Interaction Notes

Note 510

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AN FIR SYSTEM FOR TARGET DISCRIMINATION

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Abstract

The theory of synthesizing a finite-impulse response (FIR) system intended for the discrimination of radar targets is presented. The way to construct the filter is discussed in detail in its most general form, with additional mathematical items regarding solvability of the system and the way to perform quasi-inversion of the corresponding matrix provided; the use of this system for identifying different types of aircraft according to their backscattered responses to ultra wide band radars is considered.

Introduction

The method of digital signal processing presented in this paper is essentially in the class of FIR filters which in the time domain are described by a window of some chosen length $2L+1$ moving along the sequence of samples of the input signal:

$$y_i = \sum_{l=-L}^{L} \mu_l x_{i+l} \tag{1}$$

where every new retrieved sample y_i is represented by the weighted sum of $2L+1$ given input samples x_{i+l} each of them being multiplied by the corresponding weight — the filter coefficient μ_l.

The procedure of synthesizing the filter discussed is similar to that implemented by the frequency sampling method. Recall that according to the latter scheme the desired magnitude response $|H_D(e^{j\omega})|$ of the filter, which is a continuous function of frequency, is sampled providing a sequence of samples at, say, N equally spaced frequency points over $0 \leq \omega < 2\pi$: $|H_D(e^{j\omega_k})|$, where $\omega_k = 2\pi k/N$ for $0 \leq k \leq N-1$. Then an FIR filter is designed whose magnitude response exactly matches the desired response at these N points, and approximates the desired response in intervals between the points mentioned. In the approach proposed no intention exists to approximate the desired magnitude response; it can be stated that in some sense there is no frequency response specified, at least in the usually accepted meaning of the term. To the contrary, certain values of the magnitude response are specified at some points on the frequency axis given beforehand, but, however, not necessarily equally spaced. In this method the obtained filter response in undefined frequency regions, i.e. between those given points, can have any form. Points on the frequency axis where values of the magnitude response are set actually represent the frequency values of the input signal components. This item can be considered a bottleneck of the approach proposed, because the aforesaid implies *a priori* knowledge of the input signal components' frequencies. However, there are a number of problems where this information is directly available, and in those cases the advantages of the method are apparent. The problem of the discrimination of radar targets can be considered one of the possible examples of the situation mentioned.

The time-domain scattered field response of a conducting target has been observed to be composed of two distinct sets of waveforms. An early-time forced period, when the excitation field is interacting with the scatterer, and the discontinuities in electrical characteristics at the object's boundaries give rise to impulsive components in the response, is followed immediately by a late-time period during which the target oscillates freely, and the currents induced upon the object radiate energy at natural frequencies. The late-time period can be decomposed into a finite sum of damped sinusoids (excited by an incident field waveform), while their oscillating frequencies are determined purely by the target geometry.

Then for the late-time period the measured time-domain backscattered field can be represented in the form

$$r(t) = \sum_{p=1}^{P} a_p e^{\sigma_p t} \cos(\omega_p t + \varphi_p), \quad t > T_l \tag{2}$$

where a_p and φ_p are the aspect-dependent amplitude and phase of the p-th mode, T_l describes the beginning of the late-time period and P modes are taken into account. The natural frequencies $s_p = \sigma_p \pm j\omega_p$ for different aircraft can be found experimentally and are reported in several publications [1,2].

In the subsequent sections of the paper the algorithm in a general form is derived, and its operation for the needs of target discrimination is demonstrated.

Construction of filtering algorithm

Consider an input signal composed of M decaying sinusoids having damping factors $\{\sigma_i\}$ and frequencies $\{\omega_i\}$, for $1 \leq i \leq M$, and N pure sinusoids with frequencies $\{\omega_i\}$, for $M+1 \leq i \leq M+N$. The model of the input signal can be written as

$$x(t) = \sum_{i=1}^{M} 2A_i e^{\sigma_i t} \cos(\omega_i t + \varphi_i) + \sum_{i=M+1}^{M+N} A_i \cos(\omega_i t + \varphi_i) \quad (3)$$

We adopt an assumption that the entire sets of $\{\sigma_i\}$ and $\{\omega_i\}$, for $1 \leq i \leq M+N$, are known. All other parameters of the input, namely the amplitudes ($2A_i$ for the decaying sinusoids and A_i for pure ones; the convenience of the use of these notations will become apparent later) and the initial phases φ_i, for $1 \leq i \leq M+N$, are unknown. We wish to construct a symmetric FIR filter which would pass the desired $M_1 \leq M$ decaying sinusoids and $N_1 \leq N$ pure sinusoids embedded in the input and would reject all others. Under the condition of symmetry of the filter window (meaning that $\mu_l = \mu_{-l}$ in (1), for $1 \leq l \leq L$) the algorithm will be proved to be invariant to all undefined parameters of the input. For evident convenience we will leave for use only the positive index in the notation of the filter coefficient, μ_l, for $l = 1, \ldots, L$; the first coefficient is chosen to be $2\mu_0$. In accordance with (1) and the aforesaid we write down the filter operation in the form:

