Interaction Notes

Note 495

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Scattering from Finite Wedges and Half Spaces

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Abstract

The far-scattering from finite-length wedges and finite-dimensioned half spaces is developed based on dilation symmetry with respect to cones, wedges, and half spaces. This leads to a form (decomposition) of the scattering which is exact for early times, involving a few terms which are each real 2 x 2 dyadics times a simple temporal operation (integral, identity, derivative) on the incident plane-wave field. The time of validity is given by the first appearance of multiple scattering at the observer.

I. Introduction

In [3] it was shown that the class of scatterers satisfying dilation symmetry has the far-scattered fields expressable as a real dyadic (2 x 2) times a power of the complex frequency depending on the dimensions (3, 2, 1) describing the expansion of the far-scattered field. The three-dimensional case corresponds to a general cone of arbitrary cross section which can include a variety of perfect conductors, dielectrics, etc. The two- and one-dimensional cases correspond to infinite length wedges and half-spaces. In time domain the frequency part becomes a convolution which can also be thought of as fractional integration of order (d-1)/2 where d is the number of dimensions concerned. Furthermore in time domain certain truncations of these structures away from the cone tip, wedge edge, and planar boundary of the half space are allowed. This reduces the time of validity of the results from infinity to a certain clear time based on the additional time (after first scattered signal) required for the incident wave to scatter from the truncation and reach the far-field observer.

The present paper goes a step further by extending the previous results based on dilation symmetry to the cases of wedges of finite length and half spaces with the boundary plane (face) of finite extent. These truncations are made in the form of planar cuts which preserve the dilation symmetry for the various waves, at least until multiple scattering is observed. Due to the finite extent of these scatterers the far-field Green's function in a three-dimensional sense is appropriate. As in the previous paper the far-scattered fields are represented now by a finite sum of terms, each of which exactly factors into a real-valued 2 x 2 dyadic coefficient times a simple temporal operation on the incident field.

Summarizing some important equations concerning dilation (generalized-conical) symmetry we have the dilation [4]

$$\vec{r}\,' = \chi \vec{r} \;,\; t' = \chi t \;,\; s' = \frac{s}{\chi}$$

$$\begin{aligned} &\chi > 0 \text{ (scaling parameter)} \\ &\sim \;\equiv \text{Laplace Transform (two-sided)} \\ &s \equiv \Omega + j\omega \equiv \text{Laplace-transform variable} \\ &\qquad\qquad \equiv \text{complex frequency} \end{aligned} \tag{1.1}$$

We need the usual cylindrical and spherical coordinates related to cartesian coordinates as

$$x = \Psi\cos(\phi) \quad , \quad y = \Psi\sin(\phi)$$
$$z = r\cos(\theta) \quad , \quad \Psi = r\sin(\theta) \tag{1.2}$$

The fields in the Maxwell equations are invariant under the transformation (1.1) provided

$$\tilde{\vec{E}}(\vec{r},s) = \tilde{\vec{E}}(\vec{r}\,',s') \quad , \quad \tilde{\vec{H}}(\vec{r},s) = \tilde{\vec{H}}(\vec{r}\,',s')$$
$$\tilde{\vec{D}}(\vec{r},s) = \tilde{\vec{D}}(\vec{r}\,',s') \quad , \quad \tilde{\vec{B}}(\vec{r},s) = \tilde{\vec{B}}(\vec{r}\,',s')$$
$$\tilde{\vec{J}}(\vec{r},s) = \chi\,\tilde{\vec{J}}\,'(\vec{r}\,',s') \quad \text{(current density, like } s\text{)} \tag{1.3}$$
$$\tilde{\vec{J}}_s(\vec{r},s) = \tilde{\vec{J}}_s(\vec{r}\,',s') \quad \text{(sheet or surface current density, like } \vec{H}\text{)}$$

with constitutive parameters scaling like

$$\tilde{\overleftrightarrow{\epsilon}}(\vec{r},s) = \tilde{\overleftrightarrow{\epsilon}}(\vec{r}\,',s') \quad , \quad \tilde{\overleftrightarrow{\mu}}(\vec{r},s) = \tilde{\overleftrightarrow{\mu}}(\vec{r}\,',s')$$
$$\tilde{\overleftrightarrow{\sigma}}(\vec{r},s) = \chi\,\tilde{\overleftrightarrow{\sigma}}(\vec{r}\,',s) \tag{1.4}$$

including sheet admittances as

$$\tilde{\overleftrightarrow{Y}}_s(\vec{r}_s,s) = \tilde{\overleftrightarrow{Y}}_s(\vec{r}\,'_s,s')$$
$$\vec{r}_s \equiv \text{coordinates on surface} \tag{1.5}$$

Simple examples include frequency-independent parameters as

$$\overleftrightarrow{\epsilon} = \overleftrightarrow{\epsilon}(\theta,\phi) \quad , \quad \overleftrightarrow{\mu} = \overleftrightarrow{\mu}(\theta,\phi) \quad , \quad \overleftrightarrow{\sigma} = \frac{1}{r}\overleftrightarrow{\Sigma}(\theta,\phi)$$
$$\overleftrightarrow{Y}_s = \overleftrightarrow{Y}_s(\theta,\phi) \tag{1.6}$$

There is then a variety of cases with this symmetry, including perfectly conducting sheets, simple dielectrics, and hybrids of these and more complicated spatial distributions.

The incident field is taken as a plane wave of the form

$$\tilde{\vec{E}}^{(inc)}(\vec{r},s) = E_o \, \vec{1}_p \, \tilde{f}(s) \, e^{-\gamma \vec{1}_i \cdot \vec{r}} \; , \; \vec{E}^{(inc)}(\vec{r},t) = E_o \, \vec{1}_p \, f\left(t - \frac{\vec{1}_i \cdot \vec{r}}{c}\right)$$

$$\tilde{\vec{H}}^{(inc)}(\vec{r},s) = \frac{E_o}{Z_o} \, \vec{1}_i \times \vec{1}_p \, \tilde{f}(s) \, e^{-\gamma \vec{1}_i \cdot \vec{r}} \; , \; \vec{H}^{(inc)}(\vec{r},t) = \frac{E_o}{Z_o} \, \vec{1}_i \times \vec{1}_p \, f\left(t - \frac{\vec{1}_i \cdot \vec{r}}{c}\right)$$

$$\vec{1}_i \equiv \text{direction of incidence}$$

$$\vec{1}_i \cdot \vec{1}_p = 0 \quad , \quad f(t) \equiv \text{incident waveform}$$

$$\gamma \equiv \frac{s}{c} \equiv \text{propagation constant}$$

$$c = \left[\mu_o \, \epsilon_o\right]^{-\frac{1}{2}} \equiv \text{speed of light}$$

$$Z_o = \left[\frac{\mu_o}{\epsilon_o}\right]^{\frac{1}{2}} \equiv \text{wave impedance of free space} \tag{1.7}$$

$$E_o \equiv \text{amplitude factor (V/m)}$$

For convenience $\tilde{f}(s)$ is typically taken as unity for frequency-domain analysis and later reinserted in the final results. Another convenient way to view this scattering process is to let $f(t) = u(t)$ in time domain and correct for an arbitrary waveform later.

The coordinate origin can be taken at any place of convenience. As we shall see, the cases to be considered have dilation symmetry about more than one point, so we may even have various convenient selections of coordinate origin for the waves of interest.

In the previous paper [3] the scattered far fields expanded according to the number of dimensions appropriate to the geometries, some of which were infinite in the sense of edges or planar surfaces, all contributing to the far fields. In the present paper such (transverse) distances are all taken as finite. So in the limit as $r \to \infty$, the far field diverges in a three-dimensional sense giving [2]

$$\tilde{\vec{E}}_f(\vec{r},s) = -\frac{s\mu_0}{4\pi r} e^{-\gamma r} \overleftrightarrow{1}_o \cdot \int_V \tilde{\vec{J}}(\vec{r}'',s)\, e^{\gamma \vec{1}_o \cdot \vec{r}''} dV''$$

$$\overleftrightarrow{1}_o \equiv \overleftrightarrow{1} - \vec{1}_o \, \vec{1}_o$$

$$\vec{1}_o \equiv \text{direction to distant observer}$$

$$\tilde{\vec{J}}(\vec{r}'',s) = \left[\tilde{\overleftrightarrow{\sigma}}(\vec{r}'',s) + s\left[\tilde{\overleftrightarrow{\epsilon}}(\vec{r}'',s) - \epsilon_0 \overleftrightarrow{1}\right]\right] \cdot \tilde{\vec{E}}(\vec{r}'',s) \tag{1.8}$$

$$\equiv \text{total current density on/in scatterer}$$

$$V \equiv \text{domain of scatterer}$$

While (1.8) only includes the electric currents one can easily include magnetic currents as

$$\tilde{\vec{J}}_m(\vec{r}'',s) = s\left[\tilde{\overleftrightarrow{\mu}}(\vec{r}'',s) - \mu_0 \overleftrightarrow{1}\right] \cdot \tilde{\vec{H}}(\vec{r}'',s) \tag{1.9}$$

which by duality can be integrated to give, after a rotation around $\vec{1}_o$ by $\pi/2$ (magnetic to electric orientation) a contribution to (1.8). Since these magnetic currents will scale the same way as the electric currents then, without loss of generality, we can use the form in (1.8) for convenience in the derivation.

