Interaction Notes

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The SEM Representation of Acoustic and Elastodynamic Scattering

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Abstract

The singularity expansion method (SEM) has been extensively applied to electromagnetic scattering problems for target identification based on the aspect-independent natural frequencies. Similar concepts have also been developed for acoustic and elastodynamic scattering. The associated pole residues also have a useful factored form separating the dependence on incidence and scattering directions. Acoustic and elastodynamic reciprocity is used in this paper to extend the residue decomposition as done in electromagnetics to acoustic and elastodynamic pole residues. The resulting factored form of the residues shows that there two angular functions, one scalar and one vector, that characterize the elastodynamic residues.

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1. Introduction

Recent papers have considered the SEM (singularity expansion method) representation of perfectly conducting and dielectric targets in lossy media (such as soil), in particular the behavior of the pole terms (natural frequencies, natural modes, and coupling vectors) [9, 12]. As the incident field can be approximated as a plane wave in the vicinity of the target and wavelengths (in the external medium) are of the general order of the target dimensions, we can think of the determination of the SEM parameters for identifying the target as EMSI (electromagnetic singularity identification).

An alternate technique applies to metal targets noting that they are not perfectly conducting and have poles on the negative real axis of the s (complex frequency) plane, corresponding to simple decaying exponentials in time domain [10, 11]. In this case, the incident field is the near magnetic field of loop antennas and the wavelength (or skin depth) in the surrounding medium is much larger than the distances from the antennas to the target so that the external medium has negligible influence on the target response (as long as the external medium is not magnetic, i.e., has free-space permeability). The frequencies of interest are quite low, corresponding to diffusion through the target. As such, we can think of the determination of the SEM parameters of the targets (representing the magnetic polarizability dyadic) as MSI (magnetic singularity identification).

The EMSI and MSI have their various advantages and limitations, depending on the characteristics of the target and the surrounding medium. Another technique with yet different advantages and limitations involves the scattering of sound waves from the target. This involves the acoustic and elastodynamic properties of the medium surrounding the target, these properties varying considerably from water to the various kinds of soils of potential interest. Since the sound waves satisfy a wave equation in such media (approximated as uniform, isotropic, linear, and reciprocal) there are similarities in the formal structure of the acoustic and elastodynamic scattering to the electromagnetic scattering. It has been observed that there are resonances in the scattering of sound waves which can also be used to identify targets in a manner similar to that which has been observed in the electromagnetic case [27]. Since there has been a considerable development of the SEM formalism for electromagnetic scattering [1, 2, 5-8, 16, 18, 20, 22, 23, 26], it should be helpful to apply this to acoustic/elastodynamic scattering to aid in understanding the target-identification potential here. As will be exhibited in this paper, the pole terms in the acoustic/elastodynamic scattering have a similar form involving natural frequencies, natural modes, and coupling terms which are scalar or vector depending on the types of waves (p and/or s) involved. Let us call this type of target identification ASI (acoustic singularity identification).

Figure 1.1 shows the scattering geometry. The target is located inside a minimum circumscribing sphere (radius a) centered on the coordinate origin ($\vec{r} = \vec{0}$). The incident and scattered waves (whether p or s waves) propagate in directions

$$
\begin{aligned}
\vec{1}_i &\equiv \text{direction of incidence (plane wave)} \\
\vec{1}_o &\equiv \text{direction of scattering (outgoing spherical wave in the far field)} \\
&\equiv \text{direction to observer}
\end{aligned}
\tag{1.1}
$$

These also describe the polarizations of the p (pressure) waves. Transverse to these directions are the s (shear) waves, for which we can define.

$$
\begin{aligned}
\overleftrightarrow{1}_i &\equiv \overleftrightarrow{1} - \vec{1}_i\,\vec{1}_i \equiv \text{dyadic transverse to } \vec{1}_i \\
\overleftrightarrow{1}_o &\equiv \overleftrightarrow{1} - \vec{1}_o\,\vec{1}_o \equiv \text{dyadic transverse to } \vec{1}_o
\end{aligned}
\tag{1.2}
$$

Additional quantities are

$$
\begin{aligned}
\overleftrightarrow{1} &\equiv \vec{1}_x\,\vec{1}_x + \vec{1}_y\,\vec{1}_y + \vec{1}_z\,\vec{1}_z \equiv \text{identity dyadic} \\
\overleftrightarrow{1}_i \cdot \vec{1}_i &= \vec{0}\;,\quad \overleftrightarrow{1}_o \cdot \vec{1}_o = \vec{0} \\
\vec{r} &\equiv x\,\vec{1}_x + y\,\vec{1}_y + z\,\vec{1}_z \equiv (x, y, z) \\
r &\equiv |\vec{r}\,|
\end{aligned}
\tag{1.3}
$$

We also use the two-sided Laplace tranform as

$$
\begin{aligned}
\tilde{f}(s) &= \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt \\
s &\equiv \Omega + j\omega \equiv \text{Laplace-transform variable or complex frequency} \\
t &\equiv \text{time} \\
f(t) &= \frac{1}{2\pi j} \int_{Br} \tilde{f}(s)\, e^{st}\, ds \equiv \text{inverse Laplace transform} \\
Br &\equiv \text{Bromwich contour in strip of convergence parallel to } j\omega \text{ axis}
\end{aligned}
\tag{1.4}
$$

By setting $s = j\omega$ this is also called the Fourier transform.