$$y_i = 2\mu_0 x_i + \sum_{l=1}^{L} \mu_l (x_{i-l} + x_{i+l}) \quad (4)$$

Let the input (3) be sampled at the rate ω_s (i.e., the discretization interval is $\Delta t = 2\pi/\omega_s$). To simplify the calculations, we firstly consider the transition of a single k-th decaying sinusoid through this filter; this sinusoid can be written as:

$$\begin{aligned} x^{(k)}(t) &= 2A_k e^{\sigma_k t} \cos(\omega_k t + \varphi_k) = A_k e^{\sigma_k t} \left(e^{j(\omega_k t + \varphi_k)} + e^{-j(\omega_k t + \varphi_k)} \right) \\ &= A_k \left(e^{(\sigma_k + j\omega_k)t + j\varphi_k} + e^{(\sigma_k - j\omega_k)t - j\varphi_k)} \right) = \tilde{A}_k e^{s_k t} + \tilde{A}_k^* e^{s_k^* t} \end{aligned}$$

where $\tilde{A}_k = A_k e^{j\varphi_k}$, $s_k = \sigma_k + j\omega_k$, and $*$ designates the complex conjugate. Let us for the sake of simplicity consider only one term of the complex-conjugate pair of the addenda in the last expression — $\tilde{A}_k e^{s_k t}$ — forming the input to our filter. Then according to the filtering algorithm (4) and assuming that the filter's middle point lies in the time moment

$t = t_0$, the corresponding i-th output sample can be presented in the form

$$
\begin{aligned}
y_i^{(k)}\Big|_{\tilde{A}_k e^{s_k t}} &= 2\mu_0 \tilde{A}_k e^{s_k t_0} + \sum_{l=1}^{L} \mu_l \left[\tilde{A}_k e^{s_k (t_0 - l\Delta t)} + \tilde{A}_k e^{s_k (t_0 + l\Delta t)}\right] \\
&= 2\mu_0 \tilde{A}_k e^{s_k t_0} + \sum_{l=1}^{L} \mu_l \tilde{A}_k e^{s_k t_0} \left[e^{-s_k l\Delta t} + e^{s_k l\Delta t}\right] \\
&= 2\tilde{A}_k e^{s_k t_0} \left[\mu_0 + \sum_{l=1}^{L} \mu_l \cosh(2\pi l s_k/\omega_s)\right]
\end{aligned}
$$

If we want the damped sinusoid of frequency $s_k = \sigma_k + j\omega_k$ to pass through the filter without distortion, then the components $\tilde{A}_k e^{s_k t}$ and $\tilde{A}_k^* e^{s_k^* t}$ should be transmitted without change, i.e. the following condition should be fulfilled:

$$
\begin{aligned}
2\tilde{A}_k e^{s_k t_0} \left[\mu_0 + \sum_{l=1}^{L} \mu_l \cosh(2\pi l s_k/\omega_s)\right] &= \tilde{A}_k e^{s_k t_0} \\
2\tilde{A}_k^* e^{s_k^* t_0} \left[\mu_0 + \sum_{l=1}^{L} \mu_l \cosh(2\pi l s_k^*/\omega_s)\right] &= \tilde{A}_k^* e^{s_k^* t_0}
\end{aligned}
$$

or

$$
\begin{aligned}
\mu_0 + \sum_{l=1}^{L} \mu_l \cosh(2\pi l s_k/\omega_s) &= 1/2 \\
\mu_0 + \sum_{l=1}^{L} \mu_l \cosh(2\pi l s_k^*/\omega_s) &= 1/2
\end{aligned}
\tag{5}
$$

The possibility of cancelling the complex conjugate terms $\tilde{A}_k e^{s_k t}$ and $\tilde{A}_k^* e^{s_k^* t}$ actually implies the proof of the algorithm's invariance to amplitudes and initial phases of the input components. From the theory it is known that using a symmetric (or antisymmetric, i.e. when $\mu_l = -\mu_{-l}$) set of coefficients of an FIR filter ensures a linear phase response of that filter. Note, however, that in our case the algorithm's invariance to phase means not only a linear phase response, but also having a zero value, i.e. introducing no delay to any sample processed.

In the event that certain frequencies are undesirable in the output mixture, (5) should be rewritten for each such component in the form:

$$
\mu_0 + \sum_{l=1}^{L} \mu_l \cosh(2\pi l s_k/\omega_s) = 0
$$

$$\mu_0 + \sum_{l=1}^{L} \mu_l \cosh(2\pi l s_k^*/\omega_s) = 0 \tag{6}$$

From (5) and (6) it is evident that the only information needed for filter construction is knowledge of the complex frequencies $s_k = \sigma_k + j\omega_k$. Written for M damped sinusoids, the conditions (5) and (6) provide $2M$ equations.

