Interaction Notes

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Complementary Structures in Two Dimensions

Carl E. Baum
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Abstract

Self-complementary antennas are based on the Babinet principle in which electric and magnetic fields are interchanged (duality) with the structure invariant to this transformation. This paper develops self complementary structures from the electric and magnetic parts of a complex potential as used in conformal transformation. This leads to various geometries of electrically small impedance sheets with relatively simply calculable admittance properties. These are also related to the impedance properties of some TEM transmission-line structures (cylindrical and conical). The use of conformal and stereographic transformations allows one to generalize self-complementary structures on planes and spheres to structures having the self-complementary admittance properties without the restricted geometrical symmetries.

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1. Introduction

A classical concept in antenna theory is the complementary antenna derived from the Babinet principle [9]. The complement is formed by replacing an aperture in a perfectly conducting planar sheet by a perfectly conducting disc of the same shape in free space. Then with an appropriate introduction of antenna terminals the input impedance of the two antennas has the relationship

$$\tilde{Z}(s)\ \tilde{Z}^{(c)}(s) = \frac{Z_0^2}{4}$$
$$Z_0 = \left[\frac{\mu_0}{\varepsilon_0}\right]^{\frac{1}{2}} \approx 376.73\Omega \text{ (wave impedance of free space)} \tag{1.1}$$

where the superscript c denotes complement. Which of the two antennas is the original one, and which is the complement, is arbitrary. The above definition of complement can be generalized to the case of portions of the plane of interest consisting of impedance sheets of scalar or even dyadic (2 x 2) form [4, 5, 14]. The complementary sheet impedance (which may vary with position) satisfies a formula like (1.1) (with a rotation in the case of a dyadic sheet impedance). If the original and the complement are the same except for a rotation and/or reflection the antenna is said to be self complementary and (1.1) directly gives

$$\tilde{Z}(s) = \tilde{Z}^{(c)}(s) = \frac{Z_0}{2} \approx 188.4\Omega \tag{1.2}$$

which is now frequency independent. As discussed in various papers [10, 12] this relationship can be generalized to multi-terminal cases.

As observed in [10] this self complementary relationship can also be applied to the TEM modes on structures consisting of perfectly conducting cylindrical and conical sheets lying on a circular cylinder and circular cone, respectively, and having an N-fold rotation axis with alternating spaces of equal angular width to the conducting sheets. (This is described by C_{Nc} symmetry discussed in [5, 14].) Note that by the stereograph transformation [13] the conical case can be described by an equivalent cylindrical case, so that our present discussion can be carried out in terms of two dimensions on a plane.

As observed in [5, 14] the self-complementary relationship for impedance sheets, when considered for low frequencies (quasi static), implies that such structures can be used to give lumped impedances which have the same value (ohms) as the sheet impedance (ohms per square). A more classical way to calculate the impedance of such structures is via the two-dimensional Laplace equation and the common technique of conformal transformation for solving this equation.

This suggests that one might approach the concept of complementarity from the point of view of complex variables such as are used in conformal transformation. As we shall see, there is a certain duality or complementarity between the electric and magnetic potentials making up the complex potential. This can be used to define complementary and, in turn, self-complementary two-dimensional structures.

Consider, for example, a uniform impedance sheet illustrated in fig. 2.1A. Here we write this as R_s, a simple resistance sheet, but later, this can be given frequency dependence as desired. This sheet is terminated in two separate perfectly conducting terminals forming a port with (from some source) a voltage V and current I. There are also two truncations of the resistive material as indicated so that the current is limited to some domain D_{in}. The electric field $\vec{E}$ and surface current density $\vec{J}_s$ are perpendicular to the terminal conductors (electric boundaries) and parallel to the other two truncations (magnetic boundaries). For the present discussion the surface current density is limited to the interior domain D_{in}. One can also consider the exterior domain D_{ex} as having an impedance sheet with similar results. Later we shall consider the union of these.

In our two dimensional (x, y) coordinates we have

$$\begin{aligned}
\vec{E}(x,y) &= R_s \; \vec{J}_s(x,y) = -\nabla_s \, \Phi_e(x,y) \\
\Phi_e(x,y) &\equiv \text{electric potential} \\
\Phi_{e+} &= \Phi_{e-} = V \\
\nabla_s &= \vec{1}_x \frac{\partial}{\partial x} + \vec{1}_y \frac{\partial}{\partial y}
\end{aligned} \tag{2.1}$$

With

$$\nabla_s \cdot \vec{J}_s(x,y) = 0 \;\; \text{(equation of continuity, static)} \tag{2.2}$$

we have the Laplace equation

$$\begin{aligned}
\nabla_s^2 \, \Phi_e(x,y) &= -\nabla_s \cdot \vec{E}(x,y) = -\nabla_s \cdot (R_s \vec{J}_s(x,y)) \\
&= -R_s \nabla_s \cdot \vec{J}_s(x,y) = 0
\end{aligned} \tag{2.3}$$

since R_s is not a function of position. This two-dimensional potential function is to be solved subject the boundary conditions

$$\begin{aligned}
&\Phi_e(x,y) = \Phi_{e+} \, , \; \Phi_{e-} \;\; \text{on electric boundaries} \\
&\vec{1}_h \cdot \nabla_s \, \Phi_e(x,y) = 0 \;\; \text{on magnetic boundaries} \\
&\vec{1}_h \equiv \text{unit normal (in } x,\, y \text{ plane) to magnetic boundaries}
\end{aligned} \tag{2.4}$$

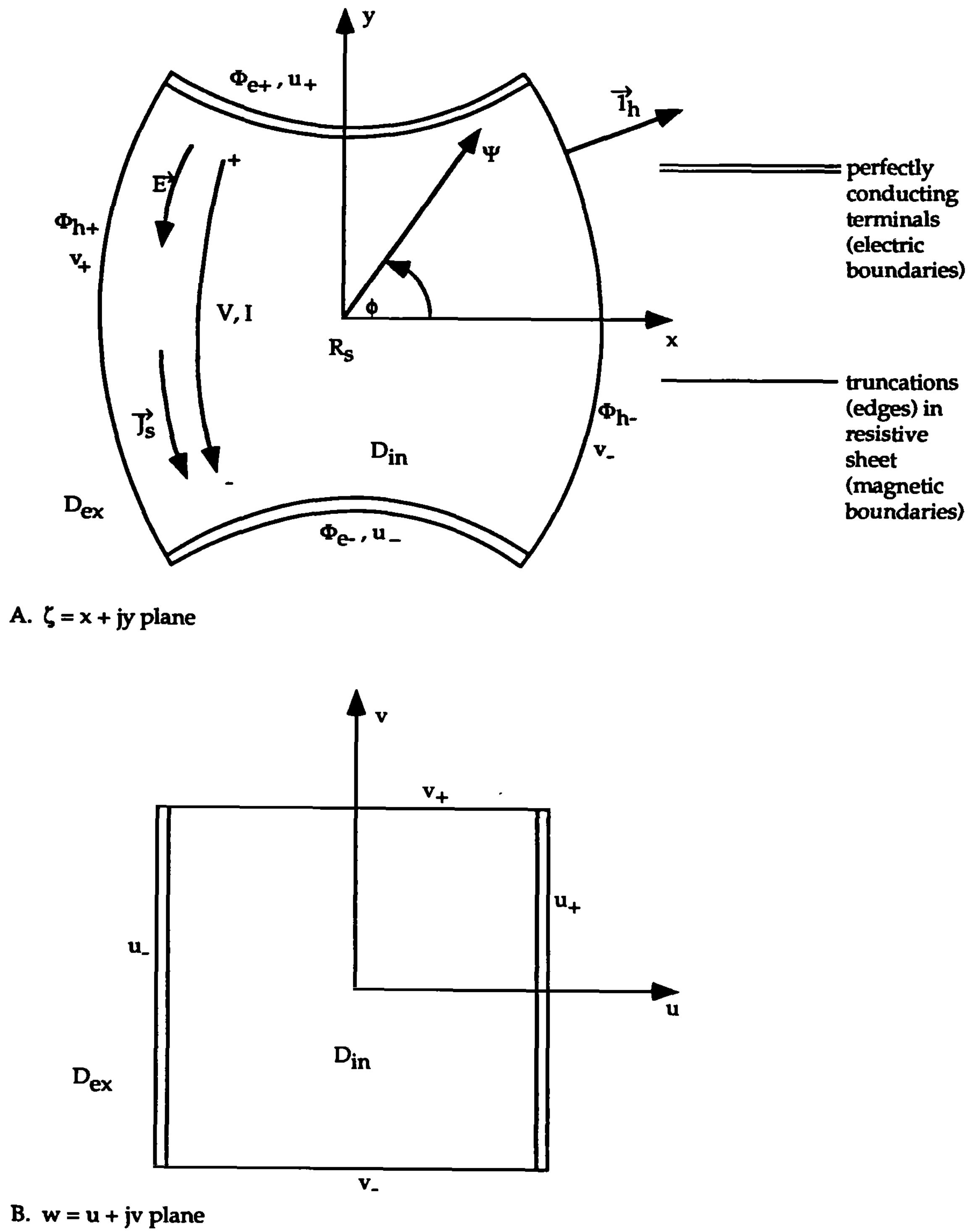


Fig. 2.1. Uniform Impedance Sheet of Finite Dimensions with Two Terminals (Single Port)

