Interaction Notes

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Extension of the BLT Equation into Time Domain

Carl E. Baum

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Directed Energy Directorate

Abstract

Various forms of the BLT equation have been developed for modelling electromagnetic interaction with complex electronic systems. This paper develops a form, BLT3, which is appropriate for efficient early-time representation based on a geometric-series expansion of a supermatrix inverse. The delays on the uniform-multiconductor-transmission-line tubes play a key role. For late-time purposes, an SEM representation is more appropriate.

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1. Introduction

For the analysis of multiconductor-transmission-line (MTL) networks there are various forms of the BLT equation that have been introduced. The original form (BLT1) has been formulated in terms of uniform MTLs for the tubes connecting the junctions characterized by scattering matrices [1]. This is the basis for the CRIPTE computer code for calculating the electromagnetic response of complex systems [6].

The tubes can be shrunk to zero length by modelling them as junctions. This gives another form called the BLT2 equation [2, 4, 7]. In this form, tubes can be replaced by more general structures which are characterized by scattering matrices, as are the original junctions. Yet another form is the NBLT equation which allows the inclusion of NMTLs (nonuniform MTLs) in the tubes with propagation on such tubes representable in terms of product integrals [7].

In the present paper the BLT1 equation is rewritten in terms of a series, giving what might be called the BLT3 equation. For lossless MTLs this gives a time-domain form which is convenient for early-time computations. For late times an SEM (singularity-expansion-method) representation is more appropriate.

2. BLT1 Equation

The BLT1 equation has the form

$$
\begin{aligned}
&[((1_{n,m})_{u,\mathrm{v}}) - ((\tilde{S}_{n,m}(s))_{u,\mathrm{v}}) \odot ((\tilde{\Gamma}_{n,m}(s))_{u,\mathrm{v}})] \odot ((\tilde{V}_n(s))_u) \\
&\qquad = ((\tilde{S}_{n,m}(s))_{u,\mathrm{v}}) \odot ((\tilde{V}_n^{(s)}(s))_u)
\end{aligned}
\tag{2.1}
$$

This is written on a graph, the interaction sequence diagram, such as the example in Fig. 2.1. In this diagram with four junctions J_v there are six tubes, but we are more concerned with the wave indices w_u where u ranges from one to twelve, as does its dummy-variable partner v in (2.1). Here u and v are topological indices ranging over the interaction sequence diagram (or the topology). Now each tube has N_u conductors (plus reference) with

$$N_u = N_\mathrm{v} \quad \text{for } u \text{ and v on the same tube} \tag{2.2}$$

One can think of the u, v indices as indicating waves with propagation directions as shown, except that for each u there are actually N_u modes (waves) based on an appropriate matrix diagonalization.

One can define an $N_w \times N_w$ wave interconnection matrix $(W_{u,\mathrm{v}})$ with

$$W_{u,\mathrm{v}} = \begin{cases} 1 & \text{if wave } w_\mathrm{v} \text{ scatters into wave } w_u \text{ at a junction} \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

The importance of this matrix is that it is the structure of the scattering supermatrix $((\tilde{S}_{n,m}(s))_{u,\mathrm{v}})$ which is block sparse with zero blocks where $w_{u,\mathrm{v}} = 0$. This is an important property in efficiently solving the BLT1 equation [8]. Note now that the $(\tilde{S}_{n,m}(s))_{u,\mathrm{v}}$ blocks are the scattering matrices for the vth wave (W_v variables) into the uth wave (N_u variables) thereby identifying it as an $N_u \times N_\mathrm{v}$ matrix.