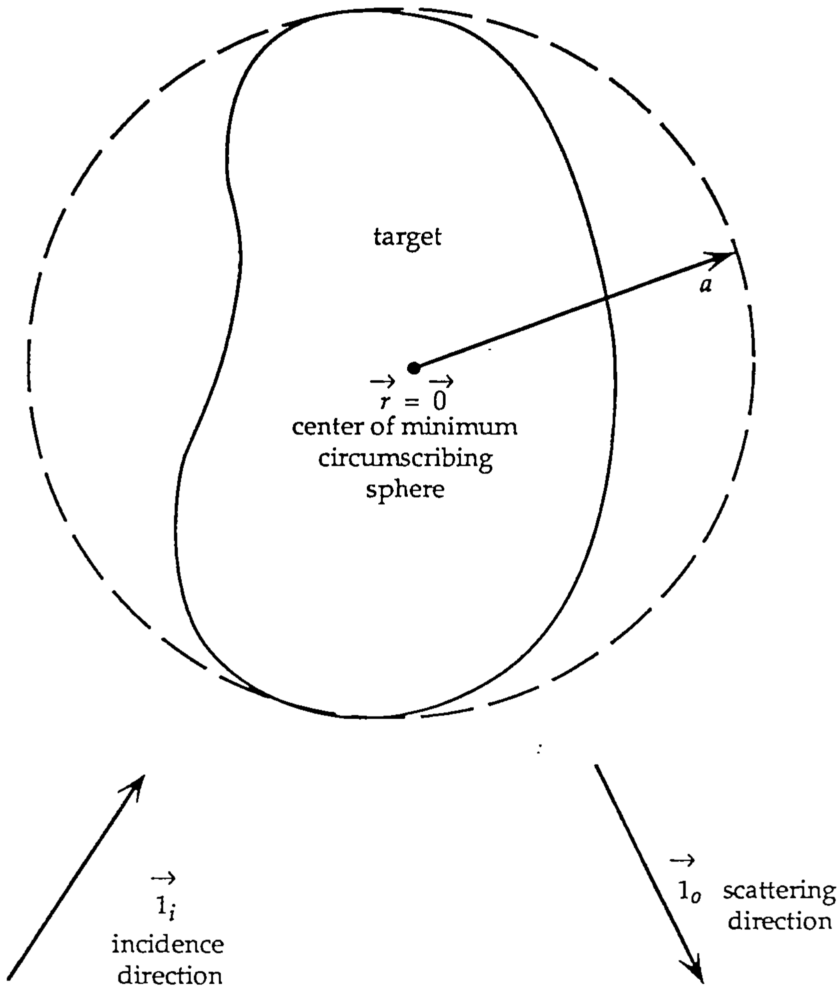

Fig. 1.1 Acoustic and Elastodynamic Scattering from a Target of Finite Linear Dimensions in an Infinite, Homogeneous, Isotropic, Reciprocal, and Linear Medium.

2. Acoustic Scattering

Consider first scalar acoustic scattering. In this case we have [21, 24, 25]

$$
\begin{aligned}
\vec{u}(\vec{r},t) &\equiv \text{particle displacement} \\
\vec{v}(\vec{r},t) &\equiv \frac{\partial}{\partial t}\vec{u}(\vec{r},t) \equiv \text{particle velocity} \\
\Phi(\vec{r},t) &\equiv \text{velocity potential} \\
\vec{v}(\vec{r},t) &= \nabla\Phi(\vec{r},t) \\
p(\vec{r},t) &\equiv \text{pressure} \\
\rho_a &\equiv \text{mass density} \\
\kappa_a &= \frac{1}{\rho_a}\frac{d\rho_a}{dp} = -\frac{1}{V}\frac{dV}{dp} = -\ \text{fractional change in volume per unit pressure} \\
&\equiv \text{compressibility} \\
p(\vec{r},t) &= -\rho_a \frac{\partial}{\partial t}\Phi(\vec{r},t)
\end{aligned}
\tag{2.1}
$$

Note that ρ_a is the undisturbed or background mass density. The wave parameters are assumed sufficiently small to give a linearized form of the equations. The scalar form of the wave is a consequence of the scalar form of the compressibility.

If the medium is dispersive, then one regards the compressibility $\tilde{\kappa}_a(s)$ as a function of frequency, since in time domain the compressibility is a convolution operator. Waves travel with speed

$$
\begin{aligned}
\tilde{v}_a(s) &= \left[\tilde{\kappa}_a(s)\ \rho_a\right]^{-\frac{1}{2}} \\
\tilde{\gamma}_a(s) &= \frac{s}{\tilde{v}_a(s)} = s\left[\tilde{\kappa}_a(s)\ \rho_a\right]^{\frac{1}{2}} \equiv \text{propagation constant}
\end{aligned}
\tag{2.2}
$$

One also have

$$
\begin{aligned}
\tilde{Z}_a(s) &\equiv \frac{\tilde{p}(\vec{r},s)}{\tilde{v}(\vec{r},s)} = \rho_a\ \tilde{v}_a(s) = \left[\frac{\rho_a}{\tilde{\kappa}_a(s)}\right]^{\frac{1}{2}} \\
&\equiv \text{characteristic acoustic impedance} \\
&\equiv \text{acoustic wave impedance}
\end{aligned}
\tag{2.3}
$$

where the pressure and velocity have are that of a uniform plane wave in the medium, $\tilde{v}(\vec{r},s)\ \vec{1}_i$ being the vector velocity.

There is a wave equation

$$\left[\nabla^2 - \tilde{\gamma}_a^2(s)\right] \tilde{\Phi}(\vec{r},s) = 0 \tag{2.4}$$

away from sources (i.e., away from the target). This can also be written in terms of the particle displacement or the particle velocity. The Green's function is given by

$$\begin{aligned} \left[\nabla^2 - \tilde{\gamma}_a^2(s)\right] \tilde{G}_0(\vec{r},\vec{r}\,';s) &= -\delta(\vec{r}-\vec{r}\,') \\ \tilde{G}_0(\vec{r},\vec{r}\,';s) &= \frac{e^{-\tilde{\gamma}_a(s)|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r},\vec{r}\,'|} \end{aligned} \tag{2.5}$$

If the medium is dispersionless, we have

$$\left[\nabla^2 - \frac{1}{v_a^2}\frac{\partial^2}{\partial t^2}\right] \Phi(\vec{r},t) = 0 \tag{2.6}$$

as the temporal wave equation, and

$$G_0(\vec{r},\vec{r}\,';t) = \frac{1}{4\pi|\vec{r}-\vec{r}\,'|}\,\delta\left(t-\frac{|\vec{r}-\vec{r}\,'|}{v_a}\right) \tag{2.7}$$

as the temporal Green's function which with convolution (over time) gives a temporal operator. This Green's function can be used to formulate integral equations for scattering, in this case for scalar acoustic scattering [25].