For various irregular boundary shapes one may wish to solve this Laplace equation numerically. For present purposes of obtaining analytic results we introduce the usual complex coordinate and potential as

$$\begin{aligned} \zeta &= x + jy \\ w(\zeta) &= u(\zeta) + j\mathrm{v}(\zeta) \end{aligned} \tag{2.5}$$

where $w(\zeta)$ is also referred to as a conformal transformation. This complex potential can be used to express the potentials and fields [2, 15] as

$$\begin{aligned} \Phi_e(x,y) &= \frac{\Phi_{e+} - \Phi_{e-}}{\Delta u}\, u(\zeta) + \text{constant} \\ \Phi_h(x,y) &= \frac{\Phi_{h+} - \Phi_{h-}}{\Delta \mathrm{v}}\, \mathrm{v}(\zeta) + \text{constant} \\ \Delta u &= u_+ - u_- \equiv \text{change in } u \text{ between electric boundaries (terminals)} \\ \Delta \mathrm{v} &= \mathrm{v}_+ - \mathrm{v}_- \equiv \text{change in v between magnetic boundaries (edges)} \\ E(\zeta) &= E_x(\zeta) - j\,E_y(\zeta) = -\frac{V}{\Delta u}\frac{dw(\zeta)}{d\zeta} \end{aligned} \tag{2.6}$$

Note that w is constrained to be an analytic function of ζ so that the derivative exists independent of the direction of approach to the point ζ. This is expressed by the Cauchy-Riemann conditions

$$\frac{\partial u(\zeta)}{\partial x} = \frac{\partial \mathrm{v}(\zeta)}{\partial y}\,, \quad \frac{\partial u(\zeta)}{\partial y} = -\frac{\partial \mathrm{v}(\zeta)}{\partial x} \tag{2.7}$$

so that the electric and magnetic potential functions are closely related to each other.

The potentials u and v can also be regarded as coordinates in a (u, v) plane as indicated in fig. 2.1B. This conformal transformation gives what are called carvilinear squares in that equal decrements of constant u and constant v contours give square patches in D_{in} in the limit of small decrements between contours. Such a small square is characterized by a resistance R_s. Counting the decrements in u (to total Δu) and in v (to total Δv) gives the resistance for the entire domain D_{in} as

$$\begin{aligned} R &= \frac{V}{I} = f_g\ R_s \\ f_g &= \frac{\Delta u}{\Delta \mathrm{v}} \equiv \text{geometrical factor} \end{aligned} \tag{2.8}$$

It is this geometrical factor which can be used to scale any sheet impedance, the wave impedance in the case of a cylindrical transmission line, and the inductance and capacitance of related two-dimensional structures (together with the length).

3. Two-Dimensional Complementary Structures

Let us now define a complementary structure and boundary-value problem by interchanging the roles of u and v, i.e.

$$w^{(c)}(\zeta) = u^{(c)}(\zeta) + j\,v^{(c)}(\zeta) = \pm\, v(\zeta) \pm ju(\zeta) + \text{constant} \tag{3.1}$$

As indicated, there are various choices of sign that one can use, and being potential functions, an arbitrary constant can be added. It is the change in the functions (taken positive) between the appropriate boundaries that is relevant. If desired, one can choose the complementary potential function as an analytic function of ζ, e.g.

$$w^{(c)}(\zeta) = -j\,w(\zeta) = v(\zeta) - ju(\zeta) \tag{3.2}$$

which is a rotation in the u, v plane.

For our example problem in fig. 2.1, this corresponds to interchanging the roles of the electric and magnetic boundaries. The perfectly conducting terminals are placed along the magnetic boundaries denoted by magnetic potentials v_+ and v_-, and removed from the electric boundaries denoted by electric potentials u_+ and u_-. This gives a complementary resistance

$$\begin{aligned} R^{(c)} &= f_g^{(c)} R \;\;,\;\; f_g^{(c)} = \frac{\Delta u^{(c)}}{\Delta v^{(c)}} \equiv \frac{\Delta v}{\Delta u} \\ f_g f_g^{(c)} &= 1 \;\;,\;\; R\,R^{(c)} = R_s^2 \end{aligned} \tag{3.3}$$

Comparing this to (1.1) we have the same form of complementary relationship with $Z_0/2$ replaced by R_s.

One way to define self complementarity is then

$$f_g \equiv f_g^{(c)} \equiv 1 \;\;,\;\; \Delta u \equiv \Delta v \;\;,\;\; R \equiv R^{(c)} \equiv R_s \tag{3.4}$$

This is rather general, but in the example in fig. 2.1A, this may still allow rather unsymmetrical shapes. Note, however, in fig. 2.1B, that in the w plane the domain D_{in} is a square. Keeping the distinction between electric and magnetic boundaries, this has a two-fold rotation axis (C_2 symmetry, equivalent to inversion in two dimensions) with two axial symmetry planes (C_{2a} symmetry) and self complementarity (C_{2ac} symmetry).

Comparing the highly symmetric form in fig. 2.1B to the generally unsymmetric form in fig. 2.1A, remember that they are related by the conformal transformation $w(\zeta)$ and have the same impedance properties. So we can take a symmetrical shape, including self complementarity, and transform it to a variety of other shapes which can be analyzed as though they had the symmetries of the original symmetrical shape.

4. Lowest-Order Self-Complementary Rotation Group; C_{2c} symmetry.

In the simplest form, C_{2c} symmetry, we have a two fold rotation axis (C_2 symmetry) and the operation of self complement on rotation by $\pi/2$. By self complement we take the interchange of the roles of electric and magnetic potentials as in Section 3, now on rotation by $\pi/2$, as

$$\begin{aligned} \zeta^{(c)} &= j\zeta = e^{j\frac{\pi}{2}}\zeta \;\; (\pi/2 \text{ rotation of coordinates}) \\ &= x^{(c)} + j\,y^{(c)} = -y + j\,x \\ w^{(c)}\left(\zeta^{(c)}\right) &= u^{(c)}\left(\zeta^{(c)}\right) + j\,\mathrm{v}^{(c)}\left(\zeta^{(c)}\right) = w(\zeta) = u(\zeta) + j\,\mathrm{v}(\zeta) \\ w^{(c)}(\zeta) &= -j\,w(\zeta) \;\; , \;\; w\left(\zeta^{(c)}\right) = j\,w(\zeta) \end{aligned} \tag{4.1}$$

Note that taking the complement also involves the interchange of the role of electric and magnetic boundaries. So the complementary structure looks just like the original structure, except rotated by $\pi/2$, and it is therefore self complementary. In terms of shape the structure has C_4 symmetry (4-fold rotation axis) except that there are two each of alternating electric and magnetic boundaries. Fig. 2.1B gives a simple example of such a self-complementary two-terminal device with a square shape. Fig. 4.1 gives a more general example. Other examples are given in [5, 14]. There are other ways to define complement, such as by interchanging u and v as $\pm$v and $\pm u$ with various combinations of signs. Which one chooses is but a matter of convenience.

Carrying the development further, the transformation in (4.1) can be carried full circle as

$$\begin{aligned} \zeta^{(n)} &\equiv x^{(n)} + jy^{(n)} \equiv e^{j\frac{n\pi}{2}}\zeta^{(0)} \\ \zeta^{(4)} &= \zeta^{(0)} \equiv \zeta \;\; , \;\; \zeta^{(1)} \equiv \zeta^{(c)} \\ w^{(n)}\left(\zeta^{(n)}\right) &= w^{(0)}\left(\zeta^{(0)}\right) \equiv w(\zeta) \;\; , \;\; w^{(c)}(\zeta) = w^{(1)}(\zeta) \end{aligned} \tag{4.2}$$

So C_{2c} symmetry has four group elements as special kinds of successive $\pi/2$ rotations. Note that

$$\begin{aligned} &\zeta^{(2)} = -\zeta^{(0)} \;\; , \;\; \zeta^{(3)} = -\zeta^{(1)} \\ &w^{(2)}\left(\zeta^{(0)}\right) = -w^{(0)}\left(\zeta^{(0)}\right) = w^{(0)}\left(-\zeta^{(0)}\right) \;\; , \;\; w^{(3)}\left(\zeta^{(0)}\right) = -w^{(1)}\left(\zeta^{(0)}\right) = w^{(1)}\left(-\zeta^{(0)}\right) \end{aligned} \tag{4.3}$$

This illustrates that rotation by $\pi/2$ merely changes signs, the structure (including respective positions of electric and magnetic boundaries) being invariant to this rotation. This gives C_2 symmetry (which is also two-dimensional inversion) as a subgroup.

