Interaction Notes

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Symmetry Analysis of Targets Near an Earth/Air Interface

Carl E. Baum
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Abstract

In the presence of a lossy half space such as earth, the point symmetry groups (rotation and reflection) of a target reduce to the two-dimensional N-fold rotation axis (perpendicular to the ground surface) giving C_N symmetry, and the adjunction of axial symmetry planes giving C_{Na} symmetry. The eigenmodes and natural modes of such targets are organized by such symmetry giving a partially factored form for their description using a generalization of circulant-matrix properties. Linear combinations of the rotational modes can be used to construct modes which are symmetric or antisymmetric with respect to a given axial symmetry plane. These properties appear in the backscattered field and can be used to aid in target identification.

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Contents

1. Introduction

For the identification of buried targets symmetry plays an important role. In this paper we restrict ourselves to the case of electromagnetic singularity identification (EMSI) for which wavelengths in the external medium (soil) are of the order of the target dimensions. For typical mines and other unexploded ordnance (UXO) this implies 100s of MHz to several GHz for the pulse or set of frequencies used to illuminate the target from some sort of ground-penetrating radar (GPR) [8]. In contrast to the low-frequency magnetic-singularity identification (MSI) for which the magnetic-polarizability dyadic and its associated symmetries are important [9], in the case of EMSI much more geometric detail and associated symmetries are important [10].

An important result is the vampire signature of zero cross polarization (hv) in the usual h,v radar coordinates for the backscattered fields from a target with O_2 $(= C_{\infty a})$ symmetry as in Fig. 1.1. [11, 15]. Note here that the rotation axis is required to be perpendicular to the surface S_g of the local earth on which or in which the target resides. Small deviations from this symmetry split the 2-fold modal degeneracy applicable to most of the resonant modes [13, 14].

The local earth is even allowed to be layered in the sense that for the permittivity

$$\overleftrightarrow{\tilde{\varepsilon}}(z,s) = \tilde{\varepsilon}_z(z,s)\,\vec{1}_z\,\vec{1}_z + \tilde{\varepsilon}_t(z,s)\,\overleftrightarrow{1}_z$$

$$\overleftrightarrow{1}_z \equiv \overleftrightarrow{1} - \vec{1}_z\,\vec{1}_z = \vec{1}_x\,\vec{1}_x + \vec{1}_y\,\vec{1}_y \equiv \text{transverse (to } z\text{) dyadic}$$

$$\sim \; \equiv \; \text{2-sided Laplace transform (over time } t) \tag{1.1}$$

$$s \equiv \Omega + j\omega \equiv \text{Laplace-transform variable or complex frequency}$$

and similarly for the conductivity and permeability. This type of earth possesses T_2 symmetry (translation in two dimensions: x and y) as well as O_2 symmetry (rotation and reflection in two dimensions) [18]. Note that C_N denotes an N-fold symmetry axis (z axis in this case) and that an axial symmetry plane can give R_a symmetry (say R_x or R_y). Combining the two gives

$$C_{Na} = C_N \otimes R_a \tag{1.2}$$

as an N-fold rotation axis with N axial symmetry planes. Letting $N \to \infty$ we have

$$O_2^+ = SO(2) = C_\infty = \text{continuous rotation in two dimensions}$$

$$O_2 = O(2) = C_{\infty a} = \text{continuous rotation plus all reflections in two dimensions} \tag{1.3}$$

So our soil has $T_2 \otimes O_2$ symmetry into which the target is to be placed. (It can have other symmetries, such as dilation, but that need not concern us here.)

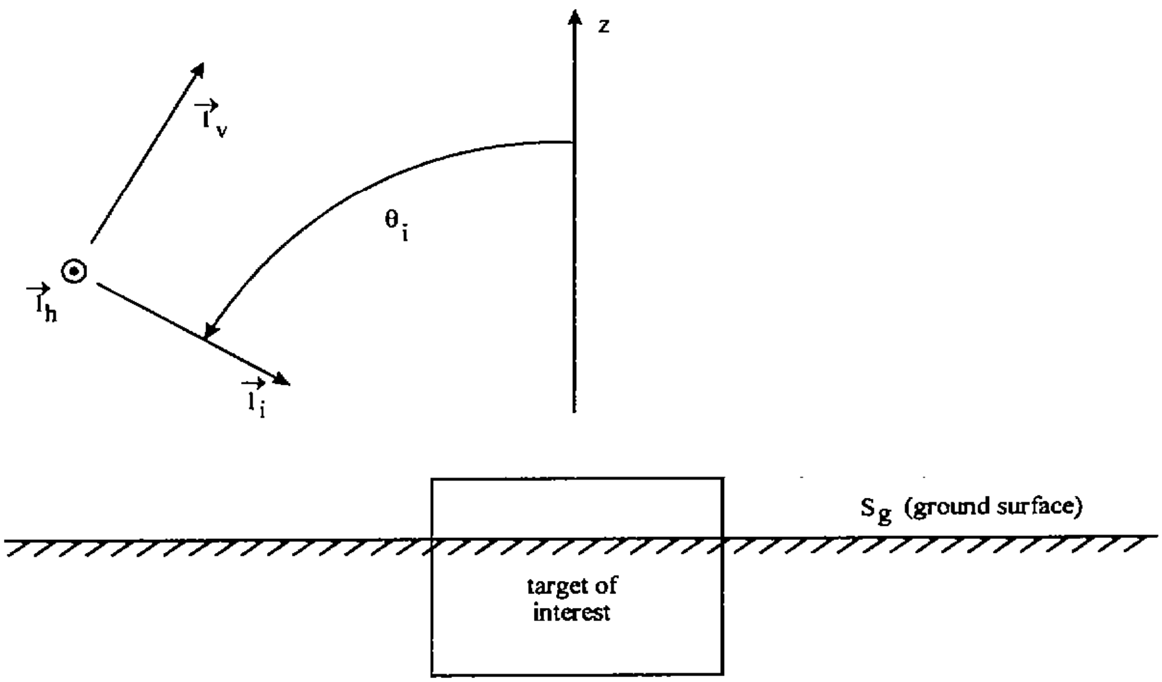


Figure 1.1. Target and Radar Coordinates.

2. Symmetries of Target in Presence of Earth/Air Interface

Since our target by hypothesis has finite linear dimensions, then translation does not apply to it, but point symmetries (rotations and reflections) potentially do. So the T_2 translation of the earth do not apply, but O_2 rotation and reflection may apply. Furthermore, any subgroup of O_2 is then of interest. The fundamental point is that *any target symmetry group which is a subgroup of the earth symmetry group is retained as a symmetry group of the target/earth combination.*

So now our attention is limited to C_N and C_{Na} symmetry. Note that the symmetry of a single axial symmetry plane, say R_a is equivalent to

$$C_{1a} = R_a \tag{2.1}$$

More axial symmetry planes (N) induce C_{Na} symmetry, e.g.,

$$C_{2a} = R_x \otimes R_y \tag{2.2}$$

where the coordinates have been rotated (about the z axis) to align the symmetry planes with the $x = 0$ and $y = 0$ planes.

3. Cross Polarization and Symmetry

3.1 One or more symmetry planes

As discussed in [2, 4, 5, 10, 11, 18, 19], the presence of a symmetry plane in a target separates the currents, fields, etc., into symmetric and antisymmetric parts. Then if the incident wave is symmetric, so are the induced currents and scattered fields, and similarly for the antisymmetric case. Referring to Fig. 3.1 if $\vec{1}_i$ (the direction of incidence) is parallel to a symmetry plane S_n then a horizontally polarized incident electric field (parallel to $\vec{1}_h$, perpendicular to $\vec{1}_z$) is antisymmetric, as is the scattered electric field on S_n. Similarly a vertically polarized electric field is the symmetric case (parallel to $\vec{1}_v$, perpendicular to $\vec{1}_h$) giving a symmetric scattered field on S_n.

For C_{Na} symmetry with N axial symmetry planes located on

$$\phi_n = \frac{2\pi n}{N} \ , \quad n = 1, 2, \cdots, N \tag{3.1}$$

the backscattering dyadic with $\phi_i = \phi_n$ or $\phi_n + \pi$ ($2N$ angles) has the property that

$$\tilde{\overleftrightarrow{\Lambda}}_b(\vec{1}_i, s) = \begin{pmatrix} \tilde{\overleftrightarrow{\Lambda}}_{b_{h,h}} & 0 \\ 0 & \tilde{\overleftrightarrow{\Lambda}}_{b_{v,v}}(\vec{1}_i, s) \end{pmatrix} \tag{3.2}$$

i.e., zero cross polarization referred to the h, v coordinates. Figure 3.1 shows the case of three symmetry planes. In general there are N such planes S_n for $n = 1, 2, \cdots, N$. In each of these $2N$ cases (3.2) holds. Note that with the azimuthal angle of incidence ϕ_i constrained as in (3.1) there is still continuous variation of the polar angle of incidence θ_i (as in Fig. 1.1) allowed over $0 \le \theta_i < \pi/2$ for plane waves incident above the ground surface S_g.

The zero crosspol now holds for discrete ϕ_i instead over all $0 \le \phi_i < 2\pi$, sort of a discrete version of the vampire signature. For large N all ϕ_i are near some ϕ_n, $\phi_n + \pi$, and one may expect the same results to approximately hold for all ϕ_i. In some sense then we can regard C_{Na} as approximating O_2 symmetry for large N.