In the case where the k-th pure sinusoid

$$x^{(k)}(t) = A_k \cos(\omega_k t + \varphi_k)$$

forms an input of our filter, then the i-th output sample of this sinusoid is

$$\begin{aligned} y_i^{(k)} &= 2\mu_0 A_k \cos(\omega_k t + \varphi_k) \\ &+ \sum_{l=1}^{L} \mu_l A_k \left[\cos(\omega_k(t - l\Delta t) + \varphi_k) + \cos(\omega_k(t + l\Delta t) + \varphi_k)\right] \\ &= 2\mu_0 A_k \cos(\omega_k t + \varphi_k) + \sum_{l=1}^{L} 2\mu_l A_k \cos(\omega_k t + \varphi_k)\cos(l\omega_k \Delta t) \\ &= 2A_k \cos(\omega_k t + \varphi_k)\left[\mu_0 + \sum_{l=1}^{L} \mu_l \cos(2\pi l \frac{\omega_k}{\omega_s})\right] \end{aligned}$$

If this component is to be passed without distortion, the following equality must be satisfied:

$$2A_k \cos(\omega_k t + \varphi_k)\left[\mu_0 + \sum_{l=1}^{L} \mu_l \cos(2\pi l \frac{\omega_k}{\omega_s})\right] = A_k \cos(\omega_k t + \varphi_k)$$

or

$$\mu_0 + \sum_{l=1}^{L} \mu_l \cos(2\pi l \frac{\omega_k}{\omega_s}) = 1/2 \tag{7}$$

In order to reject this sinusoid, 1/2 in the last expression should be replaced by 0:

$$\mu_0 + \sum_{l=1}^{L} \mu_l \cos(2\pi l \frac{\omega_k}{\omega_s}) = 0 \tag{8}$$

As before for the case of damped sinusoids, the last two expressions prove that the filter is invariant to amplitudes and initial phases of the given input components. For N pure sinusoids, expressions (7) and (8) provide N more equations. Thus, we have $2M + N$ equations for $2M + N$ variables.

As a result we will have the following matrix equation:

$$\mathbf{Z}\vec{\mu} = \vec{b} \tag{9}$$

where the matrix $\mathbf{Z}$ has the form:

$$\mathbf{Z} = \begin{bmatrix} 1 & \cosh(2\pi s_1/\omega_s) & \cosh(2\pi 2 s_1/\omega_s) & \dots & \cosh(2\pi L s_1/\omega_s) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \cosh(2\pi s_M/\omega_s) & \cosh(2\pi 2 s_M/\omega_s) & \dots & \cosh(2\pi L s_M/\omega_s) \\ 1 & \cosh(2\pi s_1^*/\omega_s) & \cosh(2\pi 2 s_1^*/\omega_s) & \dots & \cosh(2\pi L s_1^*/\omega_s) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \cosh(2\pi s_M^*/\omega_s) & \cosh(2\pi 2 s_M^*/\omega_s) & \dots & \cosh(2\pi L s_M^*/\omega_s) \\ 1 & \cos(2\pi \omega_{M+1}/\omega_s) & \cos(2\pi 2\omega_{M+1}/\omega_s) & \dots & \cos(2\pi L\omega_{M+1}/\omega_s) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \cos(2\pi \omega_{M+N}/\omega_s) & \cos(2\pi 2\omega_{M+N}/\omega_s) & \dots & \cos(2\pi L\omega_{M+N}/\omega_s) \end{bmatrix} \tag{10}$$

$\vec{\mu} = col\{\mu_i\}$ is the column vector of the filter coefficients, and $\vec{b}$ is composed of 0's and 1/2's according to (5), (6), (7) and (8). Solving the equation (9) for $\vec{\mu}$, we are able to construct the filter desired. From inspection of system (10) it is evident what the value of L, i.e. the filter window length, should be: it must be equal to $2M + N - 1$ so that the matrix $\mathbf{Z}$ be square and inversible.

It is easy to show that the components of the vector $\vec{\mu}$ are real. Indeed, consider the complex conjugate for the equation (9):

$$\mathbf{Z}^*\vec{\mu}^* = \vec{b}^* = \vec{b}$$

The complex conjugation $\mathbf{Z}^*$ results in just the transposition of rows of $\mathbf{Z}$ corresponding to conjugated frequencies s_k and s_k^* without changing the vector $\vec{b}$ itself. Thus from (10) it follows that

$$\mathbf{Z}\vec{\mu}^* = \vec{b}$$

and consequently the vector $\vec{\mu}$ is real.

Existence and uniqueness of the solution

Now let us consider the problem of solvability of the matrix equation (9). The following lemma provides the answer.

Lemma. The unique solution of the equation (9) exists if $\omega_s > 2\max\{\omega_i\}$, for $1 \leq i \leq 2M + N$.