Divide the waves into an incident plane wave (direction $\vec{1}_i$) and scattered spherical wave (direction $\vec{1}_o$ to observer) as indicated in fig. 1.1. Using superscripts inc (incident) and sc (scattered) and subscript f for far field we have

$$\begin{aligned} \tilde{\Phi}_f^{(sc)}(\vec{r},s) &= \frac{e^{-\tilde{\gamma}_a(s)r}}{4\pi r}\,\tilde{\Lambda}(\vec{1}_o,\vec{1}_i;s)\;\tilde{\Phi}^{(inc)}\left(\vec{0},s\right) \\ \tilde{\Lambda}(\vec{1}_o,\vec{1}_i;s) &\equiv \text{scattering scalar (convolution operator in time domain)} \end{aligned} \tag{2.8}$$

This scattering coefficient is merely a statement of the linear relationship between the incident wave (evaluated at $\vec{r} = \vec{0}$) and the scattered far field evaluated in the direction $\vec{1}_o$. This can also be applied to other wave variables as

$$\begin{aligned}\tilde{\vec{v}}_f^{(sc)}(\vec{r},s) &= \vec{1}_o\,\frac{e^{-\tilde{\gamma}_a(s)r}}{4\pi r}\,\tilde{\Lambda}(\vec{1}_o,\vec{1}_i;s)\,\vec{1}_i\cdot\tilde{\vec{v}}^{(inc)}(\vec{0},s)\\ \tilde{\vec{u}}_f^{(sc)}(\vec{r},s) &= \vec{1}_o\,\frac{e^{-\tilde{\gamma}_a(s)r}}{4\pi r}\,\tilde{\Lambda}(\vec{1}_o,\vec{1}_i;s)\,\vec{1}_i\cdot\tilde{\vec{u}}^{(inc)}(\vec{0},s)\end{aligned} \tag{2.9}$$

If the sound speed is independent of frequency the scattering relationship can be conveniently written in time domain as

$$\begin{gathered}\Phi_f^{(sc)}(\vec{r},t) = \frac{1}{4\pi r}\Lambda(\vec{1}_o,\vec{1}_i;t)\circ\Phi^{(inc)}(\vec{0},t-\frac{r}{v_a})\\ \circ \equiv \text{convolution with respect to time}\end{gathered} \tag{2.10}$$

where the velocity potential here can also be replaced by the particle velocity and displacement as used in (2.9).

Of fundamental importance is the acoustic reciprocity theorem, which for scattering is [17, 24]

$$\tilde{\Lambda}(\vec{1}_o,\vec{1}_i;s) = \tilde{\Lambda}(-\vec{1}_i,-\vec{1}_o;s) \tag{2.11}$$

This is similar to the case of electromagnetic scattering [5, 26], except that in electromagnetics, one has a scattering dyadic which also undergoes a transpose on the interchange

$$\vec{1}_i \to -\vec{1}_o,\quad \vec{1}_o \to -\vec{1}_i \tag{2.12}$$

i.e., an interchange of incidence and scattering directions with a change of sign (reversal of direction). For the special case of backscattering we have

$$\begin{aligned}\vec{1}_o &= -\vec{1}_i\\ \tilde{\Lambda}_b(\vec{1}_i;s) &= \tilde{\Lambda}(-\vec{1}_i,\vec{1}_i;s)\end{aligned} \tag{2.13}$$

3. Elastodynamic Scattering

Elastodynamic waves are considerably more complicated than acoustic waves. The scalar compressibility is replaced by a fourth-rank stiffness tensor as [13-15, 17, 24]

$$
\begin{aligned}
\tilde{\overleftrightarrow{\tau}}(\vec{r},s) &= \left(\tilde{C}_{n,m,\ell,\kappa}(s)\right) \cdot \nabla \tilde{\overleftrightarrow{u}}(\vec{r},s) \\
&\equiv \text{stress tensor (Laplace transformed)} \\
\left(\tilde{C}_{n,m,\ell,\kappa}(s)\right) &\equiv \text{stiffness tensor} \\
\tilde{C}_{n,m,\ell,\kappa}(s) &= \tilde{\ell}_1(s) 1_{n,m}\, 1_{\ell,\kappa} + \tilde{\ell}_2(s)\left[1_{n,\ell}\, 1_{m\kappa} + 1_{n,\kappa}\, 1_{m,\ell}\right] \\
1_{n,n'} &= \begin{cases} 1 & \text{for } n = n' \\ 0 & \text{for } n \neq n' \end{cases} \\
\tilde{\ell}_1(s),\ \tilde{\ell}_2(s) &\equiv \text{Lame constants} \\
\tilde{\overleftrightarrow{\tau}}(\vec{r},s) &= \tilde{\ell}_1(s)\left[\nabla \cdot \tilde{\vec{u}}(\vec{r},s)\right] \overleftrightarrow{1} + \tilde{\ell}_2(s)\left[\nabla \tilde{\vec{u}}(\vec{r},s) + \left[\nabla \tilde{\vec{u}}(\vec{r},s)\right]^{T}\right] \\
&= \tilde{\overleftrightarrow{\tau}}^{T}(\vec{r},s)
\end{aligned}
\tag{3.1}
$$

This leads to two wave speeds

$$
\begin{aligned}
\tilde{v}_p(s) &= \left[\frac{\tilde{\ell}_1 + 2\tilde{\ell}_2}{\rho_a}\right]^{\frac{1}{2}} \equiv p\text{-wave speed (longitudinal wave speed)} \\
\tilde{v}_s(s) &= \left[\frac{\tilde{\ell}_2}{\rho_a}\right]^{\frac{1}{2}} \equiv s\text{-wave speed (transverse wave speed)}
\end{aligned}
\tag{3.2}
$$

where p stands for pressure, and s for shear. For the case of frequency-independent (and thereby real and positive) Lame constants the speeds are real and we have

$$
v_p > v_s > 0 \tag{3.3}
$$

One can go on to set up integral equations for scattering and the associated Green's function for which one can consult the references.