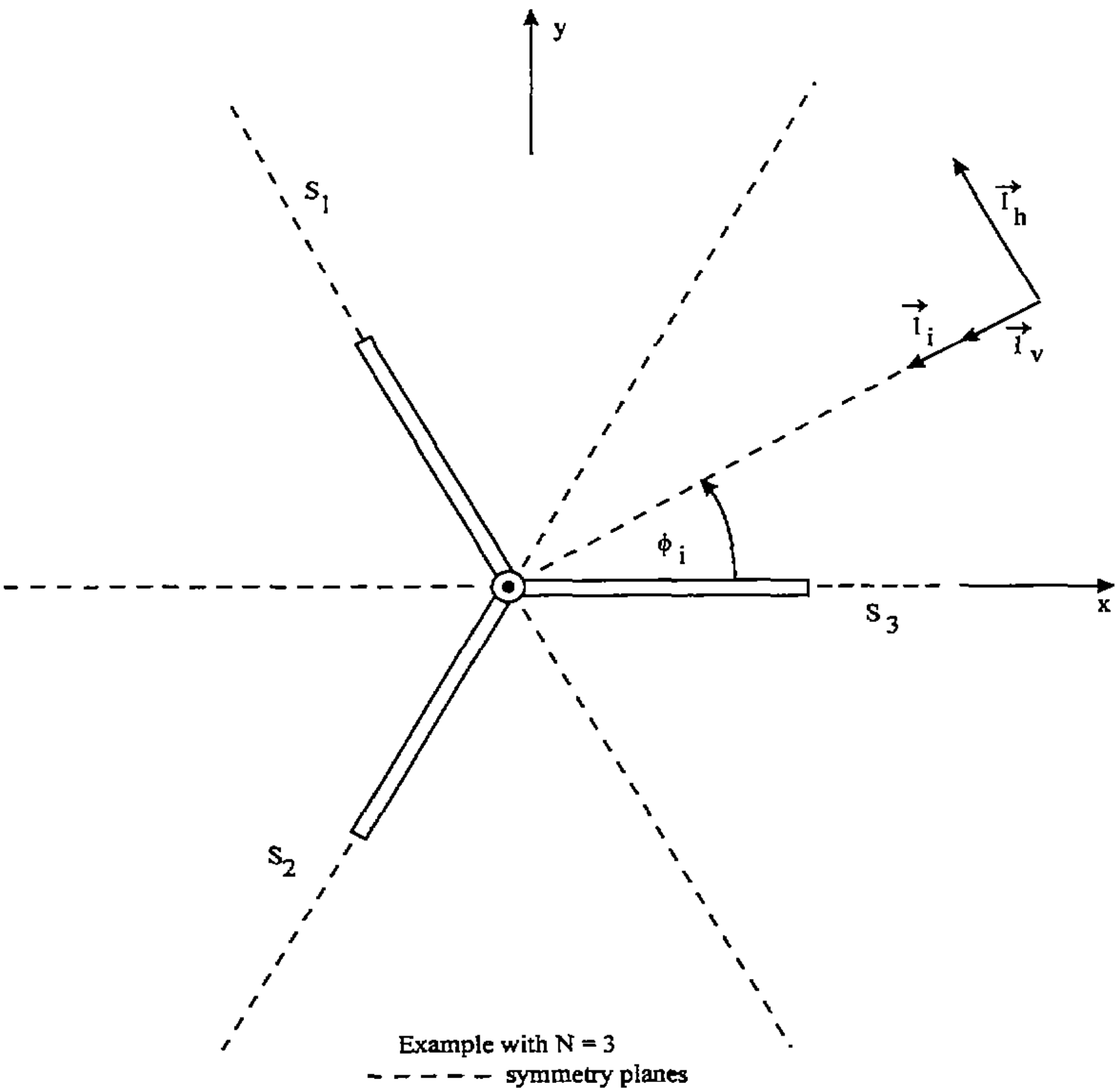


Fig. 3.1. Interaction of Incident Plane Wave With a target With Axial Symmetry Planes.

However, one needs to temper this in terms of wavelength. For wavelengths smaller than the physical spacing of these planes near the target extremities (farthest distance from the z axis) one may expect significant variation with ϕ_i. So the continuous (with ϕ_i) vampire signature should be better approximated at low frequencies than at high frequencies.

Looking at symmetry planes from the viewpoint of causality, consider an incident wavefront passing over the target. As the wave first reaches the target with S_n constrained to any of the ϕ_n or $\phi_n + \pi$, the illuminated portion of the target has a symmetry plane, one of the S_n parallel to $\vec{1}_i$. This property holds for every snapshot in time as the incident wave passes over the target.

3.2. C_N symmetry for $\theta_i \neq 0, \pi$

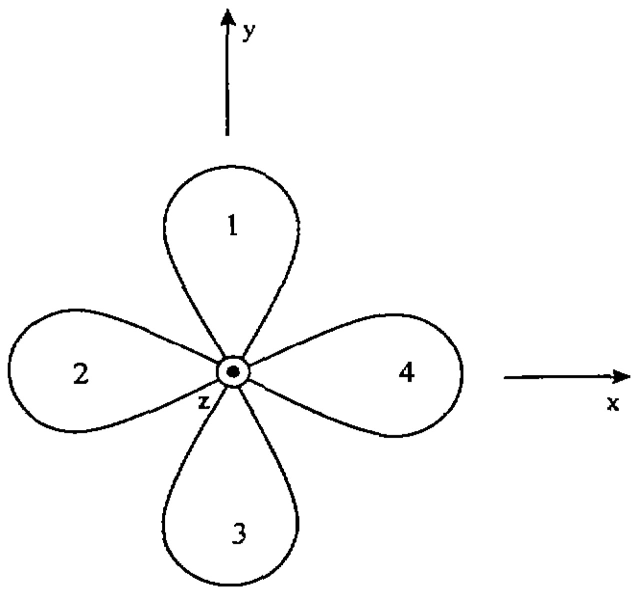
Now let there be no axial symmetry planes, leaving only discrete rotation symmetry C_N. Also restrict $\vec{1}_i$ so as not to be parallel to the z axis ($\theta_i \neq 0$ for waves incident above the ground and $\theta_i \neq \pi$ in more general cases).

From a causality viewpoint, as the incident wavefront first reaches the target the illuminated portion of the target does *not* have C_N symmetry. Only after the incident wave completely envelops the target is the C_N symmetry operative. This is in contradistinction to the case of the incident wave propagating parallel to a symmetry plane as in Section 3.1.

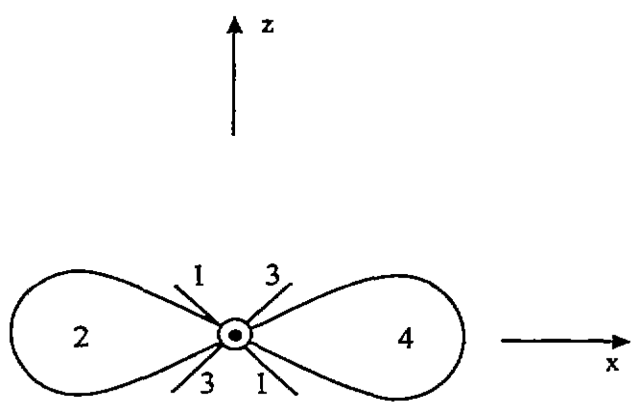
As an example consider the case of a C_4 propeller in Fig. 3.2. Let the incident wave be propagating in the $\vec{1}_y$ direction. The wave first reaches blade 3. As seen in Fig. 3.2B the blade is canted at some angle away from the x and z axes. As such the scattered fields on the $y\,z$ plane of incidence have a large crosspolarized component received back at the source on the $-y$ axis. Only later in time do blades 1, 2, and 4 enter into the scattering.

3.3 C_N symmetry for $\theta_i = 0, \pi$

If, however, $\vec{1}_i$ is made parallel to the rotation axis, a special thing happens. Now, as the incident wavefront on a plane of constant z propagates in the $\pm z$ direction, the illuminated portion of the target does have C_N symmetry (in contradistinction to Section 3.2) for every snapshot in time as the wave passes over the target. In this case we have for all time and frequency the special result for the backscattering dyadic



A. Top view



B. Side view

Fig. 3.2. C_4Symmetry: Four-Bladel Propeller.

$$\overset{\leftrightarrow}{\tilde{\Lambda}}_b(\vec{1}_i, s) = \tilde{\Lambda}_b(\vec{1}_i, s)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for} \quad N \geq 3 \tag{3.3}$$

i.e., no depolarization for any incident polarization. Note that for $\theta_i = 0, \pi$ we cannot uniquely specify $\vec{1}_h$ and $\vec{1}_v$ since both are parallel to the $z = 0$ plane (i.e., horizontal). Of course the transverse identity in (3.3) makes this unimportant.

Note that for $N = 2$ the result (3.3) does not hold as can be seen by considering a two-bladed propeller, and comparing the scattering for the incident electric field parallel versus perpendicular to the blades, especially for the case that a blade is approximately a quarter wave in length. Viewed another way an invariance in the scattering on successive rotation by $2\pi / N$ implies enough independent vectors (2) to span the two-dimensional h,v space for $N \geq 3$ [19]. So for $N = 1, 2$ one needs to consider axial symmetry planes, if any.

4. Eigenmodes and Modal Degeneracy

The target/external-medium symmetry also has a great influence on the eigenmodes and associated natural modes describing the currents. Following [9] we start with an integral equation of the general form

$$\left\langle \tilde{\overleftrightarrow{Z}}(\vec{r},\vec{r}\,';s);\tilde{\vec{J}}(\vec{r}\,',s)\right\rangle = \tilde{\vec{E}}^{(inc)}(\vec{r},s) \tag{4.1}$$

This is in the form of the E-field or impedance integral equation which has a symmetric kernel. Here integration over the common coordinates $\vec{r}\,'$ is taken in volume form, but the surface form (say for a perfectly conducting target) is similar.

Eigenmodes and eigenvalues are defined by

$$\begin{aligned}
&\left\langle \tilde{\overleftrightarrow{Z}}(\vec{r},\vec{r}\,';s);\; \tilde{\vec{j}}_\beta(\vec{r}\,',s)\right\rangle = \tilde{Z}_\beta(s)\; \tilde{\vec{j}}_\beta(\vec{r},s) \\
&\tilde{\vec{j}}_\beta(\vec{r},s) \equiv \beta\text{th eigenmode} \\
&\tilde{Z}_\beta(s) \equiv \beta\text{th eigenvalue (eigenimpedance)} \\
&\left\langle \tilde{\vec{j}}_\beta(\vec{r},s);\tilde{\overleftrightarrow{Z}}(\vec{r},\vec{r}\,';s)\right\rangle = \tilde{Z}_\beta(s)\; \tilde{\vec{j}}_\beta(\vec{r}\,',s) \\
&\tilde{\overleftrightarrow{Z}}(\vec{r},\vec{r}\,';s) = \tilde{\overleftrightarrow{Z}}^T(\vec{r}\,',\vec{r};s) \qquad \text{(reciprocity)}
\end{aligned} \tag{4.2}$$

The natural modes and natural frequencies are defined by

$$\left\langle \tilde{\overleftrightarrow{Z}}(\vec{r},\vec{r}\,';s_\alpha);\; \vec{j}_\alpha(\vec{r}\,')\right\rangle = \vec{0} = \left\langle \vec{j}_\alpha(\vec{r}\,');\; \tilde{\overleftrightarrow{Z}}(\vec{r},\vec{r}\,';s_\alpha)\right\rangle \tag{4.3}$$

These two sets of parameters are related by

$$\begin{aligned}
\alpha &= (\beta,\beta') \\
\tilde{Z}_\beta(s_{\beta,\beta'}) &= 0 \\
s_{\beta,\beta'} \equiv s_\alpha &= \beta'\text{th root of } \beta\text{th eigenvalue} \\
\vec{j}_\alpha(\vec{r}\,') &= \tilde{\vec{j}}_\beta(\vec{r},s_{\beta,\beta'}) \text{ (times an arbitrary scaling constant, if desired)}
\end{aligned} \tag{4.4}$$

One can treat the eigenmodes and how they are sorted by the symmetry, and the natural modes will have the same properties.

