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Nonuniform Multiconductor Transmission Lines

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Abstract

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1. Introduction

In [7,17] several techniques for solving the multiconductor-transmission-line (MTL) equations were discussed. The present paper continues this discussion, utilizing the product integral (Appendix A) for extending these results.

The telegrapher equations for an NMTL (nonuniform MTL) for N conductors (plus reference) are

$$\begin{aligned} \frac{\partial}{\partial z}\left(\tilde{V}_n(z,s)\right) &= -\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{I}_n(z,s)\right)+\left(\tilde{V}_n^{(s)'}(z,s)\right) \\ \frac{\partial}{\partial z}\left(\tilde{I}_n(z,s)\right) &= -\left(\tilde{Y}'_{n,m}(z,s)\right)\cdot\left(\tilde{V}_n(z,s)\right)+\left(\tilde{I}_n^{(s)'}(z,s)\right) \end{aligned} \tag{1.1}$$

The vectors have N components and the matrices are N×N with the various terms the same as previously used (e.g., [5-8]). The per-unit-length impedance and admittance matrices can be combined to give

$$\begin{aligned} \left(\tilde{\gamma}_{n,m}(z,s)\right) &= \left[\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right)\right]^{\frac{1}{2}} \quad \text{(positive real (p.r.) square root)} \\ &\equiv \text{propagation matrix} \\ \left(\tilde{Z}_{c_{n,m}}(z,s)\right) &= \left(\tilde{\gamma}_{n,m}(z,s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right)^{-1} = \left(\tilde{\gamma}_{n,m}(z,s)\right)^{-1}\cdot\left(\tilde{Z}'_{n,m}(z,s)\right) \\ &= \left(\tilde{Z}_{c_{n,m}}(z,s)\right)^{\mathsf{T}} \equiv \text{characteristic impedance matrix} \\ \left(\tilde{Y}'_{c_{n,m}}(z,s)\right) &= \left(\tilde{Z}_{c_{n,m}}(z,s)\right)^{-1} = \left(\tilde{Y}_{c_{n,m}}(z,s)\right)^{\mathsf{T}} \\ &\equiv \text{characteristic admittance matrix} \end{aligned} \tag{1.2}$$

where reciprocity has been assumed. Here the treatment is in complex-frequency domain as indicated by the two-sided Laplace-transform variable s.

2. Normalization Using Characteristic Impedance Matrix

For combining the voltage and current vectors let us use the characteristic impedance matrix, where (as contrasted to [1,8]) this is allowed to be a function of z, the coordinate along the transmission line. Noting

$$
\begin{aligned}
\frac{\partial}{\partial z}\left(\tilde{I}_n(z,s)\right) &= \frac{\partial}{\partial z}\left[\left(\tilde{Y}_{c_{n,m}}(z,s)\right)\cdot\left(\tilde{Z}_{c_{n,m}}(z,s)\right)\cdot\left(\tilde{I}_n(z,s)\right)\right] \\
&= \left[\frac{\partial}{\partial z}\left(\tilde{Y}_{c_{n,m}}(z,s)\right)\right]\cdot\left(\tilde{Z}_{c_{n,m}}(z,s)\right)\cdot\left(\tilde{I}_n(z,s)\right) + \left(\tilde{Y}_{c_{n,m}}(z,s)\right)\cdot\frac{\partial}{\partial z}\left[\left(\tilde{Z}_{c_{n,m}}(z,s)\right)\cdot\left(\tilde{I}_n(z,s)\right)\right]
\end{aligned}
\tag{2.1}
$$

and the relations

$$
\begin{aligned}
\left(0_{n,m}\right) = \frac{\partial}{\partial z}\left(1_{n,m}\right) &= \frac{\partial}{\partial z}\left[\left(\tilde{Z}_{c_{n,m}}(z,s)\right)\cdot\left(\tilde{Y}_{c_{n,m}}(z,s)\right)\right] \\
&= \left[\frac{\partial}{\partial z}\left(Z_{c_{n,m}}(z,s)\right)\right]\cdot\left(\tilde{Y}_{c_{n,m}}(z,s)\right) + \left(\tilde{Z}_{c_{n,m}}(z,s)\right)\cdot\frac{\partial}{\partial z}\left(\tilde{Y}_{c_{n,m}}(z,s)\right) \\
\left(0_{n,m}\right) = \frac{\partial}{\partial z}\left(1_{n,m}\right) &= \frac{\partial}{\partial z}\left[\left(\tilde{Y}_{c_{n,m}}(z,s)\right)\cdot\left(\tilde{Z}_{c_{n,m}}(z,s)\right)\right] \\
&= \left[\frac{\partial}{\partial z}\left(\tilde{Y}_{c_{n,m}}(z,s)\right)\right]\cdot\left(\tilde{Z}_{c_{n,m}}(z,s)\right) + \left(\tilde{Y}_{c_{n,m}}(z,s)\right)\cdot\frac{\partial}{\partial z}\left(\tilde{Z}_{c_{n,m}}(z,s)\right)
\end{aligned}
\tag{2.2}
$$

the telegrapher equations can be combined as

$$
\begin{aligned}
\frac{\partial}{\partial z}\begin{pmatrix} \left(\tilde{V}_n(z,s)\right) \\ \left(\tilde{Z}_{c_{n,m}}(z,s)\right)\cdot\left(\tilde{I}_n(z,s)\right) \end{pmatrix} &= \begin{pmatrix} \left(0_{n,m}\right) & -\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{Y}_{c_{n,m}}(z,s)\right) \\ -\left(\tilde{Z}_{c_{n,m}}(z,s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right) & -\left(\tilde{Z}_{c_{n,m}}(z,s)\right)\cdot\frac{\partial}{\partial z}\left(\tilde{Y}_{c_{n,m}}(z,s)\right) \end{pmatrix} \odot \begin{pmatrix} \left(\tilde{V}_n(z,s)\right) \\ \left(\tilde{Z}_{c_{n,m}}(z,s)\right)\cdot\left(\tilde{I}_n(z,s)\right) \end{pmatrix} \\
&\quad + \begin{pmatrix} \left(\tilde{V}_n^{(s)\prime}(z,s)\right) \\ \left(\tilde{Z}_{c_{n,m}}(z,s)\right)\cdot\left(\tilde{I}_n^{(s)\prime}(z,s)\right) \end{pmatrix} \\
&= \begin{pmatrix} \left(0_{n,m}\right) & -\left(\gamma_{n,m}(z,s)\right) \\ -\left(\tilde{\gamma}_{n,m}(z,s)\right) & \left(\tilde{g}_{n,m}(z,s)\right) \end{pmatrix} \odot \begin{pmatrix} \left(\tilde{V}_n(z,s)\right) \\ \left(\tilde{Z}_{c_{n,m}}(z,s)\right)\cdot\left(\tilde{I}_n(z,s)\right) \end{pmatrix} + \begin{pmatrix} \left(\tilde{V}_n^{(s)\prime}(z,s)\right) \\ \left(\tilde{Z}_{c_{n,m}}(z,s)\right)\cdot\left(\tilde{I}_n^{(s)\prime}(z,s)\right) \end{pmatrix}
\end{aligned}
\tag{2.3}
$$

where the supervectors have 2N components and the supermatrices are 2N×2N, and where

$$\left(\tilde{g}_{n,m}(z,s)\right) = -\left(\tilde{Z}_{c_{n,m}}(z,s)\right) \cdot \frac{\partial}{\partial z}\left(\tilde{Y}_{c_{n,m}}(z,s)\right)$$
$$= \left[\frac{\partial}{\partial z}\left(\tilde{Z}_{c_{n,m}}(z,s)\right)\right] \cdot \left(\tilde{Y}_{c_{n,m}}(z,s)\right) \tag{2.4}$$
$$= D_z\left[\left(\tilde{Z}_{c_{n,m}}(z,s)\right)\right]$$

The product derivative D_z is discussed in Appendix A.