In time domain this takes the form

$$\vec{E}_f\left(\vec{r}, t+\frac{r}{c}\right) = -\frac{\mu_0}{4\pi r} \overleftrightarrow{1}_o \cdot \frac{d}{dt} \int_V \vec{J}\left(\vec{r}'', t + \frac{\vec{1}_o \cdot \vec{r}''}{c}\right) dV'' \tag{1.10}$$

Let $t = 0$ be the time that the incident wave first touches the target. Then the time for the fields to propage on/into the scatterer, and the retarded time accounting for time for signals to reach the observer, assure that for finite times the volume over which one integrates has finite linear dimensions.

Our interest is centered on the far scattered field for which we write [2, 3]

$$\vec{E}_f(\vec{r},s) = \frac{e^{-\gamma r}}{4\pi r} \tilde{\overleftrightarrow{\Lambda}}\left(\vec{1}_o, \vec{1}_i; s\right) \cdot \tilde{\vec{E}}^{(inc)}\left(\vec{0}, s\right) \tag{1.11}$$

which in time domain takes the form

$$\vec{E}_f\left(\vec{r}, t + \frac{\vec{1}_o \cdot \vec{r}}{c}\right) = \frac{1}{4\pi r} \overleftrightarrow{\Lambda}\left(\vec{1}_o, \vec{1}_i; t\right) \circ \vec{E}^{(inc)}(\vec{0}, t) \tag{1.12}$$

$\circ \equiv$ convolution with respect to time

It is basically the properties of the scattering dyadic which is a convolution operator in time domain in which we are interested. Note the 1/r dependence (three-dimensional expansion) of the far fields used here.

For convenience the incident waveform is considered in frequency domain as $\tilde{f}(s) = 1$, and then used to multiply the final expressions. One could also consider some convenient time-domain incident waveform such as a step or ramp, noting that for general waveforms the scattered waveforms will be convolution operators which in the present case will turn out to be simple integrations/differentions with respect to time. By the dilation symmetry we will find that the volume integral in (1.8) will be proportional to a power of s, this being a more convenient form to work in.

II. Finite-Length Wedge: First Signal From One Tip

Consider the wedge geometry in fig. 2.1. The edge is aligned along the z axis with the origin of the $\vec{r}$ coordinates then being taken anywhere along the finite length ℓ of the edge. For later use the ends of this edge, P_1 and P_2, are the origin of coordinates $\vec{r}_1$ and $\vec{r}_2$ respectively. For the moment, however, let the wedge be infinitely long in the z direction.

As discussed in [3] an infinite wedge is characterized by having its constitutive parameters independent of z. Noting that $z = r\cos(\theta)$, then the dilation scaling (conical) specializes as

$$\begin{aligned}
\tilde{\overleftrightarrow{\epsilon}}(\vec{r},s) &\equiv \tilde{\overleftrightarrow{\epsilon}}(\Psi,\phi;s) = \tilde{\overleftrightarrow{\epsilon}}(\Psi',\phi;s') \\
\tilde{\overleftrightarrow{\mu}}(\vec{r},s) &\equiv \tilde{\overleftrightarrow{\mu}}(\Psi,\phi;s) = \tilde{\overleftrightarrow{\mu}}(\Psi',\phi;s') \\
\tilde{\overleftrightarrow{\sigma}}(\vec{r},s) &\equiv \tilde{\overleftrightarrow{\sigma}}(\Psi,\phi;s) = \chi\, \tilde{\overleftrightarrow{\sigma}}(\Psi',\phi;s') \\
\tilde{\overleftrightarrow{Y}}_s(\vec{r}_s,s) &\equiv \tilde{\overleftrightarrow{Y}}_s(\Psi_s,s) = \tilde{\overleftrightarrow{Y}}_s(\Psi_s',s') \text{ (on surfaces of constant } \phi_s)
\end{aligned} \tag{2.1}$$

with a special form when frequency independent as

$$\begin{aligned}
&\overleftrightarrow{\epsilon} \equiv \overleftrightarrow{\epsilon}(\phi) \;,\; \overleftrightarrow{\mu} \equiv \overleftrightarrow{\mu}(\phi) \;,\; \overleftrightarrow{\sigma} = \frac{1}{\Psi}\overleftrightarrow{\Sigma}(\phi) \\
&\overleftrightarrow{Y}_s \neq \text{function of coordinates on sheets of constant } \phi_s
\end{aligned} \tag{2.2}$$

so that the r dependence has been subsumed into the Ψ dependence, removing θ from the expressions in cylindrical coordinates. This is a special case of dilation (conical) symmetry which applies no matter where $\vec{r} = \vec{0}$ is selected along the edge. This is a consequence of the imposition of both dilation and translation (one direction) symmetry.

Now position cutting plane 1 which intersects the z axis at only P_1. As indicated in fig. 2.1 remove all wedge materials "above" this plane (side with $z \to +\infty$), replacing these with free space. Leave the rest of the wedge as before with no second cutting plane for the moment. In the $\vec{r}_1$ coordinates centered on P_1 we now have dilation (conical) symmetry. Cutting plane 1 is a conical surface, "above" is free space, and "below" is a special case of (1.4) and (1.5) in the $\vec{r}_1$ coordinate system. For some specified