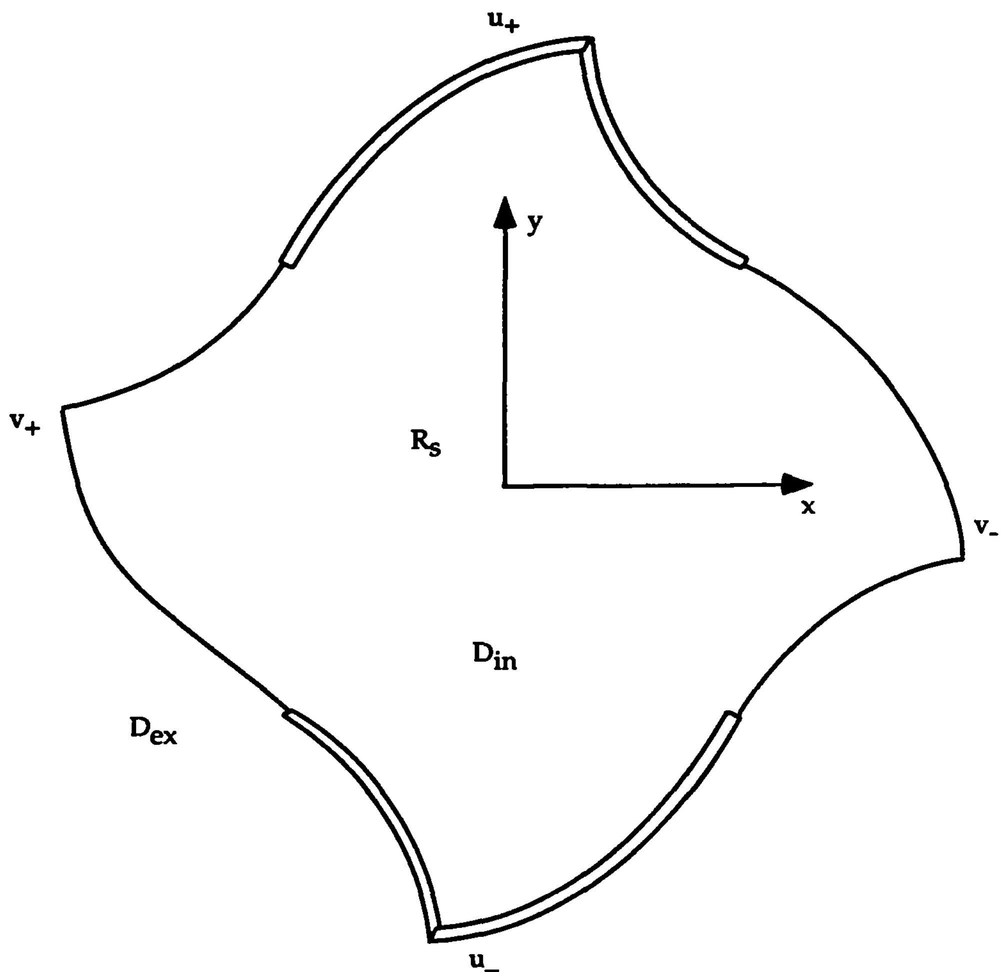


Fig. 4.1. C_{2c} Impedance Sheet

A consequence of this symmetry is that $w(\zeta)$ can be taken as an add function of ζ, i.e.

$$w(-\zeta) = -w(\zeta) \tag{4.4}$$

which specifies the arbitrary additive constant for potentials with now

$$w(0) = 0 \tag{4.5}$$

suggesting that our potential reference be at $\zeta = 0$. The boundary conditions as in fig. 4.1 now take the form

$$u_+ = -u_- = v_+ = -v_- \tag{4.6}$$

Referring back to Section 2, this suggests that one express the electric and magnetic potentials in the symmetrical form

$$\Phi_{e+} + \Phi_{e-} = 0 \quad , \quad \Phi_{h+} + \Phi_{h-} = 0 \tag{4.7}$$

so that voltages at and currents into the two terminals (electric boundaries) are equal and opposite for a completely differential system.

Here we have illustrated the case where D_{in} contains the sheet resistance. Moving R_s to D_{ex} also produces a two-terminal device with C_{2c} symmetry and resistance R_s. In this case the reference potential of zero can be taken at $\zeta = \infty$. Note that in the conformal transformations one encounters branch cuts as discontinuities in $v(\zeta)$, but these can be placed for convenience outside of D_{in} or outside of D_{ex}.

5. N-Fold Rotation Axis: C_N Symmetry

As a prelude to C_{Nc} symmetry, consider C_N symmetry (*N*-fold rotation axis) which is a subgroup. We have the cylic group with *N* elements

$$
\begin{aligned}
C_N &= \left\{(C_N)_\ell \middle| \ell = 1,2,\cdots,N\right\} \\
(C_N)_\ell &= (C_N)_1^\ell \equiv \text{rotation by } \frac{2\pi\ell}{N} \\
(C_N)_1^N &= (C_N)_N = (1) \equiv \text{identity}
\end{aligned} \tag{5.1}
$$

This has a scalar representation

$$
\begin{aligned}
(C_N)_\ell &\to C(\phi_\ell) = e^{j\phi_\ell} \\
\phi_\ell &\equiv \frac{2\pi\ell}{N} \quad , \quad \ell = 1,2,\cdots,N
\end{aligned} \tag{5.2}
$$

Being a scalar representation, the group is clearly commutative. One can also have a commonly used matrix representation [14].

Compared to the previous section, for $N \geq 3$, there is a new complication in that the boundary conditions are not neatly equal and opposite for both electric and magnetic potentials as in (4.6). With three or more terminals there are $N - 1 \geq 2$ potential differences to consider. We can generalize (4.6) and (4.7) by constraining

$$
\sum_{n=1}^{N} u_n = 0 \quad , \quad \sum_{n=1}^{N} v_n = 0 \tag{5.3}
$$

where the u_n and v_n are the boundary conditions assumed on the successive electric and magnetic boundaries. This is illustrated for the case of N = 5 in fig. 5.1 (taken as self complementary). On a physical basis, the N–1 potential differences can be chosen independently, implying that $w(\zeta)$ is not unique. One can similarly interpret this in terms of N–1 currents that can be independently chosen, subject to the Kirchoff condition that the sum of the N currents be zero. Note that the subscript index n runs from 1 to N since only the alternate boundaries (the electric (N) or the magnetic (N), but not both) are counted.

For rotating the potential functions we can define a coordinate rotation by

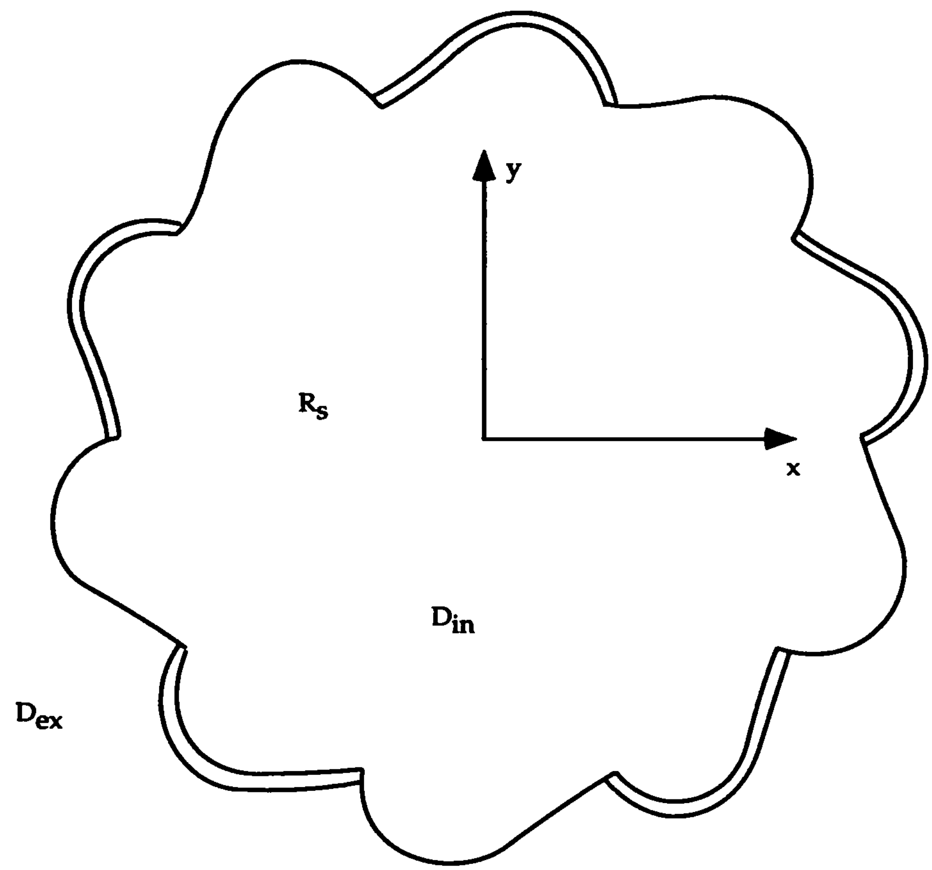


Fig. 5.1. C_{5c} Impedance Sheet

$$\zeta^{(2\ell)} \equiv e^{j\phi_\ell}\zeta^{(0)}$$
$$\zeta^{(2N)} \equiv \zeta^{(0)} \equiv \zeta \tag{5.4}$$

and rotated potentials by

$$W^{(2\ell)}\left(\zeta^{(2\ell)}\right) \equiv W^{(0)}\left(\zeta^{(0)}\right) \equiv w(\zeta) \tag{5.5}$$

The factor of two in the superscripts is to allow for later inclusion of odd indices for self-complementary rotation.