For far-field scattering we need both p and s waves. In terms of the displacement, we have the incident plane wave

$$\begin{aligned}
\vec{\tilde{u}}^{(inc)}(\vec{r},s) &= \vec{\tilde{u}}_p^{(inc)}(\vec{r},s) + \vec{\tilde{u}}_s^{(inc)}(\vec{r},s) \\
&= e^{-\tilde{\gamma}_p(s)\vec{1}_i\cdot\vec{r}}\,\vec{\tilde{u}}_p^{(inc)}(\vec{0},s) + e^{-\tilde{\gamma}_s(s)\vec{1}_i\cdot\vec{r}}\,\vec{\tilde{u}}_s^{(inc)}(\vec{0},s) \\
\vec{\tilde{u}}^{(inc)}(\vec{0},s) &= \tilde{u}_p^{(inc)}(\vec{0},s)\,\vec{1}_i \\
\vec{\tilde{u}}^{(inc)}(\vec{0},s)\cdot\vec{1}_i &= 0
\end{aligned} \tag{3.4}$$

where the two propagation constants are

$$\tilde{\gamma}_p(s) \equiv \frac{s}{\tilde{v}_p(s)} \; , \; \tilde{\gamma}_p(s) \equiv \frac{s}{\tilde{v}_s(s)} \tag{3.5}$$

The scattered far field is a spherical wave of the form

$$\begin{aligned}
\vec{\tilde{u}}_f^{(sc)}(\vec{r},s) &= \vec{\tilde{u}}_{f_p}^{(sc)}(\vec{r},s) + \vec{\tilde{u}}_{f_s}^{(sc)}(\vec{r},s) \\
\vec{r} &= r\,\vec{1}_o \\
\vec{\tilde{u}}_{f_p}^{(sc)}(\vec{r},s) &= \tilde{u}_{f_p}^{(sc)}(\vec{r},s)\,\vec{1}_o \\
\vec{\tilde{u}}_{f_s}^{(sc)}(\vec{r},s)\cdot\vec{1}_o &= 0
\end{aligned} \tag{3.6}$$

Relating the scattered far field to the incident field by linearity we have

$$\begin{aligned}
\vec{\tilde{u}}_{f_p}^{(sc)}(\vec{r},s) &= \vec{1}_o\,\frac{e^{-\tilde{\gamma}_p(s)r}}{4\pi r}\left[\tilde{\Lambda}_{p,p}(\vec{1}_o,\vec{1}_i;s)\,\vec{1}_i\cdot\vec{\tilde{u}}^{(inc)}(\vec{0},s) + \vec{\tilde{\Lambda}}_{p,s}(\vec{1}_o,\vec{1}_i;s)\cdot\vec{\tilde{u}}_s^{(inc)}(\vec{0},s)\right] \\
\vec{\tilde{u}}_{f_s}^{(sc)}(\vec{r},s) &= \frac{e^{-\tilde{\gamma}_s(s)r}}{4\pi r}\left[\vec{\tilde{\Lambda}}_{s,p}(\vec{1}_o,\vec{1}_i;s)\,\vec{1}_i\cdot\vec{\tilde{u}}_p^{(inc)}(\vec{0},s) + \overleftrightarrow{\tilde{\Lambda}}_{s,s}(\vec{1}_o,\vec{1}_i;s)\cdot\vec{\tilde{u}}_s^{(inc)}(\vec{0},s)\right] \\
\vec{\tilde{\Lambda}}_{p,s}(\vec{1}_o,\vec{1}_i;s)\cdot\vec{1}_i &= 0 \; , \; \vec{1}_o\cdot\vec{\tilde{\Lambda}}_{s,p}(\vec{1}_o,\vec{1}_i;s) = 0 \\
\overleftrightarrow{\tilde{\Lambda}}_{s,s}(\vec{1}_o,\vec{1}_i;s)\cdot\vec{1}_i &= \vec{0} \; , \; \vec{1}_o\cdot\overleftrightarrow{\tilde{\Lambda}}_{s,s}(\vec{1}_o,\vec{1}_i;s) = \vec{0}
\end{aligned} \tag{3.7}$$

Note that the four scattering coefficients are variously scalar, vector, or dyadic depending on the various combinations of p and s waves.

The reciprocity theorem for elastodynamic scattering is [17, 24]

$$
\begin{aligned}
\tilde{\Lambda}_{p,p}(\vec{1}_o,\vec{1}_i;s) &= \tilde{\Lambda}_{p,p}(-\vec{1}_i,-\vec{1}_o;s) \\
\tilde{\vec{\Lambda}}_{p,s}(\vec{1}_o,\vec{1}_i;s) &= -\frac{\tilde{v}_s^2(s)}{\tilde{v}_p^2(s)}\tilde{\vec{\Lambda}}_{s,p}(-\vec{1}_i,-\vec{1}_o;s) \\
\tilde{\overleftrightarrow{\Lambda}}_{s,s}(\vec{1}_o,\vec{1}_i;s) &= \tilde{\overleftrightarrow{\Lambda}}{}^{T}_{s,s}(-\vec{1}_i,-\vec{1}_o;s)
\end{aligned}
\tag{3.8}
$$

Note that the s, s scattering is characterized by a dyadic which is transposed on the interchange

$$
\vec{1}_i \to -\vec{1}_o \quad , \quad \vec{1}_o \to -\vec{1}_i
\tag{3.9}
$$

just as in electromagnetic scattering. Note also that the p waves are referenced positive with respect to their directions of propagation, $\vec{1}_i$ and $\vec{1}_o$. Reversing these then reverses sign for the scalar function describing the p wave. For the special case of backscattering we have

$$
\begin{aligned}
\vec{1}_o &= -\vec{1}_i \\
\tilde{\Lambda}_{b_{p,p}}(\vec{1}_i;s) &\equiv \tilde{\Lambda}_{p,p}(-\vec{1}_i,\vec{1}_i;s) \\
\tilde{\vec{\Lambda}}_{b_{p,s}}(\vec{1}_i;s) &= \tilde{\vec{\Lambda}}_{p,s}(-\vec{1}_i,\vec{1}_i;s) = -\frac{\tilde{v}_s^2(s)}{\tilde{v}_p^2(s)}\tilde{\vec{\Lambda}}_{s,p}(-\vec{1}_i,\vec{1}_i;s) = -\frac{\tilde{v}_s^2(s)}{\tilde{v}_p^2(s)}\tilde{\vec{\Lambda}}_{b_{s,p}}(\vec{1}_i;s) \\
\tilde{\overleftrightarrow{\Lambda}}_{b_{s,s}}(\vec{1}_i;s) &= \tilde{\vec{\Lambda}}_{s,s}(-\vec{1}_i,\vec{1}_i;s) = \tilde{\vec{\Lambda}}{}^{T}_{s,s}(-\vec{1}_i,\vec{1}_i;s) = \tilde{\vec{\Lambda}}{}^{T}(\vec{1}_i;s)
\end{aligned}
\tag{3.10}
$$

As in the electromagnetic case, the backscattering dyadic for s to s waves is symmetric.

