'=k(1-M/2)}^{k(1+M/2)} u_{k'x}(r_0)u_{k'y}(r)\sin\omega t \\ E_z(r,t) &= a_0 \sum_{k'=k(1-M/2)}^{k(1+M/2)} u_{k'x}(r_0)u_{k'z}(r)\sin\omega t \end{aligned} \qquad (151)$$

where $k = \omega/c$ and the constant a_0 is

$$a_0 = \frac{\pi \Delta \ell I_0 Q}{\epsilon_0 \omega}. \qquad (152)$$

It is immediately obvious from (150) that the presence of the source destroys the isotropy of the fields in the cavity. The r.v. $u_{kx}(r_0)$, $u_{ky}(r)$ and $u_{kz}(r)$ are all independent r.v. that are also independent of the wave number k. It follows that the products contained in the y and z-components of the partial field are also independent. The r.v. $u_{kx}(r_0)$ and $u_{kx}(r)$ are dependent r.v. and their correlation coefficient depends on the wave number k. Therefore, the statistics for the y and z-components of the partial fields are, in general different from the statistics for the x-component of the partial fields.

The statistics for the partial fields are significantly simplified if it is assumed that $k\,|r-r_0|$ is much greater than one. Then the correlation coefficient (143) is approximately zero and the two r.v. $u_{kx}(r_0)$ and $u_{kx}(r)$ can be treated as independent r.v. since they are normally distributed. When this is true, the partial fields are isotropic. For this case, the expressions for the cartesian components of the partial field take on the simple form

$$
\begin{aligned}
E_x(r,t) &= a_0 \sum_{m=1}^{M} \xi_{xm} \sin \omega t \\
E_y(r,t) &= a_0 \sum_{m=1}^{M} \xi_{ym} \sin \omega t \\
E_z(r,t) &= a_0 \sum_{m=1}^{M} \xi_{zm} \sin \omega t
\end{aligned}
\tag{153}
$$

where

$$
\begin{aligned}
\xi_{xm} &= u_{kx}(r_0) u_{kx}(r) \\
\xi_{ym} &= u_{kx}(r_0) u_{ky}(r) \\
\xi_{zm} &= u_{kx}(r_0) u_{kz}(r).
\end{aligned}
\tag{154}
$$

The free index k on the right hand side of (154) can be replaced with the free index m on the left hand side because the statistics of each of the ξ's are independent of the wave number k. In addition, the set of ξ's contained in (153) are all independent r.v. Since the ξ's are the product of two normally distributed independent r.v with zero mean and standard deviation $\beta = (3V)^{-1/2}$, it follows from (A-53) of the Appendix, that each of the ξ's are distributed according to the following distribution

$$
f_\xi(\xi) = \frac{1}{\pi \beta^2} K_0\left(\frac{|\xi|}{\beta^2}\right) \tag{155}
$$

where K_0 is a modified zeroth-order Bessel function of the second kind. The pdf given by (155) is valid over the range $-\infty \leq \xi \leq \infty$. The characteristic function $\Phi_\xi(\nu)$ of ξ is given by (A-55) of the Appendix or

$$
\Phi_\xi(\nu) = \frac{1}{\sqrt{1 + \beta^4 \nu^2}}. \tag{156}
$$

The derivation of the partial field pdf's is completed by noting that each component (153) of the partial field contains the sum over M identical independent r.v. (here M is the number of modes (146) excited in the cavity). Writing one of these sums as

$$\Xi = \sum_{m=1}^{M} \xi_m, \tag{157}$$

then the characteristic function $\Phi_{\Xi}(\nu)$ for Ξ follows from (A-30) and the definition of the characteristic function (A-16) of the Appendix or

$$\Phi_{\Xi}(\nu) = [\Phi_{\xi}(\nu)]^M. \tag{158}$$

Substituting (156) into (158) yields

$$\Phi_{\Xi}(\nu) = \left[1 + \beta^4\nu^2\right]^{-M/2} \tag{159}$$

where β has its usual value. Writing (159) in the form

$$\Phi_{\Xi}(\nu) = \left[\left(1 + \frac{M\beta^4\nu^2}{M}\right)^M\right]^{-1/2} \tag{160}$$

and using the limiting value definition for an exponential

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x, \tag{161}$$

the characteristic function $\Phi_{\Xi}(\nu)$ can be approximated by

$$\Phi_{\Xi}(\nu) \approx e^{-\frac{1}{2}M\beta^4\nu^2} \tag{162}$$

for large values of M, i.e., the expression (162) is valid when a large number of modes is excited in the cavity. The range of the validity of (162) is still an open question and will be discussed in more detail in the conclusion section. The characteristic function for a normal distribution with zero mean and a standard deviation of γ is given by (A-33) of the Appendix or

$$\Phi(\nu) = e^{-\frac{1}{2}\gamma^2\nu^2}. \tag{163}$$

Therefore, in the limit of large M, the pdf for Ξ is

$$f_{\Xi}(\Xi) = \frac{1}{\sqrt{2\pi M}\beta^2} e^{-\frac{\Xi^2}{2M\beta^4}} \tag{164}$$

or Ξ is distributed according to a normal distribution with a mean of zero and a standard deviation of $M^{1/2}\beta^2$ where M is given by (146) and β is given by (135). Of course, this derivation is a re-statement of the central limit theorem, however, it is now possible to determine its range of validity when applied to EM fields in cavities.

Defining a new r.v. X as

$$\mathrm{X} = a_0 \Xi, \tag{165}$$

the cartesian components of the partial field due to the u eigenvector can be written as

$$\begin{aligned} E_{ux}(r,t) &= \mathrm{X}_{ux} \sin \omega t \\ E_{uy}(r,t) &= \mathrm{X}_{uy} \sin \omega t \\ E_{uz}(r,t) &= \mathrm{X}_{uz} \sin \omega t . \end{aligned} \tag{166}$$

Similarly, the cartesian components of the partial field due to the w eigenvector can be written as

$$\begin{aligned} E_{wx}(r,t) &= \mathrm{X}_{wx} \sin \omega t \\ E_{wy}(r,t) &= \mathrm{X}_{wy} \sin \omega t \\ E_{wz}(r,t) &= \mathrm{X}_{wz} \sin \omega t . \end{aligned} \tag{167}$$

The six X's contained in (166) and (167) are identical independent r.v. distributed according to a normal distribution. Their pdf is

$$f_{\mathrm{X}}(\mathrm{X}) = \frac{1}{\sqrt{2\pi M} a_0 \beta^2} e^{-\frac{\mathrm{X}^2}{2Ma_0^2\beta^4}} \tag{168}$$

where (164), (165) and (A-19) of the Appendix have been used. This completes the development of the pdf's for the partial fields. These results are used to derive the pdf's for the total fields.

3.3.1.2 **The pdf's for the Total fields**

For an over-moded cavity (the number of modes M is large) and for positions in the cavity sufficiently far from the source, the cartesian components of the total electric field $\mathcal{E}(r,t)$

are can be expressed in the following form:

$$\in_x(r,t) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \odot \begin{pmatrix} X_{ux} \\ X_{wx} \end{pmatrix} \sin \omega t$$

$$\in_y(r,t) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \odot \begin{pmatrix} X_{uy} \\ X_{wy} \end{pmatrix} \sin \omega t \qquad (169)$$

$$\in_z(r,t) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \odot \begin{pmatrix} X_{uz} \\ X_{wz} \end{pmatrix} \sin \omega t$$

where as before the X's are all independent r.v. with the distribution given by (168). The time average of the square a component E_s^2 of the total electric field is

$$E_s^2 = \frac{1}{T}\int_0^T \in_s'(r,t) \odot \in_s(r,t)\, dt \qquad (170)$$

where T is the period of the driving frequency and the prime denotes the transpose. Upon performing the indicated operations, the time average for the square of any component of the total field is

$$E_s^2 = \frac{1}{2}(X_{us}^2 + X_{ws}^2). \qquad (171)$$

Defining a new r.v. as

$$\Upsilon = \frac{X}{\sqrt{2}} \qquad (172)$$

then (171) becomes

$$E_s^2 = \Upsilon_{us}^2 + \Upsilon_{ws}^2 \qquad (173)$$

or the magnitude of a cartesian component of the electric field is

$$E_s = \sqrt{\Upsilon_{us}^2 + \Upsilon_{ws}^2}. \tag{174}$$

The pdf of any of the r.v. Υ follows directly from (168) and (A-19) of the Appendix and is

$$f(\Upsilon) = \frac{1}{\sqrt{\pi M} a_0 \beta^2} e^{-\frac{\Upsilon^2}{M a_0^2 \beta^4}}. \tag{175}$$

The set of Υ's are identical and independent normal r.v. with a zero mean and a standard deviation γ of

$$\gamma = \sqrt{\frac{M}{2}} a_0 \beta^2. \tag{176}$$

Therefore, it follows immediately from (A-37) of the Appendix that the pdf for the time averaged magnitude of a component of the electric field is

$$f(E_s) = \frac{E_s}{2\gamma^2} e^{-\frac{E_s^2}{2\gamma^2}}, \tag{177}$$

a Rayleigh distribution (a chi-statistic with two degrees of freedom).