3. Solution for the Supermatrizant for Voltage and Current Vectors

The supermatrizant differential equation for (2.3) is

$$\frac{\partial}{\partial z}\left(\left(\tilde{\Xi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)=\left(\left(\tilde{\xi}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)\odot\left(\left(\tilde{\Xi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)$$

$$\left(\left(\tilde{\Xi}_{n,m}(z_0,z_0;s)\right)_{\sigma,\sigma'}\right)=\left(\left(1_{n,m}\right)_{\sigma,\sigma'}\right)=\begin{pmatrix}(1_{n,m}) & (0_{n,m})\\ (0_{n,m}) & (1_{n,m})\end{pmatrix} \tag{3.1}$$

$$\equiv \text{supermatrix identity}$$

$$\left(\left(\tilde{\xi}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)=\begin{pmatrix}(0_{n,m}) & -\left(\tilde{\gamma}_{n,m}(z,s)\right)\\ -\left(\tilde{\gamma}_{n,m}(z,s)\right) & \left(\tilde{g}_{n,m}(z,s)\right)\end{pmatrix}$$

Then (2.3) can be solved as

$$\begin{aligned}\begin{pmatrix}\left(\tilde{V}_n(z,s)\right)\\ \left(\tilde{Z}_{c_{n,m}}(z,s)\right)\cdot\left(\tilde{I}_n(z,s)\right)\end{pmatrix}&=\left(\left(\tilde{\Xi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)\odot\begin{pmatrix}\left(\tilde{V}_n(z,s)\right)\\ \left(\tilde{Z}_{c_{n,m}}(z,s)\right)\cdot\left(\tilde{I}_n(z,s)\right)\end{pmatrix}\\ &+\int_{z_0}^{z}\left(\left(\tilde{\Xi}_{n,m}(z,z';s)\right)_{\sigma,\sigma'}\right)\odot\begin{pmatrix}\left(\tilde{V}_n^{(s)'}(z,s)\right)\\ \left(\tilde{Z}_{c_{n,m}}(z,s)\right)\cdot\left(\tilde{I}_n^{(s)'}(z,s)\right)\end{pmatrix}dz'\end{aligned} \tag{3.2}$$

So let us consider the properties of the supermatrizant.

The supermatrizant can be represented as a product integral (Appendix A)

$$\left(\left(\tilde{\Xi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)=\prod_{z_0}^{z}e^{\left(\left(\tilde{\xi}_{n,m}(z',s)\right)_{\sigma,\sigma'}\right)dz'} \tag{3.3}$$

For convenience write

$$\left(\left(\tilde{\xi}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)=\left(\left(\tilde{g}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)+\left(\left(\tilde{h}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)$$

$$\left(\left(\tilde{g}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)=\begin{pmatrix}(0_{n,m}) & (0_{n,m})\\ (0_{n,m}) & \left(\tilde{g}_{n,m}(z,s)\right)\end{pmatrix} \tag{3.4}$$

$$\left(\left(\tilde{h}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)=\begin{pmatrix}(0_{n,m}) & -\left(\tilde{\gamma}_{n,m}(z,s)\right)\\ -\left(\tilde{\gamma}_{n,m}(z,s)\right) & (0_{n,m})\end{pmatrix}$$

Next note that

$$\left(\left(\tilde{g}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)=\begin{pmatrix}\left(0_{n,m}\right) & \left(0_{n,m}\right)\\ \left(0_{n,m}\right) & D_z\left[\left(\tilde{Z}_{c_{n,m}}(z,s)\right)\right]\end{pmatrix}$$

$$=D_z\left[\begin{pmatrix}\left(1_{n,m}\right) & \left(0_{n,m}\right)\\ \left(0_{n,m}\right) & \left(\tilde{Z}_{c_{n,m}}(z,s)\right)\end{pmatrix}\right]$$

$$\left(\left(\tilde{G}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)\equiv\prod_{z_0}^{z}e^{\left(\left(\tilde{g}_{n,m}(z',s)\right)_{\sigma,\sigma'}\right)dz'}\tag{3.5}$$

$$=\begin{pmatrix}\left(1_{n,m}\right) & \left(0_{n,m}\right)\\ \left(0_{n,m}\right) & \left(\tilde{Z}_{c_{n,m}}(z,s)\right)\end{pmatrix}\odot\begin{pmatrix}\left(1_{n,m}\right) & \left(0_{n,m}\right)\\ \left(0_{n,m}\right) & \left(\tilde{Z}_{c_{n,m}}(z,s)\right)\end{pmatrix}^{-1}$$

$$=\begin{pmatrix}\left(1_{n,m}\right) & \left(0_{n,m}\right)\\ \left(0_{n,m}\right) & \left(\tilde{Z}_{c_{n,m}}(z,s)\right)\cdot\left(\tilde{Z}_{c_{n,m}}(z_0,s)\right)^{-1}\end{pmatrix}$$

The sum rule then gives

$$\left(\left(\tilde{\Xi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)=\left(\left(\tilde{G}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)$$

$$\odot\prod_{z_0}^{z}e^{\left(\left(\tilde{G}_{n,m}(z',z_0;s)\right)_{\sigma,\sigma'}\right)^{-1}\odot\left(\left(\tilde{h}_{n,m}(z',s)\right)_{\sigma,\sigma'}\right)\odot\left(\left(\tilde{G}_{n,m}(z',z_0;s)\right)_{\sigma,\sigma'}\right)dz'}\tag{3.6}$$

This is an interesting result in that the leading supermatrizant split off from the product integral is precisely the term in the same position in the piecewise constant approximation for an NMTL in [7].

Next write

$$\left(\left(\tilde{\varphi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)\equiv\left(\left(\tilde{G}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)^{-1}\odot\left(\left(\tilde{h}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)\odot\left(\left(\tilde{G}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)$$

$$=-\begin{pmatrix}\left(0_{n,m}\right) & \left(\tilde{\gamma}_{n,m}(z,s)\right)\cdot\left(\tilde{Z}_{c_{n,m}}(z,s)\right)\cdot\left(\tilde{Z}_{c_{n,m}}(z_0,s)\right)^{-1}\\ \left(\tilde{Z}_{c_{n,m}}(z_0,s)\right)\cdot\left(\tilde{Z}_{n,m}(z,s)\right)^{-1}\cdot\left(\tilde{\gamma}_{n,m}(z,s)\right) & \left(0_{n,m}\right)\end{pmatrix}$$

$$\equiv\left(\left(\tilde{h}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)+\left(\left(\tilde{d}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)\tag{3.7}$$

Note that

$$\left(\Delta\tilde{Z}_{c_{n,m}}(z,s)\right) \equiv \left(\tilde{Z}_{c_{n,m}}(z,s)\right) - \left(\tilde{Z}_{c_{n,m}}(z_0,s)\right)$$

$$\left(\tilde{Z}_{c_{n,m}}(z,s)\right)\cdot\left(\tilde{Z}_{c_{n,m}}(z_0,s)\right)^{-1} = \left(1_{n,m}\right) + \left(\Delta\tilde{Z}_{c_{n,m}}\right)\cdot\left(\tilde{Z}_{c_{n,m}}(z_0,s)\right)^{-1}$$

$$\begin{aligned}\left(\tilde{Z}_{c_{n,m}}(z_0,s)\right)\cdot\left(\tilde{Z}_{c_{n,m}}(z,s)\right)^{-1} &= \left(1_{n,m}\right) - \left(\Delta\tilde{Z}_{c_{n,m}}\right)\cdot\left(\tilde{Z}_{c_{n,m}}(z,s)\right)^{-1} \\ &= \left(1_{n,m}\right) - \left(\Delta\tilde{Z}_{c_{n,m}}\right)\cdot\left(\tilde{Z}_{c_{n,m}}(z_0,s)\right)^{-1} + O\left(\left[\left(\Delta\tilde{Z}_{c_{n,m}}\right)\cdot\left(\tilde{Z}_{c_{n,m}}(z_0,s)\right)^{-1}\right]^2\right) \\ &\quad \text{as } \left(\Delta\tilde{Z}_{c_{n,m}}\right)\cdot\left(\tilde{Z}_{c_{n,m}}(z_0,s)\right)^{-1} \to \left(0_{n,m}\right)\end{aligned} \tag{3.8}$$

where the smallness of the relative change in the characteristic impedance matrix can be measured in various norm senses as discussed in Appendix B. The "remainder" supermatrix is then

$$\left(\left(\tilde{d}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right) = \begin{pmatrix} \left(0_{n,m}\right) & -\left(\tilde{\gamma}_{n,m}(z,s)\right)\cdot\left(\Delta\tilde{Z}_{c_{n,m}}\right)\cdot\left(\tilde{Z}_{c_{n,m}}(z,s)\right)^{-1} \\ \left(\Delta\tilde{Z}_{c_{n,m}}\right)\cdot\left(\tilde{Z}_{c_{n,m}}(z_0,s)\right)^{-1}\cdot\left(\tilde{\gamma}_{n,m}(z,s)\right) & \left(0_{n,m}\right) \end{pmatrix}$$
$$+\begin{pmatrix} \left(0_{n,m}\right) & \left(0_{n,m}\right) \\ O\left(\left[\left(\Delta\tilde{Z}_{c_{n,m}}\right)\cdot\left(\tilde{Z}_{c_{n,m}}(z_0,s)\right)^{-1}\right]^2\cdot\left(\tilde{\gamma}_{n,m}(z,s)\right)\right) & \left(0_{n,m}\right) \end{pmatrix} \tag{3.9}$$