4. The SEM Representation for Pole Terms

There can, in principle, be various kinds of singularities in the s plane, including poles, branch cuts, and singularities at infinity (or entire functions). In the electromagnetic case for finite-size objects, either perfectly conducting or comprised of suitably simple media, located in free space it is known that there are no branch terms, i.e., there are only poles in the finite s plane [1-3]. However, if the surrounding infinite medium is a lossy dielectric, a branch cut appears in the response [9]. In the acoustic case scattering from acoustically hard and soft finite size targets in a medium with a frequency independent wave speed v_a (real and positive) also has no branch cut in the scattering [4]. Similar results apply to elastodynamic scattering [19]. The important feature is that the Green's function for the external medium (three dimensional) not have any branch cut, implying that the propagation constants in (3.5) are analytic functions of s in the finite s plane. This requirement is met for the case of constant, real, and positive v_p and v_s.

Let us now consider the SEM form of the various scattering coefficients that appear in (3.7). In particular, the pole terms can be exhibited as

$$
\begin{aligned}
\tilde{\Lambda}_{p,p}(\vec{1}_o, \vec{1}_i; s) &= \sum_{\alpha} \Lambda_{\alpha}^{(p,p)}(\vec{1}_o, \vec{1}_i)\left[s - s_\alpha\right]^{-1} \\
&\quad + \text{other singularity terms} \\
\tilde{\vec{\Lambda}}_{p,s}(\vec{1}_o, \vec{1}_i; s) &= \sum_{\alpha} \vec{\Lambda}_{\alpha}^{(p,s)}(\vec{1}_o, \vec{1}_i)\left[s - s_\alpha\right]^{-1} \\
&\quad + \text{other singularity terms} \\
\tilde{\vec{\Lambda}}_{s,p}(\vec{1}_o, \vec{1}_i; s) &= \sum_{\alpha} \vec{\Lambda}_{\alpha}^{(s,p)}(\vec{1}_o, \vec{1}_i)\left[s - s_\alpha\right]^{-1} \\
&\quad + \text{other singularity terms} \\
\tilde{\overleftrightarrow{\Lambda}}_{s,s}(\vec{1}_o, \vec{1}_i; s) &= \sum_{\alpha} \overleftrightarrow{\Lambda}_{\alpha}^{(s,s)}(\vec{1}_o, \vec{1}_i)\left[s - s_\alpha\right]^{-1} \\
&\quad + \text{other singularity terms}
\end{aligned}
\tag{4.1}
$$

Here, the poles are taken as first order, the typical case. However, as observed in electromagnetic scattering, there are cases (involving resistive loading of the scatterer) when second order poles can appear; these require a more general treatment [23]. As discussed above the other singularity terms can include branch cuts in some cases, and entire functions in general (like in the electromagnetic case [7]). While (4.1) is for the general case of elastodynamic scattering as in (3.7) it includes acoustic scattering as in (2.9) as a special case, merely by considering $\tilde{\Lambda}_{p,p}$ as the only non-zero scattering coefficient.

The reciprocity relations in (3.18) can be directly applied to the pole residues as

$$
\begin{aligned}
\Lambda_\alpha^{(p,p)}(\vec{1}_o,\vec{1}_i) &= \Lambda_\alpha^{(p,p)}(-\vec{1}_i,-\vec{1}_o) \\
\vec{\Lambda}_\alpha^{(p,s)}(\vec{1}_o,\vec{1}_i) &= -\frac{\tilde{v}_s^2(s_\alpha)}{\tilde{v}_p^2(s_\alpha)}\vec{\Lambda}_\alpha^{(s,p)}(-\vec{1}_i,-\vec{1}_o) \\
\overleftrightarrow{\Lambda}_\alpha^{(s,s)}(\vec{1}_o,\vec{1}_i) &= \overleftrightarrow{\Lambda}^{(s,s)^{\mathrm{T}}}(-\vec{1}_i,-\vec{1}_o)
\end{aligned}
\tag{4.2}
$$

Consider now a non-degenerate natural mode with natural frequency s_α. From a physical point of view, we can regard a natural mode as the scattering response that can exist at $s = s_\alpha$ with no excitation (i.e., no incident field). If one considers an integral equation, then as in the electromagnetic case (e.g., see [1]) the natural mode is the solution of the integral equation in which the integral operator (evaluated at s_α) operating on the mode is zero. Note that such a mode then has no dependence on the incident field. The strength (or scalar coefficient) of the natural mode of course scales with the incident field (evaluated at s_α). In (3.7) then remove the delays, $(4\pi r)^{-1}$, and $(s - s_\alpha)^{-1}$, and write the remaining terms in the scattered far field for the αth pole as