Given a particular $w(\zeta)$ subject to (5.3) then a set of $w^{(2\ell)}(\zeta^{(2\ell)})$ can be generated as per (5.4) and (5.5). Considering first the electric potentials $u^{(2\ell)}(\zeta^{(2\ell)})$, form the sum

$$u'(\zeta) = \sum_{\ell=1}^{N} u^{(2\ell)}(\zeta) \tag{5.6}$$

which is also a solution of the Laplace equation for D_{in}. The potential at the electric boundaries is the sum of the electric boundary potentials for the $u^{(2n)}$. Define

$$\begin{aligned} \zeta_1 &\equiv \zeta_1^{(0)} \equiv \text{point on 1st electric boundary} \\ \zeta_\ell &\equiv \zeta_\ell^{(0)} \equiv e^{j\phi_{\ell-1}}\zeta_1 = e^{j\frac{2\pi}{N}(\ell-1)}\zeta_1 = e^{j\phi_\ell}\zeta_0 = e^{j\phi_\ell}\zeta_N \\ &\equiv \text{corresponding point on } \ell\text{th electric boundary} \end{aligned} \tag{5.7}$$

We then have

$$\begin{aligned} u_\ell &\equiv u_\ell^{(0)} = u^{(0)}(\zeta_\ell) \\ u'_\ell &\equiv u'(\zeta_\ell) = \sum_{n=1}^{N} u^{(2n)}(\zeta_\ell) = \sum_{n=1}^{N} u^{(2N)}(\zeta_\ell^{(2(N-n))}) \\ &= \sum_{n=1}^{N} u^{(0)}\left(e^{j\phi_1(N-h)}\zeta_\ell^{(0)}\right) = \sum_{n=1}^{N} u\left(\zeta_{\ell+N-n}^{(0)}\right) = \sum_{n'=1}^{N} u(\zeta_{n'}) \\ &= \sum_{n'=1}^{N} u'_n = 0 \end{aligned} \tag{5.8}$$

Thus we have all the electric boundaries (all ℓ) with zero potential for computing $u'(\zeta)$. The only solution is

$$u'(\zeta) = 0 = \sum_{n=1}^{N} u^{(2n)}(\zeta) \tag{5.9}$$

Similarly, considering the magnetic potential and magnetic boundaries, the same steps as above lead to the same result for $v'(\zeta)$, giving

$$w'(\zeta) = 0 = \sum_{n=1}^{N} w^{(2n)}(\zeta) = \sum_{n=1}^{N} w^{(0)}\left(\zeta^{(2(N-n))}\right) = \sum_{n'=1}^{N} w^{(0)}\left(\zeta^{(2n')}\right) \tag{5.10}$$

Applying this result to $\zeta = 0$ we have

$$\begin{aligned} \zeta = 0 &\Rightarrow \zeta^{(2m)} = 0 \text{ for all } m \\ 0 &= \sum_{n=1}^{N} w^{(0)}(0) = N\ w(0) \\ w(0) &= 0 \end{aligned} \tag{5.11}$$

Thus C_N symmetry with the sum of the terminal potentials equal to zero (as in (5.3) implies that the complex potential is zero at the coordinate origin. Considering a sheet impedance in D_{ex} instead of D_{in}, the same arguments lead to a zero of the potential at ∞. Note that only C_N (N-fold rotation axis) instead of the more restrictive C_{Nc} has been assumed for obtaining this result.

The N terminals have voltages V_n with

$$V_n = V_o\ u_n \tag{5.12}$$

and N currents with

$$I_n = I_o\left[v_n - v_{n-1}\right]\ ,\ v_0 = v_N \tag{5.13}$$

Since the ratio of voltage drop across an elementary curvilinear square (equal decrements of u and v) to the current through this curvilinear square is just R_s we have the normalizing condition

$$V_0 = I_0\ R_s \tag{5.14}$$

Note that the V_n are referenced to $\zeta = 0$ which is not a terminal, and the voltages from an experimental viewpoint are voltage differences between terminals. In some problems (such as multiconductor transmission lines [6, 7]) there is another conductor (such as a cable shield or ground plane) which is taken as zero voltage, giving a total of N + 1 conductors. If we define an impedance matrix via

$$(V_n) = (Z_{n,m}) \cdot (I_n) \quad , \quad (Z_{n,m})^T = (Z_{n,m}) \tag{5.15}$$

we have the consequence

$$Z_{n,n} = \infty \tag{5.16}$$

i.e., a single current into the nth terminal is forced to zero by the constraint in (4.15). Defining an admittance matrix we have

$$\begin{gathered} (I_n) = (Y_{n,m}) \cdot (V_n) \quad , \quad (Y_{n,m})^T = (Y_{n,m}) \\ \sum_{n=1}^{N} Y_{n,m} = 0 \text{ for } m = 1, 2, \cdots, N \end{gathered} \tag{5.17}$$

The zero row- and column-sum constraint comes from setting all voltages to zero except one and requiring the sum of the currents to be zero. Within these limitations, one can regard these two matrices as mutually inverse in a generalized sense.

As discussed in various contexts [6, 7, 10] the C_N symmetry makes $(Y_{n,m})$ *circulant*, i.e.

$$Y_{n,m} = Y_{m-n+1} \tag{5.18}$$

so that the admittance between two terminals is only a function of how far apart the two terminals are around a circle. Furthermore, due to reciprocity, the matrix is symmetric and it does not matter which direction one goes around the circle. The matrix is then *symmetric circulant* or *bicirculant* as

$$(Y_{n,m}) = \begin{pmatrix} Y_1 & Y_2 & Y_3 & \cdots & Y_N \\ Y_2 & Y_1 & Y_2 & \cdots & \vdots \\ Y_3 & Y_2 & Y_1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ Y_N & \cdots & \cdots & \cdots & Y_1 \end{pmatrix}$$

$$Y_{N+2-\ell} = Y_\ell \quad , \quad \ell = 1, 2, \cdots, N$$
$$\sum_{\ell=1}^{N} Y_\ell = 0 \tag{5.19}$$

with a similar form for the impedance matrix. Note that there are not N distinct entries, but only N/2+1 for N even and (N+1)/2 for N odd. This can be diagonalized by the Fourier matrix (unitary) [6, 7] or other related forms.

The admittance matrix has the dyadic form [7]

$$\left(Y_{n,m}\right) = \sum_{\beta=1}^{N} y_\beta (x_n)_\beta \; (x_n)_\beta^* \quad \text{(a purely real sum)}$$
$$y_\beta = \sum_{\ell=1}^{N} Y_{\ell+1} e^{j2\pi\frac{\beta}{N}} \quad , \quad Y_{N+1} = Y_1$$
$$(x_n)_\beta = N^{-\frac{1}{2}}\left(e^{j2\pi\frac{\beta}{N}}, e^{j2\pi\frac{2\beta}{N}}, e^{j2\pi\frac{3\beta}{N}}, \cdots, e^{j2\pi\beta}\right) \tag{5.20}$$
$$y_\beta = y_{N-\beta} \quad \text{(all real and } \geq 0)$$
$$(x_n)_\beta^* = (x_n)_{N-\beta}$$

with the special case for $\beta = N$

$$y_N = \sum_{\ell=1}^{N} Y_\ell = y_0 = 0$$
$$(X_n)_N = N^{-\frac{1}{2}} (1, 1, 1, \cdots, 1) + (x_n)_0 \tag{5.21}$$

For more general cases of C_N symmetry involving a reference conductor $y_N > 0$ and the diagonalization still applies. There is an alternate form of the above which can simplify matters. For an even N we have N/2+1 eigenvalues from (5.20) with

$$y_\beta = Y_1 + 2\left[\sum_{\ell=1}^{\frac{N}{2}-1} Y_{\ell+1} \cos\left(2\pi\frac{\ell\beta}{N}\right)\right] + (-1)^\beta \frac{Y_N}{2} + 1 \tag{5.22}$$
$$\beta = 0, 1, 2, \cdots, N/2 - 1$$

For an odd N, we have (N+1)/2 eigenvalues with

$$y_\beta = Y_1 + 2\left[\sum_{\ell=1}^{\frac{N-1}{2}} Y_{\ell+1}\cos\left(2\pi\frac{\ell\beta}{N}\right)\right] \tag{5.23}$$

$$\beta = 0,1,2,\cdots,(N-1)/2$$

The eigenvectors can be put in a purely real form by taking linear combinations of the two in (5.20) associated with equal (non-distinct) eigenvalues to give two real eigenvectors associated with each distinct eigenvalue as

$$\begin{aligned}(x_n)_{\beta,e} &= \left[\frac{2}{N}\right]^{\frac{1}{2}}\left(\cos\left(2\pi\frac{\beta}{N}\right),\ \cos\left(2\pi\frac{2\beta}{N}\right),\cdots,\ \cos\left(2\pi\frac{(N-1)\beta}{N}\right),\ 1\right)\\ (x_n)_{\beta,o} &= \left[\frac{2}{N}\right]^{\frac{1}{2}}\left(\sin\left(2\pi\frac{\beta}{N}\right),\ \sin\left(2\pi\frac{2\beta}{N}\right),\cdots,\ \sin\left(2\pi\frac{(N-1)\beta}{N}\right),\ 1\right)\\ \beta &= \begin{cases}1,2,\cdots,\ N/2-1 \text{ for } N \text{ even}\\ 1,2,\cdots,\ (N-1)/2-1 \text{ for } N \text{ odd}\end{cases}\end{aligned} \tag{5.24}$$

with the special case for $\beta = 0$ in (5.21) (only one eigenvector). The subscripts e (even) and o (odd) distinguish the two eigenvectors associated with a particular β. This is an example of two-fold degeneracy of eigenmodes as discussed in [8].