Now we have

$$\left(\left(\tilde{H}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right) \equiv \prod_{z_0}^{z} e^{\left(\left(\tilde{h}_{n,m}(z',s)\right)_{\sigma,\sigma'}\right)dz'}$$

$$\begin{aligned}\left(\left(\Phi_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right) &\equiv \prod_{z_0}^{z} e^{\left(\left(\tilde{\phi}_{n,m}(z',z_0;s)\right)_{\sigma,\sigma'}\right)dz'} \\ &= \left(\left(\tilde{H}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right) \\ &\quad \odot \prod_{z_0}^{z} e^{\left(\left(\tilde{H}_{n,m}(z',z_0;s)\right)_{\sigma,\sigma'}\right)^{-1}\odot\left(\left(\tilde{d}_{n,m}(z',s)\right)_{\sigma,\sigma'}\right)\odot\left(\left(\tilde{H}_{n,m}(z',z_0;s)\right)_{\sigma,\sigma'}\right)dz'}\end{aligned} \tag{3.10}$$

At this juncture we can temporarily assume that the NMTL has a z-independent propagation matrix, as in the piecewise-constant approximation as in [7], giving

$$\left(\left(\tilde{H}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)=\prod_{z_0}^{z} e^{-\begin{pmatrix}(0_{n,m}) & (\tilde{\gamma}_{n,m}(s))\\(\tilde{\gamma}_{n,m}(s)) & (0_{n,m})\end{pmatrix}dz'}$$

$$= e^{-\begin{pmatrix}(0_{n,m}) & (\tilde{\gamma}_{n,m}(s))\\(\tilde{\gamma}_{n,m}(s)) & (0_{n,m})\end{pmatrix}[z-z_0]}$$

$$=\begin{pmatrix}\cosh\left((\tilde{\gamma}_{n,m}(s))[z-z_0]\right) & -\sinh\left((\tilde{\gamma}_{n,m}(s))[z-z_0]\right)\\-\sinh\left((\tilde{\gamma}_{n,m}(s))[z-z_0]\right) & \cosh\left((\tilde{\gamma}_{n,m}(s))[z-z_0]\right)\end{pmatrix}$$

(3.11)

Concerning the remaining product integral, noting that $\left(\left(\tilde{d}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)$ is small, let us use the results of Appendix C for perturbations, giving

$$\left(\left(D_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)$$

$$\equiv\prod_{z_0}^{z} e^{\left(\left(\tilde{H}_{n,m}(z',z_0;s)\right)_{\sigma,\sigma'}\right)^{-1}\odot\left(\left(\tilde{d}_{n,m}(z',s)\right)_{\sigma,\sigma'}\right)\odot\left(\left(\tilde{H}_{n,m}(z',z_0;s)\right)_{\sigma,\sigma'}\right)dz'}$$

$$=\left(1_{n,m}\right)+\int_{z_0}^{z}\left(\left(\tilde{H}_{n,m}(z',z_0;s)\right)_{\sigma,\sigma'}\right)^{-1}\odot\left(\left(\tilde{d}_{n,m}(z',s)\right)_{\sigma,\sigma'}\right)\odot\left(\left(\tilde{H}_{n,m}(z',z_0;s)\right)_{\sigma,\sigma'}\right)dz'$$

$$+O\left(\left(\chi_{\max}|z-z_0|\right)^2\right) \text{ as } \chi_{\max}|z-z_0|\to 0$$

(3.12)

$$\chi_{\max}=\sup_{z_0\le z'\le z}\left\|\left(\left(\tilde{H}_{n,m}(z',z_0;s)\right)_{\sigma,\sigma'}\right)^{-1}\odot\left(\left(\tilde{d}_{n,m}(z',s)\right)_{\sigma,\sigma'}\right)\odot\left(\left(\tilde{H}_{n,m}(z',z_0;s)\right)_{\sigma,\sigma'}\right)\right\|$$

where the order symbol is replaced as in (C.8) for a bound on the norm of the error.

4. Conversion to Wave Variables

As in [8] convert the formulation to waves with + giving the direction of increasing z (right), and - giving the direction of decreasing z (left), as

$$\begin{pmatrix} \left(\tilde{V}_n(z,s)\right)_+ \\ \left(\tilde{V}_n(z,s)\right)_- \end{pmatrix} = \left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n(z,s)\right) \\ \left(\tilde{Z}_{c_{n,m}}(z,s)\right) \cdot \left(\tilde{I}_n(z,s)\right) \end{pmatrix}$$

$$\begin{pmatrix} \left(\tilde{V}_n^{(s)'}(z,s)\right)_+ \\ \left(\tilde{I}_n^{(s)'}(z,s)\right)_- \end{pmatrix} = \left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n^{(s)'}(z,s)\right) \\ \left(\tilde{Z}_{c_{n,m}}(z,s)\right) \cdot \left(\tilde{I}_n^{(s)'}(z,s)\right) \end{pmatrix}$$

$$\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right) = \begin{pmatrix} \left(1_{n,m}\right) & \left(1_{n,m}\right) \\ \left(1_{n,m}\right) & -\left(1_{n,m}\right) \end{pmatrix} = \left(\left(R_{n,m}\right)_{\sigma,\sigma'}\right) + \left(\left(P_{n,m}\right)_{\sigma,\sigma'}\right)$$

$$\left(\left(R_{n,m}\right)_{\sigma,\sigma'}\right) = \begin{pmatrix} \left(1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & -\left(1_{n,m}\right) \end{pmatrix} = \left(\left(R_{n,m}\right)_{\sigma,\sigma'}\right)^{-1} \equiv \text{reflection supermatrix}$$

$$\left(\left(P_{n,m}\right)_{\sigma,\sigma'}\right) = \begin{pmatrix} \left(0_{n,m}\right) & \left(1_{n,m}\right) \\ \left(1_{n,m}\right) & \left(0_{n,m}\right) \end{pmatrix} = \left(\left(P_{n,m}\right)_{\sigma,\sigma'}\right)^{-1} \equiv \text{permutation supermatrix}$$

$$\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)^{-1} = \frac{1}{2}\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right) \tag{4.1}$$

Converting (2.3) to wave variables then gives

$$\frac{\partial}{\partial z}\begin{pmatrix} \left(\tilde{V}_n(z,s)\right)_+ \\ \left(\tilde{V}_n(z,s)\right)_- \end{pmatrix} = \left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right) \odot \left(\left(\tilde{\xi}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right) \odot \left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)^{-1} \tag{4.2}$$

The supermatrizant differential equation for this is the homogeneous equation

$$\frac{\partial}{\partial z}\left(\left(\tilde{\Gamma}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right) = \left(\left(\tilde{\gamma}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right) \odot \left(\left(\tilde{\Gamma}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)$$

$$\left(\left(\tilde{\Gamma}_{n,m}(z_0,z_0;s)\right)_{\sigma,\sigma'}\right) = \left(\left(1_{n,m}\right)_{\sigma,\sigma'}\right) \tag{4.3}$$

$$\left(\left(\tilde{\gamma}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)=\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)\odot\left(\left(\xi_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)\odot\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)^{-1}$$

$$\left(\left(\tilde{\Gamma}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)=\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)\odot\left(\left(\tilde{\Xi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)\odot\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)^{-1}$$

The wave supermatrizant is then related to the previous form (for voltage and current vectors) by a constant similarity transformation. In terms of this supermatrizant (4.2) has the solution

$$\begin{pmatrix}\left(\tilde{V}_n(z,s)\right)_+\\ \left(\tilde{V}_n(z,s)\right)_-\end{pmatrix}=\left(\left(\tilde{\Gamma}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)\odot\begin{pmatrix}\left(\tilde{V}_n(z_0,s)\right)_+\\ \left(\tilde{V}_n(z_0,s)\right)_-\end{pmatrix}+\int_{z_0}^{z}\left(\left(\tilde{\Gamma}_{n,m}(z,z';s)\right)_{\sigma,\sigma'}\right)\odot\begin{pmatrix}\left(\tilde{V}_n^{(s)'}(z',s)\right)_+\\ \left(\tilde{V}_n^{(s)'}(z',s)\right)_-\end{pmatrix} \tag{4.4}$$

Using the transformations in (4.3) the results of Section 3 can all be easily transformed into the wave variables.