One usually orthonormalizes the eigenmodes as [7]

$$\left\langle \vec{\tilde{j}}_{\beta_1}(\vec{r},s) ; \vec{\tilde{j}}_{\beta_2}(\vec{r},s) \right\rangle = 1_{\beta_1,\beta_2} \tag{4.5}$$

For distinct eigenvalues the symmetric product of the modes must be zero. For degenerate modes (equal eigenvalues) the situation is more complicated [7, 9]. In the present case we have two-fold degeneracy of various modes, but these can be constructed as two orthogonal modes via Gram-Schmidt orthogonalization. Then one can write the integral-equation kernel as

$$\overset{\leftrightarrow}{\tilde{Z}}(\vec{r},\vec{r}\,';s) = \sum_{\beta} \tilde{Z}_{\beta}(s)\, \vec{\tilde{j}}_{\beta}(\vec{r},s)\, \vec{\tilde{j}}_{\beta}(\vec{r}\,',s) \tag{4.6}$$

If we have a symmetry group G with

$$G = \left\{ G_{\ell} \middle| \ell = 1,2,\cdots,\ell_0 \right\} \tag{4.7}$$

with dyadic representation

$$G_{\ell} \to \overset{\leftrightarrow}{G}_{\ell} \tag{4.8}$$

then our target symmetry implies

$$\begin{aligned} \vec{r}^{(2)} &= \overset{\leftrightarrow}{G}_{\ell} \cdot \vec{r}^{(1)} \\ \overset{\leftrightarrow}{\tilde{Z}}(\vec{r}^{(2)},\vec{r}^{(2)\prime};s) &= \overset{\leftrightarrow}{\tilde{G}} \cdot \overset{\leftrightarrow}{\tilde{Z}}(\vec{r}^{(1)},\vec{r}^{(1)\prime};s) \cdot \overset{\leftrightarrow}{G}_{\ell}^{-1} \end{aligned} \tag{4.9}$$

and $\overset{\leftrightarrow}{G}_{\ell} \cdot \vec{\tilde{j}}(\vec{r}^{(1)},s)$ is also an eigenmode (at $\vec{r}^{(2)}$) associated with eigenvalue $\tilde{Z}_{\beta}(s)$, this being true for each $\overset{\leftrightarrow}{G}_{\ell}$ in the group representation. For the point symmetry groups the dyadics are orthogonal, i.e.,

$$\overleftrightarrow{G}_\ell^{-1} = \overleftrightarrow{G}_\ell^{T} \qquad (4.10)$$

and real, and their eigenvalues are roots of one.

With n_ℓ the smallest positive integer such that

$$\overleftrightarrow{G}_\ell^{n_\ell} = \overleftrightarrow{1} = \overrightarrow{1}_x \overrightarrow{1}_x + \overrightarrow{1}_y \overrightarrow{1}_y + \overrightarrow{1}_z \overrightarrow{1}_z = \text{identity}$$

$$n_\ell \equiv \text{order of group element representation } \overleftrightarrow{G}_\ell \qquad (4.11)$$

then n_ℓ operations of $\overleftrightarrow{G}_\ell$ on the target as in (4.7) brings the target back to its original configuration. Applying this to the eigenmodes we have

$$\overleftrightarrow{G}_\ell^{n} \cdot \tilde{\vec{j}}_\beta(\vec{r},s) \equiv \text{eigenmodes for } n = 0\,0,1,\cdots,n_\ell - 1 \text{ evalauted at } \overleftrightarrow{G}_\ell^{n} \cdot \vec{r} \qquad (4.12)$$

but these n_ℓ solutions are not necessarily linearly independent. Any such linear combination is also an eigenmode with eigenvalue $\tilde{Z}_\beta(s)$. There are ν_β linearly independent eigenmodes with

$$1 \le \nu_\beta \le n_\ell \qquad (4.13)$$

which span the space of these eigenmodes. For point symmetries the dimension d_β of the irreducible representation may be 1, 2, or 3. So the eigenmodes may now be written

$$\tilde{\vec{j}}^{(\nu)}(\vec{r},s) \equiv \nu\text{th independent eigenmode associated with } \tilde{Z}_\beta(s)$$

$$\nu = 1,\cdots,d_\beta \qquad (4.14)$$

and similarly for the associated natural modes.

For present purposes we are concerned with two-dimensional rotations and reflections. The associated $d_\beta s$ are then 1 or 2 only.

5. Rotation in Two Dimensions: C_N Symmetry

Consider now C_N symmetry with [19]

$$\begin{aligned} C_N &= \left\{ (C_N)_\ell \,\middle|\, \ell = 1, 2, \cdots, N \right\} \\ (C_N)_\ell &= \text{rotation by } \frac{2\pi\ell}{N} = (C_N)_1^\ell \end{aligned} \tag{5.1}$$

with matrix (3 x 3) representation

$$\begin{aligned} \overleftrightarrow{C}(\phi_\ell) &\equiv (C_{n,m}(\phi_\ell)) = \begin{pmatrix} \cos(\phi_\ell) & -\sin(\phi_\ell) & 0 \\ \sin(\phi_\ell) & \cos(\phi_\ell) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \phi_\ell &= \frac{2\pi\ell}{N}, \quad \ell = 1, 2, \cdots, N \\ \overleftrightarrow{C}(0) &= \overleftrightarrow{C}(2\pi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \overleftrightarrow{1} = \vec{1}_x \vec{1}_x + \vec{1}_y \vec{1}_y + \vec{1}_z \vec{1}_z \equiv \text{identity} \\ \overleftrightarrow{C}^{-1}(\phi) &= \overleftrightarrow{C}^{T}(\phi) = \overleftrightarrow{C}(-\phi) \\ \det(\overleftrightarrow{C}(\phi)) &= 1 \end{aligned} \tag{5.2}$$

from which we see that inverse and transpose are the same as rotation through the negative angle. Note that the various $\overleftrightarrow{C}(\phi)$ commute with each other and have the property of adding rotations as

$$\overleftrightarrow{C}(\phi') \cdot \overleftrightarrow{C}(\phi'') = \overleftrightarrow{C}(\phi' + \phi'') \tag{5.3}$$

We also have an alternate form using [1, 3]

$$\begin{aligned} e^{\phi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} &= \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \\ \overleftrightarrow{C}(\phi) &= e^{\phi \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \end{aligned} \tag{5.4}$$

While our targets are three-dimensional, we are considering only two-dimensional rotations about the z axis. We can write our matrix representation via a direct sum of the form

$$\begin{aligned}\overleftrightarrow{C}(\phi) &= \begin{pmatrix}\cos(\phi) & -\sin(\phi)\\ \sin(\phi) & \cos(\phi)\end{pmatrix} \oplus (1_{1,1}) \\ &= e^{\phi\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}} \oplus (1) = e^{\phi\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}} \oplus e^{\phi(0_{1,1})}\end{aligned} \tag{5.5}$$

showing the separation of the transverse (x, y) and longitudinal (z) parts. As discussed in [12 (Appendix A)] one can diagonalize such a direct sum by considering the blocks separately. Simple calculations give $e^{\pm j\phi}$ as the transverse eigenvalues, and 1 as the longitudinal eigenvalue, giving three eigenvalues total. The dyadic expansion is

$$\begin{aligned}\overleftrightarrow{C}(\phi) &= \left[e^{j\phi t}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -j\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ j\end{pmatrix} + e^{-j\phi t}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ j\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -j\end{pmatrix}\right] \oplus (1_{1,1}) \\ &= e^{j\phi}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -j\\ 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ j\\ 0\end{pmatrix} + e^{-j\phi}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ j\\ 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -j\\ 0\end{pmatrix} + \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} \\ &= \left[\frac{e^{j\phi}}{2}\begin{pmatrix}1 & j\\ -j & 1\end{pmatrix} + \frac{e^{-j\phi}}{2}\begin{pmatrix}1 & -j\\ j & 1\end{pmatrix}\right] \oplus (1_{1,1}) \\ &= \frac{e^{j\phi}}{2}\begin{pmatrix}1 & j & 0\\ -j & 1 & 0\\ 0 & 0 & 0\end{pmatrix} + \frac{e^{-j\phi}}{2}\begin{pmatrix}1 & -j & 0\\ j & 1 & 0\\ 0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\end{aligned} \tag{5.6}$$

6. Rotation and Reflection in Two Dimensions: C_{Na} Symmetry.

Axial reflection symmetry R_a can be given by the group [19]

$$R_a = \{(1), \ (R_a)\} \tag{6.1}$$

Such an axial symmetry plane contains the z axis. Without loss of generality let us take such a symmetry plane to define $\phi = 0$ (thereby also lying on $\phi = \pi$). Such a reflection then consists of reversing the sign on the y coordinate. Our matrix representation (3 x 3) is then

$$\begin{aligned} &(1) \rightarrow \overset{\leftrightarrow}{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &(R_a) \rightarrow \overset{\leftrightarrow}{R}_a = \overset{\leftrightarrow}{R}_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{6.2}$$

Combining the two-dimensional rotation and reflection we have

$$C_{Na} = \left\{(C_N)_\ell , \ (R_a)(C_N)_\ell \mid \ell = 1, 2, \cdots, N\right\} \tag{6.3}$$

In the matrix representation the elements in (5.2) are augmented by

$$\begin{aligned} \overset{\leftrightarrow}{R}_a \cdot \overset{\leftrightarrow}{C}(\phi_\ell) &= \begin{pmatrix} \cos(\phi_\ell) & -\sin(\phi_\ell) & 0 \\ -\sin(\phi_\ell) & -\cos(\phi_\ell) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \left[\overset{\leftrightarrow}{R}_a \cdot \overset{\leftrightarrow}{C}(\phi_\ell)\right]^T = \overset{\leftrightarrow}{C}^T(\phi_\ell) \cdot \overset{\leftrightarrow}{R}_a \\ &= \overset{\leftrightarrow}{C}(-\phi_\ell) \cdot \overset{\leftrightarrow}{R}_a \end{aligned} \tag{6.4}$$

With N independent values of ϕ_ℓ this gives N additional group elements for a total of $2N$.

Note that the presence of one axial symmetry plane implies a total of N axial symmetry planes. The N symmetry planes are described differently for odd and even N. For odd N the planes are described by ϕ_n , $\phi_n + \pi$ for $N = 1, 2, \cdots, N$. For even N two choices of ϕ_n related by $\phi_{n+N/2}$ give the same plane ($N/2$ of them). The remaining $N/2$ planes are positioned between the first set at $\phi_n + \phi_1 / 2$, $\phi_n + \pi + \phi_1 / 2$.