$$
\begin{aligned}
\left[d_\alpha^{(p)} + d_\alpha^{(s)}\right] f_\alpha(\vec{1}_o)\,\vec{1}_o &= \vec{1}_o\left[\Lambda_\alpha^{(p,p)}(\vec{1}_o,\vec{1}_i)\,\vec{1}_i\cdot\tilde{\vec{u}}_s^{(inc)}(\vec{0},s_\alpha) + \vec{\Lambda}_\alpha^{(p,s)}(\vec{1}_o,\vec{1}_i)\cdot\tilde{\vec{u}}_s^{(inc)}(\vec{0},s_\alpha)\right] \\
\left[d_\alpha^{(p)} + d_\alpha^{(s)}\right] \vec{f}_\alpha(\vec{1}_o) &= \vec{\Lambda}_\alpha^{(s,p)}(\vec{1}_o,\vec{1}_i)\,\vec{1}_i\cdot\tilde{\vec{u}}_p^{(inc)}(\vec{0},s_\alpha) + \overleftrightarrow{\Lambda}_\alpha^{(s,s)}(\vec{1}_o,\vec{1}_i)\cdot\tilde{\vec{u}}_s^{(inc)}(\vec{0},s_\alpha)
\end{aligned}
\tag{4.3}
$$

for the p- and s-wave parts, respectively. Note the common coefficient $d_\alpha^{(p)} + d_\alpha^{(s)}$ on the left sides indicating that the p and s parts of the natural mode (i.e., f_α and $\vec{f}_\alpha$) must scale together for a non-degenerate natural mode. Depending on symmetries in the target one may have degenerate modes where two or more independent natural modes can exist for a particular s_α, in which case a linear combination of these is required. For the non-degenerate case linearity requires that the contributions of p and s incident waves separate as

$$
\begin{aligned}
d_\alpha^{(p)} f_\alpha(\vec{1}_o) &= \Lambda_\alpha^{(p,p)}(\vec{1}_o,\vec{1}_i)\,\vec{1}_i\cdot\tilde{\vec{u}}_p^{(inc)}(\vec{0},s_\alpha) = \Lambda_\alpha^{(p,p)}(\vec{1}_o,\vec{1}_i)\,\tilde{\vec{u}}_p^{(inc)}(\vec{0},s_\alpha) \\
d_\alpha^{(p)} \vec{f}_\alpha(\vec{1}_o) &= \vec{\Lambda}_\alpha^{(s,p)}(\vec{1}_o,\vec{1}_i)\,\vec{1}_i\cdot\tilde{\vec{u}}_p^{(inc)}(\vec{0},s_\alpha) = \vec{\Lambda}_\alpha^{(s,p)}(\vec{1}_o,\vec{1}_i)\,\tilde{u}_p^{(inc)}(\vec{0},s_\alpha) \\
d_\alpha^{(s)} f_\alpha(\vec{1}_o) &= \vec{\Lambda}_\alpha^{(p,s)}(\vec{1}_o,\vec{1}_i)\cdot\tilde{\vec{u}}_s^{(inc)}(\vec{0},s_\alpha) \\
d_\alpha^{(s)} \vec{f}_\alpha(\vec{1}_o) &= \overleftrightarrow{\Lambda}_\alpha^{(s,s)}(\vec{1}_o,\vec{1}_i)\cdot\tilde{\vec{u}}_s^{(inc)}(\vec{0},s_\alpha) \\
\vec{f}_\alpha(\vec{1}_o)\cdot\vec{1}_o &= 0
\end{aligned}
\tag{4.4}
$$

which gives some constraints among the various terms.

Consider first $\Lambda_\alpha^{(p,p)}$. Noting that $d_\alpha^{(p)}$ contains the dependence on $\vec{1}_i$ we can write

$$d_\alpha^{(p)} = a_\alpha\, c_\alpha(\vec{1}_i)\ \tilde{u}_p^{(inc)}(\vec{0},s_\alpha),\ a_\alpha \equiv \text{scaling constant} \tag{4.5}$$

This then gives

$$\begin{aligned} &\Lambda_\alpha^{(p,p)}(\vec{1}_o,\vec{1}_i) = a_\alpha\, f_\alpha(\vec{1}_o) c_\alpha(\vec{1}_i) \\ &= \Lambda_\alpha^{(p,p)}(-\vec{1}_i,-\vec{1}_o) = a_\alpha\, f_\alpha(-\vec{1}_i) c_\alpha(-\vec{1}_o) \end{aligned} \tag{4.6}$$

where the reciprocity relationship (4.2) has been used. This allows us to set

$$\begin{aligned} a_\alpha\, f_\alpha(\vec{1}_o) &= c_\alpha(-\vec{1}_o) \\ \Lambda_\alpha^{(p,p)} &= c_\alpha(-\vec{1}_o) c_\alpha(\vec{1}_i) \end{aligned} \tag{4.7}$$

where we have chosen the constant for convenience such that c_α can be used without an extra scaling constant. In this symmetric form c_α gives the dependence on both incidence and scattering directions. Only one scalar function with argument varying over 4π steradians is needed. This function can, in principle, be determined from detailed scattering calculations and/or experiment.

Second $\overleftrightarrow{\Lambda}_\alpha^{(s,s)}$ also has a factored dependence on the two angles with the dependence on $\vec{1}_i$ contained in $d_\alpha^{(s)}$ which we write as

$$d_\alpha^{(s)} = b_\alpha\ \vec{c}_\alpha(\vec{1}_i) \cdot \vec{\tilde{u}}(\vec{0},s_\alpha)\ ,\ b_\alpha \equiv \text{scaling constant} \tag{4.8}$$

Varying $\vec{\tilde{u}}_s^{(inc)}$ over angles (polarizations) transverse to $\vec{1}_i$ allows us to write

$$\begin{aligned} &\overleftrightarrow{\Lambda}_\alpha^{(s,s)}(\vec{1}_o,\vec{1}_i) = b_\alpha\ \vec{f}_\alpha(\vec{1}_o) \vec{c}_\alpha(\vec{1}_i) \\ &= \overleftrightarrow{\Lambda}_\alpha^{(s,s)^{\mathrm{T}}}(-\vec{1}_i,-\vec{1}_o) = b_\alpha\ \vec{c}_\alpha(-\vec{1}_o) \vec{f}_\alpha(-\vec{1}_i) \end{aligned} \tag{4.9}$$

where again reciprocity has been used. This allows us to set

$$\begin{aligned} &b_\alpha\ \vec{f}_\alpha(\vec{1}_o) = \vec{c}_\alpha(-\vec{1}_i) \\ &\overleftrightarrow{\Lambda}_\alpha^{(s,s)}(\vec{1}_o,\vec{1}_i) = \vec{c}_\alpha(-\vec{1}_o)\ \vec{c}_\alpha(\vec{1}_i) \end{aligned} \tag{4.10}$$

where the vector function $\vec{c}_\alpha$ is different from the scalar function c_α. It can also be determined from detailed scattering calculations and/or experiment. Note that this dyadic form is just like that for electromagnetic scattering [5, 26].