There is a propagation matrix for the uth wave as

$$
\begin{aligned}
&(\tilde{\gamma}_{c_{n,m}}(s))_u^2 = (\tilde{Z}'_{n,m}(s))_u \cdot (\tilde{Y}'_{n,m}(s))_u \\
&(\tilde{Z}'_{n,m}(s))_u \equiv \text{impedance-per-unit-length matrix (longitudinal) for the } u\text{th wave} \\
&(\tilde{Y}'_{n,m}(s))_u \equiv \text{admittance-per-unit-length matrix (transverse) for the uth wave}
\end{aligned}
\tag{2.4}
$$

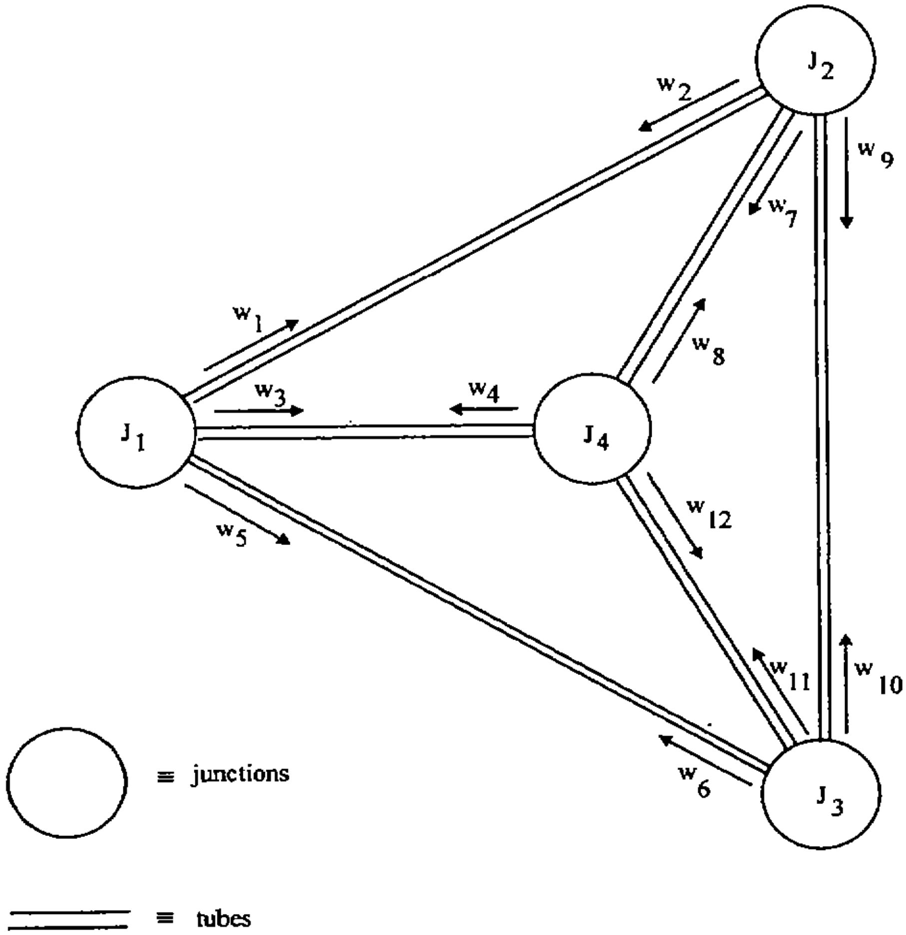

Fig. 2.1. Example Interaction Sequence Diagram

Diagonalizing the matrix product we find

$$(\tilde{\gamma}_{c_{n,m}}(s))_u = \sum_{p=1}^{N_u} \tilde{\gamma}_{p;u}(s)\,(\tilde{v}_{c_n}(s))_{p;u}(\tilde{i}_{c_n}(s))_{p;u}$$

$$(\tilde{v}_{c_n}(s))_{p_1;u} \cdot (\tilde{i}_{c_n}(s))_{p_2;u} = 1_{p_1,p_2} \quad \text{(biorthonormal)} \tag{2.5}$$

where the eigenvalues $\tilde{\gamma}_{p;u}(s)$ are computed via the positive real (p.r.) square root. These results can also be used to calculate