7. Eigenmodes for Scatterers with C_N Symmetry.

Define a set of N volumes

$$\left\{V_n \mid n = 1, 2, \cdots, N\right\} \ni V_\ell = \text{portion of scatterer contained in } \phi_\ell - \frac{1}{2}\phi_1 < \phi < \phi_\ell + \frac{1}{2}\phi_1 \tag{7.1}$$

so that V_ℓ is centered on ϕ_ℓ, with

$$\begin{aligned} \phi_0 &= 0 = \phi_N \\ \phi_{n+\ell N} &= \phi_\ell \quad \text{for all integer } \ell \end{aligned} \tag{7.2}$$

Let $\vec{r}^{(0)}$ be a position in V_N with

$$\vec{r}^{(n)} = \overleftrightarrow{C}(\phi_n) \cdot \vec{r}^{(0)} \tag{7.3}$$

Varying $\vec{r}^{(0)}$ over $V_0 = V_N$ varies $\vec{r}^{(n)}$ over V_n for all n due to the target symmetry. In cylindrical coordinates (Ψ, ϕ, z) we have

$$\begin{aligned} &\Psi^{(n)} = \Psi^{(0)}, \quad \phi^{(n)} = \phi^{(0)} + \phi_n, \quad z^{(n)} = z^{(0)} \\ &\vec{r}^{(n)} = \Psi^{(n)} \vec{1}_\Psi^{(n)} + z^{(\ell)} \vec{1}_z^{(\ell)} \\ &\vec{1}_\Psi^{(n)} = \overleftrightarrow{C}(\phi_\ell) \cdot \vec{1}_\Psi^{(0)}, \quad \vec{1}_\phi^{(n)} = \overleftrightarrow{C}(\phi_\ell) \cdot \vec{1}_\phi^{(0)}, \quad \vec{1}_z^{(n)} = \vec{1}_z^{(0)} \end{aligned} \tag{7.4}$$

See also Section 5.

Define

$$\begin{aligned} \overleftrightarrow{\tilde{Z}}_{n,m}(\vec{r}, \vec{r}\,'; s) &= \overleftrightarrow{\tilde{Z}}(\vec{r}^{(n)}, \vec{r}^{(m)\prime}; s) \\ \vec{r} \in V_n \quad &, \quad \vec{r}\,' \in V_m \end{aligned} \tag{7.5}$$

This divides the kernel into N^2 kernels so that the integral equation (4.1) can be written in the alternate form

$$\sum_{m=1}^{n}\left\langle \overleftrightarrow{Z}(\vec{r}^{(n)},\vec{r}^{(m)'};s);\ \vec{J}(r^{(m)'},s)\right\rangle = \vec{E}^{(inc)}(\vec{r}^{(n)},s)$$

$$\left\langle\left(\overleftrightarrow{Z}_{n,m}(\vec{r},\vec{r}\,';s)\right) \odot \left(\vec{J}_n(\vec{r}\,',s)\right)\right\rangle = \left(\vec{E}_n^{(inc)}(\vec{r},s)\right) \tag{7.6}$$

$$= \left\langle\left(\overleftrightarrow{Z}(\vec{r}^{(n)},\vec{r}^{(m)'};s)\right) \odot \left(\vec{J}(\vec{r}^{(n)'},s)\right)\right\rangle = \left(\vec{E}^{(inc)}(\vec{r}^{(n)},s)\right)$$

where now the vectors are supervectors of N vectors and the dyadic kernel is a supermatrix of N^2 dyadics with n, m the dummy indices as needed. Note the operation of generalized dot product together with integration over the $\vec{r}\,'$ coordinates, over each V_n as appropriate.

Similarly eigenmodes (eigensupervectors) can be defined via

$$\left\langle\left(\overleftrightarrow{Z}_{n,m}(\vec{r},\vec{r}\,';s)\right) \odot \left(\vec{j}_n(\vec{r}\,',s)\right)_\beta\right\rangle = \tilde{Z}_\beta(s)\left(\vec{j}_n(\vec{r})\right)_\beta$$

$$\left\langle\left(\vec{j}_n(\vec{r},s)\right)_\beta \odot \left(\overleftrightarrow{Z}_{n,m}(\vec{r},\vec{r}\,';s)\right)\right\rangle = \tilde{Z}_\beta(s)\left(\vec{j}_n(\vec{r}\,',s)\right)_\beta$$

$$\left\langle\left(\vec{j}_n(\vec{r},s)\right)_\beta \odot \left(\overleftrightarrow{Z}_{n,m}(\vec{r},\vec{r}\,';s)\right) \odot \left(\vec{j}_n(\vec{r}\,',s)\right)_\beta\right\rangle = \tilde{Z}_\beta(s) \tag{7.7}$$

$$\left\langle\left(\vec{j}_n(\vec{r},s)\right)_\beta \odot \left(\vec{j}_n(\vec{r},s)\right)_{\beta_2}\right\rangle = 1_{\beta_1,\beta_2} \quad \text{(orthonormal)}$$

$$\left(\vec{j}_n(\vec{r},s)\right)_\beta \equiv \text{eigenmode (supervector)}$$

$$\vec{j}_{n;\beta}(\vec{r},s) \equiv \vec{j}_\beta(\vec{r}^{(n)},s) \equiv \text{eigenmode portion in } V_n$$

Recall the symmetry condition (4.9)

$$\overleftrightarrow{Z}(\vec{r}^{(n+\ell)},\vec{r}^{(m+\ell)'};s) = \overleftrightarrow{C}(\phi_\ell)\cdot\overleftrightarrow{Z}(\vec{r}^{(n)},\vec{r}^{(m)};s) = \overleftrightarrow{C}(-\phi_\ell)$$

$$\overleftrightarrow{Z}(\vec{r}^{(n)},\vec{r}^{(m)'};s) = \overleftrightarrow{C}(-\phi_\ell)\cdot\overleftrightarrow{Z}(\vec{r}^{(n+\ell)},\vec{r}^{(m+\ell)'};s)\cdot\overleftrightarrow{C}(\phi_\ell) \tag{7.8}$$

where ℓ can be chosen as any integer. Rotation by ϕ_ℓ moves $\vec{r}$ in V_n to $V_{n+\ell}$ and $\vec{r}'$ in V_m to $V_{m+\ell}$. Special choices of ℓ give

$$\begin{aligned}\tilde{\overleftrightarrow{Z}}(\vec{r}^{(n)},\vec{r}^{(m)'};s) &= \overleftrightarrow{C}(\phi_m)\cdot\tilde{\overleftrightarrow{Z}}(\vec{r}^{(n-m)},\vec{r}^{(0)'};s)\cdot\overleftrightarrow{C}(-\phi_m)\\ &= \overleftrightarrow{C}(\phi_n)\cdot\tilde{\overleftrightarrow{Z}}(\vec{r}^{(0)},\vec{r}^{(m-n)'};s)\cdot\overleftrightarrow{C}(-\phi_n)\end{aligned} \tag{7.9}$$

More generally choosing ℓ as $-m+\ell'$ we have

$$\tilde{\overleftrightarrow{Z}}(\vec{r}^{(n)},\vec{r}^{(m)'};s) = \overleftrightarrow{C}(\phi_{m-\ell'})\cdot\tilde{\overleftrightarrow{Z}}(\vec{r}^{(n-m+\ell')},\vec{r}^{(\ell')'};s)\cdot\overleftrightarrow{C}(\phi_{\ell'-m}) \tag{7.10}$$

with ℓ' as any integer. So as a supermatrix we can see that at the level of the n, m indices the individual dyadics can be rotated into forms which are functions of n-m only. In this form and at this level (outer inddices) the supermatrix is then circulant. Note that if the dyadic kernel were instead scalar the additional rotation dyadics would not be needed.

Identifying in (7.9) (first form)

$$\begin{aligned}&n-m \equiv u-1\\ &\tilde{\overleftrightarrow{Z}}(\vec{r}^{(n-m)},\vec{r}^{(0)'};s) = \tilde{\overleftrightarrow{Z}}(\vec{r}^{(u-1)},\vec{r}^{(0)'};s) = \tilde{\overleftrightarrow{Z}}_u(\vec{r}^{(u-1)},\vec{r}^{(0)'};s)\end{aligned} \tag{7.11}$$

we have the dyadics in the circulant form in (A.11) and the results of Appendix A apply. Then the form

$$\tilde{\overleftrightarrow{Z}}(\vec{r}^{(n)},\vec{r}^{(m)'};s) = \overleftrightarrow{C}(\phi_m)\cdot\tilde{\overleftrightarrow{Z}}(\vec{r}^{(n-m)},\vec{r}^{(0)'};s)\cdot\overleftrightarrow{C}(-\phi_m) \tag{7.12}$$

is precisely that in Appendix B.

Form (as in (B.8))

$$\tilde{\overleftrightarrow{z}}_q(\vec{r}^{(0)},\vec{r}^{(0)'};s) = \sum_{u=1}^{N}\overleftrightarrow{C}(\phi_{-u})\cdot\tilde{\overleftrightarrow{Z}}_{u+1}(\vec{r}^{(u)},\vec{r}^{(0)'};s)\,e^{j2\pi\frac{uq}{N}} \tag{7.13}$$

noting that the $\vec{r}^{(0)}$ and $\vec{r}^{(0)'}$ variables extend from V_0 to all V_n via (7.3). Like (B.5) let us look for eigensuper-vectors of the form

$$\left(\tilde{\vec{J}}_n(\vec{r},s)\right)_\beta = \left(\tilde{\vec{J}}(\vec{r}^{(n)},s)\right)_\beta = \left(\overleftrightarrow{C}(\phi_n) \cdot \tilde{\vec{J}}_{\delta;q}(\vec{r}^{(0)},s)\, x_{n;q}\right) \tag{7.14}$$

where the $\overleftrightarrow{C}(\phi_n)$ rotate the current-density orientation to the various V_n and the $x_{n;q}$ weight the currents in the various V_n. Then we form (from (7.7) and (B.9))

$$\begin{aligned}
&\left(\tilde{\vec{k}}(\vec{r}^{(n)},s)\right)_\beta \\
&= \left\langle \left(\tilde{\overleftrightarrow{Z}}(\vec{r}^{(n)},\vec{r}^{(m)'};s)\right) \odot \left(\tilde{\vec{j}}(\vec{r}^{(n)'},s)\right)_\beta \right\rangle_{V_{all}} \\
&= \left\langle \left(\overleftrightarrow{C}(\phi_n) \cdot \tilde{\overleftrightarrow{Z}}_q(\vec{r}^{(0)},\vec{r}^{(0)'};s);\ \tilde{\vec{j}}_{\delta;q}(\vec{r}^{(0)'},s)x_{n;q}\right)\right\rangle_{V_0} \\
&= \left(\overleftrightarrow{C}(\phi_n) \cdot \left\langle \tilde{\overleftrightarrow{z}}_q(\vec{r}^{(0)},\vec{r}^{(0)'};s);\ \tilde{\vec{j}}_{\delta;q}(\vec{r}^{(0)'},s)x_{n;q}\right\rangle_{V_0} x_{n;q}\right)
\end{aligned} \tag{7.15}$$

$$V_{all} = \bigcup_{n=1}^{N} V_n\ , \quad V_N = V_0$$

Here the subscript V_{all} indicates that the integration is taken over the entire target volume. The various $\vec{r}^{(n)'}$ have all been rotated to V_0 so this is just N times the integral over a single V_n, but this factor is just the N appearing in the summation of the N terms in (7.13), the order of summation and integration also being interchanged.