Proof. It is known that the condition of a unique solution is defined by the value of the determinant of the matrix $\mathbf{Z}$: $D = \det(\mathbf{Z})$. If $D \neq 0$ then the solution exists and is unique: $\vec{\mu} = \mathbf{Z}^{-1}\vec{b}$. Thus our aim is to show that $D \neq 0$ when the conditions of Lemma are satisfied. For this purpose we use the following expansions [3]:

$$\begin{aligned}
\cos(nx) &= 2^{n-1}\cos^n x \\
&+ \frac{n}{2}\sum_{k=0}^{(n-2)/2}\frac{(-1)^{k+1}}{k+1}C_{n-k-2}^{k}2^{n-2k-2}\cos^{n-2k-2}x = f_n(\cos x) \\
\cosh(nx) &= 2^{n-1}\cosh^n x \\
&+ \frac{n}{2}\sum_{k=0}^{(n-2)/2}\frac{(-1)^{k+1}}{k+1}C_{n-k-2}^{k}2^{n-2k-2}\cosh^{n-2k-2}x = f_n(\cosh x) \qquad (11)
\end{aligned}$$

where

$$f_n(z) = \sum_{k=1}^{n} a_k^{(n)} z^k \qquad (12)$$

is a polynomial of order n, and

$$a_n^{(n)} = 2^{n-1}, \quad a_{2k+1}^{(n)} = 0, \quad a_{2k}^{(n)} = \frac{n}{2}\frac{(-1)^{k+1}}{k+1}C_{n-k-2}^{k}2^{n-2k-2} \qquad (13)$$

Applying (11) firstly to the third column of D, we have:

$$D = \begin{vmatrix}
1 & \cosh(2\pi s_1/\omega_s) & 2\cosh^2(2\pi s_1/\omega_s) - 1 & \dots \\
\dots & \dots & \dots & \dots \\
1 & \cosh(2\pi s_M/\omega_s) & 2\cosh^2(2\pi s_M/\omega_s) - 1 & \dots \\
1 & \cosh(2\pi s_1^*/\omega_s) & 2\cosh^2(2\pi s_1^*/\omega_s) - 1 & \dots \\
\dots & \dots & \dots & \dots \\
1 & \cosh(2\pi s_M^*/\omega_s) & 2\cosh^2(2\pi s_M^*/\omega_s) - 1 & \dots \\
1 & \cos(2\pi \omega_{M+1}/\omega_s) & 2\cos^2(2\pi \omega_{M+1}/\omega_s) - 1 & \dots \\
\dots & \dots & \dots & \dots \\
1 & \cos(2\pi \omega_{M+N}/\omega_s) & 2\cos^2(2\pi \omega_{M+N}/\omega_s) - 1 & \dots
\end{vmatrix}$$

and then adding the first column to the third, we get:

$$D = 2\begin{vmatrix} 1 & \cosh(2\pi s_1/\omega_s) & \cosh^2(2\pi s_1/\omega_s) & \dots \\ \dots & \dots & \dots & \dots \\ 1 & \cosh(2\pi s_M/\omega_s) & \cosh^2(2\pi s_M/\omega_s) & \dots \\ 1 & \cosh(2\pi s_1^*/\omega_s) & \cosh^2(2\pi s_1^*/\omega_s) & \dots \\ \dots & \dots & \dots & \dots \\ 1 & \cosh(2\pi s_M^*/\omega_s) & \cosh^2(2\pi s_M^*/\omega_s) & \dots \\ 1 & \cos(2\pi\omega_{M+1}/\omega_s) & \cos^2(2\pi\omega_{M+1}/\omega_s) & \dots \\ \dots & \dots & \dots & \dots \\ 1 & \cos(2\pi\omega_{M+N}/\omega_s) & \cos^2(2\pi\omega_{M+N}/\omega_s) & \dots \end{vmatrix}$$

Using this technique step by step for each column from the third to $(2M+N)$-th we finally obtain:

$$D = 2^{1+2+\dots+(2M+N-2)}\begin{vmatrix} 1 & \cosh(2\pi s_1/\omega_s) & \dots & \cosh^{2M+N-1}(2\pi s_1/\omega_s) \\ \dots & \dots & \dots & \dots \\ 1 & \cosh(2\pi s_M/\omega_s) & \dots & \cosh^{2M+N-1}(2\pi s_M/\omega_s) \\ 1 & \cosh(2\pi s_1^*/\omega_s) & \dots & \cosh^{2M+N-1}(2\pi s_1^*/\omega_s) \\ \dots & \dots & \dots & \dots \\ 1 & \cosh(2\pi s_M^*/\omega_s) & \dots & \cosh^{2M+N-1}(2\pi s_M^*/\omega_s) \\ 1 & \cos(2\pi\omega_{M+1}/\omega_s) & \dots & \cos^{2M+N-1}(2\pi\omega_{M+1}/\omega_s) \\ \dots & \dots & \dots & \dots \\ 1 & \cos(2\pi\omega_{M+N}/\omega_s) & \dots & \cos^{2M+N-1}(2\pi\omega_{M+N}/\omega_s) \end{vmatrix}$$

$$= 2^{\frac{(2M+N-2)(2M+N-1)}{2}}\det(\tilde{\mathbf{Z}}) \quad (14)$$

The matrix $\tilde{\mathbf{Z}}$ is the Vandermonde matrix; its determinant can be easily found [4] as the product of all possible difference combinations of its second column elements:

$$\det(\tilde{\mathbf{Z}}) = \prod_{k\neq l}\left(\cosh(2\pi\frac{\tilde{s}_k}{\omega_s}) - \cosh(2\pi\frac{\tilde{s}_l}{\omega_s})\right) \quad (15)$$

where

$$\tilde{s}_i = \begin{cases} s_i & \text{if } 1 \le i \le M \\ s_{i-M}^* & \text{if } M+1 \le i \le 2M \\ j\omega_{i-M} & \text{if } 2M+1 \le i \le 2M+N \end{cases}$$