$$\begin{aligned}
(\tilde{Z}_{c_{n,m}}(s))_u &= (\tilde{\gamma}_{c_{n,m}}(s))_u \cdot (\tilde{Y}'_{n,m}(s))_u^{-1} = (\tilde{\gamma}_{c_{n,m}}(s))^{-1} \cdot (\tilde{Z}'_{n,m}(s)) \\
&= (\tilde{Z}_{c_{n,m}}(s))^T = (\tilde{Y}_{c_{n,m}}(s))^{-1} \\
&= \sum_{p=1}^{N_u} (\tilde{v}_{c_n}(s))_{p;u}(\tilde{v}_{c_n}(s))_{p;u} \\
&\equiv \text{characteristic-impedance matrix for the } u\text{th wave} \\
(\tilde{Y}_{c_{n,m}}(s))_u &= (\tilde{Y}'_{n,m}(s))_u \cdot (\tilde{\gamma}_{c_{n,m}}(s))_u^{-1} = (\tilde{Z}'_{n,m}(s)) \cdot (\tilde{\gamma}_{c_{n,m}}(s))_u \\
&= (\tilde{Y}_{c_{n,m}}(s))^T \\
&= \sum_{p=1}^{N_u} (\tilde{i}_{c_n}(s))_{p;u}(\tilde{i}_{c_n}(s))_{p;u} \\
&\equiv \text{characteristic-admittance matrix for the uth wave} \\
(\tilde{v}_{c_n})_{p;u} &= (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{i}_n(s))_{p;u} \quad \text{(normalization between right and left eigenmodes)}
\end{aligned} \tag{2.6}$$

In turn, this propagation matrix is used to calculate another term in (2.1), the supermatrix (block diagonal)

$$((\tilde{\Gamma}_{n,m}(s))_{u,v}) = \bigoplus_{u=1}^{N_w} e^{-(\tilde{\gamma}_{c_{n,m}}(s))_u\, L_u} \equiv \text{delay supermatrix}$$

$$(\tilde{\Gamma}_{n,m}(s))_{u,v} = (0_{n,m}) \text{ for } u \neq v$$

$$L_u \equiv \text{length of tube on which the } u\text{th wave propagates} \tag{2.7}$$

where the repeated direct sum places the square blocks progressively down the diagonal of the supermatrix. Each block can furthermore be diagonalized as

$$e^{-(\tilde{\gamma}_{c_{n,m}}(s))_u L_u} = \sum_{p=1}^{N_u} e^{-\tilde{\gamma}_{p;u}(s)\, L_u} (\tilde{v}_{c_n}(s))_{p;u}(\tilde{i}_{c_n}(s))_{p;u} \tag{2.8}$$

We have the combined voltage vector for the *u*th wave as

$$
\begin{aligned}
&(\tilde{V}_n(z_u,s))_u = (\tilde{V}_n(z_u,s)) + (\tilde{Z}_{c_{n,m}}(s))_u \cdot (\tilde{I}_n(z_u,s)) \\
&z_u \equiv \text{distance travelled by the uth wave along its tube from 0 toward } L_u \\
&0 \le z_u \le L_u
\end{aligned}
\tag{2.9}
$$

The subscript u on the combined voltage is important in that it identifies the positive direction along the tube which in turn gives the convention for the positive direction for the N_u currents $\tilde{I}_n(z_u,s)$. For the other wave on the same tube, say the vth, the direction is opposite. So by sum and difference the voltages and currents can be recovered from the combined voltage vector. In (2.1) the combined voltage supervector is the collection of all the outgoing waves as they leave the junctions, i.e.,

$$
((\tilde{V}_n(s))_u) = ((\tilde{V}_n(0,s))_u)
\tag{2.10}
$$

There are, in general, per-unit-length sources along each tube, which for the *u*th wave take the combined form

$$
(\tilde{V}_n^{(s)'}(z_u,s))_u = (\tilde{V}_n^{(s)'}(z_u,s)) + (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n^{(s)'}(z_u,s))
\tag{2.11}
$$

analogous to (2.9). The *u*th wave has the solution

$$
\begin{aligned}
(\tilde{V}_n(z_u,s))_u &= e^{-(\tilde{\gamma}_{c_{n,m}}(s))_u z_u} \cdot (\tilde{V}_n(0,s))_u \\
&\quad += \int_0^{z_u} e^{-(\gamma_{c_{n,m}}(s))[z_u - z_u']} \cdot (\tilde{V}_n^{(s)'}(z_u',s))_u \; dz_u' \\
&= \sum_{p=1}^{N_u} (\tilde{\mathrm{v}}_{c_n}(s))_{p;u} (\tilde{i}_{c_n}(s))_{p;u} \cdot \Big[e^{-\tilde{\gamma}_{p;u}(s)\, z_u} (\tilde{V}_n(0,s))_u \\
&\quad + \int_0^{z_u} e^{-\tilde{\gamma}_{p,u}(s)[z_u - z_u']} (\tilde{V}_n^{(s)'}(z_u',s))_u \, dz' \Big]
\end{aligned}
\tag{2.12}
$$