5. Almost Uniform Multiconductor Transmission Line

Let us now take a different tack in solving the NMTL equations. If the MTL were perfectly uniform then both the propagation matrix and characteristic impedance matrix would not be functions of z. Then we can write from (3.1)

$$\begin{aligned}
&\left(\left(\tilde{\xi}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)=\left(\left(\tilde{h}^{(0)}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)+\left(\left(\tilde{h}^{(1)}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)\\
&\left(\left(\tilde{h}^{(0)}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)\equiv\begin{pmatrix}\left(0_{n,m}\right) & -\left(\tilde{\gamma}_{n,m}(s)\right)\\ -\left(\tilde{\gamma}_{n,m}(s)\right) & \left(0_{n,m}\right)\end{pmatrix}\\
&\left(\left(\tilde{h}^{(1)}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right)=\begin{pmatrix}\left(0_{n,m}\right) & -\left(\Delta\tilde{\gamma}_{n,m}(z,s)\right)\\ -\left(\Delta\tilde{\gamma}_{n,m}(z,s)\right) & \left(\tilde{g}_{n,m}(z,s)\right)\end{pmatrix}\\
&\qquad\left(\tilde{\gamma}_{n,m}(z,s)\right)=\left(\tilde{\gamma}_{n,m}(s)\right)+\left(\Delta\tilde{\gamma}_{n,m}(z,s)\right)
\end{aligned}\tag{5.1}$$

where $\left(\tilde{\gamma}_{n,m}(s)\right)$ is chosen to minimize $\left(\Delta\tilde{\gamma}_{n,m}(z,s)\right)$ in some sense over the (z_0,z) interval of interest. Then from Appendix B we have the perturbation solution in terms of the wave variables of Section 4 as

$$\begin{aligned}
\left(\left(\tilde{H}^{(w,0)}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)&=\prod_{z_0}^{z} e^{\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)\odot\left(\left(h^{(0)}_{n,m}(z',s)\right)_{\sigma,\sigma'}\right)\odot\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)^{-1}dz'}\\
&=\prod_{z_0}^{z} e^{\begin{pmatrix}-\left(\tilde{\gamma}_{n,m}(s)\right) & \left(0_{n,m}\right)\\ \left(0_{n,m}\right) & \left(\tilde{\gamma}_{n,m}(s)\right)\end{pmatrix}dz'}\\
&=\begin{pmatrix}e^{-\left(\tilde{\gamma}_{n,m}(s)\right)[z-z_0]} & \left(0_{n,m}\right)\\ \left(0_{n,m}\right) & e^{\left(\tilde{\gamma}_{n,m}(s)\right)[z-z_0]}\end{pmatrix}
\end{aligned}\tag{5.2}$$

$$\left(\left(\tilde{\Gamma}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)=\left(\left(\tilde{H}_{n,m}^{(w,0)}(z,z_0;s)\right)_{\sigma,\sigma'}\right)\odot$$

$$\prod_{z_0}^{z} e^{\left(\left(\tilde{H}_{n,m}^{(w,0)}(z',z_0;s)\right)_{\sigma,\sigma'}\right)^{-1}\odot\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)\odot\left(\left(\tilde{h}_{n,m}^{(1)}(z',s)\right)_{\sigma,\sigma'}\right)\odot\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)^{-1}\odot\left(\left(\tilde{H}_{n,m}^{(w,0)}(z',z_0;s)\right)_{\sigma,\sigma'}\right)dz'}$$

$$=\left(\left(\tilde{H}_{n,m}^{(w,0)}(z,z_0;s)\right)_{\sigma,\sigma'}\right)\odot\Big[\left(1_{n,m}\right)+\int_{z_0}^{z}\left(\left(\tilde{H}_{n,m}^{(w,0)}(z',z_0;s)\right)_{\sigma,\sigma'}\right)^{-1}\odot\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)$$

$$\odot\left(\left(\tilde{h}_{n,m}^{(1)}(z',s)\right)_{\sigma,\sigma'}\right)\odot\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)^{-1}\odot\left(\left(\tilde{H}_{n,m}^{(w,0)}(z',z_0;s)\right)_{\sigma,\sigma'}\right)dz'$$

$$+O\left(\left(\chi_{\max}|z-z_0|\right)^2\right)\text{ as }\chi_{\max}|z-z_0|\to 0\Big]$$

$$\chi_{\max}=\sup_{z_0\le z'\le z}\Bigg\|\left(\left(\tilde{H}_{n,m}^{(w,0)}(z',z_0;s)\right)_{\sigma,\sigma'}\right)^{-1}\odot\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)\odot\left(\left(\tilde{h}_{n,m}^{(1)}(z',s)\right)_{\sigma,\sigma'}\right)$$

$$\odot\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)^{-1}\odot\left(\left(\tilde{H}_{n,m}^{(w,0)}(z',z_0;s)\right)_{\sigma,\sigma'}\right)\Bigg\|$$

where the order symbol is replaced as in (C.8) for a bound on the norm of the error. In this form one can investigate the first order perturbation for various deviations of the propagation-constant matrix and impedance matrix.

As a simple case consider N perfect conductors (plus reference) immersed in a uniform isotropic medium. Then as in [3,9] we have

$$\left(\tilde{\gamma}_{n,m}(z,s)\right)=\tilde{\gamma}(s)\left(1_{n,m}\right)$$

$$\left(\Delta\tilde{\gamma}_{n,m}(z,s)\right)=\left(0_{n,m}\right)$$

$$\left(\tilde{Z}_{c_{n,m}}(z,s)\right)=\left[\frac{s\mu}{\sigma+s\varepsilon}\right]^{\frac{1}{2}}\left(f_{g_{n,m}}(z)\right) \tag{5.3}$$

$\mu \equiv$ permeability of medium

$\varepsilon \equiv$ permittivity of medium

$\sigma \equiv$ conductivity of medium

$$
\begin{aligned}
\left(\tilde{g}_{n,m}(z,s)\right) &= \left[\frac{\partial}{\partial z}\left(\tilde{Z}_{c_{n,m}}(z,s)\right)\right]\cdot\left(\tilde{Z}_{c_{n,m}}(z,s)\right)^{-1} \\
&= \left[\frac{d}{dz}\left(f_{g_{n,m}}(z)\right)\right]\cdot\left(f_{g_{n,m}}(z)\right)^{-1} \\
&\equiv \left(g_{n,m}(z)\right)
\end{aligned}
$$

where the constitutive parameters can be allowed to be functions of frequency without changing the form of the results (in frequency domain). In this form all the modes have the same propagation constant (same speed) and the only perturbation comes from the variation in the characteristic impedance matrix due to changes in the cross section geometry (size and spacing of the conductors) as contained in the geometric-factor matrix $\left(f_{g_{n,m}}(z)\right)$ (real, symmetric, and positive definite).

Assembling these terms gives

$$
\left(\left(\tilde{H}_{n,m}^{(w,0)}(z,z_0;s)\right)_{\sigma,\sigma'}\right) = \begin{pmatrix} e^{-\gamma[z-z_0]}\left(1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & e^{-\gamma[z-z_0]}\left(1_{n,m}\right) \end{pmatrix}
$$

$$
\left(\left(\tilde{h}_{n,m}^{(1)}(z,s)\right)_{\sigma,\sigma'}\right) \equiv \left(\left(\tilde{h}_{n,m}^{(1)}(z)\right)_{\sigma,\sigma'}\right) = \begin{pmatrix} \left(0_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(g_{n,m}(z)\right) \end{pmatrix}
$$

$$
\begin{aligned}
\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)\odot\left(\left(\tilde{h}_{n,m}^{(1)}(z)\right)_{\sigma,\sigma'}\right)\odot\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)^{-1} &\equiv \left(\left(\tilde{h}_{n,m}^{(w,1)}(z)\right)_{\sigma,\sigma'}\right) \\
&= \frac{1}{2}\begin{pmatrix} \left(g_{n,m}(z)\right) & -\left(g_{n,m}(z)\right) \\ -\left(g_{n,m}(z)\right) & \left(g_{n,m}(z)\right) \end{pmatrix}
\end{aligned}
\tag{5.4}
$$

$$
\begin{aligned}
\left(\left(\tilde{H}_{n,m}^{(w,0)}(z,z_0;s)\right)_{\sigma,\sigma'}\right)^{-1}\odot\left(\left(\tilde{h}_{n,m}^{(w,1)}(z)\right)_{\sigma,\sigma'}\right)\odot\left(\left(\tilde{H}_{n,m}^{(w,0)}(z,z_0;s)\right)_{\sigma,\sigma'}\right) &\equiv \left(\left(h_{n,m}^{(w,2)}(z,z_0;s)\right)_{\sigma,\sigma'}\right) \\
&= \frac{1}{2}\begin{pmatrix} \left(g_{n,m}(z)\right) & -e^{2\tilde{\gamma}(s)[z-z_0]}\left(g_{n,m}(z)\right) \\ -e^{-2\tilde{\gamma}(s)[z-z_0]}\left(g_{n,m}(z)\right) & \left(g_{n,m}(z)\right) \end{pmatrix}
\end{aligned}
$$

The perturbation term is then

$$
\begin{aligned}
\left(\left(\tilde{H}_{n,m}^{(w,1)}(z,z_0;s)\right)_{\sigma,\sigma'}\right) &= \int_{z_0}^{z}\left(\left(h_{n,m}^{(w,2)}(z',z_0;s)\right)_{\sigma,\sigma'}\right)dz' \\
&= \frac{1}{2}\int_{z_0}^{z}\begin{pmatrix} \left(g_{n,m}(z')\right) & -e^{2\tilde{\gamma}[z-z_0]}\left(g_{n,m}(z')\right) \\ -e^{2\tilde{\gamma}[z'-z_0]}\left(g_{n,m}(z')\right) & \left(g_{n,m}(z')\right) \end{pmatrix}dz'
\end{aligned}
\tag{5.5}
$$

The detailed form this takes depends on the form one assumes for $\left(g_{n,m}(z')\right)$; numerous choices are possible depending on the physical situation one wishes to model (including a statistical form based on random variations of the wire size and spacing).