Now consider the mixed-wave terms. From (4.4), (4.5), and (4.10) we have

$$\vec{\Lambda}_\alpha^{(s,p)}(\vec{1}_o, \vec{1}_i) = \frac{a_\alpha}{b_\alpha} \vec{c}_\alpha(-\vec{1}_o) c_\alpha(\vec{1}_i) \tag{4.11}$$

Similarly from (4.4), (4.7), and (4.8), and varying $\vec{\tilde{u}}_s^{(inc)}$ over angles transverse to $\vec{1}_i$ gives

$$\vec{\Lambda}_\alpha^{(p,s)}(\vec{1}_o, \vec{1}_i) = \frac{b_\alpha}{a_\alpha} c_\alpha(-\vec{1}_o) \vec{c}_\alpha(\vec{1}_i) \tag{4.12}$$

Applying reciprocity from (4.2) gives

$$\begin{aligned} \left[\frac{b_\alpha}{a_\alpha}\right]^2 &= -\left[\frac{\tilde{v}_s^2(s_\alpha)}{\tilde{v}_p^2(s_\alpha)}\right]^2 \\ \frac{b_\alpha}{a_\alpha} &= \pm j \frac{\tilde{v}_s(s_\alpha)}{\tilde{v}_p(s_\alpha)} \end{aligned} \tag{4.13}$$

This interesting result may look strange at first. Consider the special case that s_α is a pole on the negative real axis of the s plane ($s_\alpha = \Omega_\alpha$, $\omega_\alpha = 0$). Then since scattering of real-valued temporal pulses must give real valued temporal pulses, we conclude for such poles

$$\begin{aligned} \Lambda_\alpha^{(p,p)}(\vec{1}_o, \vec{1}_i) &= c_\alpha(-\vec{1}_o)\; c_\alpha(\vec{1}_i) = \text{real scalar} \\ \vec{\Lambda}_\alpha^{(p,s)}(\vec{1}_o, \vec{1}_i) &= \frac{a_\alpha}{b_\alpha} \vec{c}_\alpha(-\vec{1}_o)\; c_\alpha(\vec{1}_i) = \text{real vector} \\ \vec{\Lambda}_\alpha^{(s,p)}(\vec{1}_o, \vec{1}_i) &= \frac{a_\alpha}{b_\alpha} \vec{c}_\alpha(-\vec{1}_o)\; c_\alpha(\vec{1}_i) = \text{real vector} \\ \overleftrightarrow{\Lambda}_\alpha^{(s,s)}(\vec{1}_o, \vec{1}_i) &= \vec{c}_\alpha(-\vec{1}_o)\; \vec{c}_\alpha(\vec{1}_i) = \text{real dyadic} \end{aligned} \tag{4.14}$$

The wave speeds also being real then b_α/a_α must be imaginary. If we further specify to backscattering then

$$-\vec{1}_o = \vec{1}_i \tag{4.15}$$

and we can distinguish two cases. The first case has

$$
\begin{aligned}
&c_\alpha(\vec{1}_i)\, c_\alpha(\vec{1}_i) > 0 \\
&c_\alpha(\vec{1}_i) = \text{real scalar} \\
&\vec{c}_\alpha(\vec{1}_i) \cdot \vec{c}_\alpha(\vec{1}_i) < 0 \\
&\vec{c}_\alpha(\vec{1}_i) = \text{imaginary vector, i.e. } j \text{ times real vector} \\
&\qquad \frac{b_\alpha}{a_\alpha} = \text{positive or negative imaginary number depending} \\
&\qquad\qquad \text{on choice of + or -- real for } c_\alpha\text{, and which of two} \\
&\qquad\qquad \text{choices for } \vec{c}_\alpha \text{ one takes}
\end{aligned}
\tag{4.16}
$$

Note that the nondegenerate natural mode as expressed in (4.3) constrains the relation of f_α to $\vec{f}_\alpha$ and hence the relation of c_α to $\vec{c}_\alpha$ as expressed through b_α/a_α. The second case has

$$
\begin{aligned}
&c_\alpha(\vec{1}_i)\; c_\alpha(\vec{1}_i) < 0 \\
&c_\alpha(\vec{1}_i) = \text{imaginary scalar} \\
&\vec{c}_\alpha(\vec{1}_i) \cdot \vec{c}_\alpha(\vec{1}_i) > 0 \\
&\vec{c}_\alpha(\vec{1}_i) = \text{real vector}
\end{aligned}
\tag{4.17}
$$

with a similar constraint on the choice of b_α/a_α. Both of these caes satisfy (4.14). In the first case, the p-wave scattering is positive while the s-wave scattering is negative, and the second case is the converse. One can consider various simple cases of elastodynamic scattering to confirm this.

5. Concluding Remarks

The four elastodynamic scattering-coefficient residues conveniently can be expressed as products involving only two angular functions, one scalar and one vector. This simplifies the calculation of the SEM pole terms and the measurement of these parameters. Note the similarity to the electromagnetic case, except that there are now four residues for a given pole due to the combinations of p and s waves. The present results apply to the case of non-degenerate natural modes, but can be extended to cases of degenerate modes that occur with some target symmetries.