Therefore, $\det(\tilde{\mathbf{Z}}) = 0$ if and only if there exist k and l such that

$$\cosh\left(2\pi\frac{\tilde{s}_k}{\omega_s}\right) = \cosh\left(2\pi\frac{\tilde{s}_l}{\omega_s}\right)$$

Because all $\tilde{s}_i$ are different, the last equality is valid only when such an integer n exists that

$$2\pi\frac{\tilde{s}_k}{\omega_s} = \pm 2\pi\frac{\tilde{s}_l}{\omega_s} + 2\pi n j$$

or

$$\tilde{s}_k \pm \tilde{s}_l = n j \omega_s$$

$$[Re(\tilde{s}_k) \pm Re(\tilde{s}_l)] + j\,[Im(\tilde{s}_k) \pm Im(\tilde{s}_l)] = n j \omega_s \tag{16}$$

It follows from (16) that

$$Re(\tilde{s}_k) \pm Re(\tilde{s}_l) = 0$$

$$Im(\tilde{s}_k) \pm Im(\tilde{s}_l) = n\omega_s \tag{17}$$

According to the Lemma condition $|Im(\tilde{s}_k)| < \omega_s/2$, hence

$$|Im(\tilde{s}_k) \pm Im(\tilde{s}_l)| \leq |Im(\tilde{s}_k)| + |Im(\tilde{s}_l)| < \omega_s$$

thus equality (17) is valid only in the case $n = 0$, which however leads to a contradiction of the Lemma condition $\tilde{s}_k = \tilde{s}_l$. So the determinant of the system (14) is not equal to zero implying that the inverse matrix exists and the solution $\vec{\mu} = \mathbf{Z}^{-1}\vec{b}$ is unique.

The quasi-inversion method for solving the matrix equation

It is clear that the matrix $\mathbf{Z}$ in (9) may have a considerable dimension, which obviously will cause difficulties when trying to reverse it in order to solve for the coefficients vector. The problem of ill-conditioning of the system arises, sometimes making the whole task of implementing the filter impossible. Below a method allowing this difficulty to be avoided is proposed.

First of all, recall the expansions given in (11) in a modified form:

$$\cos(nx) = \sum_{k=0}^{n} a_k^{(n)} \cos^n(x)$$

$$\cosh(nx) = \sum_{k=0}^{n} a_k^{(n)} \cosh^n(x)$$

Further, applying the notation of the formulae (11) and (12), we can rewrite (10) in the form:

$$\mathbf{Z} = \begin{bmatrix} f_0\left[\cosh\left(2\pi s_1/\omega_s\right)\right] & \dots & f_{2M+N-1}\left[\cosh\left(2\pi s_1/\omega_s\right)\right] \\ \dots & \dots & \dots \\ f_0\left[\cosh\left(2\pi s_M/\omega_s\right)\right] & \dots & f_{2M+N-1}\left[\cosh\left(2\pi s_M/\omega_s\right)\right] \\ f_0\left[\cosh\left(2\pi s_1^*/\omega_s\right)\right] & \dots & f_{2M+N-1}\left[\cosh\left(2\pi s_1^*/\omega_s\right)\right] \\ \dots & \dots & \dots \\ f_0\left[\cosh\left(2\pi s_M^*/\omega_s\right)\right] & \dots & f_{2M+N-1}\left[\cosh\left(2\pi s_M^*/\omega_s\right)\right] \\ f_0\left[\cos\left(2\pi \omega_{M+1}/\omega_s\right)\right] & \dots & f_{2M+N-1}\left[\cos\left(2\pi \omega_{M+1}/\omega_s\right)\right] \\ \dots & \dots & \dots \\ f_0\left[\cos\left(2\pi \omega_{M+N}/\omega_s\right)\right] & \dots & f_{2M+N-1}\left[\cos\left(2\pi \omega_{M+N}/\omega_s\right)\right] \end{bmatrix} \quad (18)$$