For evaluating the remaining error, let us further assume that

$$\begin{aligned}
&\tilde{\gamma}(s) = \frac{s}{v} \\
&v = [\mu\varepsilon]^{-\frac{1}{2}} = \text{real, frequency - independent propagation speed} \\
&\sigma = 0
\end{aligned} \tag{5.6}$$

This corresponds to a lossless dispersionless MTL consisting of perfectly conducting wires (plus reference) in a medium with real, frequency-independent permittivity and permeability. Of practical interest, this approximately corresponds to a plastic such as polyethylene. Then let us specialize to frequencies on the $j\omega$ axis of the s plane as

$$\begin{aligned}
&s = j\omega \\
&\tilde{\gamma}(j\omega) \equiv j\frac{\omega}{v} \equiv jk \ \ (\text{k real})
\end{aligned} \tag{5.7}$$

Then we observe that we have unitary matrices

$$\begin{aligned}
\left[2^{-\frac{1}{2}}\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)\right]^{-1} &= 2^{-\frac{1}{2}}\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right) = \left[2^{-\frac{1}{2}}\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)\right]^{T} \\
&= \left[2^{-\frac{1}{2}}\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)\right]^{\dagger} \\
\left(\left(\tilde{H}_{n,m}^{(w,0)}(z',z_0;j\omega)\right)_{\sigma,\sigma'}\right)^{-1} &= \begin{pmatrix} e^{jk[z-z_0]}\left(1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & e^{-jk[z-z_0]}\left(1_{n,m}\right) \end{pmatrix} \\
&= \begin{pmatrix} e^{-jk[z-z_0]}\left(1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & e^{jk[z-z_0]}\left(1_{n,m}\right) \end{pmatrix}^{\dagger} \\
&= \left(\left(\tilde{H}_{n,m}^{(w,o)}(z',z;j\omega)\right)_{\sigma,\sigma'}\right)^{\dagger}
\end{aligned} \tag{5.8}$$

Choosing the 2-norm we then have from (B.24)

$$
\begin{aligned}
\chi^{(2)}(z') &\equiv \left\| \left(\left(\tilde{H}^{(w,2)}_{n,m}(z',z_0;j\omega) \right)_{\sigma,\sigma'} \right) \right\|_2 \\
&= \left\| \left(\left(\tilde{h}^{(w,1)}(z',z_0;j\omega) \right)_{\sigma,\sigma'} \right) \right\|_2 \\
&= \left\| \left(\left(\tilde{h}^{(1)}_{n,m}(z') \right)_{\sigma,\sigma'} \right) \right\|_2 \\
\chi^{(2)}_{\max} &\equiv \sup_{z_0 \le z' \le z} \chi^{(2)}(z')
\end{aligned}
\tag{5.9}
$$

Carrying this further as

$$
\begin{aligned}
&\left(\left(h^{(1)}_{n,m}(z') \right)_{\sigma,\sigma'} \right)^{\dagger} \odot \left(\left(h^{(1)}_{n,m}(z') \right)_{\sigma,\sigma'} \right) = \left(\left(h^{(1)}_{n,m}(z') \right)_{\sigma,\sigma'} \right)^{\mathsf{T}} \odot \left(\left(h^{(1)}_{n}(z') \right)_{\sigma,\sigma'} \right) \\
&= \begin{pmatrix} (0_{n,m}) & (0_{n,m}) \\ (0_{n,m}) & (g_{n,m}(z'))^{\mathsf{T}} \cdot (g_{n,m}(z')) \end{pmatrix}
\end{aligned}
\tag{5.10}
$$

gives the result for this block-diagonal supermatrix

$$
\begin{aligned}
\chi^{(2)}(z') &= \left[\rho\left((g_{n,m}(z'))^{\mathsf{T}} \cdot (g_{n,m}(z')) \right) \right]^{\frac{1}{2}} \\
&= \left\| (g_{n,m}(z')) \right\|_2
\end{aligned}
\tag{5.11}
$$

since the 2-norm a block-diagonal supermatrix is the maximum of the 2-norms of the blocks on the diagonal [2]. The spectral radius ρ (or maximum eigenvalue magnitude) is discussed in Appendix B.

The smallness of the correction term and remaining error then depends on the smallness of

$$
\begin{aligned}
(g_{n,m}(z)) &= \left[\frac{d}{dz} \left(f_{g_{n,m}}(z) \right) \right] \cdot \left(f_{g_{n,m}}(z) \right)^{-1} \\
&= -\left(f_{g_{n,m}}(z) \right) \cdot \frac{d}{dz} \left(\left(f_{g_{n,m}}(z) \right)^{-1} \right)
\end{aligned}
\tag{5.12}
$$

Note that while

$$
\left(f_{g_{n,m}}(z) \right) = \left(f_{g_{n,m}}(z) \right)^{\mathsf{T}} \tag{5.13}
$$

this does not in general make $(g_{n,m}(z))$ symmetric. Of course we have the bound

$$\chi^{(2)}(z') = \left\| (g_{n,m}(z')) \right\|_2 \le \left\| \frac{d}{dz'} \left(f_{g_{n,m}}(z') \right) \right\|_2 \left\| \left(f_{g_{n,m}}(z') \right)^{-1} \right\|_2$$

$$\begin{aligned} \left\| \left(f_{g_{n,m}}(z') \right)^{-1} \right\|_2 &= \rho\left(\left(f_{g_{n,m}}(z') \right)^{-1} \right) \\ &= \left[\text{ minimum eigenvalue magnitude of } \left(f_{g_{n,m}}(z') \right) \right]^{-1} \\ \left\| \frac{d}{dz} \left(f_{g_{n,m}}(z') \right) \right\|_2 &= \rho\left(\frac{d}{dz} \left(f_{g_{n,m}}(z') \right) \right) \end{aligned} \qquad (5.14)$$

So, as we would expect, the smallness of the perturbation is proportional to the perturbation of the characteristic impedance matrix.

6. Concluding Remarks

The product integral then gives some interesting ways to represent and compute the supermatrizants representing waves on NMTLs. In particular one can examine perturbations of the transmission lines away from ideal cases including those here and elsewhere [7].

Appendix A. The Product Integral

From the matrix differential equation (N×N matrices)

$$\frac{d}{dz}\left(X_{n,m}(z,z_0)\right) = \left(a_{n,m}(z)\right)\cdot\left(X_{n,m}(z,z_0)\right)$$
$$\left(X_{n,m}(z_0,z_0)\right) = \left(1_{n,m}\right) \qquad \text{(boundary condition, identity)} \tag{A.1}$$
$$\left(X_{n,m}(z,z_0)\right) \equiv \text{ matrizant}$$

we have what is called a matrizant as the solution [3,5,8,11-13,15]. This has the property

$$\det\left[\left(X_{n,m}(z,z_0)\right)\right] = e^{\int_{z_0}^{z} tr\left[\left(a_{n,m}(z')\right)\right]dz'} \tag{A.2}$$

so that

$$\det\left[\left(X_{n,m}(z,z_0)\right)\right] \neq 0,\infty \tag{A.3}$$

for reasonably well behaved $\left(a_{n,m}(z)\right)$. Noting that the boundary condition is imposed for arbitrary z_0 we also have

$$\left(X_{n,m}(z,z_0)\right) = \left(X_{n,m}(z,z_1)\right)\cdot\left(X_{n,m}(z_1,z_0)\right)$$
$$\left(X_{n,m}(z,z_0)\right)^{-1} = \left(X_{n,m}(z_0,z)\right) \tag{A.4}$$

The matrizant is constructed so that the vector/matrix differential equation (N component vectors)

$$\frac{d}{dz}\left(f_n(z)\right) = \left(a_{n,m}(z)\right)\cdot\left(f_n(z)\right) + \left(F_n(z)\right) \tag{A.5}$$

has the solution

$$\left(f_n(z)\right) = \left(X_{n,m}(z,z_0)\right)\cdot\left(f_n(z_0)\right) + \int_{z_0}^{z}\left(X_{n,m}(z,z')\right)\cdot\left(F_n(z')\right)dz' \tag{A.6}$$

where $\left(F_n(z)\right)$ makes the differential equation inhomogeneous, but allows for sources in our physical problems.