The pdf for the magnitude of the time averaged total field E can be derived in a similar manner. The total field is

$$E = \left[\sum_{s=x,y,z} (\Upsilon_{us}^2 + \Upsilon_{ws}^2) \right]^{1/2}. \tag{178}$$

Again, since the Υ are identical and independent normal r.v. with a zero mean and a standard deviation of γ, it follows from (A-39) of the Appendix that the pdf for E is

$$f(E) = \frac{E^5}{8\gamma^6} e^{-\frac{E^2}{2\gamma^2}}, \tag{179}$$

a chi-statistic with 6 degrees of freedom.

3.3.2 The pdf's for the Power Density

The pdf's for the power density at a point in a cavity follows immediately from the results for the fields presented above. The s-component of the power density P_s follows from the definition of the Poynting vector and Maxwell's equations and is given by

$$P_s = \frac{\epsilon_0 \omega}{4\pi} E_s^2 = \eta E^2 \tag{180}$$

where Z_0, the impedance of free space, has been used. Using Z_0 for the impedance of the propagating fields is valid since the partial magnetic fields are everywhere normal to the partial electric fields. Defining a new r.v. q_s as

$$q_s = E_s^2, \tag{181}$$

it follows immediately from (168), (173) and (A-38) of the Appendix that the pdf for q_s is

$$f(q_s) = \frac{1}{2\gamma^2} e^{-\frac{q_s}{2\gamma^2}}, \tag{182}$$

an exponential distribution (a χ^2-statistic with two degrees of freedom). Using (A-19) of the Appendix, the pdf for P_s can be written as

$$f(P_s) = \frac{1}{2\gamma^2/\eta} e^{-\frac{P_s}{2\gamma^2/\eta}} \tag{183}$$

which again is an exponential distribution.

Similarly, the total power density P at a point in the cavity is expressed in terms of the total electric field E as

$$P = \eta E^2. \tag{184}$$

Again, defining a new r.v. q as

$$q = E^2, \tag{185}$$

the pdf for q is

$$f(q) = \frac{q^2}{16\gamma^6} e^{-\frac{q}{2\gamma^2}} \tag{186}$$

or a χ^2-statistic with 6 degrees of freedom. This distribution follows immediately from (168), (178) and (A-40) of the Appendix. Using (A-19) of the Appendix, the pdf for P is

$$f(P) = \frac{P^2}{16(\gamma^2/\eta)^3} e^{-\frac{P}{2\gamma^2/\eta}} \tag{187}$$

which is also a χ^2-statistic with 6 degrees of freedom.

3.3.3 **The Correlation Functions**

To complete the statistics for the cavity, the correlation functions for the field and power variables are needed. For the fields and field variables there will be two kinds of correlation. The first kind is referred to as spatial correlation and results because the fields are continuous functions of position, just as was the case for the eigenvectors. The second kind is referred to as temporal correlation. This kind of correlation will occur when the bandwidths of two driving frequencies, say ω and ω', overlap. The statistics for the two frequencies will not be independent because some of the same modes are excited by each driving frequency. The spatial correlation functions are addressed first.

3.3.3.1 **The Spatial Correlation Functions**

Spatial correlation functions are developed for the amplitude of the components of the time averaged electric field, the square of the components of the amplitude of the components of the time averaged electric field and the power density associated with a field component. To derive these correlation functions, it is necessary to develop the correlation coefficients for the two partial fields E_u and E_w.

3.3.3.1.1 **The Spatial Correlation Functions for the Partial Fields**

From (166) and (167), the partial fields are

$$E_{us}(r,t) = X_{us}(r)\sin\omega t \tag{188}$$

and

$$E_{ws}(r,t) = X_{ws}(r)\sin\omega t \tag{189}$$

where $s=x,y,z$. For an over-moded cavity and large separation distances between the source point r_0 and the measurement point r, the X's are independent and identical normal r.v. with

zero mean and a standard deviation Γ of

$$\Gamma = \sqrt{M}a_0\beta^2 = \sqrt{2}\gamma. \tag{190}$$

The quantities in (190) are as previously defined. Since the X's are independent r.v. with zero mean, the only non-zero correlation functions are

$$\kappa[X_{us}(r_1),X_{us}(r_2)] = \frac{< X_{us}(r_1),X_{us}(r_2) >}{\Gamma^2} \tag{191}$$

and

$$\kappa[X_{ws}(r_1),X_{ws}(r_2)] = \frac{< X_{ws}(r_1)X_{ws}(r_2) >}{\Gamma^2} \tag{192}$$

where the definition of correlation given by (A-43) has been used.

To evaluate the expectation value in (191) and (192), recall that in terms of the eigenvectors

$$X_{us}(r) = a_0\sum_k u_{kx}(r_0)u_{ks}(r). \tag{193}$$

Then

$$< X_{us}(r_1)X_{us}(r_2) > = a_0^2\sum_{k'}\sum_{k''} < u_{k'x}(r_0)u_{k''x}(r_0) >< u_{k's}(r_1)u_{k''s}(r_2) > \tag{194}$$

where the approximation that $u_{kx}(r_0)$ and $u_{ks}(r)$ are independent r.v. has been used. Since

$$< u_{k'x}(r_0)u_{k''x}(r_0) > = \beta^2\delta_{k'k''} \tag{195}$$

it follows that

$$< X_{us}(r_1)X_{us}(r_2) > = a_0^2\beta^2\sum_{k'} < u_{k's}(r_1)u_{k's}(r_2) >. \tag{196}$$

From (140) and (143), the expectation value on the right hand side of (196) is

$$< u_{k's}(r_1)u_{k's}(r_2) > = \beta^2\kappa_S(r_1,r_2) \tag{197}$$

where

$$\kappa_S(r_1,r_2) = \frac{\sin k'|r_1-r_2|}{k'|r_1-r_2|}. \tag{198}$$

Using $k' \approx k$ and letting the sum in (197) go from $m=1$ to M yields

$$< X_{us}(r_1)X_{us}(r_2) > = Ma_0^2\beta^4\kappa_S(r_1,r_2). \tag{199}$$

Substituting (199) into (191), it follows that

$$\kappa[X_{us}(r_1),X_{us}(r_2)] = \kappa_S(r_1,r_2) \tag{200}$$

and

$$\kappa[X_{ws}(r_1),X_{ws}(r_2)] = \kappa_S(r_1,r_2). \tag{201}$$

To find the correlation functions for the amplitude of the fields and the power, the expectation value

$$< X_{us}^2(r_1)X_{us}^2(r_2) > \tag{202}$$

must be evaluated. From [15], the expectation value of the product of the square of two normal r.v. x and y is given as

$$< x^2y^2 > = <x^2><y^2> + 2<xy>^2. \tag{203}$$

Therefore, it follows from (199) that

$$< X_{us}^2(r_1)X_{us}^2(r_2) > = \Gamma^4[1 + 2\kappa_S^2(r_1,r_2)]. \tag{204}$$

A similar result can be shown to be true for the w expectation values.

Another expectation value of interest is

$$< |X_{us}(r_1)X_{us}(r_2)| >. \tag{205}$$

From [15], the expectation value of the absolute magnitude of the product of two normal r.v. x and y each with a zero mean and a standard deviation of Γ is given by

$$<|xy|> = \frac{2\Gamma^2}{\pi}[r\sin^{-1}r + \sqrt{1-r^2}] \tag{206}$$

where r is the correlation coefficient of the two r.v. Therefore, it follows that

$$< |X_{us}(r_1)X_{us}(r_2)| > = \frac{2\Gamma^2}{\pi}\left[\kappa_S(r_1,r_2)\sin^{-1}\kappa_S(r_1,r_2) + \sqrt{1-\kappa_S^2(r_1,r_2)}\right]. \tag{207}$$

The correlation coefficients for the w components are the same.