Now require that the $\tilde{\vec{j}}_{\delta;q}$ diagonalize the integral operator, i.e.,

$$\left\langle \tilde{\overleftrightarrow{z}}_q(\vec{r}^{(0)},\vec{r}^{(0)'};s);\ \tilde{\vec{j}}_{\delta;q}(\vec{r}^{(0)'},s)\right\rangle = \tilde{z}_{\delta;q}(s)\, \tilde{\vec{j}}_{\delta;q}(\vec{r}^{(0)},s) \tag{7.16}$$

The index δ now generally has an infinite range (infinite number of eigenvalues for $\delta = 1, 2, \ldots$), instead of just three, but this is not important for the present discussion which concentrates on q and n. If one wishes one can orthonormalize the modes based on either (7.7) or 7.16).

Returning to (7.15) we have

$$\left(\vec{\tilde{k}}(\vec{r}^{(n)},s)\right)_\beta = \tilde{z}_{\delta;q}(s)\left(\overleftrightarrow{C}(\phi_n)\cdot\vec{\tilde{j}}_{\delta;q}(\vec{r}^{(0)'},s)\,x_{n;q}\right)$$
$$= \tilde{z}_{\delta;q}(s)\left(\vec{\tilde{j}}(\vec{r}^{(n)},s)\right)_\beta \tag{7.17}$$

so that in (7.7) we have

$$\tilde{Z}_\beta(s) = \tilde{z}_{\delta;q}(s) \tag{7.18}$$

giving now both the eigenvalues and eigensupervectors of our scattering problem for C_N symmetry.

To illustrate the spatial properties of these modes, consider the special $N = 3$ target in Fig. 7.1. This is composed of thin wires, one on the z axis, and three carved wires all connected together at the coordinate origin. Consider first the properties of the Nth or 0th mode for which we have

$$(x_n)_N = (x_n)_0 = \frac{1}{\sqrt{N}}(e^{-j2\pi n}) = \frac{1}{\sqrt{N}}(1, 1, \cdots, 1)$$
$$(\vec{\tilde{j}}(\vec{r}^{(n)},s))_{\delta,N} = (\vec{\tilde{j}}(\vec{r}^{(n)},s))_{\delta;0} = \frac{1}{\sqrt{N}}\left(\overleftrightarrow{C}(\phi_n)\cdot\vec{\tilde{j}}_{\delta;0}(\vec{r}^{(0)'},s)\right) \tag{7.19}$$

Interpreting these modes as currents on thin wires near the origin we have

$$I_n = I_m \text{ for } n,m = 1,2,\cdots,N$$
$$I_0 = -N\ I_n \text{ for all } n \tag{7.20}$$

The other modes have (see (A.5), (B.7))

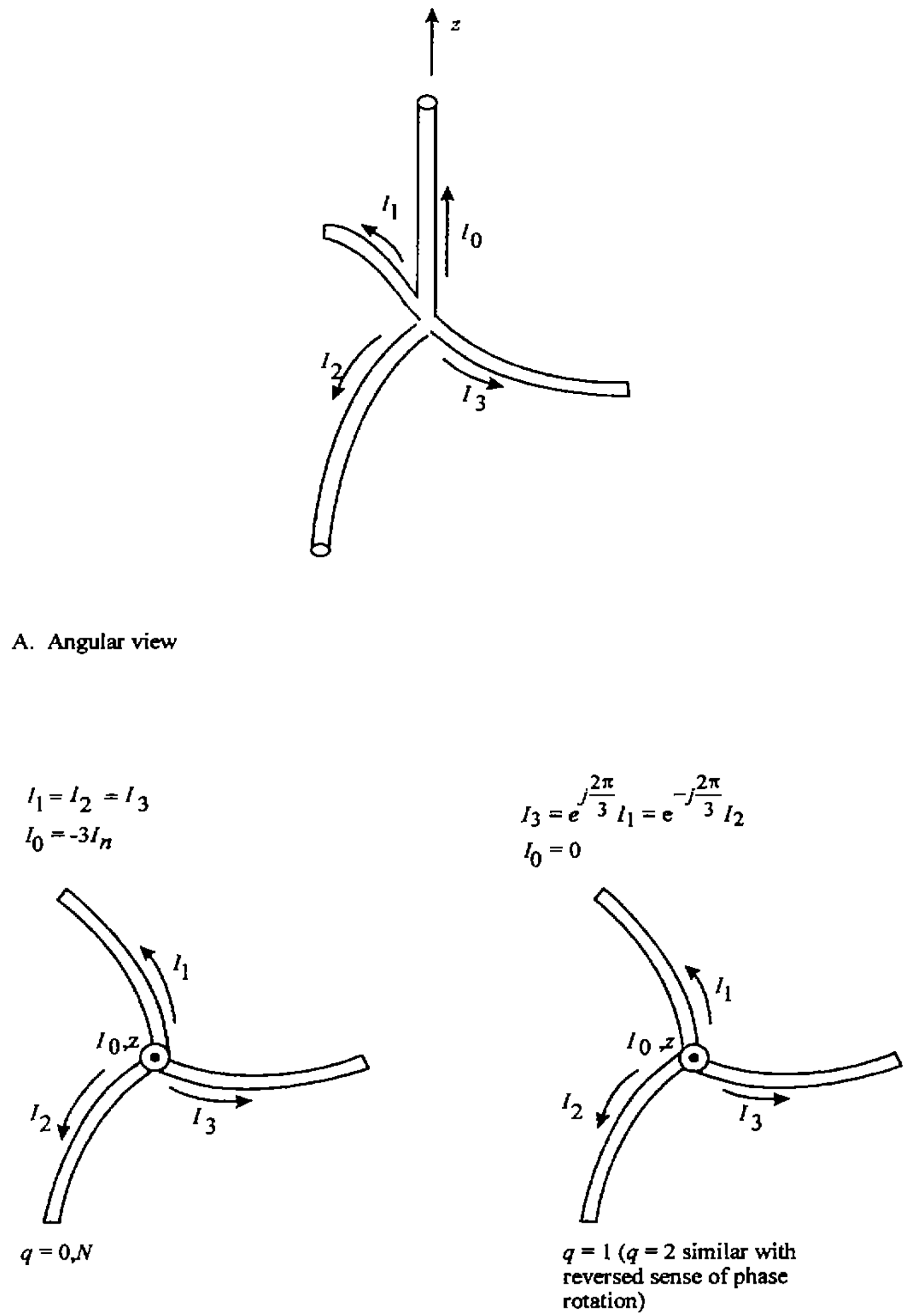


Fig. 7.1. Characteristics of Eigenmodes for $N = 3$.

$$
\begin{aligned}
(x_n)_q &= \frac{1}{\sqrt{N}} \left(e^{-j2\pi\frac{nq}{N}}\right) \\
\sum_{q=1}^{N} x_{n;q} &= \sqrt{N} \; 1_{q,N} = 0 \;\text{ for } q \neq N, 0
\end{aligned}
\tag{7.21}
$$

Thus only the $n = 0, N$ mode has nonzero net current on the z-axis wire, as indicated for $N = 3$ in Fig. 7.1

For the case of $N = 3$ we have

$$
\begin{aligned}
(x_n)_1 &= \frac{1}{\sqrt{3}}\left(e^{-j\frac{2\pi}{3}}, e^{j\frac{2\pi}{3}}, 1\right) \\
(x_n)_2 &= \frac{1}{\sqrt{3}}\left(e^{j\frac{2\pi}{3}}, e^{-j\frac{2\pi}{3}}, 1\right) \\
(x_n)_3 &= (x_n)_0 = \frac{1}{\sqrt{3}}(1, \; 1, \; 1)
\end{aligned}
\tag{7.22}
$$

giving the structure (with respect to n) of all three modes. The $q = 1$ mode has no (net) component on the z wire, and has the three wire currents (sum zero) with relative values (phases) as in $(x_n)_1$ so that

$$
I_1 = e^{-j\frac{2\pi}{3}} \, I_3 \, , \quad I_2 = e^{j\frac{2\pi}{3}} \, I_3
\tag{7.23}
$$

This is shown in Fig. 7.1B. The $q = 2$ mode has a similar structure, considering only the $(x_n)_q$. There is, however, also the question of the $\vec{j}^{(0)}_{\delta;q}(\vec{r}^{(0)},s)$, particularly whether any two q values give the same current density distribution in V_0. For the moment modes such as $(x_n)_1$ and $(x_n)_2$, which are reflections (not rotations) of each other, will be considered to have generally different current distributions in V_0 such as $\vec{j}^{(0)}_{\delta;1}(\vec{r}^{(0)},s)$ and $\vec{j}^{(0)}_{\delta;2}(\vec{r}^{(0)},s)$. We may also expect that natural frequencies associated with such modes are, in general, different. (An exception to this occurs if there is a transverse symmetry plane say $z = 0$. Then reversing the direction of rotation of the modes is the same as looking from opposite sides ($\pm$ z directions) for which spiraling in one direction (say $+ \phi$) is reversed when viewed from the opposite side.) Diagrams like those in Fig. 7.1 can also be constructed for other N using (7.21).

8. Eigenmodes for Scatterers with C_{Na} Symmetry

Now add N axial symmetry lanes as discussed in Section 6 to give C_{Na} symmetry. Without loss of generality one of the planes is taken on $\phi = 0, \pi$.

The case of $q = 1$ in Fig. 7.1 is now extended to include symmetry planes. In Fig. 8.1 consider the symmetry plane described by $\phi = 0, \pi$. Reflection through this plane is described by

$$\overrightarrow{R}_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \overleftrightarrow{1} - 2 \overrightarrow{1}_y \overrightarrow{1}_y \tag{8.1}$$

which corresponds to inversion of the target with respect to the y coordinate. As is well known [2, 19] the eigenmodes and natural modes can be decomposed into two sets with respect to such a symmetry plane: symmetric (sy) and antisymmetric (as). These have the property for the current density

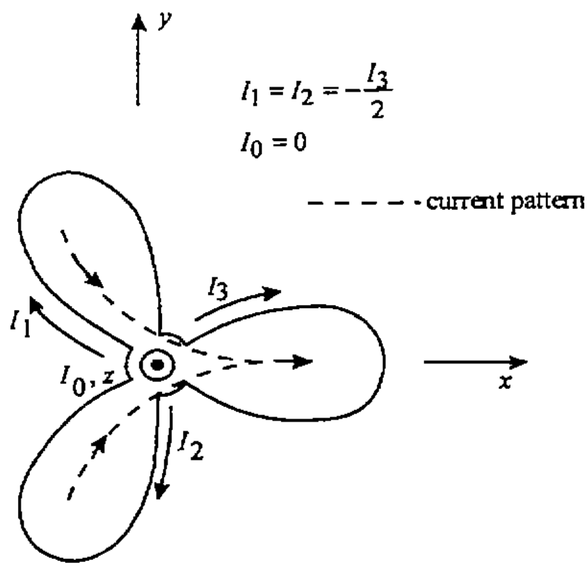
$$\begin{aligned} \overrightarrow{r}_m &= \overleftrightarrow{R}_y \cdot \overrightarrow{r} \\ \left(\overrightarrow{j}^{(sy)}(\overrightarrow{r}_m, s) \right)_{\beta'} &= \overleftrightarrow{R}_y \cdot \left(\overrightarrow{j}^{(sy)}(\overrightarrow{r}, s) \right)_{\beta'} \\ \left(\overrightarrow{j}^{(as)}(\overrightarrow{r}_m, s) \right)_{\beta'} &= -\overleftrightarrow{R}_y \cdot \left(\overrightarrow{j}^{(as)}(\overrightarrow{r}, s) \right)_{\beta'} \end{aligned} \tag{8.2}$$

An example of such modes is given in Fig. 8.1.