Now any acceptable $w(\zeta)$ can be constructed in terms of the eigenvectors above. For each eigenvector $(x_n)_{\beta,{e \atop o}}$, there is a complex (analytic) mode function $w_{\beta,{e \atop o}}(\zeta)$ satisfying the electric boundary conditions (at ζ_n) given by the eigenvectors. A linear combination of these will give the desired electric boundary values subject to (5.3). For this purpose we have the identity

$$\left(1_{n,m}\right) = \sum_{\beta=1}^{N}(x_n)_\beta^*(x_n)_\beta = \sum_{\beta,{e \atop o}}(x_n)_{\beta,{e \atop o}}(x_n)_{\beta,{e \atop o}} \tag{5.25}$$

for calculating the coefficients of each of the eigenmodes via

$$(u_n) = \sum_{\beta=1}^{N}(x_n)_\beta^*\left[(x_n)_\beta\cdot(u_n)\right] = \sum_{\beta,{e \atop o}}(x_n)_{\beta,{e \atop o}}\left[(x_n)_{\beta,{e \atop o}}\cdot(u_n)\right] \tag{5.26}$$

The coefficient of the N (or zero) mode is zero due to (5.3).

6. Self-Complementary Rotation Group: C_{Nc} Symmetry

For constructing the self-complementary rotation group C_{Nc} begin with the self-complementary angle

$$\phi_c \equiv \frac{\phi_1}{2} = \frac{\pi}{N} \tag{6.1}$$

so that twice application of this rotation gives rotation by ϕ_1, the elementary rotation for C_N. Consistent with this define rotated coordinates

$$\zeta^{(n)} \equiv e^{jn\phi_c}\zeta^{(0)} \quad , \quad \zeta^{(2N)} = \zeta^{(0)} \equiv \zeta \ , \ \zeta^{(N)} = -\zeta \tag{6.2}$$

Taking the complement as in (4.1) we have the first application of complement

$$w^{(1)}\left(\zeta^{(1)}\right) = w^{(0)}\left(\zeta^{(0)}\right) \tag{6.3}$$

which interchanges the electric and magnetic boundaries. The second application of complement,

$$w^{(2)}\left(\zeta^{(2)}\right) = w^{(1)}\left(\zeta^{(1)}\right) = w^{(0)}\left(\zeta^{(0)}\right) \tag{6.4}$$

returns the boundaries to their original type, but with the potentials rotated by $2\phi_c = \phi_1$. Generalizing this we have

$$\begin{aligned} w^{(n)}\left(\zeta^{(n)}\right) &= w^{(0)}\left(\zeta^{(0)}\right) \quad , \quad n = 1, 2, \cdots, 2N \\ w^{(2N)}(\zeta) &= w^{(0)}(\zeta) \end{aligned} \tag{6.5}$$

The group structure of C_{Nc} is the same as S_{2N}, rotation reflection symmetry where interchange of electric and magnetic boundaries is replaced by reflection through a transverse plane, this being appropriate to a three-dimensional structure where rotation by ϕ_c is accompanied by reflection $(z \to -z)$ through the x, y symmetry plane. These two groups are said to be *isomorphic*, and C_N is a subgroup of both.

Concentrating on the admittance form (to avoid unbounded elements) we have the original problem

$$
\begin{aligned}
(I_n) &= (Y_{n,m}) \cdot (V_n) \\
(V_n) &= V_0\,(u_n) \\
(I_n) &= I_0\,(v_1 - v_N\,,\ v_2 - v_1\,,\cdots,\ v_N - v_{N-1}) \\
V_0 &= I_0\ R_s
\end{aligned}
\tag{6.6}
$$

The complementary problem is found by interpreting the v_ns for voltage and u_ns for current (thus interchanging the roles of electric and magnetic boundaries) as

$$
\begin{aligned}
\left(I_n^{(c)}\right) &= \left(Y_{n,m}^{(c)}\right) \cdot \left(V_n^{(c)}\right) \\
(V_n) &= V_0\,(v_n) \\
(I_n) &= I_0\,(u_2 - u_1\,,\ u_3 - u_2\,,\ \cdots,\ u_N - u_{N-1}\,,\ u_1 - u_N) \\
V_0 &= I_0\ R_s
\end{aligned}
\tag{6.7}
$$

Imposing self complementarity implies (from (5.20))

$$
\left(Y_{n,m}^{(c)}\right) = \left(Y_{n,m}\right) = \sum_{\beta=1}^{N} y_\beta\,(x_n)_\beta\,(x_n)_\beta^{*}
\tag{6.8}
$$

Here we have used the complex form of the eigenvectors in (5.20) for the reason that

$$
\begin{aligned}
(x_n)_\beta &= N^{-\frac{1}{2}}\left(e^{j2\pi\frac{\beta}{N}},\ e^{j2\pi\frac{2\beta}{N}},\ \cdots,\ e^{j2\pi\frac{(N-1)\beta}{N}},\ e^{j2\pi\frac{N\beta}{N}}\right) \\
&= N^{-\frac{1}{2}}e^{j2\pi\frac{\beta}{N}}\left(e^{j2\pi\frac{N\beta}{N}},\ e^{j2\pi\frac{\beta}{N}},\ \cdots,\ e^{j2\pi\frac{(N-2)\beta}{N}},\ e^{j2\pi\frac{(N-1)\beta}{N}}\right) \\
&= N^{-\frac{1}{2}}e^{-j2\pi\frac{\beta}{N}}\left(e^{j2\pi\frac{2\beta}{N}},\ e^{j2\pi\frac{3\beta}{N}},\ \cdots,\ e^{j2\pi\frac{N\beta}{N}},\ e^{j2\pi\frac{\beta}{N}}\right)
\end{aligned}
\tag{6.9}
$$

So, for these eigenvectors, shifting an index merely multiplies the eigenvector by a complex constant of magnitude one.

Considering the voltages and currents in (6.6), then let us specify the (u_n) as proportional to one of the $(x_n)_\beta$ and obtain

$$
\begin{aligned}
(V_n) &= V_0\Big[(x_n)^*_\beta \cdot (u_n)\Big]\,(x_n)_\beta \\
(I_n) &= V_0\Big[(x_n)^*_\beta \cdot (u_n)\Big]\,y_\beta\,(x_n)_\beta
\end{aligned}
\tag{6.10}
$$

so that the current is proportional to this same eigenvector. Then from the shifting property in (6.9) the $\mathrm{v}_n s$ in (6.6) can be expressed as

$$
\begin{aligned}
(\mathrm{v}_n) &\equiv (\mathrm{v}_1\,,\,\mathrm{v}_2\,,\cdots,\mathrm{v}_N) = \Big[(x_n)^*_\beta \cdot (\mathrm{v}_n)\Big]\,(x_n)_\beta \\
(\mathrm{v}_N\,,\,\mathrm{v}_1\,,\cdots,\mathrm{v}_{N-1}) &= e^{-j2\pi\frac{\beta}{N}}(\mathrm{v}_n) = e^{-j2\pi\frac{\beta}{N}}\Big[(x_n)^*_\beta \cdot (\mathrm{v}_n)\Big]\,(x_n)_\beta \\
(I_n) &= I_0\Big[(x_n)^*_\beta \cdot (\mathrm{v}_n)\Big]\left[1 - e^{j2\pi\frac{\beta}{N}}\right](x_n)_\beta
\end{aligned}
\tag{6.11}
$$

Equating the two forms for the current vector gives

$$
I_0\Big[(x_n)^*_\beta \cdot (\mathrm{v}_n)\Big]\left[1 - e^{-j2\pi\frac{\beta}{N}}\right] = V_0\Big[(x_n)^*_\beta \cdot (u_n)\Big]y_\beta
\tag{6.12}
$$

with the eigenvector removed as a common factor. Turning to (6.7) expand the (v_n) as proportional to one of the $(x_n)_\beta$ and obtain

$$
\begin{aligned}
\left(V_n^{(c)}\right) &= V_0\Big[(x_n)^*_\beta \cdot (\mathrm{v}_n)\Big]\,(x_n)_\beta \\
\left(I_n^{(c)}\right) &= V_0\Big[(x_n)^*_\beta \cdot (\mathrm{v}_n)\Big]\,y_\beta\,(x_n)_\beta
\end{aligned}
\tag{6.13}
$$

Again using the shifting property the $u_n s$ in (6.7) can be expressed as

$$
\begin{aligned}
(u_n) &\equiv (u_1\,,\,u_2\,,\cdots,u_N) = \Big[(x_n)^*_\beta \cdot (u_n)\Big]\,(x_n)_\beta \\
(u_2\,,\,u_3\,,\cdots,u_N,\,u_1) &= e^{j2\pi\frac{\beta}{N}}\,(u_n) = e^{j2\pi\frac{\beta}{N}}\Big[(x_n)^*_\beta \cdot (u_n)\Big]\,(x_n)_\beta \\
\left(I_n^{(c)}\right) &= I_0\Big[(x_n)^*_\beta \cdot (u_n)\Big]\left[e^{j2\pi\frac{\beta}{N}} - 1\right](x_n)_\beta
\end{aligned}
\tag{6.14}
$$