3.3.3.1.2 **The Spatial Correlation Function for the Magnitude of an Electric Field Component**

From (171), the magnitude E_s of a component of the time averaged electric field is

$$E_s(r) = \frac{1}{\sqrt{2}}\sqrt{X_{us}^2(r) + X_{ws}^2(r)}. \tag{208}$$

Defining a complex r.v. Ξ as

$$\Xi(r) = \frac{1}{\sqrt{2}}[X_{us}(r) + jX_{ws}(r)], \tag{209}$$

it follows that

$$E_s(r) = |\Xi(r)|. \tag{210}$$

The mean of Ξ is

$$<\Xi> = \frac{1}{\sqrt{2}}(<X_{us}> + j<X_{ws}>) = 0 \tag{211}$$

and the variance of Ξ is

$$< |\Xi|^2 > = \frac{1}{2}[<X_{us}^2> + <X_{ws}^2>] = \Gamma^2. \tag{212}$$

The covariance of the two complex r.v. $\Xi(r_1)$ and $\Xi(r_2)$ is

$$\mathrm{cov}[\Xi(r_1)\Xi(r_2)] = <[\Xi^*(r_1)-<\Xi^*(r_1)>][\Xi(r_2)-<\Xi(r_2)>]> = <\Xi^*(r_1)\Xi(r_2)> \tag{213}$$

where the asterisk * denotes the complex conjugate. The expectation value on the right hand side of (213) is easily evaluated using (190) and (199) or

$$<\Xi^*(r_1)\Xi(r_2)> \;=\; \frac{1}{2}[<X_{us}(r_1)X_{us}(r_2)>+<X_{ws}(r_1)X_{ws}(r_2)>] \;=\; \Gamma^2\kappa_S(r_1,r_2). \qquad (214)$$

Using (210) it follows that

$$<E_s(r_1)E_S(r_2)> \;=\; <|\Xi(r_1)||\Xi(r_2)|> \;=\; <|\Xi^*(r_1)\Xi(r_2)|>. \qquad (215)$$

From (207) and (215), the desired expectation value is

$$<E_s(r_1)E_s(r_2)> \;=\; \frac{2\Gamma^2}{\pi}\left[\kappa_S(r_1,r_2)\sin^{-1}\kappa_S(r_1,r_2) \;+\; \sqrt{1-\kappa_S^2(r_1,r_2)}\right]. \qquad (216)$$

The magnitude of a component of the time averaged electric field $E_s(r)$ is distributed according to the Rayleigh distribution of (177). Its mean is

$$<E_s> \;=\; \sqrt{\frac{\pi}{2}}\,\gamma \;=\; \sqrt{\frac{\pi}{4}}\,\Gamma \qquad (217)$$

and its variance is

$$<E_s^2> \;-\; <E_s>^2 \;=\; \left(2-\frac{\pi}{2}\right)\gamma^2 \;=\; \left(1-\frac{\pi}{4}\right)\Gamma^2. \qquad (218)$$

The covariance of $E_s(r_1)$ and $E_s(r_2)$ is

$$\mathrm{cov}[E_s(r_1),E_S(r_2)] \;=\; <E_s(r_1)E_s(r_2)> \;-\; <E_s(r_1)><E_s(r_2)>. \qquad (219)$$

Using (216)-(219) and the definition for correlation, it follows that

$$\kappa[E_s(r_1),E_s(r_2)] \;=\; \frac{(2/\pi)\left[\kappa_S(r_1,r_2)\sin^{-1}\kappa_S(r_1,r_2) \;+\; \sqrt{1-\kappa_S^2(r_1,r_2)}\right] \;-\; \pi/4}{1 \;-\; \pi/4}. \qquad (220)$$

3.3.3.1.3 **The Spatial Correlation Function for the Field Squared and Power Density**

From (171), the square of a component of the time averaged electric field is

$$E_s^2(r) \;=\; \frac{1}{2}\left[X_{us}^2(r) \;+\; X_{ws}^2(r)\right]. \qquad (221)$$

The expectation value of this r.v. evaluated at two different positions is

$$<E_s^2(r_1)E_s^2(r_2)> = \frac{1}{4}<X_{us}^2(r_1)X_{us}^2(r_2)> + \frac{1}{4}<X_{us}^2(r_1)X_{ws}^2(r_2)> + \frac{1}{4}<X_{ws}^2(r_1)X_{us}^2(r_2)> + \frac{1}{4}<X_{ws}^2(r_1)X_{ws}^2(r_2)>. \quad (222)$$

Using

$$<X_{us}^2(r_1)X_{ws}^2(r_2)> = <X_{ws}^2(r_1)X_{us}^2(r_2)> = \Gamma^4 \quad (223)$$

and (204), then

$$<E_s^2(r_1)E_s^2(r_2)> = \Gamma^4\left[1 + \kappa_S^2(r_1,r_2)\right]. \quad (224)$$

Since the square of a field component is distributed according to the exponential distribution (175), its mean is

$$<E_s^2> = 2\gamma^2 = \Gamma^2 \quad (225)$$

and its variance is

$$<E_s^4> = 4\gamma^4 = \Gamma^4. \quad (226)$$

The covariance of the r.v evaluated at two different position is

$$\mathrm{cov}\left[E_s^2(r_1)E_s^2(r_2)\right] = <E_s^2(r_1)E_s^2(r_2)> - \Gamma^4. \quad (227)$$

Then it follows from (224) and the definition of correlation, that the spatial correlation function is

$$\kappa\left[E_s^2(r_1)E_s^2(r_2)\right] = \kappa_S^2(r_1,r_2). \quad (228)$$

By inspection, the spatial correlation function for a component of the power density is the same or

$$\kappa\left[P_s(r_1)P_s(r_2)\right] = \kappa_S^2(r_1,r_2). \quad (229)$$

3.3.3.2 **The Temporal Correlation Functions**

The temporal correlation functions for the field variables are derived in a manner similar to the derivation of the spatial correlation functions. Since most of the steps are the

same, much of the detail is omitted. Again, the correlation function for the partial fields is derived first and these results are used to derive the correlation functions for the measurable field variables.

3.3.3.2.1 **The Temporal Correlation Function for the Partial Fields**

A component of the partial field due to the u eigenvectors can be written in the form

$$E_{us}(r,t) = a_0 X_{us}(k)\sin\omega t \tag{230}$$

where the wave number k now corresponds to the driving frequency or

$$k = \frac{\omega}{c} = \frac{2\pi f}{c} \tag{231}$$

and $X(k)$ is written in the form

$$X_{us}(k) = \sum_{k'=k(1-M/2)}^{k(1+M/2)} u_{k'x}(r_0)u_{k's}(r). \tag{232}$$

The expectation value of the X's evaluated at two different frequencies f_1 and f_2 is

$$\langle X_{us}(k_1)X_{us}(k_2)\rangle = a_0^2 \sum_{k'=k_1(1-M/2)}^{k_1(1+M/2)} \sum_{k''=k_2(1-M/2)}^{k_2(1+M/2)} \langle u_{k's}(r_0)u_{k''s}(r_0)\rangle\langle u_{k's}(r)u_{k''s}(r)\rangle \tag{233}$$

where it has been assumed that both the cavity Q and the number of modes M excited are approximately the same for f_1 and f_2. Using

$$\langle u_{k'x}(r_0)u_{k''x}(r_0)\rangle = \langle u_{k's}(r)u_{k''s}(r)\rangle = \beta^2\delta_{k'k''}, \tag{234}$$

then it follows that

$$\langle X_{us}(k_1)X_{us}(k_2)\rangle = \alpha M a_0^2\beta^4 = \alpha\Gamma^2 \tag{235}$$

where αM is the number modes that are excited by both f_1 and f_2. Obviously, if

$$f_2 - f_1 \geq \frac{f_1 + f_2}{2Q} \approx \frac{f_1}{Q}, \tag{236}$$

then

$$\alpha = 0. \tag{237}$$

To estimate α, it is first assumed that the mode density in the bandwidth is constant or

$$\frac{dM}{df} = \frac{M}{\Delta f} \approx \frac{QM}{f_1}. \tag{238}$$

If $f_2 \geq f_1$, then the bandwidth δf in which modes are excited by both f_1 and f_2 is

$$\delta f = f_1\left(1 + \frac{1}{2Q}\right) - f_2\left(1 - \frac{1}{2Q}\right) \approx \frac{f_1}{Q} - \Delta f \tag{239}$$

where

$$\Delta f = f_2 - f_1. \tag{240}$$

From (238) and (239), it follows that

$$\alpha \approx 1 - Q\frac{\Delta f}{f_1}. \tag{241}$$

Redefining α as

$$\alpha \equiv \kappa_T(f_1,f_2), \tag{242}$$

then it follows from the spatial correlation development that

$$\kappa[X_{ws}(f_1),X_{ws}(f_2)] = \kappa_T(f_1,f_2). \tag{243}$$

3.3.3.2.2 **The Temporal Correlation Function for the Magnitude of the Electric Field**

Following the same steps outlined in Section 3.3.3.1.2, it can be shown that

$$\kappa[E_s(f_1),E_s(f_2)] = \frac{(2/\pi)\left[\kappa_T(f_1,f_2)\sin^{-1}\kappa_T(f_1,f_2) + \sqrt{1-\kappa_T^2(f_1,f_2)}\right] - \pi/4}{1 - \pi/4}. \tag{244}$$

3.3.3.2.3 **The Temporal Correlation Functions for the Field Squared and the Power Density**

Following the same steps outline in Section 3.3.3.1.3, it can be shown that

$$\kappa[E_s^2(f_1),E_s^2(f_2)] = \kappa_T^2(f_1,f_2) \tag{245}$$

and

$$\kappa[P_s(f_1),P_s(f_2)] = \kappa_T^2(f_1,f_2). \tag{246}$$

3.3.4 Energy Considerations

From total energy considerations, the statistical parameters derived above can be expressed in terms of measurable parameters. In addition, the probability density function of the energy density in the cavity can be determined. The total energy arguments are addressed first.