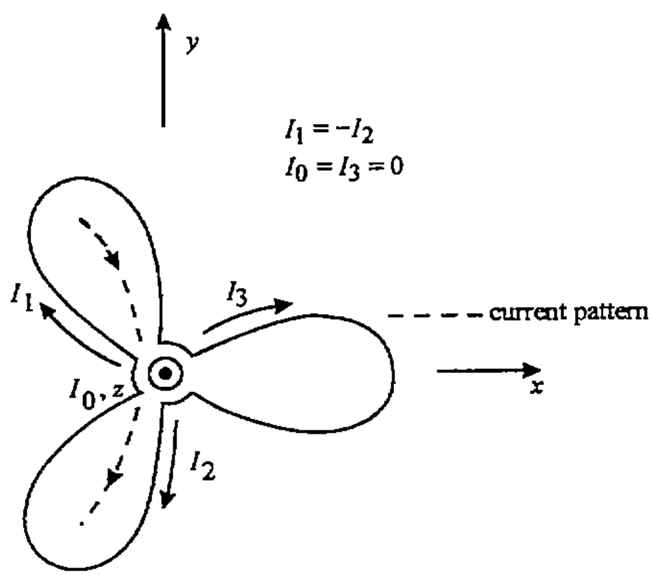
The mode index in (8.2) is given as β' to distinguish it from β for the rotational modes in Section 7. However, they can be related to the previous modes.

The axial symmetry planes allow us to reflect the rotational modes in Section 6 to give other rotational modes. Specifically an eigenmode as in (7.14) gives another eigenmode, noting that $\overleftrightarrow{R}_y$ maps V_n into $V_{-n} = V_{N-n}$ with the transformations

$$\begin{aligned} \overrightarrow{r}_m^{(0)} = \overleftrightarrow{R}_y \cdot \overrightarrow{r}^{(0)} \quad &, \quad \overrightarrow{r}_m^{(-n)} = \overrightarrow{r}_m^{(N-n)} = \overleftrightarrow{R}_y \cdot \overrightarrow{r}^{(n)} \\ \overleftrightarrow{R}_y \cdot \overrightarrow{j}_{\delta;q}(\overrightarrow{r}_m^{(0)}, s) &= \overrightarrow{j}_{\delta;-q}(\overrightarrow{r}^{(0)}, s) = \overrightarrow{j}_{\delta;N-q}(\overrightarrow{r}^{(0)}, s) \end{aligned} \tag{8.4}$$



A. Symmetric mode with respect to $\phi = 0, \pi$



B. Antisymmetric mode with respect to $\phi = 0, \pi$

Fig. 8.1. Characteristics of Eigenmodes With Axial Symmetry Planes for $N = 3$, Combining $q = 1, 2$ Modes.

This corresponds to reversal of the direction of rotation: $\phi \to -\phi,\ 2\pi-\phi;\ \ n \to -n\,,\ N-n;\ \ q \to -q,\ N-q$. The transformed eigenmode is

$$
\begin{aligned}
&(\overleftrightarrow{R}_y \cdot \overleftrightarrow{C}(-\phi_n) \cdot \vec{j}_{\delta;q}(\overleftrightarrow{R}_y \cdot \vec{r}^{\,(0)}, s) x_{-n;q}) \\
&= (\overleftrightarrow{C}(\phi_n) \cdot \overleftrightarrow{R}_y \cdot \vec{j}_{\delta,q}(\vec{r}_m^{\,(0)}, s) x_{-n;q}) \\
&= (\overleftrightarrow{C}(\phi_n) \cdot \vec{j}_{\delta;-q}(\vec{r}^{\,(0)}, s) x_{n;-q}) \\
&= (\overleftrightarrow{C}(\phi_n) \cdot \vec{j}_{\delta;N-q}(\vec{r}^{\,(0)}, s) x_{n;N-q})
\end{aligned}
\tag{8.5}
$$

Note that for $q = 0, N$ this is a symmetric mode (with respect to all N symmetry planes). For the case of distinct q and $N-q$ there are two distinct modes with the same eigenvalue

$$
\widetilde{Z}_{\delta;q}(s) = \widetilde{Z}_{\delta;-q}(s) = \widetilde{Z}_{\delta;N-q}(s) \tag{8.6}
$$

This is the case of two-fold degeneracy [9]. For special cases there is only one eigenmode with a particular eigenvalue. One case is $q = 0, N$. A second case occurs for even N with $q = N/2$. This is analogous to the eigenvalues of bicirculant matrices discussed in [6].

For cases of two-fold degeneracy we can take any linear combination of the two eigenmodes as an eigenmode with the same eigenvalue. We can form symmetric and antisymmetric modes (with respect to the $\phi = 0, \pi$ plane) via

$$
\begin{aligned}
(\vec{j}^{\,(sy)}(\vec{r}, s))_{\delta;\pm q} &= \frac{1}{2}\Big[(\overleftrightarrow{C}(\phi_n) \cdot \vec{j}_{\delta;q}(\vec{r}^{\,(0)}, s) x_{n;q}) \\
&\qquad + (\overleftrightarrow{R}_y \cdot \overleftrightarrow{C}(\phi_{-n}) \cdot \vec{j}_{\delta;q}(\vec{R}_y \cdot \vec{r}^{\,(0)}) x_{-n;q})\Big] \\
(\vec{j}^{\,(as)}(\vec{r}, s))_{\delta;\pm q} &= \frac{1}{2}\Big[(\overleftrightarrow{C}(\phi_n) \cdot \vec{j}_{\delta;q}(\vec{r}^{\,(0)}, s) x_{n;q}) \\
&\qquad - (\overleftrightarrow{R}_y \cdot \overleftrightarrow{C}(\phi_{-n}) \cdot \vec{j}_{\delta;q}(\vec{R}_y \cdot \vec{r}^{\,(0)}) x_{-n;q})\Big]
\end{aligned}
\tag{8.7}
$$

These can be left as above or renormalized in whatever form one likes.

Figure 8.1 illustrates the symmetric and antisymmetric modes for the case of C_{3a} symmetry formed from *the $q = 1$ and $q = 2$ modes. There can be a conductor along the z axis as in Fig. 7.1, but the net current (as distinct* from the detailed distribution of the current density) is zero here. Note the three axial symmetry planes bisecting the three arms (which in general have no transverse symmetry plane).

Referring back to (7.22) we have the $(x_n)_q$ part of the modes for this symmetry. The symmetric mode is formed as

$$\begin{aligned} \frac{1}{2}\left[(x_n)_1 + (x_n)_2\right] &= \frac{1}{\sqrt{3}}\left(\cos\left(\frac{2\pi}{3}\right),\ \cos\left(\frac{2\pi}{3}\right),\ 1\right) \\ &= \frac{1}{\sqrt{3}}(-0.5,\ -0.5,\ 1) \end{aligned} \tag{8.8}$$

which is illustrated in Fig. 8.1A. Note that (8.8) only gives the relative amplitude of the net currents I_n on the three arms, the current-density distribution from the $\vec{j}_{\delta;q}(\vec{r}^{(0)}, s)$ requiring a more detailed calculation. The antisymmetric mode is formed as

$$\begin{aligned} \frac{1}{2}\left[(x_n)_1 - (x_n)_2\right] &= \frac{j}{\sqrt{3}}\left(-\sin\left(\frac{2\pi}{3}\right),\ \sin\left(\frac{2\pi}{3}\right),\ 0\right) \\ &= \frac{j}{2}(-1,\ 1,\ 0) \end{aligned} \tag{8.9}$$

which is illustrated in Fig. 8.1B. Note that the net current $I_3 = 0$ on the third arm, while the current density is not in general zero there, merely being antisymmetric with respect to the $y = 0$ plane.

9. Natural Modes Appearing in Backscattered Fields

Section 3 discusses the polarization properties of the backscattered fields in the h,v radar coordinates (Fig. 1.1). As discussed in Section 4, the natural modes are special cases of the eigenmodes. So the natural modes have the same symmetry properties. The presence of a natural mode in the scattering can be detected by the presence of the associated natural frequency (pole in the s plane).

The symmetry properties of the natural modes are given by the geometrical symmetries, i.e., C_N and C_{Na} in the present case. From a polarization point of view one can distinguish between C_N and C_{Na} symmetries for $\theta_i > 0$ by the lack of N symmetry planes for which the crosspolarization is zero. This is also detected by the lack of the target natural frequencies in the crosspolarization when the observer lies on such symmetry planes (also thereby discovering the locations of these symmetry planes). Note that a target in real earth (as distinguished from the ideal layered earth in Section 1) lies in the presence of various clutter (rocks, etc.). Nulling the target natural frequencies in the crosspol signal is then a potential technique for reducing the influence of clutter in the target identification.

10. Concluding Remarks

Here we have considered the cases of C_N and C_{Na} symmetries in the targets of interest, as appropriate to targets in the presence of a layered medium such as near the earth surface. The symmetry of the target appears directly in the symmetry and associated description of the eigenmodes. This does not actually give a detailed calculation of the eigenmodes, but decomposes the problem in a way that reduces the computation to a subset of the target geometry. The present approach should be applicable to targets with other symmetries such as can exist in the absence of a halfspace such as an ideal earth. In free space targets can have symmetries corresponding to the various point symmetry groups (rotations and reflections in three dimensions) [10, 19].

While we have been considering electromagnetic scattering and modes, the same techniques should be applicable in quantum mechanics. The quantum wave functions of molecules exhibit the point symmetries of the molecules [16, 17], and the energy levels are analogous to electromagnetic natural frequencies. Quantum wave functions (Schrödinger eqn.) are scalar rather than vector functions, thereby being simpler in at least one respect.