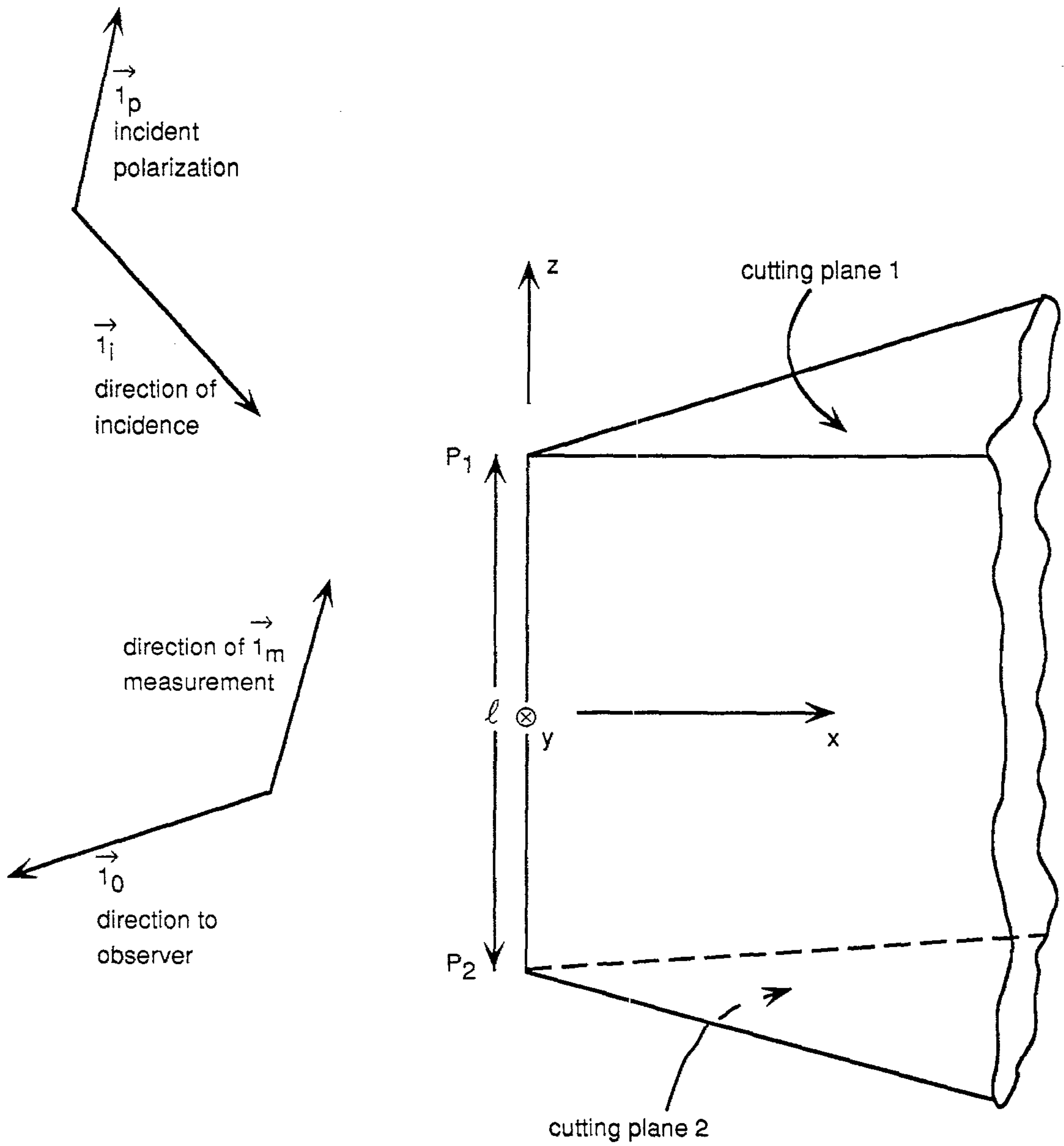


Fig. 2.1. Finite-Length Wedge

direction of incidence $\vec{1}_i$, let the direction to the observer $\vec{1}_o$ be chosen such that the first signal to reach the observer comes from currents near P_1. This is the conical case considered in [3] for which one can quote the result from there. However, let us now derive the result using the procedure discussed in Section I involving the currents on the scatterer.

Picking some incident polarization $\vec{1}_p$ and setting $\tilde{f}(s) = 1$ produces some current density $\tilde{\vec{J}}_1(\vec{r},s)$ on the target which scales according to (1.3). The integral in (1.8) then scales as

$$\begin{aligned}\int_V \tilde{\vec{J}}_1(\vec{r},s)\ e^{\gamma \vec{1}_o \cdot \vec{r}} dV &= \int_V \tilde{\vec{J}}_1(\vec{r},s)\ e^{\gamma \vec{1}_o \cdot \vec{r}} dx\,dy\,dz \\ &= \chi \int_V \tilde{\vec{J}}_1(\vec{r}\,',s')\ e^{\gamma' \vec{1}_o \cdot \vec{r}\,'} dx\,dy\,xz \\ &= \chi^{-2} \int_V \tilde{\vec{J}}_1(\vec{r}\,',s')\ e^{\gamma' \vec{1}_o \cdot \vec{r}\,'} dx'\,dy'\,xz' \\ &= \left[\frac{s'}{s}\right]^2 \int_V \tilde{\vec{J}}_1(\vec{r}\,',s')\ e^{\gamma' \vec{1}_o \cdot \vec{r}\,'} dx'\,dy'\,xz'\end{aligned} \tag{2.3}$$

Choosing any particular value of s' we find that

$$\int_V \tilde{\vec{J}}_1(\vec{r}\,',s)\ e^{\gamma \vec{1}_o \cdot \vec{r}\,'} dV = s^{-2}\ \vec{u}_1^{(p)}$$

(2.4)

$\vec{u}_1^{(p)}$ = some constant (frequency - independent) vector associated with $\vec{1}_p$ incidence

Combining this with the factor of s in the far-field expression gives

$$\tilde{\vec{E}}_f(\vec{r},s) = -\frac{\mu_o}{4\pi r}\ \frac{e^{-\gamma r}}{s}\ \overleftrightarrow{1}_o \cdot \vec{u}_1^{(p)} \tag{2.5}$$

Considering two orthogonal incident polarizations then gives a scattering dyadic in (1.11) as

$$\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o, \vec{1}_i; s) = \frac{c}{s} \overleftrightarrow{K}_1^{(c)}(\vec{1}_o, \vec{1}_i)$$

$$\overleftrightarrow{K}_1^{(c)}(\vec{1}_o, \vec{1}_i) = 2 \times 2 \text{ real dyadic (dimensionless)} \tag{2.6}$$

$$= \overleftrightarrow{K}_1^{(c)^T}(-\vec{1}_i, -\vec{1}_o) \text{ (reciprocity)}$$

Here the superscript is for cone-type scattering (three dimensional), while the subscript 1 is associated with cone tip 1. Note that powers of c are introduced to keep these real dyadics dimensionless.

In time domain we have

$$\overleftrightarrow{\Lambda}(\vec{1}_o, \vec{1}_i; t) = c\ u(t)\ \overleftrightarrow{K}_1^{(c)}(\vec{1}_o, \vec{1}_i) \tag{2.7}$$

which can be used in a convolution integral as in (1.12). Reinserting the waveform we have in time domain the exact result

$$\vec{E}_f\left(\vec{r}, t + \frac{\vec{1}_o \cdot \vec{r})}{c}\right) = \frac{c}{4\pi r} \overleftrightarrow{K}_1^{(c)}(\vec{1}_o, \vec{1}_i) \cdot \int_{-\infty}^{t} \vec{E}^{(inc)}\left(\vec{0}, t''\right) dt'' \tag{2.8}$$

valid until the time some deviation from the conical scaling (about P_1) can scatter a signal to the observer.

Now introduce cutting plane 2 passing through the point P_2 on the z axis. Removing the wedge material "below" ($z \to -\infty$) this cutting plane breaks the conical dilation symmetry with respect to P_1. Let $t_{2,1} > 0$ be the time that a signal from P_2 can reach the observer after the signal from P_1. Then for $f(t) = 0$ for $t < 0$, the result in (2.8) is valid in general out to times just less than $t_{2,1}$.

Were it not for cutting plane 1 one could regard P_2 as a cone tip with dilation symmetry of the truncated wedge in the $\vec{r}_2$ coordinate system. Since it takes some time for signals from the wedge truncation on cutting plane 1 to reach cutting plane 2 (e.g. ℓ/c on the z-axis edge), the effect of the plane 1 termination will not be evident during this time window, during which we can apply dilation symmetry to the currents. Let us call $\vec{J}_1$ as before the current density on the wedge before the truncation by cutting plane 2. After this truncation call the current density $\vec{J}$ and define

$$\vec{J}_2(\vec{r},t) = \vec{J}(\vec{r},t) - \vec{J}_1(\vec{r},t) \tag{2.9}$$

Note that "below" the truncation

$$\vec{J}(\vec{r},t) = 0, \ \vec{J}_2(\vec{r},t) = -\vec{J}_1(\vec{r},t) \tag{2.10}$$

while "above" the truncation the situation is more complicated.

In the $\vec{r}_2$ system shift the time by the arrival time -cos(θ_i) ℓ/c of the incident wave at P_2 so that we can think of the coordinate/time dilation. Now $\vec{J}$ has such symmetry (conical variety) as does $\vec{J}_1$ (wedge variety) and hence so does $\vec{J}_2$, again before signals from cutting plane 1 reach cutting plane 2. We have already the volume integral of $\vec{J}_1$ in (2.4). Now we need the same for $\vec{J}_2$ which we evaluate in the same scaling procedure as in (2.3). Noting the time shift (above) we have

$$\int_V \tilde{\vec{J}}_2(\vec{r},s)\, e^{\gamma \vec{1}_o \cdot \vec{r}} dV = e^{\frac{\ell}{c}\cos(\theta_i)}\, s^{-2}\, \vec{u}_2^{(p)}$$

$$\cos(\theta_i) = \vec{1}_i \cdot \vec{1}_z \quad \text{(negative)} \tag{2.11}$$

The volume integral now is a region starting from P_2 and spreading outward in time over the wedge, both "above and "below" cutting plane 2, noting that $\vec{J}_2$ is not zero in this latter case due to (2.10).