3.3.4.1 Total Energy Considerations

Using the expression for the electromagnetic energy density

$$\varepsilon(r,t) = \frac{1}{2}\left[\epsilon_0 E^2(r,t) + \mu_0 H^2(r,t)\right], \tag{247}$$

the expression for the total electric field (111) and Maxwell's equations, it can be shown that the energy density of the propagating fields in the cavity is not a function of time and is simply

$$\varepsilon(r) = \frac{\epsilon_0}{2}E^2(r) = \frac{\epsilon_0}{2}\sum_{s=x,y,z} E_s^2(r). \tag{248}$$

Therefore, the total energy in the cavity is

$$\varepsilon_T = \frac{\epsilon_0}{2}\int_V E^2(r)dV = \frac{3\epsilon_0 V}{2}\langle E_s^2\rangle. \tag{249}$$

Since the expectation value of a component squared of the total electric field is Γ^2, the total energy in the cavity is

$$\varepsilon_T = \frac{3\epsilon_0 V}{2}\Gamma^2. \tag{250}$$

From antenna theory, the power radiated by an infinitesimal dipole source of length $\Delta\ell$ is

$$P_{in} = \frac{Z_0\omega^2}{12\pi c^2}(\Delta\ell I_0)^2 \tag{251}$$

or

$$(\Delta U_0)^2 = \frac{12\pi c^2}{Z_0 \omega^2} P_{in}. \tag{252}$$

Using (252) and the definition of Γ, it can be shown that in terms of the power radiated into the cavity

$$\Gamma^2 = \frac{4\pi Q}{3\epsilon_0 \omega V} P_{in}. \tag{253}$$

Substituting (253) into (250) yields for the total energy in the cavity

$$\varepsilon_T = \frac{2\pi Q}{\omega} P_{in}. \tag{254}$$

From conservation of energy, the energy lost per cycle due to Joule heating is

$$\varepsilon_D = P_{in} T \tag{255}$$

where T is the period of oscillation of the source or

$$T = \frac{1}{f}. \tag{256}$$

Substituting (255) into (254) and solving for Q yields

$$Q = \frac{\varepsilon_T}{\varepsilon_D} \tag{257}$$

which is the standard definition of Q (the ratio of the total energy in the cavity to the energy lost per cycle).

3.3.4.2 **The Statistical Model for Energy Density**

The pdf for the energy density follows immediately from (185) and (248) or

$$f(\varepsilon) = \frac{27\varepsilon^2}{2\bar{\varepsilon}^3} e^{-\frac{3\varepsilon}{\bar{\varepsilon}}} \tag{258}$$

where

$$\bar{\varepsilon} = \frac{\varepsilon_T}{V} \tag{259}$$

is the average energy density in the cavity. Similarly, the correlation functions can be shown to be

$$\kappa[\varepsilon(r_1)\varepsilon(r_2)] = \kappa_S^2(r_1,r_2) \tag{260}$$

for the energy density at two different points in the cavity and

$$\kappa[\varepsilon(f_1)\varepsilon(f_2)] = \kappa_T^2(f_1,f_2) \tag{261}$$

for the energy density at the same point for two different cavity excitation frequencies.

3.4 Summary of Statistical Model

The statistical model of the fields in a large over-moded, complex cavity can be summarized as follows:

(1) The cartesian components of the steady state electric fields are random variables defined by

$$\in_s(r,t) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \odot \begin{pmatrix} X_{us}(r) \\ X_{ws}(r) \end{pmatrix} \sin \omega t \tag{262}$$

where $s=x,y,z$ and the X's are identical independent normal r.v. with zero mean and a standard deviation of Γ or

$$f(X) = \frac{1}{\sqrt{2\pi}\Gamma} e^{-\frac{X^2}{2\Gamma^2}} \tag{263}$$

with

$$\Gamma^2 = \frac{4\pi Q}{3\epsilon_0 \omega V} P_{in} \tag{264}$$

and P_{in} is the power delivered to the cavity by an external source of energy. The correlation coefficients for the random variables measured at different positions and the correlation coefficients for the random variables measured at the same position for two different driving frequencies are

$$\kappa[X_{us}(r_1),X_{us}(r_2)] = \kappa_S(r_1,r_2) \tag{265}$$

and

$$\kappa[X_{us}(f_1),X_{us}(f_2)] = \kappa_T(f_1,f_2) \tag{266}$$

where

$$\kappa_S(r_1,r_2) = \frac{\sin k|r_1-r_2|}{k|r_1-r_2|} \tag{267}$$

and

$$\kappa_T(f_1,f_2) = \begin{cases} 1 - \dfrac{f_2-f_1}{f_1}Q & f_2-f_1 \le \dfrac{f_1}{Q} \\ 0 & \text{elsewhere.} \end{cases} \tag{268}$$

(2) The magnitude of a cartesian component of the time averaged electric field is a r.v. distributed according to a Rayleigh distribution or

$$f(E_s) = \frac{E_s}{\Gamma^2} e^{-\frac{E_s^2}{\Gamma^2}} \tag{269}$$

where $s=x,y,z$. The spatial correlation function for E_s is

$$\kappa[E_s(r_1),E_s(r_2)] = \frac{(2/\pi)\left[\kappa_S(r_1,r_2)\sin^{-1}\kappa_S(r_1,r_2) + \sqrt{1-\kappa_S^2(r_1,r_2)}\right] - \pi/4}{1 - \pi/4} \tag{270}$$

and its temporal correlation function is

$$\kappa[E_s(f_1),E_s(f_2)] = \frac{(2/\pi)\left[\kappa_T(f_1,f_2)\sin^{-1}\kappa_T(f_1,f_2) + \sqrt{1-\kappa_T^2(f_1,f_2)}\right] - \pi/4}{1 - \pi/4}. \tag{271}$$

(3) The square of a component of the time averaged electric field is a random variable distributed according to an exponential distribution or

$$f(E_s^2) = \frac{1}{\Gamma^2} e^{-\frac{E_s^2}{\Gamma^2}}. \tag{272}$$

The spatial correlation function for this random variable is

$$\kappa[E_s^2(r_1),E_S^2(r_2)] = \kappa_S^2(r_1,r_2) \tag{273}$$

and its temporal correlation function is

$$\kappa[E_s(f_1),E_s(f_2)] = \kappa_T^2(f_1,f_2). \tag{274}$$

(4) A component of the power density is a random variable distributed according to an exponential distribution or

$$f(P_s) = \frac{3}{QP_0} e^{-\frac{3P_s}{QP_0}} \tag{275}$$

where

$$P_0 = \frac{P_{in}}{V}. \tag{276}$$

The spatial correlation function for this random variable is

$$\kappa[P_s(r_1),P_s(r_2)] = \kappa_S^2(r_1,r_2) \tag{277}$$

and its temporal correlation function is

$$\kappa[P_s(f_1),P_s(f_2)] = \kappa_T^2(f_1,f_2). \tag{278}$$

The probability density function (275) best illustrates the role that the damping factor Q has in characterizing the field distributions in cavities. The mean value $<P_s>$ is

$$< P_s > = \left(\frac{P_{in}}{3V}\right) Q. \tag{279}$$

The quantity in the brackets on the right hand side of (279) is power density available to one component of the electric field due to the external source. The damping factor Q behaves as

a gain or amplification factor for the power density in the cavity.

(5) The energy density in the cavity is a random variable distributed according to a χ^2-statistic with 6 degrees of freedom or

$$f(\varepsilon) = \frac{27\varepsilon^2}{2\bar{\varepsilon}^3} e^{-\frac{3\varepsilon}{\bar{\varepsilon}}} \tag{280}$$

where

$$\bar{\varepsilon} = \frac{\varepsilon_T}{V} \tag{281}$$

is the average energy density in the cavity. The spatial correlation function for this random variable is

$$\kappa[\varepsilon(r_1),\varepsilon(r_2)] = \kappa_S^2(r_1,r_2) \tag{282}$$

and its temporal correlation function is

$$\kappa[\varepsilon(f_1),\varepsilon(f_2)] = \kappa_T^2(f_1,f_2). \tag{283}$$

This completes the development of the statistical model for the electromagnetic fields in complex cavities. Some potential applications of theory as well as limitations of the theory are discussed in the following section.