Appendix A. Diagonalization of Supermatrices Circulant in Outermost Indices

Following [6 (Appendices A and B)] circulant matrices ($N \times N$)

$$\left(X_{n,m}\right) = \begin{pmatrix} X_1 & X_2 & X_3 & \cdots & X_N \\ X_N & X_1 & X_2 & \cdots & X_{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ X_2 & X_3 & X_4 & \cdots & X_N\ X_1 \end{pmatrix} \tag{A.1}$$

are diagonalized with the aid of the Fourier matrix

$$\left(U_{n,m}\right) = \frac{1}{\sqrt{N}} \begin{pmatrix} e^{j\frac{2\pi}{N}} & e^{j2\frac{2\pi}{N}} & \cdots & e^{j[N-1]\frac{2\pi}{N}} & 1 \\ e^{j2\frac{2\pi}{N}} & e^{j4\frac{2\pi}{N}} & \cdots & e^{j2[N-1]\frac{2\pi}{N}} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ e^{j[N-1]\frac{2\pi}{N}} & e^{j2[N-1]\frac{2\pi}{N}} & \cdots & e^{j[N-1]^2\frac{2\pi}{N}} & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix} \tag{A.2}$$

$$= \frac{1}{\sqrt{N}}\left(e^{jnm\frac{2\pi}{N}}\right) = \left(U_{n,m}\right)^T$$

$$\left(U_{n,m}\right)^{-1} = \left(U_{n,m}\right)^{\dagger} \quad \text{(unitary)}$$

as

$$\left(X_{n,m}\right) = \left(U_{n,m}\right) \cdot \left[\bigoplus_{q=1}^{N} x_q\right] \cdot \left(U_{n,m}\right)^{-1} \tag{A.3}$$

where the direct sum gives a diagonal matrix ($N \times N$) with the x_q running down the diagonal with

$$x_q = \sum_{u=1}^{N} X_{u+1}\ e^{j2\pi\frac{uq}{N}} \equiv \text{eigenvalues} \tag{A.4}$$

Note that the cyclic nature of such a matrix allows us to interpret indices $u > N$ by subtraction of integer powers of N and for $u < 1$ by the addition of integer powers of N to place X_u with $1 \le u \le N$. For convenience the Nth term, mode, etc., may also be interpreted as the 0th.

Continuing, we can write

$$\begin{aligned}
(X_{n,m}) \cdot (X_n)_q &= x_q (x_n)_q \\
(x_n)_q^* \cdot (X_{n,m}) &= x_q (x_n)_q^* \\
(x_n)_q &= \frac{1}{\sqrt{N}} \left(e^{-j2\pi \frac{nq}{N}} \right) \equiv \text{ right eigenvectors (n intexing N components)} \\
(x_n)_{q_1}^* \cdot (x_n)_{q_2} &= 1_{q_1,q_2} \quad \text{(biorthonormal)}
\end{aligned} \tag{A.5}$$

where the right eigenvectors are columns (also rows) of $(U_{n,m})^\dagger$ and the left eigenvectors are rows (also columns) of $(U_{n,m})$. We now have the dyadic representation

$$(\mathrm{X}_{n,m}) = \sum_{q=1}^{N} x_q \, (x_n)_q (x_n)_q^* \tag{A.6}$$

Consider now a supermatrix of the form

$$\left(\overleftrightarrow{X}_{n,m} \right) = \begin{pmatrix} \overleftrightarrow{\mathrm{X}}_1 & \overleftrightarrow{\mathrm{X}}_2 & \overleftrightarrow{\mathrm{X}}_3 \cdots & \overleftrightarrow{\mathrm{X}}_N \\ \overleftrightarrow{\mathrm{X}}_N & \overleftrightarrow{\mathrm{X}}_1 & \overleftrightarrow{\mathrm{X}}_2 \cdots & \overleftrightarrow{\mathrm{X}}_{N-1} \\ \vdots & \vdots & \vdots & \vdots \\ \overleftrightarrow{\mathrm{X}}_2 & \overleftrightarrow{\mathrm{X}}_3 & \overleftrightarrow{\mathrm{X}}_4 \cdots & \overleftrightarrow{\mathrm{X}}_1 \end{pmatrix} \tag{A.7}$$

This is like a circulant matrix except that now the elements are dyadics or matrices (3 x 3). Consider a scalar form by considering one of 9 elements as

$$\begin{aligned}
\mathrm{X}_{p,p';n,m} &= \vec{1}_p \cdot \overleftrightarrow{\mathrm{X}}_{n,m} \cdot \vec{1}_{p'} \\
\mathrm{X}_{p,p';q} &= \vec{1}_p \cdot \overleftrightarrow{\mathrm{X}}_q \cdot \vec{1}_{p'} \quad , \quad q = n - m + 1
\end{aligned} \tag{A.8}$$

Here we constrain the unit vectors to be at most a function of p so that $(X_{p,p';n,m})$ is circulant (in n,m indices). Then (A.1) through (A.6) apply.

Let now the unit vectors take on Cartesian interpretation as $\vec{1}_x$, $\vec{1}_y$, and $\vec{1}_z$ in all combinations with $p, p' = 1, 2, 3$. Then we have

$$\begin{aligned} \sum_{p=1}^{3} \sum_{p'=1}^{3} \vec{1}_p X_{p,p';n,m} \vec{1}_{p'} &= \sum_{p=1}^{3} \sum_{p'=1}^{3} \vec{1}_p \vec{1}_p \cdot \overleftrightarrow{X}_{n,m} \cdot \vec{1}_{p'} \vec{1}_{p'} \\ &= \sum_{p=1}^{3} \vec{1}_p \vec{1}_p \cdot \overleftrightarrow{X}_{n,m} \cdot \overleftrightarrow{1} = \overleftrightarrow{1} \cdot \overleftrightarrow{X}_{n,m} \cdot \overleftrightarrow{1} = \overleftrightarrow{X}_{n,m} \end{aligned} \tag{A.9}$$

as a way of reconstructing the dyadic from the scalar form. We can rewrite the dyadics in (A.7) in matrix form as

$$\overleftrightarrow{X}_{n,m} = \left(X_{p,p'}\right)_{n,m} \quad , \quad \overleftrightarrow{X}_u = \left(X_{p,p'}\right)_u \tag{A.10}$$

Now we can write

$$\begin{aligned} \left(\overleftrightarrow{X}_{n,m}\right) &= \left(\left(X_{p,p'}\right)_{n,m}\right) \\ &= \begin{pmatrix} (X_{p,p'})_1 & (X_{p,p'})_2 & (X_{p,p'})_3 \cdots (X_{p,p'})_N \\ (X_{p,p'})_N & (X_{p,p'})_1 & (X_{p,p'})_2 \cdots (X_{p,p'})_{N-1} \\ \vdots & \vdots & \vdots \quad\quad \vdots \\ (X_{p,p'})_2 & (X_{p,p'})_3 & (X_{p,p'})_4 \cdots (X_{p,p'})_1 \end{pmatrix} \end{aligned} \tag{A.11}$$

in supermatrix (dimatrix) form. The matrix elements are written from (A.6) as

$$X_{p,p';n,m} = \sum_{q=1}^{N} x_{p,p';q} x_{n;q} x_{m;q}^{*} \tag{A.12}$$

Conveniently the eigenvector elements are common to (not a function of) all p, p'. The matrix blocks are then

$$\left(X_{p,p'}\right)_{n,m} = \sum_{q=1}^{N} \left(x_{p,p'}\right)_q x_{n;q} x_{m;q}^{*}$$

$$\left(\mathrm{X}_{p,p'}\right)_q = \sum_{u=1}^{N}\left(\mathrm{X}_{p,p'}\right)_{u+1} e^{j2\pi\frac{uq}{N}} \quad \text{(like a matrix-valued eigenvalue)} \tag{A.13}$$

$$= \sum_{u=1}^{N} \overset{\leftrightarrow}{\mathrm{X}}_{u+1}\, e^{j2\pi\frac{uq}{N}} \quad \text{(like a dyadic-valued eigenvalue)}$$

and the full supermatrix is

$$\left(\left(\mathrm{X}_{p,p'}\right)_{n,m}\right) = \sum_{i=1}^{N}\left(\left(x_{p,p'}\right)_q x_{n;q}\, x^{*}_{m;q}\right)$$

$$= \sum_{q=1}^{N}\left(x_{p,p'}\right)_q \otimes \left[\left(x_n\right)_q \left(x_n\right)^{*}_q\right] \tag{A.14}$$

with the direct product $\otimes$ convention as in [12 (Appendix A)]. Note that we can also write

$$\overset{\leftrightarrow}{x}_q = \left(x_{p,p'}\right)_q \tag{A.15}$$

in dyadic form in all of the expressions.

Now we can recognize the circulant eigenvectors in the eigendyads separated out by the direct product. One can also diagonalize (usually)

$$\overset{\leftrightarrow}{x}_q = \left(x_{p,p'}\right)_q = \sum_{q=1}^{3} x_{\delta;q}\left(r_p\right)_{\delta;q}\left(\ell_p\right)_{\delta;q}$$

$$\left(r_p\right)_{\delta;q} \equiv \text{right eigenvectors}$$

$$\left(\ell_p\right)_{\delta;q} \equiv \text{left eigenvectors} \tag{A.15}$$

$$\left(r_p\right)_{\delta_1;q} \cdot \left(\ell_p\right)_{\delta_2;q} = 1_{\delta_1,\delta_1} \quad \text{(biorthonormal)}$$

Then for $((\mathrm{X}_{p,p'})_{n,m})$ we have $3N$ of each of

$$\left(r_p\right)_{\delta;q} \otimes \left(x_p\right)_q \equiv \text{right eigensupervectors}$$

$$\left(\ell_p\right)_{\delta;q} \otimes \left(x_p\right)_q^* \equiv \text{left eigensupervectors} \tag{A.16}$$

$$x_{\delta;q} \equiv \text{eigenvalues } (3N \text{ of them})$$

This is found from more detailed derivations in [12 (Appendix A)]. The superdyadic expansion is then

$$\left(\left(\mathrm{X}_{p,p'}\right)_{n,m}\right) = \sum_{q=1}^{N} \sum_{\delta=1}^{3} x_{\delta;q} \left[\left(r_n\right)_{\delta;q} \left(\ell_n\right)_{\delta;q}\right] \otimes \left[\left(x_n\right)_q \left(x_n\right)_q^*\right] \tag{A.17}$$

Consider supermatrices of the form

$$(\overleftrightarrow{Y}_{n,m}) = (\overleftrightarrow{C}(\phi_m) \cdot \overleftrightarrow{X}_{n,m} \cdot \overleftrightarrow{C}(-\phi_m)) \tag{B.1}$$

with $(\overleftrightarrow{X}_{n,m})$ as in Appendix A. Essentially the elementary matrices (dyadics) of the mth supercolumn have been rotated by ϕ_m. Recalling (A.7) we have

$$\overleftrightarrow{X}_{n,m} = \overleftrightarrow{X}_u \quad , \quad u = n - m + 1 \tag{B.2}$$

exhibiting the circulant property (at the outer indices) of $(\overleftrightarrow{X}_{n,m})$. The rotation dyadic $\overleftrightarrow{C}(\phi_m)$ is discussed in Section 5.