This allows for the construction of the voltages and currents at arbitrary positions on the tubes. For use in (2.1) this gives the source supervector as

$$
\begin{aligned}
((\tilde{V}_n^{(s)}(s))_u) &= \left(\int_0^{L_u} e^{-(\tilde{\gamma}_{c_{n,m}}(s))[L_u - z_u']} \cdot (\tilde{V}_n^{(s)}(z_u', s))_u \, dz_u' \right) \\
&= \left(\sum_{p=1}^{N_u} (\tilde{v}_{c_n}(s))_{p;u} (\tilde{i}_{c_n}(s))_{p;u} \cdot \int_0^{L_u} e^{-\tilde{\gamma}_{p;u}(s)[L_u - z_u']} \cdot (\tilde{V}^{(s)'}(z_u', s))_u \, dz' \right)
\end{aligned}
\tag{2.13}
$$

Defining now the *interaction supermatrix* as

$$
\begin{aligned}
((\tilde{I}_{n,m}(s))_{u,v}) &\equiv ((1_{n,m})_{u,v}) - ((\tilde{S}_{n,m}(s))_{u,v}) \odot ((\tilde{\Gamma}_{n,m}(s))_{u,v}) \\
((1_{n,m})_{u,v}) &\equiv \text{identity supermatrix} \\
1_{n,m;u,v} &= \begin{cases} 1 & \text{for } n = m \text{ and } u = v \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
\tag{2.14}
$$

we have the formal solution of the BLT1 equation as

$$
((\tilde{V}_n(s))_u) = ((\tilde{I}_{n,m}(s))_{u,v})^{-1} \odot ((\tilde{S}_{n,m}(s))_{u,v}) \odot ((\tilde{V}_n^{(s)}(s))_u)
\tag{2.15}
$$

We will later explore some of the properties of the inverse of the interaction supermatrix. While (2.15) is expressed in complex frequency domain it can, of course be represented in time domain via

$$
\begin{aligned}
((V_n(t))_u) &= [((K_{n,m}(t))_{u,v}) \circ] \odot [((S_{n,m}(t))_{u,v}) \circ] \odot ((V_n^{(s)}(t))_u) \\
\circ &\equiv \text{convolution with respect to time t} \\
\sim &\equiv \text{two-sided Laplace transform over time} \\
s &\equiv \Omega + j\omega \equiv \text{Laplace-transform variable or complex frequency} \\
((\tilde{K}_{n,m}(s))_{u,v}) &\equiv ((\tilde{I}_{n,m}(s))_{u,v})^{-1}
\end{aligned}
\tag{2.16}
$$

noting that the supermatrices are also convolution operators in time domain.

Note at this point that the block-diagonal form of the delay supermatrix permits combining two of the terms in the interaction supermatrix as

$$
((\tilde{S}_{n,m}(s))_{u,v}) \odot ((\tilde{\Gamma}_{n,m}(s))_{u,v}) = \left(\sum_{i=1}^{N_w} (\tilde{S}_{n,m}(s)_{u,u}) \cdot (\tilde{\Gamma}_{n,m}(s))_{u',v} \right)
$$

$$
\begin{aligned}
&= ((\widetilde{\Sigma}_{n,m}(s))_{u,\mathrm{v}}) \\
(\widetilde{\Sigma}_{n,m}(s))_{u,\mathrm{v}} &= (\widetilde{S}_{n,m}(s))_{u,\mathrm{v}} \cdot (\widetilde{\Gamma}_{n,m}(s))_{\mathrm{v},\mathrm{v}}
\end{aligned}
\tag{2.17}
$$

So this supermatrix has the same block sparse form as the scattering supermatirx.

So much for preliminaries. Additional details are found in [1].