Then the far field as in (2.5) has a new term giving (with $\tilde{f}(s) = 1$)

$$\tilde{\vec{E}}_f(\vec{r},s) = -\frac{\mu_o}{4\pi r}\frac{e^{-\gamma r}}{s}\, \overleftrightarrow{1}_o \cdot \left[\vec{u}_1^{(p)} + e^{-st_{2,1}}\vec{u}_2^{(p)}\right] \tag{2.12}$$

noting the arrival of the first signal from P_2 at the observer is just $t_{2,1}$. The scattering dyadic is now

$$\overset{\leftrightarrow}{\tilde{\Lambda}}(\vec{1}_o, \vec{1}_i; s) = \frac{c}{s}\left[\overset{\leftrightarrow}{K}_1^{(c)}(\vec{1}_o, \vec{1}_i) + e^{-st_{2,1}}\, \overset{\leftrightarrow}{K}_2^{(c)}(\vec{1}_o, \vec{1}_i)\right]$$

$$\overset{\leftrightarrow}{K}_2^{(c)}(\vec{1}_o, \vec{1}_i) = 2\times 2 \text{ real dyadic (dimensionless)} \tag{2.13}$$

$$= \overset{\leftrightarrow}{K}_2^{(c)^T}(-\vec{1}_i, -\vec{1}_o) \text{ (reciprocity)}$$

The subscript 2 is now associated with cone tip 2.

In time domain we have

$$\overset{\leftrightarrow}{\Lambda}(\vec{1}_o, \vec{1}_i; t) = c\, u(t)\, \overset{\leftrightarrow}{K}_1^{(c)}(\vec{1}_o, \vec{1}_i) + c\, u\left(t - t_{2,1}\right) \overset{\leftrightarrow}{K}_2^{(c)}(\vec{1}_o, \vec{1}_i) \tag{2.14}$$

which can be used in a convolution integral. Reinserting the waveform we have in time domain the exact result

$$\begin{aligned}\vec{E}_f\left(\vec{r}, t + \frac{\vec{1}_o \cdot \vec{r}}{c}\right) = \frac{c}{4\pi r}\Bigg\{&\overset{\leftrightarrow}{K}_1^{(c)}(\vec{1}_o, \vec{1}_i) \cdot \int_{-\infty}^{t} \vec{E}^{(inc)}(\vec{0}, t'')\, dt'' \\ &+ \overset{\leftrightarrow}{K}_2^{(c)}(\vec{1}_o, \vec{1}_i) \cdot \int_{-\infty}^{t - t_{2,1}} \vec{E}^{(inc)}(\vec{0}, t'')\, dt''\Bigg\}\end{aligned} \tag{2.15}$$

Comparing this to (2.8) the time of validity is extended beyond $t_{2,1}$ (after incident field arrival at P_1) to another time, which we might call t_v, a time of validity. This longer time is determined by when the assumptions concerning the dilation scaling break down, and when the effects of this can reach the observer. As a simple example let $\vec{1}_o = -\vec{1}_i$ and let the first multiple scattering signal observed follow the edge from P_1 to P_2 giving

$$\begin{aligned} c\, t_v &= \ell\left[1 - \cos(\theta_i)\right] \text{ for } \frac{\pi}{2} \le \theta_i \le \pi \\ \cos(\theta_i) &= \vec{1}_i \cdot \vec{1}_z \text{ (negative)} \end{aligned} \tag{2.16}$$

Depending on the orientation of the cutting planes this time can become smaller. Furthermore the finite wedge can be truncated at some distance away from the edge (say by a plane parallel to the z axis) and the scattering from this truncation could also shorten this time.

III. Finite-Length Wedge: First Signal from Edge

In Section II it was assumed that the first signal to reach the observer came from P_1 only. Now consider the special case that $t_{2,1} = 0$, i.e. that the signal comes from both P_1 and P_2, and hence from the entire z-axis edge at zero retarded time. This gives a constraint on the direction to the observer as

$$\cos(\theta_o) = \vec{1}_o \cdot \vec{1}_z = \vec{1}_i \cdot \vec{1}_z = \cos(\theta_i)$$
$$\theta_o = \theta_i \tag{3.1}$$

while ϕ_0 can still have some range of acceptable values, based on other aspects of the wedge geometry.

Noting that the incident electric field in (1.7) can be written as

$$\tilde{\vec{E}}^{(inc)}(\vec{r},s) = E_o\,\vec{1}_p\,\tilde{f}(s)\,e^{-\gamma\,\vec{1}_i\cdot\vec{r}}$$
$$= E_o\,\vec{1}_p\,\tilde{f}(s)\,e^{-\gamma\,z\cos(\theta_i)}e^{-\gamma\Psi\,\vec{1}_i\cdot\vec{1}_\Psi} \tag{3.2}$$

then let us consider the case of translational symmetry along the z axis for all the fields, e.g.

$$\tilde{\vec{E}}^{(e)}(\vec{r},s) = e^{-\gamma\,z\cos(\theta_i)}\,\tilde{\vec{E}}^{(e)}(x,y;s)$$
$$\tilde{\vec{J}}^{(e)}(\vec{r},s) = e^{-\gamma\,z\cos(\theta_i)}\,\tilde{\vec{J}}^{(e)}(x,y;s) \tag{3.3}$$

the superscript e identifying the part associated with the edge of the wedge. If the wedge were infinitely long this would give the exact form of the current density. So let us consider this as one part of the current density and correct for the two cone tips.

With P_1 at (0, 0, z) and P_2 at (0, 0, z) then consider the integral of $\vec{J}^{(e)}$ over the domain $z_2 < z < z_1$. Noting that in (1.8) the integrand is

$$\tilde{\vec{J}}(\vec{r}\,'',s)\,e^{\gamma\,\vec{1}_o\cdot\vec{r}\,''} = \tilde{\vec{J}}(\vec{r}\,'',s)\,e^{\gamma z\cos(\theta_o)}\;e^{\gamma\Psi\,\vec{1}_o\cdot\vec{1}_\Psi} \tag{3.4}$$

with the z variation cancelling that in (3.3). The current density integral then takes the form

$$\int_V \tilde{\vec{J}}^{(e)}(\vec{r},s)\, e^{\gamma \vec{1}_o \cdot \vec{r}}\, dV = \ell \int_S \tilde{\vec{J}}^{(e)}(x,y;s)\, e^{\gamma \Psi \vec{1}_o \cdot \vec{1}_\Psi}\, dS \tag{3.5}$$

where S is the intersection of the $z = 0$ plane (anywhere between z_1 and z_2) with the wedge. The volume integral is ℓ times a surface integral which scales as

$$\begin{aligned}
\int_S \tilde{\vec{J}}^{(e)}(x,y;s)\; e^{\gamma \Psi \vec{1}_o \cdot \vec{1}_\Psi} dS &= \int_S \tilde{\vec{J}}^{(e)}(x,y;s)\; e^{\gamma \Psi \vec{1}_o \cdot \vec{1}_\Psi} dx\,dy \\
&= \chi \int_S \tilde{\vec{J}}^{(e)}(x',y';s')\; e^{\gamma' \Psi' \vec{1}_o \cdot \vec{1}_\Psi}\, dx\,dy \\
&= \chi^{-1} \int_S \tilde{\vec{J}}^{(e)}(x',y';s')\; e^{\gamma' \Psi' \vec{1}_o \cdot \vec{1}_\Psi}\, dx'\,dy' \\
&= \frac{s'}{s} \int_S \tilde{\vec{J}}^{(e)}(x',y';s')\; e^{\gamma' \Psi' \vec{1}_o \cdot \vec{1}_\Psi}\, dx'\,dy'
\end{aligned} \tag{3.6}$$

Choosing any particular value of s' we find that

$$\begin{aligned}
&\int_V \tilde{\vec{J}}(\vec{r},s)\; e^{\gamma \vec{1}_o \cdot \vec{r}}\, dV = \ell\, s^{-1}\, \vec{u}_e^{(p)} \\
&\vec{u}_e^{(p)} = \text{some constant (frequency-independent) vector associated with } \vec{1}_p \text{ incidence}
\end{aligned} \tag{3.7}$$

Considering two orthogonal incident polarizations this gives a contribution to the scattering dyadic as

$$\begin{aligned}
&\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o, \vec{1}_i; s)\Big|_{\substack{\text{edge}\\ \text{term}}} = \ell\, \overleftrightarrow{K}^{(e)}(\vec{1}_o, \vec{1}_i) \\
&\overleftrightarrow{K}^{(e)}(\vec{1}_o, \vec{1}_i) = 2 \times 2 \text{ real dyadic (dimensionless)} \\
&\qquad = \overleftrightarrow{K}^{(e)^T}(-\vec{1}_i, -\vec{1}_o) \text{ (reciprocity)}
\end{aligned} \tag{3.8}$$

In time domain this is

$$\overleftrightarrow{\Lambda}(\vec{1}_o,\vec{1}_i;t)\Big|_{\substack{\text{edge}\\ \text{term}}} = \delta(t)\,\ell\,\overleftrightarrow{K}^{(e)}(\vec{1}_o,\vec{1}_i) \tag{3.9}$$

which, as a convolution operator, just replicates the incident wave. Note the factor ℓ, indicating that the strength is proportional to the length of the edge.