One can readily verify that $\mathbf{Z}$ as written in (18) equals

$$\mathbf{Z} = \mathbf{VA}$$

where

$$\mathbf{V} = \begin{bmatrix} 1 & \cosh\left(2\pi s_1/\omega_s\right) & \cosh^2\left(2\pi s_1/\omega_s\right) & \dots & \cosh^{2M+N-1}\left(2\pi s_1/\omega_s\right) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \cosh\left(2\pi s_M/\omega_s\right) & \cosh^2\left(2\pi s_M/\omega_s\right) & \dots & \cosh^{2M+N-1}\left(2\pi s_M/\omega_s\right) \\ 1 & \cosh\left(2\pi s_1^*/\omega_s\right) & \cosh^2\left(2\pi s_1^*/\omega_s\right) & \dots & \cosh^{2M+N-1}\left(2\pi s_1^*/\omega_s\right) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \cosh\left(2\pi s_M^*/\omega_s\right) & \cosh^2\left(2\pi s_M^*/\omega_s\right) & \dots & \cosh^{2M+N-1}\left(2\pi s_M^*/\omega_s\right) \\ 1 & \cos\left(2\pi \omega_{M+1}/\omega_s\right) & \cos^2\left(2\pi \omega_{M+1}/\omega_s\right) & \dots & \cos^{2M+N-1}\left(2\pi \omega_{M+1}/\omega_s\right) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \cos\left(2\pi \omega_{M+N}/\omega_s\right) & \cos^2\left(2\pi \omega_{M+N}/\omega_s\right) & \dots & \cos^{2M+N-1}\left(2\pi \omega_{M+N}/\omega_s\right) \end{bmatrix}$$

is the Vandermonde matrix, and

$$\mathbf{A} = \begin{bmatrix} a_0^{(1)} & a_0^{(2)} & \dots & a_0^{(2M+N-1)} \\ 0 & a_1^{(2)} & \dots & a_1^{(2M+N-1)} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{2M+N}^{(2M+N-1)} \end{bmatrix}$$

where the elements of this matrix are as defined in (13).

Then (9) can be rewritten in the form:

$$\mathbf{VA}\vec{\mu} = \vec{b}$$

or

$$\mathbf{A}\vec{\mu} = \mathbf{V}^{-1}\vec{b}$$

The matrix **A** is upper triangular; its inversion can be carried out using the Gauss method and is not very difficult. Moreover, the leading elements of the rows of **A** are powers of 2, which in principle allows us to avoid the problem of the matrix be ill-conditioned during inversion. So what remains to be done is to define the matrix inverse to the Vandermonde one. The determinant of the Vandermonde matrix is given by (15); thus we are to find cofactors Q_{im} to each element V_{im} of the Vandermonde matrix:

$$Q_{im} = (-1)^{i+m} \begin{vmatrix} 1 & q_1 & \dots & q_1^{m-1} & q_1^{m+1} & \dots & q_1^{2M+N-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ 1 & q_{i-1} & \dots & q_{i-1}^{m-1} & q_{i-1}^{m+1} & \dots & q_{i-1}^{2M+N-1} \\ 1 & q_{i+1} & \dots & q_{i+1}^{m-1} & q_{i+1}^{m+1} & \dots & q_{i+1}^{2M+N-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ 1 & q_{2M+N-1} & \dots & q_{2M+N-1}^{m-1} & q_{2M+N-1}^{m+1} & \dots & q_{2M+N-1}^{2M+N-1} \end{vmatrix}$$

This determinant is known [4]; it can be found using the procedure presented below. In order to calculate the cofactors, let us firstly prove the following equality:

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{m-1} & a_1^{m+1} & \dots & a_1^{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & a_i & a_i^2 & \dots & a_i^{m-1} & a_i^{m+1} & \dots & a_i^{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & a_{n-1} & a_{n-1}^2 & \dots & a_{n-1}^{m-1} & a_{n-1}^{m+1} & \dots & a_{n-1}^{n-1} \end{vmatrix}$$

$$= \left(\sum_{k_1,k_2,\dots,k_{n-1-i}} a_{k_1}a_{k_2}\dots a_{k_{n-1-i}}\right) \begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-2} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_{n-1} & a_{n-1}^2 & \dots & a_{n-1}^{n-2} \end{vmatrix} \qquad (19)$$

$$= F_{n-1}(a_1, a_2, \dots, a_{n-1})$$

where $0 \le i \le n-1$, $1 \le m \le n-1$ and we sum over all possible combinations of $n-1-i$ numbers from the total quantity of $(n-1)$ numbers $1, 2, 3, \dots, (n-1)$.

Let us note that the both matrices in (19) have the dimension $(n-1)\times(n-1)$, and the matrix in the righthand part of the equality is the Vandermonde one having the elements $\{a_1, a_2, \dots, a_{n-1}\}$.

To prove (19) consider the following determinant:

$$W = \begin{vmatrix} 1 & x & x^2 & \dots & x^{n-1} \\ 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_{n-1} & a_{n-1}^2 & \dots & a_{n-1}^{n-1} \end{vmatrix}$$

On the one hand, this is the Vandermonde determinant and can be calculated as

$$\begin{aligned} W &= (x-a_1)(x-a_2)\dots(x-a_{n-1})(a_1-a_2)\dots(a_{n-2}-a_{n-1}) \\ &= (x-a_1)(x-a_2)\dots(x-a_{n-1}) \begin{vmatrix} 1 & a_1 & \dots & a_1^{n-2} \\ 1 & a_2 & \dots & a_2^{n-2} \\ \dots & \dots & \dots & \dots \\ 1 & a_{n-1} & \dots & a_{n-1}^{n-2} \end{vmatrix} \end{aligned} \qquad (20)$$