Equating the two forms for the complementary current gives

$$I_0\left[(x_n)^*_\beta \cdot (u_n)\right]\left[e^{j2\pi\frac{\beta}{N}} - 1\right] = V_0\left[(x_n)^*_\beta \cdot (v_n)\right] y_\beta (x_n)_\beta \tag{6.15}$$

Combining this with (6.12) gives

$$\begin{aligned} \frac{V_0^2}{I_0^2} y_\beta^2 &= R_s^2 \; y_\beta^2 = \left[e^{j2\pi\frac{\beta}{N}} - 1\right]\left[1 - e^{-j2\pi\frac{\beta}{N}}\right] = 4\sin^2\left(\pi\frac{\beta}{N}\right) \\ R_s y_\beta &= 2\,\sin\left(\pi\frac{\beta}{N}\right) \end{aligned} \tag{6.16}$$

where the positive square root has been chosen since the eigenvalues (admittances) must be real and non-negative. This is in agreement with [10] noting the replacement of $Z_0/2$ by R_s. Note that

$$y_N = y_0 = 0 \tag{6.17}$$

as required in (5.21). By the symmetry in the sine function about $\pi/2$ the eigenvalues pair as in (5.22) and (5.23) reducing the number of distinct ones to roughly N/2 as previously discussed.

Having the eigenvalues of the admittance matrix, as well as the eigenvectors, the elements of this bicirculant matrix may be obtained as

$$\begin{aligned} R_s\; Y_\ell &= R_s\, Y_{1,\ell} = R_s \sum_{\beta=1}^{N} y_\beta \;\; x_{1;\beta}\, x^*_{\ell;\beta} \\ &= -jN^{-1}\sum_{\beta=1}^{N}\left[e^{j\pi\frac{\beta}{N}} - e^{-j\pi\frac{\beta}{N}}\right] e^{j2\pi\frac{\beta}{N}} - e^{-j2\pi\ell\frac{\beta}{N}} \\ &= -jN^{-1}\sum_{\beta=1}^{N}\left[e^{j\pi[-2\ell+3]\frac{\beta}{N}} - e^{j\pi[-2\ell+1]\frac{\beta}{N}}\right] \\ &= -jN^{-1}\left\{e^{j[-2\ell+3]\frac{\pi}{N}}\,\frac{1-e^{j[-2\ell+3]\pi}}{1-e^{j[-2\ell+3]\frac{\pi}{N}}} - e^{j[-2\ell+1]\frac{\pi}{N}}\,\frac{1-e^{j[-2\ell+1]\pi}}{1-e^{j[-2\ell+1]\frac{\pi}{N}}}\right\} \\ &= N^{-1}\left\{-\frac{e^{j[-2\ell+3]\frac{\pi}{2N}}}{\sin\left([2\ell-3]\frac{\pi}{2N}\right)} + \frac{e^{j[-2\ell+1]\frac{\pi}{2N}}}{\sin\left([2\ell-1]\frac{\pi}{2N}\right)}\right\} \end{aligned}$$

$$= -N^{-1}\frac{\sin\left(\frac{\pi}{N}\right)}{\sin\left([2\ell-3]\frac{\pi}{2N}\right)\sin\left([2\ell-1]\frac{\pi}{2N}\right)} = \frac{2}{N}\,\frac{\sin\left(\frac{\phi_1}{2}\right)}{\cos\left([\ell-1]\frac{2\pi}{N}\right)-\cos\left(\frac{\pi}{N}\right)}$$

$$= \frac{2}{N}\,\frac{\sin\left(\frac{\phi_1}{2}\right)}{\cos([\ell-1]\phi_1)-\cos\left(\frac{\phi_1}{2}\right)} \tag{6.18}$$

From these matrix elements various combinations of connections of the N terminals may be connected to give a single port with various impedances of the form of a constant times R_s. Examples of the same are included in [10] where R_s is replaced by $Z_0/2$.

7. Reciprocation of Two-Dimensional Structures

The self-complementary impedance sheet has been considered for the interior domain D_{in}. However, it applies equally well to the exterior domain D_{ex} (e.g., in figs. 4.1 and 5.1). One way to see this is by the operation of reciprocation [15]. For this it is convenient to introduce complex cylindrical coordinates as

$$\zeta = x + jy = \Psi e^{j\phi} \tag{7.1}$$

Then introduce the analytic transform

$$\begin{aligned} \zeta_1 &= x_1 + jy_1 = \Psi_1 e^{j\phi_1} = b^2/\zeta \\ \Psi_1 &= b^2/\Psi \quad , \quad \phi_1 = \phi \end{aligned} \tag{7.2}$$

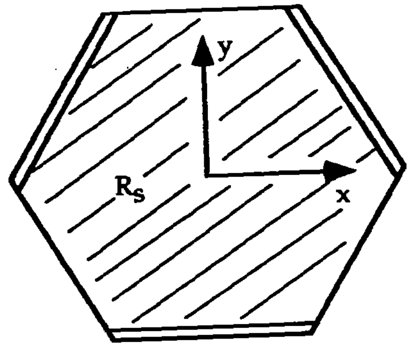
which is a conformal transformation. The operation of reciprocation (non analytic) is given by

$$\begin{aligned} \zeta_2 &= x_2 + jy_2 = \Psi_2 e^{j\phi_2} = b^2/\zeta^* = \zeta_1^* \\ \Psi_1 &= b^2/\Psi \quad , \quad \phi_1 = \phi \end{aligned} \tag{7.3}$$

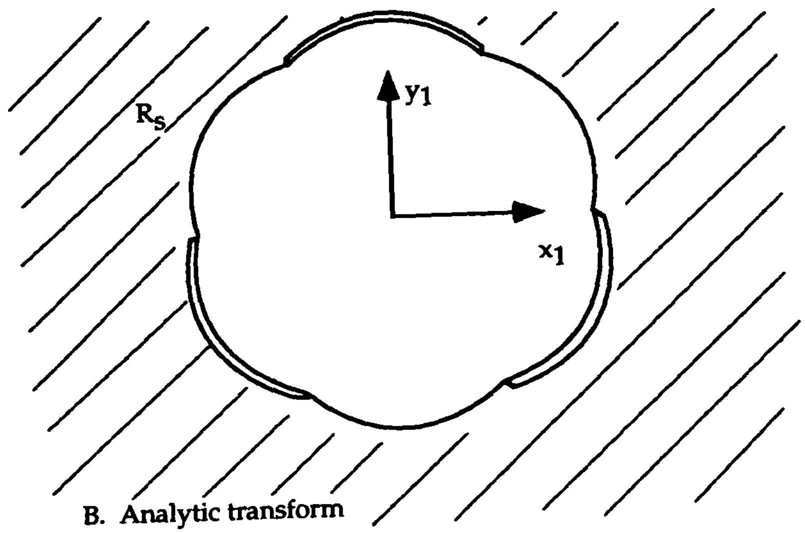
These formulas describe the simple transformation by which the angle ϕ is kept the same, but the radius Ψ is replaced by its reciprocal, appropriately scaled by some radius b.

As illustrated in fig. 7.1A, consider some impedance sheet (here illustrated as C_{3c} self complementary), where the sheet occupies D_{in} as previously discussed. Accomplishing the analytic transform as in (7.2) produces the configuration in fig. 7.1B. Since this is an analytic transformation $(\zeta \to \zeta_1)$ then the values of u on the electric boundaries and v on the magnetic boundaries are unchanged and the admittance calculations as before are still applicable. Hence fig. 7.1B also has C_{3c} symmetry and the previous results apply. Going to fig. 7.1C the reciprocation transformation $(\zeta \to \zeta_2)$ is just a reflection through the x axis. This is equivalent to looking at the figure from the backside and the admittance results are the same. So both transformations of a self-complementary structure with C_{Nc} symmetry produce other C_{Nc} self-complementary structures and the same admittance results apply.

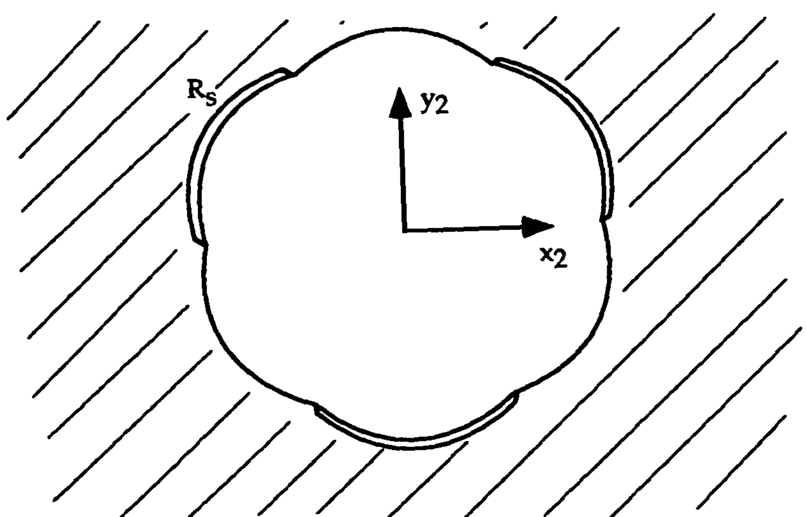
Going a step further, as in fig. 7.2A, combine two self-complementary structures with the same C_{Nc} symmetry. Let one have $R_s^{(in)}$ on the original D_{in}. Let the second have $R_s^{(out)}$ occupying a subset of D_{ex} (by making the dimensions of the interior boundaries sufficiently large so as not to intersect D_{in}). Note that the outer structure need not be related to the inner one (or even have the same value for N).