The wedge symmetry being broken by the two cutting planes, there is then the contribution from conical structures at the wedge ends. Considering the cone tip at P_1, for example, write as in (2.9)

$$\begin{aligned} \vec{J}_1(\vec{r},t) &= \vec{J}(\vec{r},t) - \vec{J}^{(e)}(\vec{r},t) \\ \vec{J}^{(e)}(\vec{r},t) &= \vec{0} \quad \text{for } z > z_1 \end{aligned} \tag{3.10}$$

with $\vec{J}$ as the correct current density. Then with $\vec{J}_e$ and $\vec{J}$ both having dilation symmetry with respect to P_1, so does $\vec{J}_1$ which is a wave expanding outwards from P_1. This conical dilation wave produces a volume integral term just like that in (2.4). Expanding out from P_2 is another current density wave like that in (3.10) giving another term as in (2.4). Since they both reach the observer at the same time they can be written as a single term giving

$$\begin{aligned} \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o,\vec{1}_i;s)\Big|_{\substack{\text{cone}\\ \text{term}}} &= \frac{c}{s}\overleftrightarrow{K}^{(c)}(\vec{1}_o,\vec{1}_i) \\ \overleftrightarrow{K}^{(c)}(\vec{1}_o,\vec{1}_i) &= 2\times 2 \ \text{real dyadic (dimensionless)} \\ &= \overleftrightarrow{K}^{(c)^T}(-\vec{1}_i,-\vec{1}_o) \ \text{(reciprocity)} \end{aligned} \tag{3.11}$$

Combining the edge and cone terms gives

$$\begin{aligned} \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o,\vec{1}_i;s) &= \ell\,\overleftrightarrow{K}^{(e)}(\vec{1}_o,\vec{1}_i) + \frac{c}{s}\overleftrightarrow{K}^{(c)}(\vec{1}_o,\vec{1}_i) \\ \overleftrightarrow{\Lambda}(\vec{1}_o,\vec{1}_i;t) &= \delta(t)\,\ell\,\overleftrightarrow{K}^{(e)}(\vec{1}_o,\vec{1}_i) + c\,u(t)\overleftrightarrow{K}^{(c)}(\vec{1}_o,\vec{1}_i) \end{aligned} \tag{3.12}$$

Reinserting the waveform in the incident field gives in time domain the exact result

$$\vec{E}_f\left(\vec{r},t+\frac{\vec{1}_o\cdot\vec{r}}{c}\right) = \frac{1}{4\pi r}\left\{\ell \overleftrightarrow{K}^{(e)}(\vec{1}_o,\vec{1}_i)\cdot\vec{E}^{(inc)}(\vec{0},t)\right.$$

$$\left. + c\overleftrightarrow{K}^{(c)}(\vec{1}_o,\vec{1}_i)\cdot\int_{-\infty}^{t}\vec{E}^{(inc)}(\vec{0},t'')\,dt''\right\} \tag{3.13}$$

this being valid up to some time t_V after first incident-field arrival at the edge. This time is based on the time at which the dilation scaling is no longer valid due to multiple scattering (e.g. one cone wave reaching opposite cutting plane) and scattering from other truncations as they reach the observer. The backscattering case gives a limiting case of (2.16), i.e.

$$c\,t_V = \ell \quad , \quad \theta_i = \theta_o = \frac{\pi}{2} \tag{3.14}$$

provided other truncations are sufficiently far from the z axis.

IV. Finite-Length Wedge: Transition from Tip to Edge Case

Comparing the results of Sections II and III, let us consider what happens as $t_{2,1} \to 0+$, i.e. how the two cone terms transition to an edge term plus a cone term. For this purpose let the incident waveform be

$$f(t) = u(t) \tag{4.1}$$

which as a step function will integrate to give a ramp $t\,u(t)$.

For this kind of excitation (2.15) gives

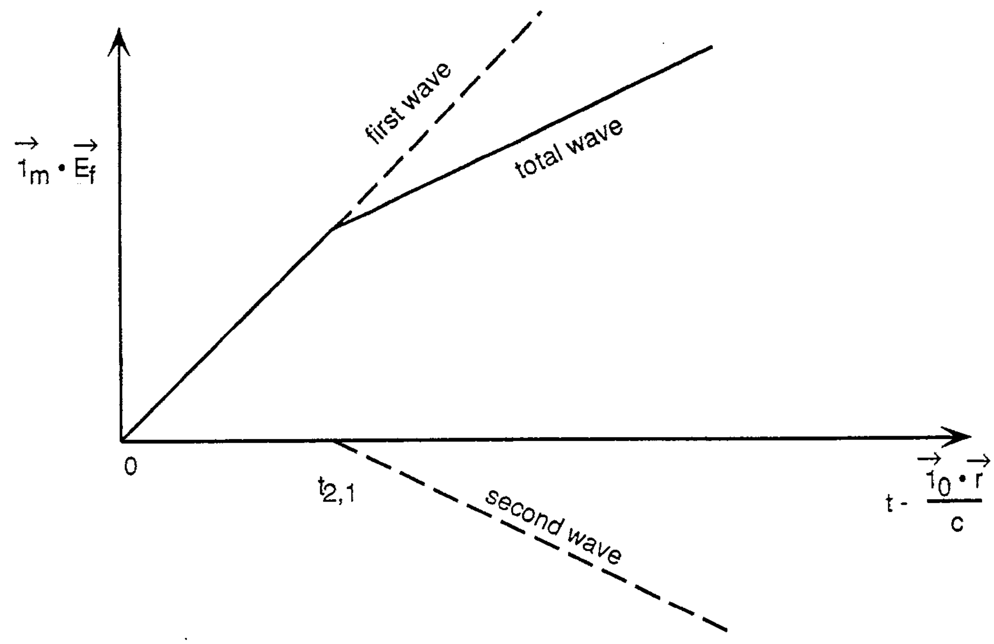
$$\begin{aligned}\vec{E}_f(\vec{r},t) = \frac{cE_o}{4\pi r}\Bigg\{&\overset{\leftrightarrow(c)}{K_1}(\vec{1}_o,\vec{1}_i)\left[t-\frac{\vec{1}_o\cdot\vec{r}}{c}\right]u\left(t-\frac{\vec{1}_o\cdot\vec{r}}{c}\right)\\ &+\overset{\leftrightarrow(c)}{K_2}(\vec{1}_o,\vec{1}_i)\left[t-\frac{\vec{1}_o\cdot\vec{r}}{c}-t_{2,1}\right]u\left(t-\frac{\vec{1}_o\cdot\vec{r}}{c}-t_{2,1}\right)\Bigg\}\cdot\vec{1}_p\end{aligned} \tag{4.2}$$

This is illustrated in Fig. 4.1A where we take a component of $\vec{E}_f$ in the $\vec{1}_m$ direction to scalarize the problem. Here the first wave starts at 0 in retarded time with some slope. The second wave starts at $t_{2,1}$ in retarded time with (in general) a different slope. These two waves combine to give a wave with a different slope after $t_{2,1}$.