There is a representation of the matrizant as

$$\begin{aligned}
\left(X_{n,m}(z,z_0)\right) &= \sum_{\ell=0}^{\infty}\left(X_{n,m}(z,z_0)\right)_\ell \\
\left(X_{n,m}(z,z_0)\right)_0 &= \left(1_{n,m}\right) \\
\left(X_{n,m}(z,z_0)\right)_{\ell+1} &= \int_{z_0}^{z}\left(a_{n,m}(z')\right)\cdot\left(X_{n,m}(z',z_0)\right)_\ell dz' \\
\left(X_{n,m}(z,z_0)\right)_\ell &= \int_{z_0}^{z}\left(a_{n,m}(z')\right)\cdot\int_{z_0}^{z_1}\left(a_{n,m}(z_2)\right)\cdot\int_{z_0}^{z_2}\cdots\int_{z_0}^{z_{\ell-1}}\left(a_{n,m}(z_\ell)\right)dz_\ell dz_{\ell-1}\cdots dz_1
\end{aligned} \tag{A.7}$$

This is an explicit series involving repeated integrals, but it can involve much computation.

Another representation of the matrizant is the product integral [16] (also called the multiplicative integral [13]). This has various notations in the literature; the modern form is

$$\left(X_{n,m}(z,z_0)\right) = \prod_{z_0}^{z} e^{\left(a_{n,m}(z')\right)dz'} \tag{A.8}$$

This is compared to the usual or sum integral by replacing the sum of small terms by the product of terms, each close to the identity, as

$$\begin{aligned}
&\prod_{z_0}^{z} e^{\left(a_{n,m}(z')\right)dz'} \\
&= \lim_{\Delta z\to 0} e^{\left(a_{n,m}(z_0+(N-1)\Delta z)\right)\Delta z}\cdot e^{\left(a_{n,m}(z_0+(N-2)\Delta z)\right)\Delta z}\cdot\ldots\cdot e^{\left(a_{n,m}(z_0+\Delta z)\right)\Delta z}\cdot e^{\left(a_{n,m}(z_0)\right)\Delta z} \\
&\equiv \lim_{\Delta z\to 0} \bigodot_{\ell=1}^{N} e^{\left(a_{n,m}(z_0+(N-\ell)\Delta z)\right)\Delta z} \\
&\Delta z = \frac{z-z_0}{N}
\end{aligned} \tag{A.9}$$

Normally z is taken as a real variable so that the path of integration is the real axis. For complex z the path must generally be specified. While (A.9) uses evenly spaced increments of z (uniform Δz between z_0 and z) this is only a convenience. Nonuniform Δz is possible with all the $\Delta z \to 0$ on the interval [16]. Note that as one proceeds from z_0 toward z each successive term is dot multiplied on the left. This order is important since matrices need not commute. For the degenerate case of scalars (1×1 matrices) we have

$$\prod_{z_0}^{z} e^{a(z')dz'} = \lim_{\Delta z \to 0} e^{\sum_{\ell=1}^{N} a(z_0 + (N-\ell)\Delta z)\Delta z}$$

$$= \lim_{\Delta z \to 0} e^{\sum_{\ell=0}^{N-1} a(z_0 + \ell \Delta z)\Delta z}$$

$$= e^{\int_{z_0}^{z} a(z')dz'} \tag{A.10}$$

which is just the usual sum integral. So the product integral is introduced to include the order of multiplication (successive terms on the left) required for non-commuting matrices. If all the $\left(a_{n,m}(z')\right)$ on the interval $z_0 \le z' \le z$ commute then (A.10) applies to such matrices as well. Note that (A.2) gives a similar relationship for general non-commuting matrices as

$$\det\left[\prod_{z_0}^{z} e^{\left(a_{n,m}(z')\right)dz'}\right] = e^{\int_{z_0}^{z} tr\left[\left(a_{n,m}(z')\right)\right]dz'} \tag{A.11}$$

The product integral has the property

$$\prod_{z_0}^{z_0} e^{\left(a_{n,m}(z')\right)dz'} = \left(1_{n,m}\right) \tag{A.12}$$

not $\left(0_{n,m}\right)$ as in the case of the usual (sum) integral. The various properties of the matrizant discussed above are of course properties of the product integral. In fact, repeated application of the formula in (A.4) with the interval (z, z_0) decomposed into (z, z_1) and (z_1, z_0) can be used to generate the continued product in (A.9) by assuming $\left(a_{n,m}(z')\right)$ to be constant in each interval. Approximating the exponential of a small argument to first order gives alternate representations of the product integral as

$$\prod_{z_0}^{z} e^{\left(a_{n,m}(z')\right)dz'} = \prod_{z_0}^{z}\left[\left(1_{n,m}\right) + \left(a_{n,m}(z')\right)dz'\right]$$

$$= \prod_{z_0}^{z}\left[\left(1_{n,m}\right) - \left(a_{n,m}(z')\right)dz'\right]^{-1} \tag{A.13}$$

Other properties include

$$\prod_{z_0}^{z} e^{(a_{n,m}(z'))dz'} = \left[\prod_{z_1}^{z} e^{(a_{n,m}(z'))dz'}\right] \cdot \left[\prod_{z_0}^{z_1} e^{(a_{n,m}(z'))dz'}\right]$$

$$\left[\prod_{z_0}^{z} e^{(a_{n,m}(z'))dz'}\right]^{-1} = \prod_{z}^{z_0} e^{(a_{n,m}(z'))dz'} \tag{A.14}$$

In the ordinary calculus the inverse operation to the (sum) integral is the derivative. For the product integral one can similarly define a product (or multiplicative) derivative as [13,16]

$$\begin{aligned} D_z\left[\prod_{z_0}^{z} e^{(a_{n,m}(z'))dz'}\right] &= D_z\left[\left(X_{n,m}(z,z_0)\right)\right] \\ &\equiv \left[\frac{\partial}{\partial z}\left(X_{n,m}(z,z_0)\right)\right] \cdot \left(X_{n,m}(z,z_0)\right)^{-1} \\ &= \left(a_{n,m}(z)\right) \end{aligned} \tag{A.15}$$

Note for scalars that this is just the logarithmic derivative as

$$\begin{aligned} D_z\left[X(z,z_0)\right] &= \left[\frac{\partial}{\partial z}X(z,z_0)\right] \cdot X^{-1}(z,z_0) = X^{-1}(z,z_0)\frac{\partial}{\partial z}X(z,z_0) \\ &= \frac{\partial}{\partial z}\,\ell n\left[X(z,z_0)\right] = a(z) \end{aligned} \tag{A.16}$$

Again the order in (A.15) is important for non-commuting matrices. There are various product derivative formulas, including

$$\begin{aligned} D_z\left[\left(X_{n,m}(z)\right)\cdot\left(Y_{n,m}(z)\right)\right] &= D_z\left[\left(X_{n,m}(z)\right)\right] + \left(X_{n,m}(z)\right)\cdot D_z\left[\left(Y_{n,m}(z)\right)\right]\cdot\left(X_{n,m}(z)\right)^{-1} \\ D_z\left[\left(X_{n,m}(z)\right)^{\mathsf{T}}\right] &= \left(X_{n,m}(z)\right)^{\mathsf{T}} \cdot \left[D_z\left[\left(X_{n,m}(z)\right)\right]\right]^{\mathsf{T}} \cdot \left(X_{n,m}(z)\right)^{\mathsf{T}^{-1}} \\ D_z\left[\left(X_{n,m}(z)\right)^{-1}\right] &= -\left(X_{n,m}(z)\right)^{-1} \cdot D_z\left[\left(X_{n,m}(z)\right)\right]\cdot\left(X_{n,m}(z)\right) \\ &= -\left[D_z\left[\left(X_{n,m}(z)\right)^{\mathsf{T}}\right]\right]^{\mathsf{T}} \\ &= -\left(X_{n,m}(z)\right)^{-1} \cdot \frac{\partial}{\partial z}\left(X_{n,m}(z)\right) \\ D_z\left[\left(X_{n,m}(z)\right)^{\mathsf{T}^{-1}}\right] &= -\left[D_z\left[\left(X_{n,m}(z)\right)\right]\right]^{\mathsf{T}} \end{aligned} \tag{A.17}$$

While the matrices in the above formulas are written as functions of z, they can be matrizants (functions of z_0 and z) or constant matrices to give special results.