3. BLT3 Equation

Now expand the inverse of the interaction supermatrix in a geometric series as

$$\begin{aligned}((\tilde{K}_{n,m}(s))_{u,v}) &= ((\tilde{I}_{n,m}(s))_{u,v})^{-1} \\ &= [((1_{n,m})_{u,v}) - ((\tilde{\Sigma}_{n,m}(s))_{u,v})]^{-1} \\ &= ((1_{n,m})_{u,v}) + \sum_{\ell=1}^{\infty} ((\tilde{\Sigma}_{n,m}(s))_{u,v})^{\ell}\end{aligned} \tag{3.1}$$

For this to make sense $((\tilde{\Sigma}_{n,m}(s))_{u,v})$ must be small in some sense of an appropriate norm small compared to 1. One can verify the above formula by multiplying (dot-product sense) by $((\tilde{I}_{n,m}(s))_{u,v})$ and rearranging terms to cancel all but the remaining identity.

If we truncate the series in (3.1) with the ℓ_0th term, we still have the exact result

$$\begin{aligned}((\tilde{K}_{n,m}(s))_{u,v}) &= ((1_{n,m})_{u,v}) + \sum_{\ell=1}^{\ell_0}((\tilde{\Sigma}_{n,m}(s))_{u,v})^{\ell} \\ &\quad + ((\tilde{\Sigma}_{n,m}(s))_{u,v})^{\ell_0+1} \odot \left[((1_{n,m})_{u,v}) + \sum_{\ell=1}^{\infty}((\tilde{\Sigma}_{n,m}(s))_{u,v})^{\ell}\right] \\ &= ((1_{n,m})_{u,v}) + \sum_{\ell=1}^{\ell_0}((\tilde{\Sigma}_{n,m}(s))_{u,v})^{\ell} \\ &\quad + ((\tilde{\Sigma}_{n,m}(s))_{u,v})^{\ell_0+1} \odot ((\tilde{K}_{n,m}(s))_{u,v})\end{aligned} \tag{3.2}$$

which can be verified by dot multiplying and collecting terms. Rearranging we have an alternate form

$$\begin{aligned}((\tilde{K}_{n,m}(s))_{u,v}) &= [((1_{n,m})_{u,v}) - ((\tilde{\Sigma}_{n,m}(s))_{u,v})^{\ell_0+1}]^{-1} \\ &\quad \odot \left[((1_{n,m})_{u,v}) + \sum_{\ell=1}^{\ell_0} ((\tilde{\Sigma}_{n,m}(s))_{u,v})^{\ell}\right]\end{aligned} \tag{3.3}$$

Here we have examples of the usual geometric-series identities extended to matrices.

To understand the convergence better, consider the propagation matrix. From (2.4) and (2.5) we have that for a passive tube (perhaps loss, but certainly no gain), the eigenvalues $\tilde{\gamma}_{p;u}(s)$ must be positive (or more precisely non-negative) real functions, i.e., in the right half plane (RHP)

$$\mathrm{Re}[\,\tilde{\gamma}_{p;u}(s)\,] \geq 0 \quad \text{for } s \text{ in RHP} \tag{3.4}$$

More importantly as $s \to \infty$ in the RHP the speed of propagation is limited by the speed of light c (causality). For our purposes we can limit this even more by some v_u of the fastest propagating mode in the uth wave (given some model for $(\tilde{Z}'_{n,m}(s))_u$ and $(\tilde{Y}'_{n,m}(s))_u$ using idealized dielectrics, conductors, etc.). We then have

$$\begin{aligned} \mathrm{Re}[\tilde{\gamma}_{p;u}(s)] &\geq \frac{\Omega}{v_u} \quad \text{as } s \to \infty \text{ in RHP} \\ v_u &\leq c = [\mu_0 \varepsilon_0]^{-\frac{1}{2}} \end{aligned} \tag{3.5}$$