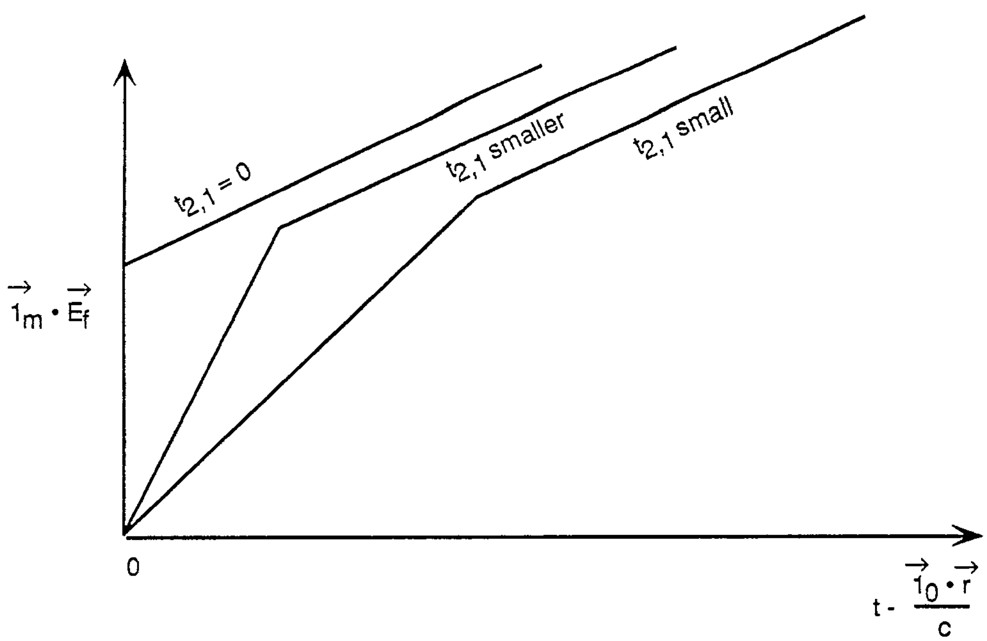
For $t_{2,1} = 0$ then (2.15) gives

$$\begin{aligned}\vec{E}_f(\vec{r},t) = \frac{E_o}{4\pi r}\Bigg\{&\ell\,\overset{\leftrightarrow(e)}{K_1}(\vec{1}_o,\vec{1}_i)\;u\left(t-\frac{\vec{1}_o\cdot\vec{r}}{c}\right)\\ &+c\,\overset{\leftrightarrow(c)}{K}(\vec{1}_o,\vec{1}_i)\left[t-\frac{\vec{1}_o\cdot\vec{r}}{c}\right]u\left(t-\frac{\vec{1}_o\cdot\vec{r}}{c}\right)\Bigg\}\cdot\vec{1}_p\end{aligned} \tag{4.3}$$

This is one of the cases in fig. 4.1B, consisting of a step at zero retarded time followed by a ramp (in general non-zero slope).



A. Combination of Two Ramp Terms



B. Merging Two Ramps into a Step Plus a Ramp

Fig. 4.1 Transition from Two Cone Terms to an Edge Term Plus a Cone Term

As $t_{2,1}$ becomes smaller the initial scattering is expected to increase as one is approaching a wedge-like case. So as $t_{2,1} \to 0$ we expect the initial slope (in magnitude) to tend to ∞. For times large compared to $t_{2,1}$ (but smaller than t_v) we expect the wedge result (3.2) to approximate the exact cone case in (3.3). So for small $t_{2,1}$ we expect the slope after $t_{2,1}$ to change little as $t_{2,1} \to 0$. If one wished to approximate the cone result by the wedge result for small $t_{2,1}$, then some kind of average of the arrival time over the length ℓ of the edge might be appropriate to define zero time, in which case for $t_{2,1} << t < t_v$ the agreement should be good. Fig. 4.1B summarizes these points qualitatively.

V. Finite-Face Half Space: First Signal from One Tip or Edge

Now extend the previous results to the half-space geometry in fig. 5.1. This half-space is the region $z \le 0$. It is finite in the sense that there are cutting planes which intersect the $z = 0$ plane (each in one straight line) to remove the half-space material on one side, away from what can be called the face, the portion of the remaining half space on the $z = 0$ plane. Furthermore let this face of area A have finite linear dimensions. With N cutting planes there are N edges of length ℓ_n for $n = 1,2,\cdots,N$ which correspond to N finite-length wedges. These edges met at N cone tips labelled P_n.

As discussed in [3] an infinite half space is characterized by having its constitutive parameters independent of both x and y, two directions of translation symmetry. Noting that $z = r\cos(\theta)$ then the general conical form of the dilation scaling in (1.4) reduces to

$$\tilde{\overleftrightarrow{\epsilon}}(\vec{r},s) \equiv \tilde{\overleftrightarrow{\epsilon}}(z,s) = \tilde{\overleftrightarrow{\epsilon}}(z',s')$$

$$\tilde{\overleftrightarrow{\mu}}(\vec{r},s) \equiv \tilde{\overleftrightarrow{\mu}}(z,s) = \tilde{\overleftrightarrow{\mu}}(z',s') \tag{5.1}$$

$$\tilde{\overleftrightarrow{\sigma}}(\vec{r},s) \equiv \tilde{\overleftrightarrow{\sigma}}(z,s) = \chi\,\tilde{\overleftrightarrow{\sigma}}(z',s')$$

with a special form when frequency independent as

$$\overleftrightarrow{\epsilon} = \text{constant dyadic}, \qquad \overleftrightarrow{\mu} = \text{constant dyadic}$$
$$\overleftrightarrow{\sigma} = -\frac{1}{z}\overleftrightarrow{\Sigma} \quad (z \text{ negative}) \tag{5.2}$$

While this last form of the conductivity is strange both 0 and ∞ are acceptable choices for the components of $\overleftrightarrow{\sigma}$. With this restriction the special case in (5.2) corresponds to a uniform anistropic medium. This general kind of half-space scaling does not admit in general of admittance sheets since they would be on planes of constant z (two-dimentional translation symmetry) and would be inconsistent with scaling $z' = \chi z$. A special exception to this is the $z = 0$ plane (the face) at which a frequency independent $\overleftrightarrow{Y}_s$ is allowed. For example the scatterer in fig. 5.1 might be a uniform dielectric (say frequency independent) with a perfectly conducting sheet ($Y_s = \infty$) on the face.

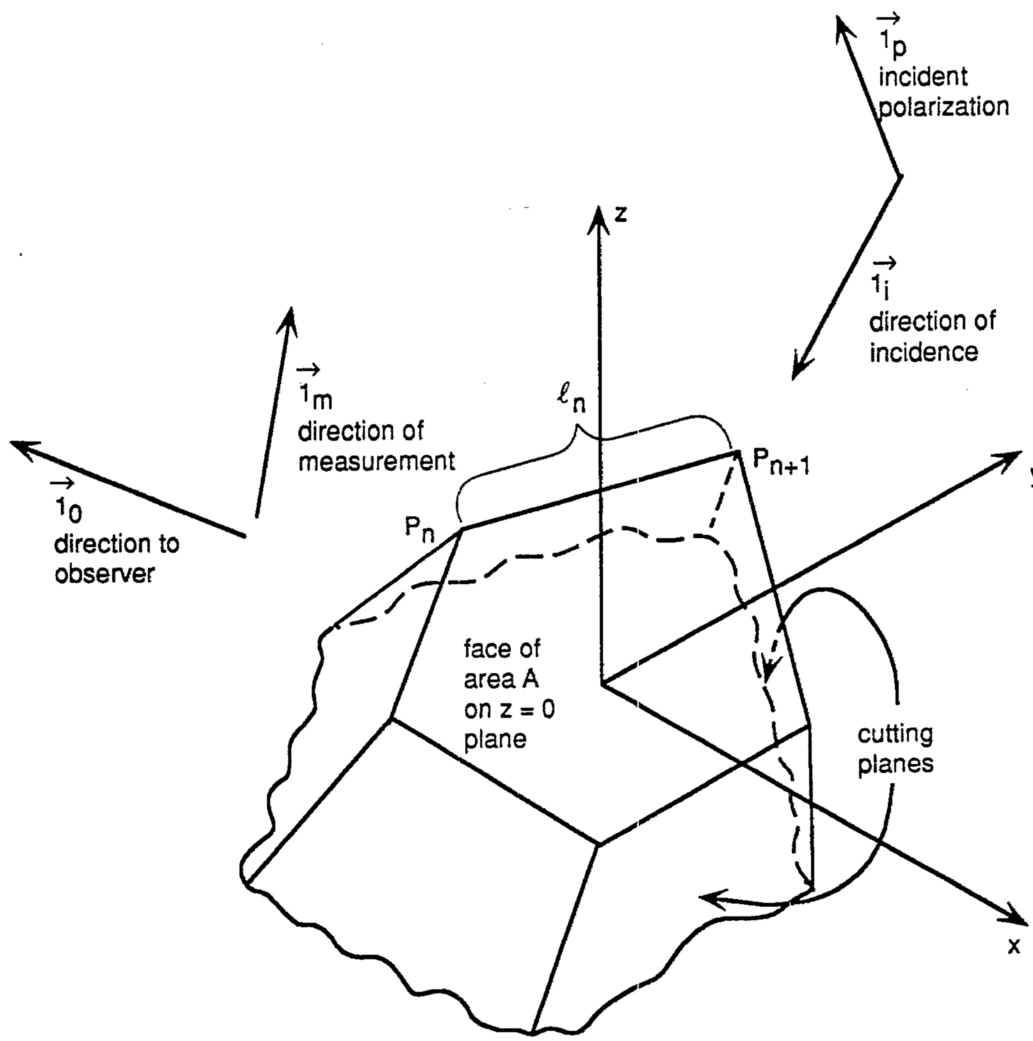


Fig. 5.1. Finite-Size Half Space

Now let

$$\vec{1}_o \neq \left[\vec{1}_x\,\vec{1}_x + \vec{1}_y\,\vec{1}_y - \vec{1}_z\,\vec{1}_z\right]\cdot\vec{1}_i \tag{5.3}$$

so that the wave scattered from the face does not reach the observer from every position on the face at the same time. For this case the first signal to reach the observer comes from either one of the cone tips or one of the edges.