Using the product differentiation formulas other product integral formulas can be derived. The sum rule is

$$\left(A_{n,m}(z,z_0)\right) \equiv \prod_{z_0}^{z} e^{\left(a_{n,m}(z')\right)dz'}$$

$$\prod_{z_0}^{z} e^{\left[\left(a_{n,m}(z')\right)+\left(b_{n,m}(z')\right)\right]dz'} = \left(A_{n,m}(z,z_0)\right)\cdot\prod_{z_0}^{z} e^{\left(A_{n,m}(z',z_0)\right)^{-1}\cdot\left(b_{n,m}(z')\right)\cdot\left(A_{n,m}(z',z_0)\right)dz'} \tag{A.18}$$

This is considered to be an analogue of integration by parts for sum integrals [13]. The similarity rule is

$$\left(P_{n,m}(z)\right)\cdot\left[\prod_{z_0}^{z} e^{\left(b_{n,m}(z')\right)dz'}\right]\cdot\left(P_{n,m}(z_0)\right)^{-1} = \prod_{z_0}^{z} e^{\left[D_z\left[\left(P_{n,m}(z)\right)\right]+\left(P_{n,m}(z')\right)\cdot\left(b_{n,m}(z')\right)\cdot\left(P_{n,m}(z')\right)^{-1}\right]dz'} \tag{A.19}$$

A special case of this has

$$\left(C_{n,m}\right) \equiv \text{constant matrix}$$

$$\left(C_{n,m}\right)\cdot\left[\prod_{z_0}^{z} e^{\left(b_{n,m}(z')\right)dz'}\right]\cdot\left(C_{n,m}\right)^{-1} = \prod_{z_0}^{z} e^{\left(C_{n,m}\right)\cdot\left(b_{n,m}(z')\right)\cdot\left(C_{n,m}\right)^{-1}dz'} \tag{A.20}$$

Appendix B. Matrix Norms and Product Integrals

Summarizing [4,10] we have vector norms defined by the properties

$$\begin{aligned} \|(x_n)\| &= 0 \quad \text{iff} \quad (x_n) \equiv 0_n \\ \|\alpha\,(x_n)\| &= |\alpha|\,\|(x_n)\| \\ \|(x_n)+(y_n)\| &\le \|(x_n)\| + \|(y_n)\| \end{aligned} \tag{B.1}$$

where the vectors have N components. For N×N matrices we have matrix norms defined by

$$\begin{aligned} \|(a_{n,m})\| &= 0 \quad \text{iff} \quad (a_{n,m}) \equiv (0_{n,m}) \\ \|\alpha\,(a_{n,m})\| &= |\alpha|\,\|(a_{n,m})\| \\ \|(a_{n,m})+(b_{n,m})\| &\le \|(a_{n,m})\| + \|(b_{n,m})\| \\ \|(a_{n,m})\cdot(b_{n,m})\| &\le \|(a_{n,m})\|\,\|(b_{n,m})\| \end{aligned} \tag{B.2}$$

where there is the additional bound for dot products. Note that all vector and matrix elements as well as scalars (α) are arbitrary complex numbers. An important class of matrix norms are associated matrix norms (associated with particular vector norms) defined by

$$\|(a_{n,m})\| \equiv \sup_{\|(x_n)\| \neq 0} \frac{\|(a_{n,m})\cdot(x_n)\|}{\|(x_n)\|} \tag{B.3}$$

immediately giving the bound

$$\|(a_{n,m})\cdot(x_n)\| \le \|(a_{n,m})\|\,\|(x_n)\| \tag{B.4}$$

Using the properties of vector norms in (B.1) one can establish that associated matrix norms have the properties in (B.2). Furthermore, for all associated matrix norms

$$\|(1_{n,m})\| = 1 \tag{B.5}$$

following directly from (B.3).

The sum bound for matrix norms is readily applied to sum integrals by repeated application for each dz' in

$$\left\| \int_{z_0}^{z} (a_{n,m}(z'))dz' \right\| \le \left| \int_{z_0}^{z} \|(a_{n,m}(z'))\| dz' \right|$$

$$\le |z - z_0| \chi_{\max} \qquad \text{(B.6)}$$

$$\chi(z') \equiv \|(a_{n,m}(z'))\| \quad , \quad \chi_{\max} \equiv \sup_{z_0 \le z' \le z} \chi(z')$$

Similarly, the sum and product rules for matrix norms are readily applied to product integrals. Note first that

$$\left\| e^{(b_{n,m})} \right\| = \left\| \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (b_{n,m})^{\ell} \right\|$$

$$\le \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \|(b_{n,m})\|^{\ell} = e^{\|(b_{n,m})\|} \qquad \text{(B.7)}$$

and similarly

$$\left\| e^{(b_{n,m})} - \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (b_{n,m})^{\ell} \right\| \le e^{\|(b_{n,m})\|} - \sum_{\ell=0}^{L} \frac{1}{\ell!} \|(b_{n,m})\|^{\ell} \qquad \text{(B.8)}$$

Writing out the exponential in series form in the product integral gives

$$\left\| \prod_{z_0}^{z} e^{(a_{n,m}(z'))dz'} \right\| = \left\| \prod_{z_0}^{z} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} [(a_{n,m})dz']^{\ell} \right\|$$

$$\le \prod_{z_0}^{z} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \|(a_{n,m}(z'))\|^{\ell} |dz'|^{\ell}$$

$$= \prod_{z_0}^{z} e^{\|(a_{n,m}(z'))\| |dz'|}$$

$$= e^{\left| \int_{z_0}^{z} \|(a_{n,m}(z'))\| dz' \right|} \qquad \text{(B.9)}$$

with the sum integral in turn bounded by (B.6). For associated matrix norms (B.5) with (B.8) subtracts the leading term off the exponential series in (B.9) giving

$$\left\| \prod_{z_0}^{z} e^{(a_{n,m}(z'))dz'} - (1_{n,m}) \right\| \le e^{\left|\int_{z_0}^{z} \|(a_{n,m}(z'))\| dz'\right|} - 1 \tag{B.10}$$

For a more general bound consider truncating the series for the matrizant in (A.7) and consider the remainder for the ℓth term onward. Here we have

$$\begin{aligned}
\prod_{z_0}^{z} e^{(a_{n,m}(z'))dz'} &= (X_{n,m}(z,z_0)) = \sum_{\ell=0}^{\infty} (X_{n,m}(z,z_0))_\ell \\
(X_{n,m}(z,z_0))_0 &= (1_{n,m}) \\
(X_{n,m}(z,z_0))_\ell &= \int_{z_0}^{z} (a_{n,m}(z_1)) \cdot \cdots \cdot \int_{z_0}^{z_{\ell-1}} (a_{n,m}(z_\ell)) dz_\ell \cdots dz_1
\end{aligned} \tag{B.11}$$

Then we have

$$\begin{aligned}
\|(X_{n,m}(z,z_0))_\ell\| &\le \left| \int_{z_0}^{z} \|(a_{n,m}(z_1))\| \ldots \int_{z_0}^{z_{\ell-1}} \|(a_{n,m}(z_\ell))\| dz_\ell \ldots dz_1 \right| \\
&\le \left| \int_{z_0}^{z} \chi_{\max} \cdots \int_{z_0}^{z_{\ell-1}} \chi_{\max} dz_\ell \ldots dz_1 \right| \\
&\le \chi_{\max}^{\ell} \left| \int_{z_0}^{z} \cdots \int_{z_0}^{z_{\ell-1}} dz_\ell \ldots dz_1 \right| \\
&= \chi_{\max}^{\ell} \frac{|z-z_0|^\ell}{\ell!}
\end{aligned} \tag{B.12}$$

The remainder for all terms from ℓ onward is

$$\begin{aligned}
\left\| \sum_{\ell'=\ell}^{\infty} (X_{n,m}(z,z_0))_{\ell'} \right\| &\le \sum_{\ell'=\ell}^{\infty} \chi_{\max}^{\ell'} \frac{|z-z_0|^{\ell'}}{\ell'!} \\
&= e^{\chi_{\max}|z-z_0|} - \sum_{\ell'=0}^{\ell-1} \chi_{\max}^{\ell} \frac{|z-z_0|^{\ell'}}{\ell'!} \\
&= \chi_{\max}^{\ell} \frac{|z-z_0|^{\ell}}{\ell!} \cdots \\
&= O\left((\chi_{\max}|z-z_0|)^\ell \right) \text{ as } \chi_{\max}|z-z_0| \to 0
\end{aligned} \tag{B.13}$$

This gives an error bound for computing the product integral via the series in (B.11).

A related concept is the spectral radius

$$\rho\left(\left(a_{n,m}\right)\right) = \max_{\beta=1,\cdots,N} \left|\lambda_\beta\left(\left(a_{n,m}\right)\right)\right|$$
$$\lambda_\beta\left(\left(a_{n,m}\right)\right) = \beta\text{th eigenvalue of } \left(a_{n,m}\right) \tag{B.14}$$

A general result has for associated matrix norms [14]

$$\left\|\left(a_{n,m}\right)\right\| \le \rho\left(\left(a_{n,m}\right)\right) \tag{B.15}$$

This is readily seen from substituting (x_n) as any eigenvector $(x_n)_\beta$ of $(a_{n,m})$ into (B.3), noting that

$$\left(a_{n,m}\right)\cdot\left(x_n\right)_\beta = \lambda_\beta\left(x_n\right)_\beta \tag{B.16}$$

Note, however, that the spectral radius is not a norm for general complex or real N×N matrices (excluding (N = 1) as can be readily established by counter examples.