4.0 CONCLUSIONS

The distributions and correlation functions for the field variables in lossy, complex and electrically large cavities containing sources have been derived. These results are summarized by (262)-(283). A discussion of these results is presented in Section 4.1. The derivation of the statistical model required the introduction of a number of assumptions. The most critical of these assumptions is the complex cavity assumption (49). A discussion of this assumption is presented in Section 4.2. Other assumptions used in the derivation of the statistical model included the large number of modes assumption (162) and the assumption that all of the excited mode have equal energy, (147) and (148). These assumptions and their potential impact on the tails of the distributions are discussed in Section 4.3. Finally, potential applications of the statistical models are discussed in Section 4.4.

4.1 Discussion of Results

All of the field variables considered herein are shown to be distributed according to know distributions. The amplitude of a component of a partial field is shown to be distributed according to a normal distribution (262) and hence, the amplitude of the electric field associated with one of the polarizations is also distributed according to a normal distribution (the sum of two normal r.v. is also normal). This result verifies the assumption used in [10]. However, this result is only valid in the limit of an infinite number of modes being excited in the cavity. Since this is never the case, the error introduced by the assumption (162) must be evaluated. This impact of this assumption and ways of bounding the error are discussed in Section 4.3.

The magnitude of a component of the total electric field in a complex cavity is shown to be distributed according to a Rayleigh distribution (269). This result is also in agreement with the distribution reported in [10]. Since this distribution is derived from the distribution for the amplitudes of the fields associated with the each polarization, it is only valid when a large number of modes are excited in the cavity (an over-moded cavity).

The square of a component of the time averaged total electric field is shown to be distributed according to a χ^2-distribution with two degrees of freedom (272) (an exponential distribution). Since this is a derived distribution, its validity is, of course, dependent upon the same caveats as the previous distributions. This distribution is in agreement with the distribution reported in [10] where it was shown that mode-stirred chamber data supported this distribution at a 95% confidence level using chi-square goodness-of-fit tests. It was also reported in [10] that "outliers" (values higher than those predicted by an exponential distribution) were found for all test configurations. Outliers have not been observed elsewhere [8]. One possible explanation for the existence of these outliers is discussed in

Section 4.2.

The power as measured along an axis of the coordinate system is shown to be distributed according to a χ^2-distribution with two degrees of freedom (275) in agreement with the results reported in [9]. In [9], probability plots are presented for 7 different cavities representing a wide class of different shapes and Q's (these include cavities found in aircraft and ground vehicles). For each cavity considered, it was reported that the measured power levels fell within the 90% confidence limits derived for the theoretical distribution.

The energy density in the cavity is distributed according to a χ^2-distribution with six degrees of freedom (279). In this form (279), the distribution for the energy density is not easily recognized. A more familiar form is obtained by considering the contribution to the energy density from only one partial field or

$$\varepsilon_u(r) = \frac{\epsilon_0}{2}\left[X_{ux}^2(r) + X_{uy}^2(r) + X_{uz}^2(r)\right] \tag{284}$$

where ε_u is the energy density associated with the u partial field. Since the X's are independent normal r.v. with zero mean and the same standard deviation Γ, it follows from (A-36) of the Appendix that the pdf for the partial energy density is

$$f(\varepsilon_u) = \frac{2\pi}{(\pi\tau)^{3/2}}\,\varepsilon_u^{1/2}\,e^{-\frac{\varepsilon_u}{\tau}} \tag{285}$$

where

$$\tau = \epsilon_0\,\Gamma^2 \tag{286}$$

and Γ is defined by (264). Thus, the pdf for the energy density associated with a partial field has the same form as the pdf for the kinetic energy of a gas atom [16]. The pdf for the kinetic energy of a gas atom is derived directly from the Maxwell velocity distribution.

In addition to the pdf's for the five field variables discussed above, the spatial and temporal correlation functions for these variables have also been derived. Knowledge of these correlation functions is required in most applications of the theory. This will be discussed in more detail in Section 4.4. Correlation functions are extremely difficult to determine using empirical statistics and virtually no data are available to support or confirm the theoretically derived expressions. Currently, an experiment has been designed to indirectly measure the spatial correlation functions.

4.2 The Complex Cavity Assumption

The statistical model presented in Section 3.0 is only valid for electrically large cavities with complex shapes. A complex cavity was mathematically defined by (49). As previously stated, it will not be possible to *a priori* validate the complex cavity assumption for every cavity of interest. The only practical way to proceed is to assume complexity based on experience with other cavities and then use experimental data to justify the assumption.

This does not mean that no attempt should be made to validate the assumption. One possible way to validate this assumption would be to numerically determine the mode shapes of cavities with complex shapes. This is not a particularly difficult computation since the walls of the cavity can be assumed to be perfectly conducting. The moment integrals of Section 2.4 could then be numerically evaluated and the results compared to the theoretical predictions. These numerical predictions would be useful for establishing the range of validity of the complex cavity assumption. In addition, these numerical predictions would be useful for providing estimates of the residual error introduced by the complex cavity assumption. Estimates of the residual error in the moment integrals could then be used to bound the errors in the tails of the derived distributions. Knowledge of these error bounds is required for many applications of the theory.

In the previous section, the existence of outliers in measured mode-stirred chamber distributions was discussed. The source of these outliers has not been established. Since the mode-stirred chamber used for these measurements is essentially a rectangular enclosure with a mechanical stirrer added to introduce complexity, it is not difficult to envision that there could be some stirrer orientations for which the complex cavity assumption is not valid for all the excited modes. These "outlier" modes could, in some sense, correspond to modes that do not "see" the complexity of the cavity, i.e., the stirrer. The coherent contribution from these "outlier" modes would dominate the measured response and the derived distributions would not be valid for this case. These modes could, in principal, be eliminated by designing the mode-stirred chamber enclosure so that all modes "see" the complexity of the cavity, i.e., design an irregular shaped enclosure with rough walls.

In any event, the definition of cavity complexity is still an open issue. More work is required before the concept of cavity complexity is fully understood. It is essential that the error introduced by this assumption, particularly in the tails of the distributions, is bounded before predictions based on the theory are accepted as gospel.

4.3 The Number of Modes and the Equal Energy Assumptions

In addition to the complex cavity assumption, two other assumptions were introduced in the derivation of the statistical model. The assumption introduced into (162) (a large

number of modes is excited in the cavity) is used to show that the amplitudes of the partial fields are distributed according to a normal distribution. Also, it is assumed in (147) and in (148) that the energy of every excited mode is the same. This assumption was used to simplify the expressions for the partial fields. Since each assumption is applied to an "exact" expression, one can, in principal, bound the error introduced by these assumptions.

For the number of modes assumption, the concern again is the error introduced in the tails of the distribution. For even a very few number of modes, say $M=3$, the assumption is probably valid if one is only concerned with values of the r.v. near the mean of the distribution. However, if one is concerned with values of the r.v. in the tails of the distribution, the approximation is probably not valid unless M is very large. Since the "exact" value of the distribution for one mode is known, it is possible to estimate the error introduced by this assumption and the magnitude of the error will be a function of the number of modes M. One way to estimate this error is to express the error as a series in terms of Hermite polynomials [15]. An effort to quantify this error will be initiated in the near future.

The second assumption is probably not as critical as the first assumption providing that the cavity Q is large. However, this requirement is in conflict with the previous assumption since the number of modes M is inversely proportional to Q (146). Therefore, the impact of this assumption is not independent of the previous assumption and the two assumptions must be addressed in parallel. Again, since "exact" expressions for the partial fields are known, the error introduced in the distributions by this assumption can be estimated.

4.4 Potential Applications

There are a number of potential applications of the derived statistical model. Two obvious applications are: (1) the analysis of mode-stirred chamber test data and (2) the survivability/vulnerability assessments of systems containing large complex cavities subjected to high frequency electromagnetic environments. These applications of the statistical model are briefly discussed below.

4.4.1 Mode-Stirred Chamber Application

A mode-stirred chamber (MSC) is a test facility consisting of a metallic enclosure, a source antenna located inside the enclosure and an electrically large and irregularly shaped paddle wheel (stirrer) also located inside the enclosure. A test object is placed inside of the enclosure and the response of the test object is measured for a large number of positions of the stirrer. The location of the test object in the enclosure is unimportant as long as it is at least one-half wavelength away from the walls of the enclosure. The costs associated with constructing and operating this type of test facility are considerable less than comparable costs associated with anechoic chamber testing. This includes source costs since the field strengths

in the cavity are enhanced by the Q of the cavity.