In the electromagnetic context, the properties of the residue dyadic have been determined from an integral equation for the currents on the scatterer [5, 6, 26]. The incident wave propagating in the direction $\vec{1}_i$ is multiplied by the current density and integrated over the object to give a coupling vector $\vec{C}_\alpha(\vec{1}_i)$. This has the same role as the $\vec{c}_\alpha(\vec{1}_i)$ and $c_\alpha(\vec{1}_i)$ here. The current density is then multiplied by a far-field Green's function for scattering in the direction $\vec{1}_o$ to form a recoupling vector $\vec{C}_{r_\alpha}(\vec{1}_o) = \vec{C}_\alpha(-\vec{1}_o)$ which is found to be the same as the coupling vector evaluated at $-\vec{1}_o$, the reciprocity being implicit in the integral equation. This recoupling vector has the same role as the $\vec{c}_\alpha(-\vec{1}_o)$ and $c_\alpha(-\vec{1}_o)$ here. Allowing for scaling constants that come from the scattering integral equations, we can regard the vector and scalar functions of $\vec{1}_i$ as coupling functions, and of $-\vec{1}_o$ as recoupling functions. As this paper has demonstrated, this general decomposition can also be derived without appeal to scattering integral equations. One needs, however, the general property of natural modes being independent of excitation (direction of incidence and polarization) and reciprocity.

Here, we have not calculated the natural frequencies and angular functions for any specific target. Rather we have developed the general form that the poles take under quite general scattering conditions (linearity, reciprocity, passivity, and infinite uniform isotropic external medium). This needs to be applied to various targets in media of interest.

References

1. C. E. Baum, On the Singularity Expansion Method for the Solution of Electromagnetic Interaction Problems, Interaction Note 88, December 1971.

2. L. Marin and R. W. Latham, Analytical Properties of the Field Scattered by a Perfectly Conducting, Finite Body, Interaction Note 92, January 1972, and Representation of Transient Scattered Fields in Terms of Free Oscillations of Bodies, Proc. IEEE, 1972, p. 640, and L. Marin, Natural-Mode Representation of Transient Scattered Fields, IEEE Trans. Antennas and Propagation, 1973, pp. 809-818.

3. L. Marin, Application of the Singularity Expansion Method to Scattering From Imperfectly Conducting Bodies and Perfectly conducting Bodies Within a Parallel Plate Region, Interaction Note 116, June 1972.

4. M. I. Sancer and A. D. Varvatsis, Toward an Increased Understanding of the Singularity Expansion Method, Interaction Note 398, May 1980.

5. C. E. Baum, Scattering, Reciprocity, Symmetry, EEM, and SEM, Interaction Note 475, May 1989.

6. C. E. Baum, SEM Backscattering, Interaction Note 476, July 1989.

7. C. E. Baum, Representation of Surface Current Density and Far Scattering in EEM and SEM With Entire FUnctions, Interaction Note 486, February 1992.

8. C. E. Baum, Properties of Eigenterms of the Impedance Integral Equation, Interaction Note 487, April 1992.

9. C. E. Baum, The SEM Representation of Scattering From Perfectly Conducting Targets in Simple Lossy Media, Interaction Note 492, April 1993.

10. C. E. Baum, Low-Frequency Near-Field Magnetic Scattering from Highly, But Not Perfectly, Conducting Bodies, Interaction Note 499, November 1993.

11. C. E. Baum, The Magnetic Polarizability Dyadic and Point Symmetry, Ineraction Note 502, May 1994.

12. C. E. Baum, Concerning the Identification of Buried Dielectric Targets, Interaction Note 504, July 1994.

13. Y.-H. Pao and V. Varatharajulu, Huygen's Principle, Radiation Conditions, and Integral Formulas for the Scattering of Elastic Waves, J. Acoust. Soc. Am., Vol. 59, No. 6, 1976, pp. 1361-1371.

14. V. Varatharajulu and Y.-H. Pao, Scattering Matrix for Elastic Waves, I. Theory, J. Acoust. Soc. Am., Vol. 60, No. 3, 1976, pp. 556-566.

15. P. C. Waterman, Matrix Theory of Elastic Wave Scattering, J. Acoust. Soc. Am., Vol. 60, No. 3, 1976, pp. 567-580.

16. C. E. Baum, Emerging Technology for Transient and Broad-Band Analysis and Synthesis of Antennas and Scatterers, Proc. IEEE, 1976, pp. 1598-1616.

17. V. Varatharajulu, Reciprocity Relations and Forward Amplitude Theorems for Elastic Waves, J. Math. Phys., 1977, pp. 537-543.

18. C. E. Baum, The Singularity Expansion Method: Background and Developments, Electromagnetics, 1981, pp. 351-360, and IEEE Antennas and Propagation Newsletter/Magazine, Vol. 28, No. 4, August 1986, pp. 15-23.

19. G. Bollig and K. J. Langenberg, The Singularity Expansion Method as Applied to the Elastodynamic Scattering Problem, Wave Motion, Vol. 5, 1983, pp. 331-354.

20. C. E. Baum, E. J. Rothwell, K.-M. Chen, and D. P. Nyquist, The Singularity Expansion Method and Its Application to Target Identification, Proc. IEEE, 1991, pp. 1481-1492.

21. P. M. Morse and K. U. Ingard, *Theoretical Acoustics*, McGraw Hill, 1968.

22. C. E. Baum, The Singularity Expansion Method, ch. 3, pp. 129-179, in L. B. Felsen (ed.), *Transient Electromagnetic Fields*, Springer-Verlag, 1976.

23. C. E. Baum, Toward an Engineering Theory of Electromagnetic Scattering: The Singularity and Eigenmode Expansion Methods, in P. L. E. Uslenghi (ed.), *Electromagnetic Scattering*, Academic Press, 1978.

24. V. V. Varadan and V. K. Varadan, Acoustic, Electromagnetic and Elastic Field Equations Pertinent to Scattering Problems, pp. 3-18, in V. K. Varadan and V. V. Varadan, *Acoustic, Electromagnetic and Elastic Wave Scattering—Focus on the T-Matrix Approach*, Pergamon Press, 1980.

25. J. J. Bowman, T. B. A. Senior, and P. L. E. Uslenghi, *Electromagnetic and Acoustic Scattering by Simple Shapes*, Hemisphere Publishing Corp. (Taylor & Francis), 1987.

26. C. E. Baum, SEM and EEM Scattering Matrices, and Time-Domain Scatterer Polarization in the Scattering Residue Matrix, pp. 427-486, in W.-M. Boerner et al. (eds.), *Direct and Inverse Methods in Radar Polarimetry*, Kluwer Academic Publishers, 1992.

27. H. Überall (ed.), *Acoustic Resonance Scattering*, Gordon and Breach, 1992.