In time domain this means that the fastest signal to reach the end of the tube takes a time

$$t_u = \frac{L_u}{v_u} \tag{3.6}$$

(There may also be some attenuation.). This then says that for the associated delay matrix

$$\begin{aligned} (\tilde{\Gamma}_{n,m}(s))_{u,u} &= e^{-(\tilde{\gamma}c_{n,m}(s))_u \, L_u} = \sum_{p=1}^{N_u} e^{-\tilde{\gamma}_{p;u}(s)\, L_u} (\tilde{v}_{c_n}(s))_{p;u} (\tilde{\iota}_{c_n}(s))_{p;u} \\ &= O_e(-st_u) \quad \text{as } s \to \infty \text{ in RHP} \end{aligned} \tag{3.7}$$

where the exponential order O_e (or O_{e+}) gives a bound as [5]

$$\begin{aligned} O_e(\zeta(s)) &= e^{\chi|s|} O(e^{\zeta(s)}) \quad \text{as } s \to s_0 \\ &\text{for all } \chi > 0 \end{aligned} \tag{3.8}$$

Again since we are considering matrices with the bound taken in a norm sense.

Consider now the scattering matrix blocks. Assuming that the junctions are passive the associated scattering-matrix blocks are bounded (in 2-norm sense) in the RHP [3]. Such matrices are also called bounded-real scattering matrices [10]. Then we have

$$
\begin{aligned}
(\tilde{S}_{n,m}(s))_{u,v} &= O(1) \quad \text{for } s \text{ in RHP} \\
&= O_e(1) \quad \text{for } s \text{ in RHP}
\end{aligned}
\tag{3.9}
$$

If there are delays in propagating through a junction an even tighter bound is obtained in the RHP, but the above will do for present purposes. Combining (3.9) with (3.7) now gives

$$
\begin{aligned}
(\tilde{\Sigma}_{n,m}(s))_{u,v} &= (\tilde{S}_{n,m}(s))_{u,v} \cdot (\tilde{\Gamma}_{n,m}(s))_{v,v} \\
&= O_e(-st_v) \quad \text{as } s \to \infty \text{ in RHP} \\
((\tilde{\Sigma}_{n,m}(s))_{u,v}) &= O_e(-st_0) \quad \text{as } s \to \infty \text{ in RHP} \\
t_0 &\equiv \min_{v} t_v \quad \text{for } v = 1, 2, \cdots, N_w
\end{aligned}
\tag{3.10}
$$

In the series in (3.2) we then have

$$
\begin{aligned}
((\tilde{\Sigma}_{n,m}(s))_{u,v})^{\ell} &= O_e(-\ell s t_0) \\
((\tilde{K}_{n,m}(s))_{u,v}) &= ((1_{n,m})_{u,v}) + \sum_{\ell=1}^{\ell_0} ((\tilde{\Sigma}_{n,m}(s))_{u,v})^{\ell} \\
&\quad + O_e(-[\ell_0 + 1] s t_0) \quad \text{as } s \to 0 \text{ in RHP}
\end{aligned}
\tag{3.11}
$$

Writing the solution to (2.1) in the form

$$
\boxed{((\tilde{V}_n(s))_u) = \left[((1_{n,m})_{u,v}) + \sum_{\ell=1}^{\infty} ((\tilde{\Sigma}_{n,m}(s))_{u,v})^{\ell}\right] \odot ((\tilde{S}_{n,m}(s))_{u,v}) \odot ((\tilde{V}_n^{(s)}(s))_u)}
\tag{3.12}
$$

we have what can be called the BLT3 equation. The infinite series can be truncated with remainder (error) as given in (3.11).

An important aspect of this result is the form it takes in time domain, i.e.,

$$
((V_n(t))_u) = \left[((1_{n,m})_{u,v})\delta(t)\circ + \sum_{\ell=1}^{\infty} [((\Sigma_{n,m}(t))_{u,v})\circ]^{\ell}\right] \odot [((S_{n,m}(t))_{u,v})\circ] \odot ((V_n^{(s)}(t))_u)
\tag{3.13}
$$

Noting that the exponential order in (3.11) represents time delay we have

$$((K_{n,m}(t))_{u,\text{v}})\circ = ((1_{n,m})_{u,\text{v}})\,\delta(t)\circ + \sum_{\ell=1}^{\ell_0}[((\Sigma_{n,m}(t))_{u,\text{v}})\circ] \quad \text{for } t < [\ell_0 + 1]t_0 \tag{3.14}$$

So in time domain the truncated series is exact up to some time dependent on the number of terms one takes. *The BL3 equation is then appropriate for early-time results.* One also needs to know the sources $((V_n(t))_u)$ from their beginning (which we can take as $t = 0$) out to the same time. Note that $[\ell_0 + 1]t_0$ is a bound, and depending on the location of sources and observer in the network the time of validity can be somewhat longer.