A. Original geometry

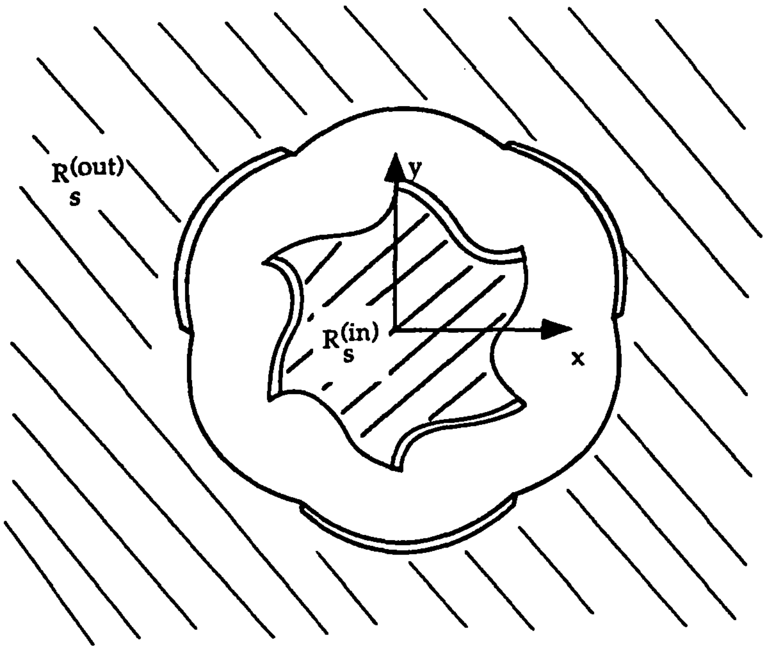


B. Analytic transform

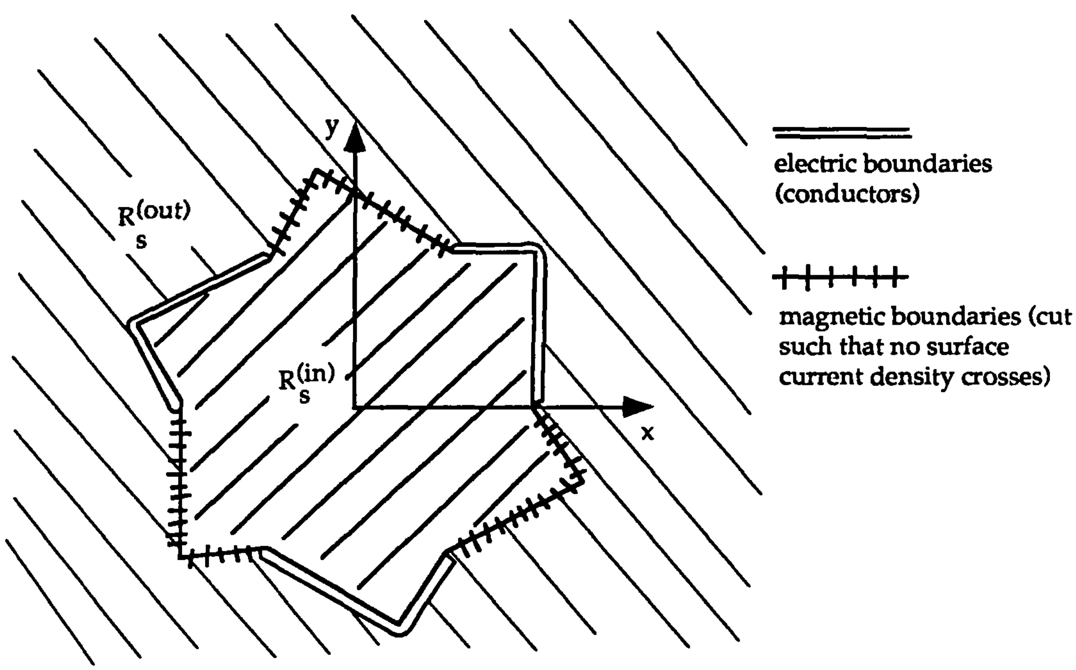


C. Reciprocal geometry

Fig 7.1. Reciprocation of C_{3c} Impedance Sheet



A. Disjoint impedance sheets



B. Special case: $D_{in} \cup D_{ex}$

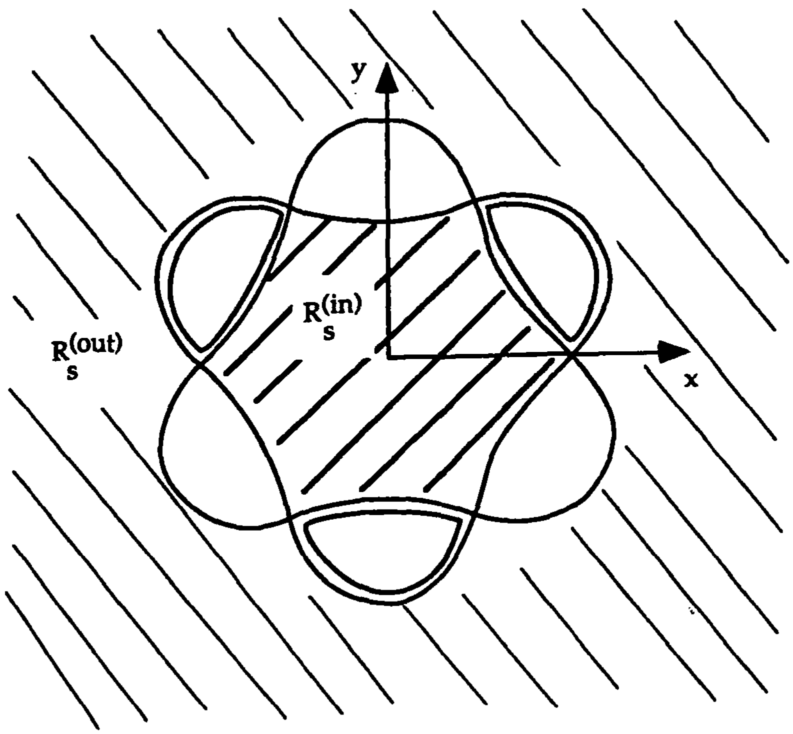
Fig. 7.2. Self-Complementary Combination of Internal and External Impedance Sheets with C_{3c} Symmetry

With the same values for N the inner and outer structures can have their corresponding electrical boundaries (terminals) connected together to give a combined structure with C_{Nc} symmetry where the previous admittance formulas can be used with R_s replaced by $R_s^{(in)} // R_s^{(out)}$ (parallel combination), or by $R_s/2$ if $R_s^{(in)} = R_s^{(out)}$. A limiting case of this combination is seen in fig. 7.2B where the two sheets are made to occupy the entire plane with respective boundaries coinciding so that $R_s^{(out)}$ now occupies all of D_{ex}. Note that the magnetic boundaries now correspond to a cut (small separation) between the two resistive sheets so that no surface current density can cross such a magnetic boundary. The electric boundaries (terminals) are now common to both resistive sheets.