Note that the dilation scaling of the half-space variety is a subset of the more general conical form. Thus if the first signal to reach the observer is from a cone tip P_m then the results of Section II directly apply. Letting

$$t_{n,m} \equiv \text{additional delay for scattered signal from } m\text{th tip to reach observer} \tag{5.4}$$

then as in (2.15) we can write

$$\vec{E}_f\left(\vec{r},t+\frac{\vec{1}_o\cdot\vec{r}}{c}\right) = \frac{c}{4\pi r}\sum_{m=1}^{N}\overleftrightarrow{K}_m^{(c)}(\vec{1}_o,\vec{1}_i)\cdot\int_{-\infty}^{t-t_{n,m}}\vec{E}^{(inc)}(\vec{0},t'')\,dt'' \tag{5.5}$$

assuming no edge has scattering arrive at the same time from all along the edge. There is again a limited time of validity t_v corresponding to the arrival of multiple scattered signals (waves outgoing from one cone tip scattering from other cone tips, edges, cutting planes, or other truncations at negative z).

Next note that the dilation scaling of the half-space variety is also a subset of the wedge form. If the signal from an edge reaches the observer from all points of the edge simultaneously, then the results of Section III apply. The form in (3.13) was derived for the case that the edge gave the first signal (instead of, say, non adjacent tips). However, it also applies more generally by observing that we can treat the current density near the edge as some $\vec{J}_1$ before this edge is introduced by the associated cutting plane and that this $\vec{J}_1$ has the requisite wedge dilation scaling (before scattering reaches the edge from other tips etc.). Then as in (2.9) one can take the difference of $\vec{J}_1$ from the correct current density giving a difference current density which also has wedge dilation symmetry. Integrating over this current expanding outward from the edge (including the region of half-space material removed by the cutting plane) gives the edge result with the form in (3.8). Similar derivation concerning the two adjacent tips which end this edge gives the form of the scattering as in (3.12) and (3.13). It is possible that there can be more than one edge (e.g. parallel edges) which meet this simultaneous-arrival-at-observer criterion. Then combining results from (3.13) terms as in (5.5) we have

$$\vec{E}_f\left(\vec{r}, t+\frac{\vec{1}_o\cdot\vec{r}}{c}\right) = \frac{1}{4\pi r}\left\{\sum_{\substack{m\varepsilon\\ \text{edge set}}} \ell_m \overset{\leftrightarrow(e)}{K}_m(\vec{1}_o,\vec{1}_i)\cdot\vec{E}^{(inc)}(\vec{0},t-t_{n,m})\right. \tag{5.6}$$

$$\left. c\sum_{\substack{m\not\varepsilon\\ \text{edge set}}} \overset{\leftrightarrow(c)}{K}_m(\vec{1}_o,\vec{1}_i)\cdot\int_{-\infty}^{t-t_{n,m}} \vec{E}^{(inc)}(\vec{0},t'')\,dt''\right\}$$

where the edge set is as discussed above. Note that when a cone tip is associated with one of these edges (m and m+1 for $m\varepsilon$ edge set with N+1 equivalent to 1), the form of the term (2 x 2 dyadic times time integral) is the same as for those not so associated. However, as discussed in Section IV, the presence of an edge term changes the cone-tip terms by merging two into one. Hence we need account for only one cone term (instead of two) when such edge terms are present. For M edge terms there are then N-M cone terms.

VI. Finite-Face Half Space: First Signal from Face

Now let the signal scattering from every element of the face reach the observer at the same time. This is the converse condition to (5.3), i.e.

$$\vec{1}_o = \left[\vec{1}_x \vec{1}_x + \vec{1}_y \vec{1}_y - \vec{1}_z \vec{1}_z\right] \cdot \vec{1}_i = \left[\overleftrightarrow{1} - 2\, \vec{1}_z \vec{1}_z\right] \cdot \vec{1}_i$$

$$\theta_o = \pi - \theta_i \quad , \quad \phi_o = \phi_i \tag{6.1}$$

In this case consider fields with translational symmetry with respect to both x and y, i.e.

$$\tilde{\vec{E}}^{(f)}(\vec{r},s) = e^{-\gamma \vec{r} \cdot \overleftrightarrow{1}_z \cdot \vec{1}_i}\, \tilde{\vec{E}}^{(f)}(z;s)$$

$$\tilde{\vec{J}}^{(f)}(\vec{r},s) = e^{-\gamma \vec{r} \cdot \overleftrightarrow{1}_z \cdot \vec{1}_i}\, \tilde{\vec{J}}^{(f)}(z;s)$$

$$\vec{r} \cdot \overleftrightarrow{1}_z \cdot \vec{1}_i = \vec{r} \cdot \overleftrightarrow{1}_z \cdot \vec{1}_o = \left[x \cos(\phi_i) + y \sin(\phi_i)\right] \sin(\theta_i) \tag{6.2}$$

$$\overleftrightarrow{1}_z \equiv \overleftrightarrow{1} - \vec{1}_z \vec{1}_z = \text{transverse dyad (to } \vec{1}_z)$$

the superscript f identifying the part associated with the face of the half space. With the factoring out of the x and y dependence (due to translation symmetry) the remaining part of the current density, etc. is a function of only z and s. So let us consider this as one part of the current density and correct for the edges and cone tips.

Consider the integral of $\vec{J}^{(f)}$ over a domain bounded by the face and its translation in the -z direction. This is a cylinder of polygonal cross section. Noting that in (1.8) the integrand is

$$\tilde{\vec{J}}(\vec{r}'',s)\, e^{\gamma \vec{1}_o \cdot \vec{r}''} = \tilde{\vec{J}}(\vec{r}'',s)\, e^{\gamma \vec{r}'' \cdot \overleftrightarrow{1}_z \cdot \vec{1}_o}\, e^{\gamma z'' \cos(\theta_o)}$$

$$\overleftrightarrow{1}_z \cdot \vec{1}_o = \overleftrightarrow{1}_z \cdot \vec{1}_i \tag{6.3}$$

the x and y variation cancels that in (6.2). The current density integral then takes the form

$$\int_V \tilde{\vec{J}}^{(f)}(\vec{r},s)\; e^{\gamma \vec{1}_o \cdot \vec{r}} = A \int_{-\infty}^{0} \tilde{\vec{J}}^{(f)}(z;s)\; e^{\gamma z \cos(\theta_o)}\, dz \tag{6.4}$$

The volume integral is A times a line integral which scales as

$$\begin{aligned} \int_{-\infty}^{0} \tilde{\vec{J}}^{(f)}(z;s)\; e^{\gamma z \cos(\theta_o)}\, dz &= \chi \int_{-\infty}^{0} \tilde{\vec{J}}^{(f)}(z';s')\; e^{\gamma z' \cos(\theta_o)}\, dz \\ &= \int_{-\infty}^{0} \tilde{\vec{J}}^{(f)}(z';s')\; e^{\gamma z' \cos(\theta_o)}\, dz' \end{aligned} \tag{6.5}$$

Choosing any particular value of s' we find that

$$\begin{aligned} \int_V \tilde{\vec{J}}^{(f)}(z;s)\; e^{\gamma z \cos(\theta_o)}\, dV &= A\; \vec{u}_f^{(p)} \\ \vec{u}_f^{(p)} &= \text{some constant (frequency-independent)} \\ &\quad \text{vector associated with } \vec{1}_p \text{ incidence} \end{aligned} \tag{6.6}$$

Considering two orthogonal incident polarizations this gives a contribution to the scattering dyadic as

$$\begin{aligned} \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o;s)\Big|_{\substack{\text{face}\\ \text{term}}} &= \frac{s}{c} A\, \overleftrightarrow{K}^{(f)}(\vec{1}_o) \\ \overleftrightarrow{K}^{(f)}(\vec{1}_o) &= 2\times 2 \text{ real dyadic (dimensionless)} \\ &= \overleftrightarrow{K}^{(f)^T}(-\vec{1}_1) \text{ (reciprocity)} \end{aligned} \tag{6.7}$$

In time domain this is

$$\overleftrightarrow{\Lambda}(\vec{1}_o;t)\Big|_{\substack{\text{face}\\ \text{term}}} = \frac{d\delta(t)}{dt} \frac{A}{c} \overleftrightarrow{K}^{(f)}(\vec{1}_o) \tag{6.8}$$

which, as a convolution operator, differentiates the incident wave. Note the factor A, indicating that the strength is proportional to the area of the face.