A deterministic characterization of the response of test objects to the MSC fields is not feasible due to the geometric complexity of the cavity. As a result, a statistical characterization is attractive since the MSC in some sense can be represented as a large complex cavity.

Although, the statistical model developed herein is based on the assumption that the position vector $\boldsymbol{r}$ in the cavity is a random variable, it easy to demonstrate that these statistical models also apply to the fields in mode-stirred chambers. The probability density functions and the correlation functions are shown to be independent of the shape of the cavity as long as it satisfies the definition of a complex cavity. Therefore, the statistical model is valid for the set of all complex cavities with constant volume V and constant Q. All of the cavities belonging to this set of constant V and Q complex shaped cavities is called an ensemble and the volume averages can be replaced by ensemble averages. For each stirrer position, a MSC represents a different shaped complex cavity having the same volume and Q as any other stirrer position. Therefore, measurements performed at different stirrer positions correspond to measurements performed for different members of the ensemble of cavities. As a result, averages over stirrer positions are equivalent to ensemble averages which in turn are equivalent to volume averages and the statistical models are applicable to the analysis of mode-stirred chamber test data.

It can also be argued that since the volume of the cavity is not a function of frequency and the theoretical value [8] of the Q of the cavity is a slowly varying function of frequency $(Q \sim f^{1/2})$, then the cavity statistics can be generated by varying the frequency of the source antenna over a limited frequency range (frequency sweeping). While this may be true for the probability density functions, it is not true for the correlation functions. The correlation functions dependent on the source frequency and as a result, the test object response in a MSC is also a function of frequency. Therefore, the response of the test object must also be a slowly varying function of frequency before frequency sweeping is a valid approach for generating cavity statistics. If the test object's response is not a slowly varying function over the range of the frequency sweep, then the statistical models developed herein are no longer valid.

Although many predictions made using the statistical model developed herein are in excellent agreement with measured MSC data, there are still a number of open issues associated with the statistics of mode-stirred chambers. These include: the optimum design of the MSC stirrer and the enclosure itself, the effect of the test object on the Q of the cavity, the validation of the correlation functions and the relationship, if any, of mode-stirred

chamber test data to free field or anechoic chamber test data.

4.4.2 Survivability/Vulnerability Assessment Applications

There are two areas where applications of the statistical model to S/V assessment problems may be valuable. These are: (1) test design and data analysis and (2) predictions of probability of effect of EM environments on large systems.

Measurements of the high frequency response of complex systems and the analysis of the resulting data have proved be notoriously difficult tasks to perform. Small changes in the test configuration and/or small changes in the test object geometry often result in very large changes in the measured responses. Statistical test techniques may prove to be useful for resolving many of these difficulties. While the result of a given experimental configuration may not be repeatable, the average value of many suitably randomized configurations may provide accurate estimates of the invariant system parameters. Other than mode-stirred chamber testing, the concept of system level EM statistical testing has not received much attention. This concept needs to be explored in more detail before more concrete statements concerning its applicability can be made.

In the past, predictions of the probability of effect of EM environments on large complex systems have usually not provided very meaningful or satisfactory results. For very large systems, the most common result obtained is that the system will be adversely affected at very low field levels of the EM environment. Experimental evidence usually does not support these predictions. For most of these predictions, log-normal distributions have been used to characterize both the stress and strength of the system elements. Log-normal distributions are attractive choices for these distributions because the two parameters associated with the log-normal distribution are shape and scale parameters and thus, it is possible to approximate many measured distributions with this distribution. It has been well-known for many years that this distribution is probably not correct and that the error introduced by the tails of these distributions is a major contributor to the meaningless and unsatisfactory probability of effect predictions.

The consequence of choosing log-normal distributions is more apparent when the stress or strength are expressed in the units of dB. Under this transformation of variables, the new distribution is a normal distribution and consequently, the random variable extends over the range from minus infinity to plus infinity. Since most physical variables usually have upper or lower limits (sometimes both), it is clear that for many applications. the normal distribution is inadequate, particularly for describing the tails of distributions.

From a physical point of view, the derived distributions for the cavity fields are much better behaved. For example, the distribution for a component of the power density is an

exponential distribution. Under the transformation of the variable from watts per meter^3 to dB-watts per meter^3, the new distribution is given by the smallest element Type I extreme value distribution defined by (A-42) of the Appendix. Although the random variable for this distribution also extends from minus infinity to plus infinity, in the limit of large power densities, the distribution $f(P)$ is approximately

$$f(P) \approx e^{-e^{P}}. \tag{287}$$

Thus for large values of power density, the probability that power density equals or exceeds a given level approaches zero much faster than for the case of a normal distribution or for that matter most conventional distributions. In effect, the derived distributions have a built-in "cut-off" in agreement with the physics of the interaction. As a consequence, the derived distributions potentially have the capability to resolve the issue associated with the low field level predictions of adverse system effects. More work is required before this can be demonstrated. This will also require the development of strength distributions that are in better agreement with the physics of the failure mechanisms.

4.5 **Concluding Remarks**

A statistical theory of the electromagnetic fields in complex cavities has been presented. Their are many open issues and questions associated with this theory which remain to be resolved and answered. These include the range of validity of the assumptions used in the derivations and the impact of these assumptions on applications of the theory. Notwithstanding these questions and issues, the models resulting from a statistical characterization of the electromagnetic fields in complex cavities are relatively simple when compared to their deterministic counterparts. Admittedly, the statistical models are not capable of predicting the values of field variables at every instant and location within the cavity; however, in the absence of perfect data sets, they provide a simple alternative to more conventional, deterministic approaches. The robustness of this theory suggests the potential applicability of statistical characterizations to other classes of EM interaction problems. This would include experiment design and the analysis and interpretation of test data.

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APPENDIX

PROBABILITY THEORY CONSIDERATIONS

A.1 INTRODUCTION

This appendix is intended to help readers familiarize themselves with some of the techniques of probability theory used in this paper. It is not intended to be a complete dissertation on probability theory; but rather, an aid to readers with some familiarity with probability theory. For a more complete treatment of probability theory, it is recommended that the readers refer to the references provided at the end of this appendix.

It should be made clear at the beginning of this appendix that probability theory is an axiomatic theory and does not depend upon a particular definition of probability. The calculus of probability theory can be developed by requiring probability to satisfy three postulates [A.1]. In this paper, there is no need to become intertwined in the controversy between the frequency and subjective interpretations of probability.

A.2 DEFINITION OF A RANDOM VARIABLE

A real random variable (r.v.) X is a real function whose domain is the space ζ [i.e. a process of assigning a real number $X(\zeta)$ to every outcome ζ of an experiment Ξ] such that:

1. The set $\{X \leq x\}$ is an event for any real number x.
2. The probability of the events $\{X=+\infty\}$ and $\{X=-\infty\}$ equals zero:

$$P\{X=+\infty\} = P\{X=-\infty\} = 0. \tag{A-1}$$

The distribution function of the r.v. X is the function

$$F(x) = P\{X \leq x\} \tag{A-2}$$

defined for any number x from $-\infty$ to ∞. Thus, for a given x, $F(x)$ equals the probability of the event $\{X \leq x\}$ consisting of all outcomes ζ such that $X(\zeta) \leq x$. For brevity, it is usually said that $F(x)$ equals the probability that $X \leq x$. The distribution function has the following properties:

(a) At $x=-\infty$ and $x=-\infty$ its values are:

$$F(-\infty) = 0 \qquad F(+\infty) = 1. \tag{A-3}$$

(b) It is a nondecreasing function of x:

$$F(x_1) \leq F(x_2) \qquad for \quad x_1<x_2. \tag{A-4}$$

(c) It is continuous from the right:

$$F(x^+) = F(x). \tag{A-5}$$

The distribution function is often referred to as the cumulative density function (cdf).

The derivative

$$f(x) = \frac{dF(x)}{dx} \tag{A-6}$$

of the distribution function is called the density (function) of the r.v. x (it is also known as the frequency function but is usually referred to as the probability density function or pdf). Since the derivative of $F(x)$ may not exist everywhere, one can distinguish several types of random variables. In this paper, only distribution functions that are continuous functions of x are considered. For this case, the number of points at which $F(x)$ is not differentiable need only be countable. From the monoticity of $F(x)$, it follows that $f(x)$ is nonnegative or

$$f(x) \geq 0 \tag{A-7}$$

and from (A-3) and (A-6) that

$$\int_{-\infty}^{\infty} f(x)dx = F(\infty) - F(-\infty) = 1. \tag{A-8}$$

It also follows that

$$F(x) = \int_{-\infty}^{x} f(u)du. \tag{A-9}$$

A.3 THE EXPECTED VALUE AND MOMENTS OF RANDOM VARIABLES

The expected value of a real r.v. X is defined by the integral

$$e\{X\} = \int_{-\infty}^{\infty} xf(x)dx. \tag{A-10}$$

If the r.v. Y is a function of r.v. X or $Y=g(X)$, then the expected value of Y is given by

$$e\{Y\} = \int_{-\infty}^{\infty} yf_Y(y)dy \tag{A-11}$$

or

$$e\{Y\} = e\{g(X)\} = \int_{-\infty}^{\infty} g(x) f_X(x) dx \tag{A-12}$$

where $f_X(x)$ and $f_Y(y)$ are the pdf's of the r.v.'s X and Y, respectively.