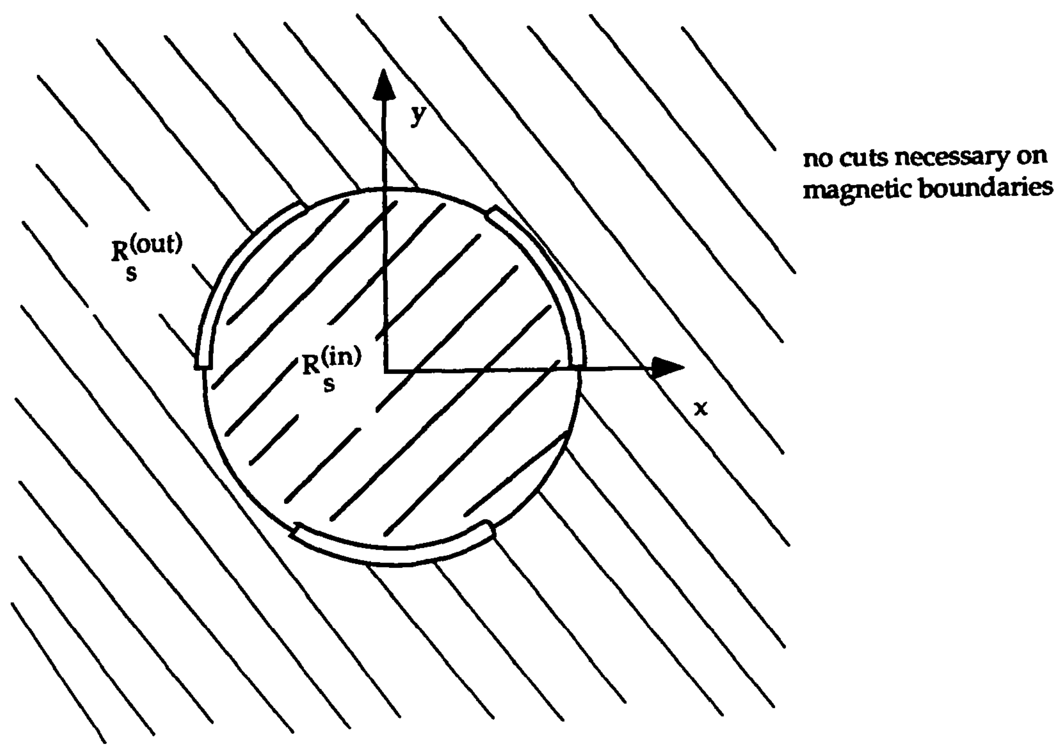
As discussed in [15], it is possible to have a self-reciprocal structure, i.e., one which maps into itself in the transformation (7.2). As shown in fig. 7.3A, this could involve two resistive sheets with at least partly disjoint boundaries. The portion of the plane between the electric boundaries can be enclosed by a closed conducting boundary as in fig. 7.3A. In the case of a TEM transmission line such closed electric boundaries correspond to a (perfectly) conducting tube. However, the corresponding portion of the plane between corresponding magnetic boundaries has no such simple analog in two-dimensional TEM transmission lines due to the difficulty in realizing magnetic boundaries in such a case. If we are to have the magnetic boundaries one and the same for the two resistive sheets under reciprocation then the magnetic boundaries become arcs of a circle of radius b as in fig. 7.3B. Self complementarity then requires that the electric boundaries also lie on the same circle. Note that applied to TEM transmission-line configurations this last type of structure is that discussed in [10].

The self-reciprocal self-complementary configuration in fig. 7.3B is a special case in that no cuts in the resistive sheets are needed to stop the current flow across the portions of the circle that are magnetic boundaries. The reciprocation principle assures us that the mapping from D_{in} to D_{ex} preserves electric potential (voltage) in mapping from ζ to ζ_2. On the circle $\zeta = be^{j\phi} = \zeta_2$ and the electric potential is the same on both sides of the circular boundary implying no current flow through the circle where there is to be a magnetic boundary. Note, however, that one can still have different values for $R_s^{(in)}$ and $R_s^{(out)}$.

The conformal transformation for C_{2a} symmetry (two-fold rotation axis with two axial symmetry planes) for two conducting plates on the circular boundary is known explicitly [1]. (The self complementary configuration C_{2ac} is a special case of this.) Via the stereographic transformation [13] a cylindrical TEM transmission line (plane wave) can be converted to an equivalent conical TEM transmission line (spherical wave), so the foregoing results can also be applied to conical-transmission-line structures. (They can also be so applied to sheets on a spherical surface.)



A. Empty regions between magnetic boundaries



B. No empty regions between magnetic boundaries

Fig. 7.3. Self-Reciprocal Self-Complementary Structures

8. Reflection Self Complementarity

Another type of self-complementary structure is one which is its own complement on reflection. As illustrated in fig. 8.1, let us take the reflection with respect to the x axis. The reflection group is then

$$R_y = \left\{\left(R_y\right), (1)\right\} \quad , \quad \left(R_y\right)^2 = (1) \tag{8.1}$$

If ζ_3 is the transformed complex coordinate, we have

$$\zeta_3 = x_3 + jy_3 = \zeta^* = x - jy \tag{8.2}$$

Thus reflection is the same as conjugation, or reversing the y coordinate.

With electric and magnetic boundaries around D_{in} as in Section 2, we have the interchange on reflection

$$\begin{aligned} u_+^{(c)} &= v_+ \quad , \quad u_-^{(c)} = v_- \\ v_+^{(c)} &= u_+ \quad , \quad v_-^{(c)} = u_- \end{aligned} \tag{8.3}$$

So we have

$$w^{(c)}(\zeta) = u^{(c)}(\zeta) + jv^{(c)}(\zeta) = v(\zeta) + ju(\zeta) = jw^*(\zeta) \tag{8.4}$$

This is one of the choices in (3.1), one which is appropriate to reflection (an improper rotation). Except for the coefficient, then conjugation of the complex coordinate results in conjugation of the complex potential. Furthermore from (8.3) we have

$$\Delta u = \Delta v \quad , \quad f_g = 1 \tag{8.5}$$

as discussed in Section 3.

In addition, the reflection symmetry gives

$$\begin{aligned} w\left(\zeta^*\right) &= u\left(\zeta^*\right) + jv\left(\zeta^*\right) = v(\zeta) + ju(\zeta) = jw^*(\zeta) \\ w^{(c)}(\zeta) &= w\left(\zeta^*\right) \end{aligned} \tag{8.6}$$

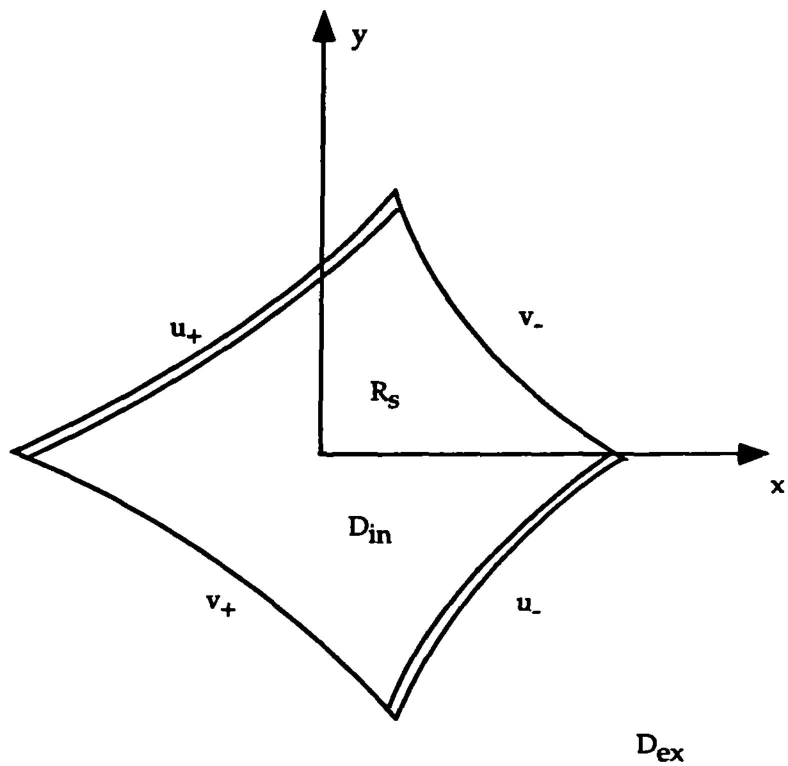


Fig. 8.1. Reflection Self Complementary

As previously discussed, this type of self-complementary structure can also be transformed by conformal, reciprocal, and/or stereographic transformations to give other structures with the same impedance properties.

9. Concluding Remarks

Complex potentials and the associated conformal transformations are then another way to define and analyze self complementary structures. The impedance sheets with uniform R_s (or even with frequency dependence) are appropriate to electrically small structures which can be analyzed via the two-dimensional Laplace equation. However, the results are the same as those derived from the Babinet principle for more general electromagnetic scattering and antennas involving planar structures (not in general electrically small).

This interchange of electric and magnetic potential functions for defining self complementarity leads to some generalization of the concept of self-complementary structures. Starting from some self complementary structure involving geometrical symmetry (e.g., rotation, reflection, etc.), the original geometry can be changed to a generally non-symmetrical one via a conformal transformation. The values of the electric and magnetic potentials are unchanged by the transformation, thereby leaving the terminal impedance properties unchanged. So one can think of any single-port device (two terminals) with $f_g = 1$ (or $\Delta u = \Delta v$) as self complementary, and similarly for N-terminal devices with an admittance matrix described as in Section 6.

By including both $R_s^{(in)}$ and $R_s^{(out)}$ to fill the entire plane, and invoking reciprocation symmetry so as to make the electric and magnetic boundaries all lie on a common circle, one has a self-complementary geometry applicable to a special cylindrical TEM transmission line (plane wave). The N terminals occupy equal arcs, equally spaced around the circle. By a stereographic transformation (plane to sphere) a conical TEM transmission line (spherical wave) is defined. However, there are various possible stereographic projections depending on the selected radius of the sphere, and point of tangency of the sphere to the plane (i.e., not necessarily in the center of the circle). This gives various possible conical transmission lines, all with the same admittance properties. If desired, one can apply the stereographic projection back to a different plane to obtain yet other geometries for a cylindrical transmission line. All of these can be considered self complementary in the generalized sense discussed above.

Here we have considered rotation and reflection symmetry, but various kinds of translation symmetry can also be included [3, 11, 14]. In terms of the complex-variable approach studied here, one can also approach periodic arrays of resistive sheets with appropriate conformal transformations. In [4, 5, 14] special kinds of self-complementary dyadic admittance sheets are found. It would be interesting if they had some relation to the present complex-variable approach.

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