Special cases can simplify the time-domain results somewhat. If the uniform MTLs are lossless and dispersionless then we have in (2.4) and (2.5)

$$(\tilde{Z}'_{n,m}(s))_u = s\,(L'_{n,m})_u \quad , \quad (\tilde{Y}'_{n,m}(s)) = s(C'_{n,m})$$

$(L'_{n,m})_u \equiv$ inductance-per-unit-lwngth matrix for the uth wave

$(C'_{n,m})_u \equiv$ capacitance-per-unit-length matrix for the uth wave

$$\begin{aligned}
(\tilde{\gamma}_{c_{n,m}}(s))_u &= s\,[(L'_{n,m})_u \cdot (C'_{n,m})_u]^{\frac{1}{2}} \\
&= \sum_{p=1}^{N_u} \tilde{\gamma}_{p;u}(s)\ (\text{v}_{c_n})_{p;u}(i_{c_n})_{p;u}
\end{aligned} \tag{3.15}$$

$$\tilde{\gamma}_{p;u}(s) = \frac{s}{\text{v}_{p;u}} \quad , \quad 0 < \text{v}_{p;u} \le c$$

$\text{v}_{p;u} \equiv$ speed (real) of the pth mode (wave) in the uth wave

$$\text{v}_u \equiv \max_p \text{v}_{p;u} \text{ for } p = 1, 2, \cdots, N_u$$

Various terms (including the characteristic-impedance matrix) are now frequency independent. The delay supermatrix now becomes

$$
\begin{aligned}
((\Gamma_{n,m}(t))_{u,\mathrm{v}}) &= \bigoplus_{u=1}^{N_w}\left[\sum_{p=1}^{N_u}(\mathrm{v}_{c_n})_{p;u}(i_{c_n})_{p;u}\delta(t-t_{p;u})\right] \\
t_{p;u} &= \frac{L_u}{\mathrm{v}_{p;u}} \\
t_u &= \frac{L_u}{\mathrm{v}_u} = \min_{p}\ t_{p;u} \quad \text{for } p = 1, 2, \cdots, N_u
\end{aligned}
\tag{3.16}
$$

so that each of the N_u modes in the uth wave gives a simple delay.

Another simplification occurs in the cases of scattering matrices of ideal junctions [9]. This corresponds to simple connections of the MTL conductors through the junction to each other as short or open circuits implying no loss and no delay. For such a junction we have

$$
\begin{aligned}
(\tilde{S}_{n,m}(s))_{u,\mathrm{v}} &= (S_{n,m})_{u,\mathrm{v}} \quad \text{(constant)} \\
(S_{n,m}(t))_{u,\mathrm{v}} &= (\tilde{S}_{n,m})_{u,\mathrm{v}}\delta(t)
\end{aligned}
\tag{3.17}
$$

One can note that this property also holds if the junction includes resistors (frequency independent).

Combining these two results gives

$$
(\Sigma_{n,m}(t))_{u,\mathrm{v}} = (S_{n,m})_{u,\mathrm{v}} \cdot \left[\sum_{p=1}^{N_u}(\mathrm{v}_{c_n})_{p;\mathrm{v}}(i_{c_n})_{p;\mathrm{v}}\delta(t-t_{p;\mathrm{v}})\right]
\tag{3.18}
$$

This can be in turn substituted in (3.13) with various of the convolutions assuming the form of the addition of delays.

4. Late-Time Behavior

For late-time behavior it is efficient to utilize the system natural frequencies s_α (resonances) described by

$$\det(((\tilde{I}_{n,m}(s_\alpha))_{u,v})) = 0 \tag{4.1}$$

at which frequencies the system can have a nonzero response $((\tilde{V}_n(s_\alpha))_u)$ with zero forcing function $((\tilde{S}_{n,m}(s_\alpha))_{u,v}) \odot ((\tilde{V}_n^{(s)}(s_\alpha))_{u,v})$. The interaction matrix as in (2.14) corresponds to the BLT I form, but one can also use other forms as in [7] (along with other forms of the forcing function).