The moments m_k of a r.v. X are defined by

$$m_k = e\{X^k\} = \int_{-\infty}^{\infty} x^k f(x) dx \tag{A-13}$$

where k is a non-zero positive integer.

A.4 THE CHARACTERISTIC FUNCTION OF A RANDOM VARIABLE

The characteristic function $\Phi(\nu)$ of a r.v. X is the Fourier transform of its density function $f(x)$ and is used to simplify certain operations involving x. Therefore,

$$\Phi(\nu) = \int_{-\infty}^{\infty} e^{j\nu x} f(x) dx \tag{A-14}$$

or

$$\Phi(\nu) = e\{e^{j\nu x}\}. \tag{A-15}$$

The density function $f(x)$ can be expressed in terms of $\Phi(\nu)$ by the integral

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\nu) e^{-j\nu x} d\Omega \tag{A-16}$$

known as the inversion formula.

A.5 THE MOMENT THEOREM

The derivatives of the characteristic function of a r.v. X are related to its moments by

$$\frac{d^n \Phi(0)}{d\nu^n} = j^n m_n. \tag{A-17}$$

The proof of (A-17) follows by expanding the exponential in (A-14) or

$$\Phi(\nu) = \int_{-\infty}^{\infty} f(x)(1 + j\nu m_1 + \cdots + \frac{(j\nu x)^n}{n!} + \cdots) dx.$$

Assuming that term by term integration is valid, then

$$\Phi(\nu) = 1 + j\nu m_1 + \cdots + \frac{(j\nu)^n}{n!} m_n + \cdots \tag{A-18}$$

and (A-17) follows.

The above operation holds if the moments m_n are finite and the series converges absolutely near $\nu = 0$. In this case $\Phi(\nu)$ is given by (A-18) and therefore $f(x)$ is uniquely determined from its moments.

A.6 FUNCTIONS OF A SINGLE RANDOM VARIABLE

Consider the case where the r.v. Y is a function of a single r.v. X such that $Y = g(X)$. If both $g(X)$ and the pdf $f_X(x)$ are known, then it is possible to derive the pdf $f_Y(y)$. Some examples that are used in the text of this paper are presented below:

Example 1: The pdf of y when $y = a + bx$ where a and b are constants

$$f_Y(y) = \frac{1}{|b|} f_X\left(\frac{y - a}{b}\right) \tag{A-19}$$

Example 2: The pdf of y when $y = 1/x$

$$f_Y(y) = \frac{1}{y^2} f_X\left(\frac{1}{y}\right) \tag{A-20}$$

Example 3: The pdf of y when $y = e^x$

$$f_Y(y) = \frac{1}{|y|} f_X(\ln y) \tag{A-21}$$

Example 4 : The pdf of y when $y = \ln x$

$$f_Y(y) = e^y f(e^y) \tag{A-22}$$

Example 5: The pdf of y when $y = x^2$

$$f_Y(y) = \frac{1}{2\sqrt{y}} [f_X(+\sqrt{y}) + f_X(-\sqrt{y})] \tag{A-23}$$

A.7 UNCORRELATED, ORTHOGONAL, INDEPENDENT RANDOM VARIABLES

Two r.v. X and Y are called uncorrelated if

$$e\{XY\} = e\{X\}e\{Y\}. \tag{A-24}$$

They are called orthogonal if

$$e\{XY\} = 0 \tag{A-25}$$

and independent if

$$f(x,y) = f_X(x)f_Y(y). \tag{A-26}$$

If X and Y are independent, then they are also uncorrelated. However, if X and Y are uncorrelated, they are not necessarily independent.

A.8 FUNCTIONS OF MORE THAN ONE INDEPENDENT RANDOM VARIABLE

In the main text, it will be necessary to determine the pdf of a random variable that is a function of more than one independent random variable. In general, it is usually possible to derive the pdf of a r.v. Z that is a function of two random variables X and Y when the pdf's for both X and Y are known. Some useful formulas for implementing these derivations are given below:

Example 1: The pdf of the sum of two independent random variables $(z = x + y)$ is

$$f_Z(z) = \int f_X(u)f_Y(z - u)du = \int f_X(z - u)f_Y(u)du \tag{A-27}$$

Example 2: The pdf of the product of two independent random variables $(z = xy)$ is

$$f_Z(z) = \int \frac{1}{|u|} f_X(u) f_Y\left(\frac{z}{u}\right) du = \int \frac{1}{|u|} f_X\left(\frac{z}{u}\right) f_Y(u)du \tag{A-28}$$

Example 3: The pdf of the quotient of two independent random variables $(z = x/z)$ is

$$f_Z(z) = \int |u| f_X(zu)f_Y(u)du = \int \frac{|u|}{z^2} f_X(u) f_Y\left(\frac{u}{z}\right) du \tag{A-29}$$

Example 4: The pdf of a sum of n independent random variables $(z = x_1 - \cdots - x_n)$ is

$$f_Z(z) = f_1(z) * \cdots * f_n(z) \tag{A-30}$$

i.e. the pdf of z is the convolution of the pdf's of $x_1, \ldots, x_n$.

A.9 THE NORMAL DISTRIBUTION

The normal (or Gaussian) distribution is the most frequently used statistical model. Its pdf is

$$f(x) = \frac{1}{\sqrt{2\pi}\beta} e^{-\frac{(x - x_0)^2}{2\beta^2}}, \qquad (A\text{-}31)$$

$$-\infty < x < \infty, \; -\infty < x_0 < \infty, \; \beta > 0$$

where x_0 and β are the mean and standard deviation respectively of the distribution. If $x_0=0$, then the moments m_n of the normal distribution are given by

$$m_n = \begin{cases} 1 \cdot 3 \cdot \cdots \cdot (n - 1)\beta^n & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases} \qquad (A\text{-}32)$$

The characteristic function $\Phi(\nu)$ of a normal distribution with zero mean is

$$\Phi(\nu) = e^{-\frac{1}{2}\beta^2\nu^2}. \qquad (A\text{-}33)$$

A.10 CHI AND CHI-SQUARED STATISTICS

Consider n independent normal r.v., $x_1, \ldots, x_n$, with a mean of zero and a standard deviation of β. Their χ and χ^2 statistics are defined by

$$\chi = \sqrt{x_1^2 + \cdots + x_n^2} \qquad y = \chi^2 = x_1^2 + \cdots + x_n^2. \qquad (A\text{-}34)$$

The number n is called the number of degrees of freedom of these statistics. Their respective densities are given by

$$f_\chi(\chi) = \frac{2}{2^{n/2}\beta^n\Gamma(n/2)} \chi^{n-1} e^{-\frac{\chi^2}{2\beta^2}} \qquad (A\text{-}35)$$

and

$$f_y(y) = \frac{1}{2^{n/2}\beta^n\Gamma(n/2)} y^{(n-2)/2} e^{-\frac{y}{2\beta^2}} \qquad (A\text{-}36)$$

where $\Gamma(z)$ is the usual gamma function [A.3].

In the text of this paper, two values of n are of interest, namely $n = 2$ and $n = 6$. For the case when $n = 2$ i.e. two degrees of freedom, the χ-statistics is

$$f_\chi(\chi) = \frac{\chi}{\beta^2} e^{-\frac{\chi^2}{2\beta^2}}, \tag{A-37}$$

a Rayleigh distribution. Similarly, the χ^2-statistics is

$$f_y(y) = \frac{1}{2\beta^2} e^{-\frac{y}{2\beta^2}}, \tag{A-38}$$

an exponential distribution.