A commonly used vector norm is the p norm

$$\left\|\left(x_n\right)\right\|_p \equiv \left\{\sum_{n=1}^{N} \left|x_n\right|^n\right\}^{\frac{1}{n}} \tag{B.17}$$

with oft-used special cases

$$\left\|\left(x_n\right)\right\|_1 \equiv \sum_{n=1}^{N} \left|x_n\right|$$
$$\left\|\left(x_n\right)\right\|_2 \equiv \left\{\sum_{n=1}^{N} \left|x_n\right|^2\right\}^{\frac{1}{2}} \equiv \text{vector magnitude} \tag{B.18}$$
$$\left\|\left(x_n\right)\right\|_\infty \equiv \max_{1\le n\le N} \left|x_n\right|$$

The matrix norms associated with these are

$$\|(a_{n,m})\|_1 = \max_{1\le m\le N} \sum_{n=1}^{N} |a_{n,m}| \equiv \text{ maximum column magnitude sum}$$

$$\|(a_{n,m})\|_2 = \left[\rho\left((a_{n,m})^\dagger \cdot (a_{n,m})\right)\right]^{\frac{1}{2}} \tag{B.19}$$

$$\|(a_{n,m})\|_\infty = \max_{1\le n\le N} \sum_{m=1}^{N} |a_{n,m}| \equiv \text{ maximum row magnitude sum}$$

$$\dagger \equiv \text{ adjoint} \equiv T* \equiv \text{ transpose conjugate}$$

As a special case for Hermitian matrices we have

$$(a_{n,m})^\dagger \equiv (a_{n,m})$$
$$\|(a_{n,m})\|_2 = \rho((a_{n,m})) \tag{B.20}$$

Since real symmetric matrices are also Hermitian, this result applies here as well.

Consider a unitary matrix $(u_{n,m})$ defined by

$$(u_{n,m})^{-1} \equiv (u_{n,m})^\dagger \tag{B.21}$$

This is also referred to as length preserving since

$$\|(u_{n,m})\cdot(x_n)\|_2 = \left[(x_n)^* \cdot (u_{n,m})^\dagger \cdot (u_{n,m})\cdot(x_n)\right]^{\frac{1}{2}}$$
$$= \left[(x_n)^* \cdot (x_n)\right]^{\frac{1}{2}}$$
$$= \|(x_n)\|_2$$
$$\|(u_{n,m})\|_2 = 1 \tag{B.22}$$

For a general NxN matrix $(a_{n,m})$ one can take a special similarity transformation (a unitary transformation) of it and evaluate the 2-norm as

$$\left\|(u_{n,m})\cdot(a_{n,m})\cdot(u_{n,m})^{\dagger}\right\|_2$$

$$=\left[\rho\left(\left((u_{n,m})\cdot(a_{n,m})\cdot(u_{n,m})^{\dagger}\right)^{\dagger}\cdot\left((u_{n,m})\cdot(a_{n,m})\cdot(u_{n,m})^{\dagger}\right)\right)\right]^{\frac{1}{2}}$$

$$=\left[\rho\left((u_{n,m})\cdot(a_{n,m})^{\dagger}\cdot(u_{n,m})^{\dagger}\cdot(u_{n,m})\cdot(a_{n,m})\cdot(u_{n,m})^{\dagger}\right)\right]^{\frac{1}{2}}$$

$$=\left[\rho\left((u_{n,m})\cdot(a_{n,m})^{\dagger}\cdot(a_{n,m})\cdot(u_{n,m})^{\dagger}\right)\right]^{\frac{1}{2}} \quad \text{(B.23)}$$

Now $(a_{n,m})^{\dagger}\cdot(a_{n,m})$ is a Hermitian matrix, the eigenvalues of which are preserved in a similarity transformation. Hence we have

$$\left\|(u_{n,m})\cdot(a_{n,m})\cdot(u_{n,m})^{\dagger}\right\|_2=\left[\rho\left((a_{n,m})^{\dagger}\cdot(a_{n,m})\right)\right]^{\frac{1}{2}}$$

$$=\left\|(a_{n,m})\right\|_2 \quad \text{(B.24)}$$

So the 2-norm is preserved under unity transformation. A special case of a unitary transformation is an orthogonal transformation in which case $(u_{n,m})$ is real and transpose is the same as adjoint.

Appendix C. Use of the Sum Rule for Perturbations

The sum rule from (A.18) has

$$\left(A_{n,m}(z,z_0)\right) \equiv \prod_{z_0}^{z} e^{\left(a_{n,m}(z')\right)dz'}$$

$$\left(b_{n,m}^{(a)}(z,z_0)\right) \equiv \left(A_{n,m}(z,z_0)\right)^{-1} \cdot \left(b_{n,m}(z)\right) \cdot \left(A_{n,m}(z,z_0)\right)$$

$$\left(B_{n,m}(z,z_0)\right) \equiv \prod_{z_0}^{z} e^{\left(b_{n,m}^{(a)}(z',z_0)\right)dz'} \tag{C.1}$$

$$\prod_{z_0}^{z} e^{\left[\left(a_{n,m}(z')\right)+\left(b_{n,m}(z')\right)\right]dz'} = \left(A_{n,m}(z,z_0)\right) \cdot \left(B_{n,m}(z,z_0)\right)$$

Here the superscript a refers to the similarity transform with respect to the product integral of $\left(a_{n,m}(z)\right)$. Let us assume that $\left(b_{n,m}^{(a)}(z',z_0)\right)$ is, in some sense, small on the interval (z_0,z). Then from (A.7) let us write

$$\left(B_{n,m}(z,z_0)\right) = \prod_{z_0}^{z} e^{\left(b_{n,m}^{(a)}(z',z_0)\right)dz'}$$

$$= \left(1_{n,m}\right) + \int_{z_0}^{z} \left(b_{n,m}^{(a)}(z',z_0)\right)dz' + \left(r_{n,m}(z,z_0)\right) \tag{C.2}$$

where the remainder term $\left(r_{n,m}(z,z_0)\right)$ represents the terms in the series for $\ell \geq 2$.

Consider the relative magnitudes of these terms. Using the sum bound for matrix norms we have (Appendix B)

$$\left\| \int_{z_0}^{z} \left(b_{n,m}^{(a)}(z',z_0)\right)dz' \right\| \leq \left| \int_{z_0}^{z} \left\| \left(b_{n,m}^{(a)}(z',z_0)\right) \right\| dz' \right| \tag{C.3}$$

Defining

$$\chi(z') \equiv \left\| \left(b_{n,m}^{(a)}(z',z_0)\right) \right\|$$

$$\chi_{\max} \equiv \sup_{z_0 \leq z' \leq z} \chi(z') \tag{C.4}$$

we then have for the second term in the series

$$\left\| \int_{z_0}^{z} \left(b_{n,m}^{(a)}(z', z_0) \right) dz' \right\| \le \chi_{\max} |z - z_0| \tag{C.5}$$

which can be compared to the leading term (the identity) which has

$$\left\| \left(1_{n,m} \right) \right\| = 1 \tag{C.6}$$

for all associated matrix norms (and spectral radius as well). This then gives a sense of how small $\left(b_{n,m}^{(a)}(z', z_0) \right)$ should be, i.e.,

$$\chi_{\max} |z - z_0| << 1 \tag{C.7}$$

for the second term in the series to represent a correction term or perturbation. Consider the remainder term in the same norm sense. From (B.13) we have

$$\begin{aligned} \left\| \left(r_{n,m}(z, z_0) \right) \right\| &\le e^{\chi_{\max}|z - z_0|} - 1 - \chi_{\max}|z - z_0| \\ &= \chi_{\max}^2 \frac{|z - z_0|^2}{2} + \cdots \\ &= O\left(\left(\chi_{\max}|z - z_0| \right)^2 \right) \text{ as } \chi_{\max}|z - z_0| \to 0 \end{aligned} \tag{C.8}$$

Our perturbation formula is then, to first order,

$$\begin{aligned} \left(B_{n,m}(z, z_0) \right) = \left(1_{n,m} \right) + \int_{z_0}^{z} b_{n,m}^{(a)}(z', z_0) dz' \\ + O\left(\left(\chi_{\max}|z - z_0| \right)^2 \right) \text{ as } \chi_{\max}|z - z_0| \to 0 \end{aligned} \tag{C.9}$$

where the order symbol can be taken in a bound sense from (C.8), this bound depending on which matrix norm is being used. Combining (C.9) with (C.1) gives

$$\begin{aligned} \prod_{z_0}^{z} e^{\left[\left(a_{n,m}(z') \right) + \left(b_{n,m}(z') \right) \right] dz'} = \left(A_{n,m}(z, z_0) \right) \cdot \left[\left(1_{n,m} \right) + \int_{z_0}^{z} \left(b_{n,m}^{(a)}(z', z_0) \right) dz' \right. \\ \left. + O\left(\left(\chi_{\max}|z - z_0| \right)^2 \right) \text{ as } \chi_{\max}|z - z_0| \to 0 \right. \end{aligned} \tag{C.10}$$

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