The half-space symmetry is broken by the N cutting planes. Considering the nth edge (length ℓ_n), for example we can write

$$\vec{J}_n^{(e)}(\vec{r},t) = \vec{J}(\vec{r},t) - \vec{J}^{(f)}(\vec{r},t) \tag{6.9}$$

with $\vec{J}^{(f)}$ zero outside the polygonal cylindrical volume discussed above and $\vec{J}$ the correct current density. Like previously, $\vec{J}_n^{(e)}$ has dilation symmetry with respect to the nth edge and is a wave propagating outward from the edge. Integrate this over a region bounded by two planes perpendicular to this edge and passing through the cone tips at each end (P_n and P_{n+1}). This gives an edge term of the same form as (3.8). There are N such terms, all arriving at the observer at zero retarded time. Being of the same form they can all be summed as a single dyadic which can be written as

$$\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o;s)\Big|_{\substack{\text{edge}\\ \text{term}}} = \ell_o \overleftrightarrow{K}^{(e)}(\vec{1}_o)$$

$$\overleftrightarrow{K}^{(e)}(\vec{1}_o) = 2\times 2 \text{ real dyadic (dimensionless)}$$

$$= \overleftrightarrow{K}^{(e)^T}(-\vec{1}_1) \text{ (reciprocity)} \tag{6.10}$$

$$\ell_o \equiv \text{perimeter of face} = \sum_{n=1}^{N} \ell_n$$

and the dyadic can be regarded as an appropriate weighted average of the N individual edge dyadics.

We still have the N cones to consider. Centering our attention on the nth cone with apex P_n we can write

$$\vec{J}_n^{(c)}(\vec{r},t) = \vec{J}(\vec{r},t) - \vec{J}^{(f)}(\vec{r},t) - \vec{J}_n^{(e)}(\vec{r},t) - \vec{J}_{n-1}^{(e)}(\vec{r},t) \tag{6.11}$$

The cone contribution $\vec{J}_n^{(c)}$ is associated with the correct current density $\vec{J}$, less the terms that we have already integrated to obtain the previous scattering terms. These include the face part over the polygonal cylinder and two wedge terms associated with the two adjacent edges and bounded by planes normal to the edges at P_n as discussed above. Then this current density $\vec{J}_n^{(c)}$ is a wave propagating outward from the cone tip with the dilation symmetry appropriate to cones. This gives a tip term of the same form as (2.6). Like the two cone terms combined in (3.11) there are now N such terms, all arriving at the observer at zero retarded time. They can then all be summed as a single dyadic as

$$\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o;s)\Big|_{\substack{\text{tip}\\ \text{term}}} = \frac{c}{s}\overleftrightarrow{K}^{(c)}(\vec{1}_o)$$

$$\overleftrightarrow{K}^{(c)}(\vec{1}_o) = 2\times 2 \text{ real dyadic (dimensionless)} \tag{6.12}$$

$$= \overleftrightarrow{K}^{(c)^T}(-\vec{1}_1) \text{ (reciprocity)}$$

Combining the three terms gives

$$\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o;s) = \frac{s}{c} A\overleftrightarrow{K}^{(f)}(\vec{1}_o) + \ell_o \overleftrightarrow{K}^{(e)}(\vec{1}_o) + \frac{c}{s}\overleftrightarrow{K}^{(c)}(\vec{1}_o)$$

$$\overleftrightarrow{\Lambda}(\vec{1}_o;t) = \frac{d\delta(t)}{dt}\frac{A}{c}\overleftrightarrow{K}^{(f)}(\vec{1}_o) + \delta(t)\,\ell_o \overleftrightarrow{K}^{(e)}(\vec{1}_o) + c\,u(t)\overleftrightarrow{K}^{(c)}(\vec{1}_o) \tag{6.13}$$

Reinserting the waveform in the incident field in time domain gives the exact result

$$\vec{E}_f\left(\vec{r},t+\frac{\vec{1}_o\cdot\vec{r}}{c}\right) = \frac{1}{4\pi r}\left\{\frac{A}{c}\overleftrightarrow{K}^{(f)}(\vec{1}_o)\cdot\frac{d}{dt}\vec{E}^{(inc)}(\vec{0},t)\right.$$

$$\left. + \ell_o \overleftrightarrow{K}^{(e)}(\vec{1}_o)\cdot\vec{E}^{(inc)}(\vec{0},t) + c\,\overleftrightarrow{K}^{(c)}(\vec{1}_o)\cdot\int_{-\infty}^{t}\vec{E}^{(inc)}(\vec{0},t'')\,dt''\right\} \tag{6.14}$$

this being valid up to some time t_v after first incident-field arrival at the face. This time is based on when the dilation symmetry of the various waves is no longer valid due to multiple scattering of the waves and the arrival of this at the observer. For example, the signal from cone tip P_n reaches P_{n+1} after a time ℓ_n/c, but depending on the shape of the face could reach some other edge across the face even sooner. For backscattering $(\vec{1}_o = -\vec{1}_i)$ the smallest of these times is t_v.

If, for example, the incident waveform is a step function then the face term is a delta function. This is not strictly the case unless one observes the limit of $r \to \infty$ (far field) before evaluating this term. For finite r this can be better viewed as an approximate delta function of small but non-zero width (tending to zero as $r \to \infty$ [1]). Noting that realistic waveforms have finite rate of rise then (6.14) does not produce an ideal delta function. Another canonical waveform one can use is

$$f(t) = c\,t\,u(t) \tag{6.15}$$

which is a ramp function (dimensionless) giving

$$\begin{aligned}\vec{E}_f(\vec{r},t) = \frac{E_o}{4\pi r}\Bigg\{ & A \overleftrightarrow{K}^{(f)}(\vec{1}_o)\, u\left(t - \frac{\vec{1}_o \cdot \vec{r}}{c}\right) \\ & + \ell_o \overleftrightarrow{K}^{(e)}(\vec{1}_o)\, c\,t\; u\left(t - \frac{\vec{1}_o \cdot \vec{r}}{c}\right) \\ & + \overleftrightarrow{K}^{(c)}(\vec{1}_o)\, \frac{(ct)^2}{2}\, u\left(t - \frac{\vec{1}_o \cdot \vec{r}}{c}\right)\Bigg\} \cdot \vec{1}_p\end{aligned} \tag{6.16}$$

The transition from the case in which the first signal comes to the observer from one cone tip or edge in Section V, to the face case in this section, is like the cone to edge transition in Section IV, except that it is more complex in its details. As the condition in (6.1) is approached the individual edge and cone terms become larger and larger, but begin to partially cancel after shorter and shorter times (similar to fig. 4.1B).

VII. Concluding Remarks

This paper extends the results of [3] to the case of finite-size wedges and half spaces. The basic concept here might be that of multiply centered dilation symmetry. A hierarchy of dilation symmetries: conical, wedge, and half space, can be constructed where the latter are subsets of the former by the additional imposition of translation symmetry. For times before the dilating waves from faces, edges, and corners as appropriate reach discontinuities which destroy the corresponding dilation symmetry (and propagate thence to the observer), the scattering takes a simple form as the sum of a few terms each consisting of a 2 x 2 real dyadic times simple operations on the incident waveform (such as time derivative and integral).

Note that it is the form of the time-domain scattering which is exact before some $t_v > 0$. The 2 x 2 dyadics depend on the details of the geometry and media, and these could be computed (analytically or numerically) or even measured for various cases of interest. Also important is the fact that this form of the scattering separates the angular from the temporal parts, thereby giving a useful parameterization for use in target identification.

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