Define right and left natural-mode vectors via

$$\begin{aligned} &((\tilde{I}_{n,m}(s_\alpha))_{u,v}) \odot ((r_n)_u)_\alpha = ((0_{n,m})_{u,v}) \\ &((\ell_n)_u)_\alpha \odot ((\tilde{I}_{n,m}(s_\alpha))_{u,v}) = ((0_{n,m})_{u,v}) \\ &((r_n)_u)_\alpha \odot ((\ell_n)_u)_\alpha = 1 \text{ (optional normalization)} \end{aligned} \tag{4.2}$$

Following the general development for poles in the singularity expansion method (SEM) [11], we have

$$\begin{aligned} &((\tilde{K}_{n,m}(s))_{u,v}) = ((\tilde{I}_{n,m}(s))_{u,v})^{-1} \\ &= \sum_\alpha W_\alpha ((r_n)_u)_\alpha ((\ell_n)_u)_\alpha [s - s_\alpha]^{-1} \\ &+ \text{possible entire function} \end{aligned} \tag{4.3}$$

$$W_\alpha^{-1} = ((\ell_n)_u)_\alpha \odot \frac{d}{ds} ((\tilde{I}_{n,m}(s))_{u,v}) \Big|_{s=s_\alpha} \odot ((r_n)_u)_\alpha$$

where we have assumed first-order poles. Higher-order poles, when present, can be included using results in [12].

In time domain (4.3) becomes

$$\begin{aligned} ((K_{n,m}(t))_{u,v}) &= \sum_\alpha W_\alpha ((r_n)_u)_\alpha ((\ell_n)_u)_\alpha \, e^{s_\alpha t} u(t) \\ &+ \text{possible entire function (temporal form)} \end{aligned} \tag{4.4}$$

The early-time behavior, including any entire function, is discussed in Section 3. Convolving this termwise with the forcing function gives what is called the class-2 form of the poles as

$$
\begin{aligned}
&((V_n(t))_u) \\
&= \sum_{\alpha} W_\alpha ((r_n)_u)_\alpha ((\ell_n)_u)_\alpha \odot \; [((S_{n,m}(t))_{u,v})\circ] \odot \; [((V_n(t))_u) \circ e^{s_\alpha t} u(t)] \\
&\quad + \left[[\text{possible entire function}]\circ\right] \odot \; [((S_{n,m}(t))_{u,v})\circ] \odot \; ((V_n^{(s)}(t))_u)
\end{aligned}
\tag{4.5}
$$

After the time that all the convolution integrals have been completed (after all the sources have been turned off plus some additional time) one is left with a sum of damped sinusoids to characterize the network response. This is related to the class-1 form of the poles as

$$
\begin{aligned}
&((V_n(t))_u) \\
&= \sum_{\alpha} W_\alpha ((r_n)_u)_\alpha ((\ell_n)_u)_\alpha \odot \; ((\tilde{S}_{n,m}(s_\alpha))_{u,v}) \odot \; ((V_n(s_\alpha))_u) \, e^{s_\alpha t} u(t) \\
&\quad + \text{ singularity terms from forcing function} \\
&\quad + \text{ possible entire function}
\end{aligned}
\tag{4.6}
$$

This form is generally easier to compute than the class-2 form due to the elimination of various convolution integrals. However, as one backs up toward early time more terms are required than in the class- 2 case.

The properties of the two forms and associated entire functions have been studied at length for the case of electromagnetic scattering as described by appropriate integral equations [5]. This may give us some insight into the possible role of entire functions in the temporal forms of the various BLT equations.

5. Concluding Remarks

The BLT3 equation developed here by manipulation of the BLT1 equation into a series form is appropriate for early-time and associated high-frequency (RHP) computations. Perhaps computer codes like CRIPTE will someday implement this alternate form.

For late-time computations the SEM forms of appropriate BLT equations is more efficient. This leaves some theoretical questions concerning how early in time SEM can be appropriately used. This also involves the entire-function issues.

Perhaps yet more BLT forms will emerge with their own special applications.

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