For the case when $n = 6$ i.e. six degrees of freedom, the χ-statistics is

$$f_\chi(\chi) = \frac{\chi^5}{8\beta^6} e^{-\frac{\chi^2}{2\beta^2}} \tag{A-39}$$

and the χ^2-statistics is

$$f_y(y) = \frac{y^2}{16\beta^6} e^{-\frac{y}{2\beta^2}}. \tag{A-40}$$

A.11 A USEFUL EXTREME VALUE STATISTICS

Many electrical engineering parameters are expressed in term of dB. For example, power is often expressed in terms of dB-watts. In this case, the relationship between power p in watts and power P in dB-watts is given by $P = 10 \log p$. Consider the case when the power p is a r.v. that is exponentially distributed or

$$f_p(p) = \frac{1}{p_0} e^{-\frac{p}{p_0}} \tag{A-41}$$

where p_0 is the mean of p. With the help of (A-19) and (A-22) and the identity $\log x = 0.43429 \ln x$, it can be shown that the power P in dB-watts is distributed according to a smallest element Type I extreme value distribution [A.2] or

$$f_P(P) = \frac{1}{\beta}\exp\left[\frac{1}{\beta}(P - P_0) - e^{(1/\beta)(P - P_0)}\right]. \qquad \text{(A-42)}$$

In the above equation P_0 and β are the location and scale parameters and are given by $P_0 = \beta ln\, p_0$ and $\beta = 4.3429$. The mean P' of the distribution is given by $P' = P_0 - 0.557\beta$ and the standard deviation for the distribution β' is given by $\beta' = 1.283\beta$. Thus, the standard deviation for this distribution (an extreme value distribution derived from an exponential distribution) is always 5.57 dB. This is an interesting distribution because for large values of P, the distribution behaves as $exp(-e^P)$. Therefore, for large values of power (in dB), the probability density function approaches zero much faster than the more conventional distributions. Effectively, the distribution has a built-in "cut-off."

A.12 THE CORRELATION COEFFICIENT

In order to complete the statistics of more than one r.v., it is necessary to specify the linear relationship between the random variables. A standardized measure of this relationship is called the correlation coefficient. For the two random variables, x and y, the correlation coefficient κ is defined by

$$\kappa = \frac{e\{(x - x_0)(y - y_0)\}}{\sqrt{e\{(x - x_0)^2\}e\{(y - y_0)^2\}}} = \frac{cov(x,y)}{\beta_x\beta_y} \qquad \text{(A-43)}$$

where (x_0,y_0) are the means and (β_x, β_y) are the standard deviations of the two r.v. When $\kappa = 1$, the two r.v. are said to be positively perfectly correlated i.e $y = kx + c$, where k and c are constants. When $\kappa = -1$, the two r.v. are said to be negatively perfectly correlated i.e. $y = -kx + c$. When $\kappa = 0$, the two r.v. are said to be uncorrelated.

A.13 A USEFUL THEOREM FOR NORMAL RANDOM VARIABLES

In section A.7 of this appendix, it was stated that two uncorrelated random variables are not necessarily independent. A theorem in probability theory states: "If two jointly normal r.v x and y are uncorrelated, then they are independent." The proof of this theorem follows trivially from the expression for the joint density $f(x,y)$ of two normal r.v. with zero means or

$$f(x,y) = \frac{1}{2\pi\beta_x\beta_y\sqrt{1 - \kappa^2}}\exp\left[-\frac{1}{2(1 - \kappa^2)}\left(\frac{x^2}{\beta_x^2} - \frac{2\kappa xy}{\beta_x\beta_y} + \frac{y^2}{\beta_y^2}\right)\right] \qquad \text{(A-44)}$$

If $\kappa = 0$ i.e. the two r.v. are uncorrelated, then $f(x,y) = f_x(x)f_y(y)$ and from (A-26) if follows that x and y are independent. The importance of this theorem will become evident in the main text where it is easy to demonstrate that two r.v. are uncorrelated. However, the much stronger statement of independence follows only because the r.v. are normal.

A.14 THE PDF OF THE PRODUCT OF TWO INDEPENDENT NORMAL R.V.

In the main body of the text, the pdf for the product of two independent and normal r.v. is required. Since (a not very intensive) literature search failed to unveil such a pdf , the derivation of the distribution is presented here. The derivation is not very difficult and must exists somewhere in the open literature.

Consider the r.v. z defined by $z = xy$ where x and y are independent normal r.v each having a zero mean and the same standard deviation β. Then from (A-29), the pdf of z in terms of the pdf's for x and y is given by

$$f_z(z) = \frac{1}{2\pi\beta^2}\int_{-\infty}^{\infty}\frac{1}{|u|}\exp\left[-\frac{1}{2\beta^2}\left(u^2 + \frac{z^2}{u^2}\right)\right]du. \tag{A-45}$$

Since the above integral is an even function of u, the pdf can be written as

$$f_z(z) = \frac{1}{\pi\beta^2}I(z) \tag{A-46}$$

where the integral $I(z)$ is given by

$$I(z) = \int_0^{\infty}\frac{1}{u}\exp\left[-\frac{1}{2\beta^2}\left(u^2 + \frac{z^2}{u^2}\right)\right]du. \tag{A-47}$$

The above integral is evaluated by first taking its derivative with respect to z and integrating by parts to yield

$$I'(z) = -\frac{1}{z\beta^2}\int_0^{\infty}u\,\exp\left[-\frac{1}{2\beta^2}\left(u^2 + \frac{z^2}{u^2}\right)\right]. \tag{A-48}$$

Taking the second derivative of $I(z)$ and setting $v = z/\beta^2$ yields the second order differential equation for $I(v)$:

$$v^2 I''(v) + vI'(v) - v^2 I(v) = 0. \tag{A-49}$$

The solution to this equation is given in [A-3] as

$$I(z) = AI_0\left(\frac{z}{\beta^2}\right) + BK_0\left(\frac{z}{\beta^2}\right) \tag{A-50}$$

where A and B are constants and $I_0(x)$ and $K_0(x)$ are modified zeroth-order Bessel functions of the first and second kind, respectively. Using the fact that the pdf of z must be symmetric about $z = 0$ yields the following result

$$f_z(z) = \frac{1}{\pi\beta^2}\left[AI_0\left(\frac{|z|}{\beta^2}\right) + BK_0\left(\frac{|z|}{\beta^2}\right)\right]. \tag{A-51}$$

The constant A can be set to zero since the pdf must be finite at $z = \pm\infty$. The constant B can be determined by using (A-8) or

$$\frac{2B}{\pi\beta^2}\int_0^\infty K_0\left(\frac{z}{\beta^2}\right)dz = 1$$

where the fact that the integrand is an even function in z has been used. The above integral can be evaluated by using the identity [A.3]

$$\frac{1}{c} = \frac{2}{\pi}\int_0^\infty \cos(at)K_0(bt)dt \tag{A-52}$$

where $c = (a^2 - b^2)^{1/2}$. Setting $a = 0$ and $b = \beta^{-2}$, it can be shown that $B = 1$, so that $f_z(z)$ is

$$f_z(z) = \frac{1}{\pi\beta^2}K_0\left(\frac{|z|}{\beta^2}\right). \tag{A-53}$$

The characteristic function $\Phi_z(\nu)$ of $f_z(z)$ is given by

$$\Phi_z(\nu) = \frac{2}{\pi\beta^2}\int_0^\infty \cos(\nu z)K_0\left(\frac{z}{\beta^2}\right)dz \tag{A-54}$$

where $cos(\nu z)$ and $sin(\nu z)$ have been used as even and odd functions of z, respectively. The integral of (A-54) is evaluated using the identity (A-52) with $a = \nu$ and $b = \beta^{-2}$. This yields

$$\Phi_z(v) = \frac{1}{\sqrt{1 + \beta^4 v^2}} \qquad \text{(A-55)}$$

for the characteristic function of the product of two independent normal random variables.

A.15 ORDER STATISTICS

In evaluating test data, the distribution of the largest and/or smallest observation is often of interest. These distributions belong to a class of statistics referred to as order statistics.

Order statistics is defined as follows: Let $X_1, X_2, \ldots, X_n$ be a random sample from a probability density function $f_X(x)$. The n observations are arranged in ascending order so that $X_{(1)} \leq X_{(2)} \leq \cdots\cdots \leq X_{(n)}$ where $X_{(1)}$ is the smallest sample from the observation and $X_{(n)}$ is the largest. $X_{(1)}$ is called the first order statistic and $X_{(n)}$ is called the nth order statistic. In general, $X_{(m)}$ is called the mth order statistic and it has $m-1$ observations smaller than it. It can be shown [A.4] that the pdf for the mth order statistic is

$$f_{X_{(m)}}(x) = m\binom{n}{m}[F_X(x)]^{m-1}[1 - F_X(x)]^{n-1} f_X(x) \qquad \text{(A-56)}$$

where $F_X(x)$ is the distribution function of $f_X(x)$ and the quantity in the brackets is the binomial coefficient. For $m = 1$, (A-56) provides the pdf of the first (smallest) order statistic as

$$f_{X_{(1)}}(x) = n[1 - F_X(x)]^{n-1} f_X(x). \qquad \text{(A-57)}$$

For $m = n$. (A-56) provides the pdf of the last (largest) order statistic as

$$f_{X_{(n)}}(x) = n[F_X(x)]^{n-1} f_X(x). \qquad \text{(A-58)}$$

A.16 REFERENCES

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