Note that the product of factors preceeding the determinant in (20) is a polynomial in x of the order $(n-1)$ and can be presented as

$$x^{n-1}-(a_1+a_2+\dots+a_{n-1})x^{n-2}+(a_1a_2+a_1a_3+\dots+a_1a_{n-1})x^{n-3}-\dots+(-1)^{n-1}(a_1a_2\dots a_{n-1})$$

On the other hand, expanding the original determinant by the elements of the first row, we have:

$$W = \sum_{i=0}^{n-1}(-1)^{n-i}x^{n-1-i} \begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{m-1} & a_1^{m+1} & \dots & a_1^{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & a_i & a_i^2 & \dots & a_i^{m-1} & a_i^{m+1} & \dots & a_i^{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & a_{n-1} & a_{n-1}^2 & \dots & a_{n-1}^{m-1} & a_{n-1}^{m+1} & \dots & a_{n-1}^{n-1} \end{vmatrix} \qquad (21)$$

Comparing coefficients of the same powers of x in (20) and (21), we prove the desired equality (19).

Assuming in (19) $n = 2M + N$ and

$$\begin{aligned} a_1 &= q_1 \\ a_2 &= q_2 \\ &\dots \\ a_{m-1} &= q_{m-1} \\ a_m &= q_{m+1} \\ &\dots \\ a_{n-1} &= q_{2M+N-1} \end{aligned}$$

we get

$$Q_{im} = (-1)^{i+m} F_{2M+N-1}(q_1, q_2, \ldots, q_{m-1}, q_{m+1}, \ldots, q_{2M+N-1})$$

$$= (-1)^{i+m} P_{im} \begin{vmatrix} 1 & q_1 & q_1^2 & \ldots & q_1^{2M+N-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & q_{m-1} & q_{m-1}^2 & \ldots & q_{m-1}^{2M+N-2} \\ 1 & q_{m+1} & q_{m+1}^2 & \ldots & q_{m+1}^{2M+N-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & q_{2M+N-2} & q_{2M+N-2}^2 & \ldots & q_{2M+N-2}^{2M+N-2} \end{vmatrix}$$

$$= (-1)^{i+m} P_{im} \prod_{k\neq l, k\neq m, l\neq m} (q_k - q_l)$$

where we introduce the notation

$$P_{im} = \sum_{k_1, k_2, \ldots, k_{2M+N-l}} (q_{k_1} \cdots q_{2M+N-l})$$

Thus, elements of the matrix $\mathbf{V}^{-1}$ inverse to the Vandermonde one can be easily calculated according to the formula:

$$V_{mi}^{-1} = \frac{Q_{im}}{det(\mathbf{V})} = \frac{(-1)^{i+m} P_{im} \prod_{k\neq l, k\neq m, l\neq m}(q_k - q_l)}{\prod_{k\neq l}(q_k - q_l)} = \frac{(-1)^{i+m} P_{im}}{\prod_{k\neq m}(q_m - q_k)}$$

The proved feature allows a simple iterative procedure for calculating the factors P_{im} from the last equation to be realized. To do so, we will sequentially compute products composed of one, two, and so on until we have $2M+N-l$ components of the form $(x-q_i)$, for $i = 1, 2, \ldots, (2M+N)$. For the k-th sequential step we will get:

$$P_k = \prod_{i=1}^{k}(x - q_i) = \sum_{i=0}^{k} x^i q_i^{(k)}$$

Expressing P_{k+1} in terms of P_k and q_{k+1}, we have:

$$P_{k+1} = P_k(x - q_{k+1}) = \left(\sum_{i=0}^{k} x^i q_i^{(k)}\right)(x - q_{k+1})$$

$$= \sum_{i=0}^{k} x^{i+1} q_i^{(k)} - \sum_{i=0}^{k} x^i q_i^{(k)} q_{k+1} = \sum_{i=0}^{k+1} x^i q_i^{(k+1)}$$

Setting coefficients of the same powers of x to be equal, we get the following rule for calculating the coefficients:

$$\begin{aligned} q_0^{(i+1)} &= q_0^{(i)} q_{k+1} \\ q_k^{(i+1)} &= q_k^{(i)} = 1 \\ q_m^{(i+1)} &= q_{m+1}^{(i)} - q_m^{(i)} q_{k+1} \end{aligned} \tag{22}$$

where $1 < m < k$.

After the necessary number of iterations (steps) the desired coefficients are obtained.

In the above sections we discussed the case of the unique solution of the matrix equation (9). In principle, however, there is no apparent objection to choosing the sampling frequency to be less than that dictated by the Lemma. This could lead to the appearance of a certain non-trivial solution of the homogeneous equation $\mathbf{Z}\vec{\mu} = \vec{0}$ for some value of ω_s. (Under the conditions of the Lemma, the solution of the homogeneous equation is always unique and trivial.) In the next section we will consider the choice of ω_s less than that according to Lemma.

Relation to the E-pulse technique

Different methods for identifying targets by their backscattered time-domain responses exist; their discussion is beyond the scope of the present paper. An interested reader can address a number of references; many techniques are mentioned and briefly discussed in [5]. Certain discrimination and idetification procedures can be interpreted as linear filtering, which will be demonstrated in this section. The particular case of a discrimination technique which we will consider is known as the extinction pulse.

The so-called extinction pulse (E-pulse) method is one of the most important techniques in the field of radar-target discrimination. An E-pulse is defined as a waveform of a finite duration which, when interacting with the radar response of a particular target, eliminates a preselected number of of its natural resonances in the late time response (2). In this section we show that this technique can be considered a particular case of the approach discussed in the present paper.

Let us consider the case of choosing the sampling frequency $\omega_s < 2\max\{Im(s_i)\}$. Besides this, we assume that we are only concerned with eliminating a set of frequencies $\{s_i = \sigma_i + j\omega_i\}$ given beforehand. For any σ_i the only one corresponding value of ω_i is assumed. In this event the vector $\vec{b}$ of free terms in equation (9) consists only of zeros and the corresponding matrix equation acquires the homogeneous form:

$$\mathbf{Z}\vec{\mu} = \vec{0}$$

This equation has a non-trivial solution only when $\det(\mathbf{Z}) = 0$. ¿From the proof of the Lemma it follows that this is the case only when numbers k and l together with the sampling frequency ω_s can be found such that

$$\cosh\left(2\pi\frac{s_k}{\omega_s}\right) = \cosh\left(2\pi\frac{s_l}{\omega_s}\right)$$

This is possible provided s_k and s_l have the same real parts, which in turn leads to the conclusion that the condition $s_l = s_k^*$ should be satisfied. If so, then equation (22) holds when

$$2\pi\frac{s_k}{\omega_s} = \pm 2\pi\frac{s_k^*}{\omega_s} + 2\pi n j$$

or

$$\frac{\omega_k}{\omega_s} = -\frac{\omega_k}{\omega_s} + n$$

which means that $\omega_s = 2\omega_k/n$.

These results coincide with the expressions corresponding to the E-pulses obtained in [6]. In the references dealing with this subject elimination of a DC component from the target response has not been considered, and the E-pulse itself contains some constant level. If in addition to the set of N frequencies $\{s_i\}$, for $1 \leq i \leq N$ we want to suppress the DC term, we will get $2N + 1$ equations and in the E-pulse the constant component will be absent.

¿From all the aforesaid it follows that the E-pulse technique can be technologically realized in the form of an FIR filter using delay elements. A possibility of implementing the filter in the analogue form is noteworthy, since responses to be processed are high-frequency which demands an even higher sampling rate.

Application to target discrimination

It should be mentioned here that an approach to radar target discrimination similar to that described in the present paper had already been proposed in the past. In [2] the concept of a resonanse annihilation filter (RAF) was introduced and its design based on an FIR system was discussed. Two RAF examples were considered: differential operators to cancel individual natural frequencies being the construction basis for the first one, and their digital counterpart, namely the appropriate difference equations — for the second. Thus, the principal difference between the two approaches is the way the filter is constructed. Besides this, it is noteworthy that the RAF is intended just to cancel a chosen set of natural modes. This leads to the only decision scheme where one decides to which target the response observed corresponds on the basis of the minimal energy contained in the filter response during the late-time period. It would be interesting to point out that the idea of rejecting a signal which is the solution of a suitable differential equation is pretty old; these rejection filters were proposed by Plotkin and named Function Elimination Filters (FEF); their theory and applications can be found, for instance, in [7].

Differing from the RAF approach, our filter enables the rejection of a given set of natural frequencies as well as letting them pass without distortion. So, if the case of the identification of any target of K possible is considered, the problem is to construct K

filters, each of which is designed to identify (to pass without distortion) all the natural frequencies of one given target and to reject both the frequency components of all other targets and the interference. Then it is possible to decide which target is observed when comparing the responses of all the filters to the backscattered response coming from the target. As a simple and obvious means of comparing, and subsequently identifying the target, one can use for instance, the energetic characteristics of the processed response, which was used also in [2]. In fact, let $\{a_m^{(1)}\}, \ldots, \{a_m^{(k)}\}, \ldots, \{a_m^{(K)}\}$, for $m = 1, \ldots, M$, be the measured instantaneous values of the target response processed by the first, the second, and so on till the K-th filters, respectively. Then the decision is made depending on the value of

$$\eta = \max_k \left\{ \sum_{m=1}^{M} \left(a_m^{(k)} \right)^2 \right\}$$

where $k = 1, \ldots, K$ as stated above is the number of the filter or, which is the same, the number of the corresponding target.

Conclusions

In the paper the design theory of an FIR filter which can be used for radar target identification and discrimination is presented. Here we would like to point out what, in our opinion, are the advantages of using this approach for the goal mentioned over other methods described in the literature:

1) The filter's invariance to amplitudes and initial phases of the mixture's frequency components, which principally allows the reconstruction of a useful signal with high accuracy, avoiding the necessity of taking these parameters into consideration.

2) The possibility of eliminating a narrowband interference, removing it from the natural frequency components of the backscattered response.

3) The possibility of making a decision without using the whole set of the natural

frequencies of a given target; the possibility of choosing the number of considered natural frequencies depending on circumstances.

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