From (A.13) and (A.14) we have

$$\begin{aligned}
\overleftrightarrow{X}_{n,m} &= (X_{p,p'})_{n,m} = \sum_{q=1}^{N} \overleftrightarrow{x}_q \, x_{n;q} \, x_{m;q}^{*} \\
\overleftrightarrow{x}_q &= (x_{p,p'})_q = \sum_{u=1}^{N} \overleftrightarrow{X}_{u+1} \, e^{j2\pi\frac{uq}{N}} \\
(\overleftrightarrow{X}_{n,m}) &= \left((X_{p,p'})_{n,m}\right) = \sum_{q=1}^{N} \overleftrightarrow{x}_q \otimes \left[(x_n)_q \, (x_n)_q^{*}\right] \\
(x_n)_q &= \frac{1}{\sqrt{N}} (e^{-j2\pi\frac{nq}{N}})
\end{aligned} \tag{B.3}$$

This, of course, gives

$$\begin{aligned}
\overleftrightarrow{Y}_{n,m} &= \overleftrightarrow{C}(\phi_m) \cdot \overleftrightarrow{X}_{n,m} \cdot \overleftrightarrow{C}(-\phi_m) \\
&= \overleftrightarrow{C}(\phi_m) \cdot \left[\sum_{q=1}^{N} \overleftrightarrow{x}_q \, x_{n;q} \, x_{m;q}^{*}\right] \cdot \overleftrightarrow{C}(-\phi_m) \\
&= \sum_{q=1}^{N} \overleftrightarrow{C}(\phi_m) \cdot \overleftrightarrow{x}_q \cdot \overleftrightarrow{C}(-\phi_m) \, x_{n;q} \, x_{m;q}^{*}
\end{aligned}$$

$$(\overleftrightarrow{Y}_{n,m}) = \sum_{q=1}^{N}\left(\overleftrightarrow{C}(\phi_m) \cdot \overleftrightarrow{x}_q \cdot \overleftrightarrow{C}(-\phi_m)\, x_{n;q}\, x^*_{m;q}\right) \tag{B.4}$$

with $\overleftrightarrow{x}_q = (x_{p,p'})_q$ giving the inner indices and n, m the outer indices.

Now try potential eigensupervectors of the form

$$\begin{aligned}
(\vec{y}_n)_\beta &= ((y_p)_n)_\beta = (\vec{y}_{\delta;n})_q = (\overleftrightarrow{C}(\phi_n \cdot \vec{y}^{(0)}_{\delta;q}\, x_{n;q}) \\
\beta &= (\delta,q) \equiv \text{eigensupervector index} \\
\delta &= 1,2,3,\ \ ,\quad q = 1,2,\cdots,N
\end{aligned} \tag{B.5}$$

Note the rotation of the vector $\vec{y}^{(0)}_{\delta;q}$ as one progresses along increasing n. In addition, there is the complex factor $x_{n;q}$ associated with the circulant eigenvectors in Appendix A.

So now form

$$\begin{aligned}
(\vec{k}_n)_\beta &= (\overleftrightarrow{Y}_{n,m}) \odot (\vec{y}_n)_\beta \\
&= \left[\sum_{q'=1}^{N}(\overleftrightarrow{C}(\phi_m) \cdot \overleftrightarrow{x}_{q'} \cdot \overleftrightarrow{C}(-\phi_n) x_{n;q'} x^*_{m;q'})\right] \odot (\overleftrightarrow{C}(\phi_n) \cdot \vec{y}^{(0)}_{\delta;q}\, x_{n;q}) \\
&= \sum_{q'=1}^{N}\left(\sum_{m=1}^{N}\overleftrightarrow{C}(\phi_m) \cdot \overleftrightarrow{x}_{q'} \cdot \overleftrightarrow{C}(-\phi_m) x_{n;q'} x^*_{m;q'} \cdot \overleftrightarrow{C}(\phi_m) \cdot \vec{y}^{(0)}_{\delta;q}\, x_{m;q}\right) \\
&= \sum_{q'=1}^{N}\left(\sum_{m=1}^{N}\overleftrightarrow{C}(\phi_m) \cdot \overleftrightarrow{x}_{q'} \cdot \vec{y}^{(0)}_{\delta;q}\, x_{n;q'} \cdot x^*_{m;q'}\ x_{m,q}\right) \\
&= \left(\left[\overleftrightarrow{C}(\phi_n)\, x_{n;q}\right] \cdot \left[N\sum_{q'=1}^{N}\ \sum_{m=1}^{N}\overleftrightarrow{C}(\phi_{m-n}) \cdot \overleftrightarrow{x}_{q'}\, x^*_{n;q}\ x_{n;q'}\, x^*_{m;q'}\ x_{m;q}\right] \cdot \vec{y}^{(0)}_{\delta;q}\right) \\
& x_{n;q}\ x^*_{n;q} = \frac{1}{N}
\end{aligned} \tag{B.6}$$

First we have

$$x^*_{n;q}\, x_{n;q'}\, x^*_{m;q'}\, x_{m;q} = \frac{1}{N^2} e^{j2\pi\frac{nq}{N}} e^{-j2\pi\frac{nq'}{N}} e^{j2\pi\frac{mq'}{N}} e^{-j2\pi\frac{mq}{N}}$$

$$= \frac{1}{N^2} e^{j2\pi\frac{n[q-q']}{N}} e^{-j2\pi\frac{m[q-q']}{N}}$$

$$= \frac{1}{N^2} e^{j2\pi\frac{[n-m][q-q']}{N}} \tag{B.7}$$

$$\frac{1}{N^2}\sum_{m'=m_0}^{N-1+m_0} e^{j2\pi\frac{m'q'}{N}} = \frac{1}{N^2}\sum_{m'=0}^{N-1} e^{j2\pi\frac{m'q''}{N}} \quad \text{(cyclic property)}$$

$$= \frac{1}{N} 1_{q'',0}$$

using a geometric series identity [6(B.10)]. Continuing we next have

$$N\sum_{q'=1}^{N}\sum_{m=1}^{N}\overleftrightarrow{C}(\phi_{m-n})\cdot\overleftrightarrow{x}_{q'}\, x^*_{n;q}\, x_{n;q'}\ \ x^*_{m;q'}\, x_{n;q}$$

$$= \frac{1}{N}\sum_{u=1}^{N}\sum_{m=1}^{N}\sum_{q'=1}^{N}\overleftrightarrow{C}(\phi_{m-n})\cdot\overleftrightarrow{X}_{u+1}\, e^{j2\pi\frac{uq}{N}}\, e^{j2\pi\frac{u[q'-q]}{N}}\, e^{j2\pi\frac{[n-m][q-q']}{N}}$$

$$= \frac{1}{N}\sum_{u=1}^{N}\sum_{m=1}^{N}\overleftrightarrow{C}(\phi_{m-n})\cdot\overleftrightarrow{X}_{u+1}\, e^{j2\pi\frac{uq}{N}}\sum_{q'=1}^{N} e^{j2\pi\frac{[u-n+m]\,[q'-q]}{N}} \tag{B.8}$$

$$= \sum_{u=1}^{N}\sum_{m=1}^{N}\overleftrightarrow{C}(\phi_{m-n})\cdot\overleftrightarrow{X}_{u+1}\, e^{j\frac{2\pi uq}{N}}\, 1_{u,n-m}$$

$$= \sum_{u=1}^{N}\overleftrightarrow{C}(\phi_{-u})\cdot\overleftrightarrow{X}_{u+1}\, e^{j\frac{2\pi uq}{N}}$$

$$\equiv \overleftrightarrow{y}_q$$

So now we have

$$(\vec{k}_n)_\beta = (\overleftrightarrow{C}(\phi_n)\cdot\overleftrightarrow{y}_q\cdot\vec{y}_{0;\delta}\, x_{n;q}) \tag{B.9}$$

which is starting to look like (B.5).

Next let the $\vec{y}^{(0)}_{\delta;q}$ diagonalize the $\overleftrightarrow{y}_q$. This has the same form replacing the $\overleftrightarrow{x}_q$ in (A.15) and (A.16) by $\overleftrightarrow{y}_q$. Then we have

$$\overleftrightarrow{y}_q \cdot \vec{y}^{(0)}_{\delta;q} = y_{\delta;q} \; \vec{y}^{(0)}_{\delta;q}$$

$y_{\delta;q}$ = eigenvalues, in general 3 for each q (B.10)

Finally we have

$$(\vec{k}_n)_\beta = y_{\delta;q} \, (\vec{C}(\phi_n) \cdot \vec{y}^{(0)}_{\delta;q} \; x_{n;q})$$
$$= y_{\delta;q} \, (\vec{y}_n)_\beta \qquad \text{(B.11)}$$

demonstrating that the $(\vec{y}_n)_\beta$ are indeed eigensupervectors. Completeness reuires the complete diagonalization of the $\overleftrightarrow{y}_q$.

References

1. C. E. Baum, Canonical Examples for High-Frequency Propagation on Unit Cell of Wave-Launcher Array, Sensor and Simulation Note 317, April 1989.

2. C. E. Baum, Interaction of Electromagnetic Fields with an Object which has an Electromagnetic Symmetry Plane, Interaction Note 63, March 1971.

3. C. E. Baum, High-Frequency Propagation on Nonuniform Multiconductor Transmission Lines in Uniform Media, Interaction Note 463, March 1988; Int'l. J. Numerical Modelling: Electronic Networks, Devices, and Fields, 1988, pp. 175-188.

4. C. E. Baum, Scattering, Reciprocity, Symmetry, EEM, and SEM, Interaction Note 475, May 1989.

5. C. E. Baum, SEM Backscattering, Interaction Note 476, July 1989.

6. J. Nitsch, C. E. Baum, and R. Sturm, The Treatment of Commuting Nonuniform Tubes in Multiconductor-Transmission-Line Theory, Interaction Note 481, May 1990; Analytical Treatment of Circulant Nonuniform Multiconductor Transmission Lines, IEEE Trans. EMC, 1992, pp. 28-38.

7. C. E. Baum, Properties of the Eigenterms of the Impedance Integral Equation, Interaction Note 487, April 1992; Ch. 2, pp. 39-91, in A. Guran, R. Mittra, and P. J. Moser (ed.), *Electromagnetic Wave Interactions*, World Scientific, 1996.

8. C. E. Baum, Low-Frequency Near-Field Magnetic Scattering from Highly, But Not Perfectly Conducting Bodies, Interaction Note 499, November 1993; Ch. 6, pp. 163-218, in C. E. Baum (ed.), *Detection and Identification of Visually Obscured Targets*, Taylor & Francis, 1998.

9. C. E. Baum, The Magnetic-Polarizability Dyadic and Point Symmetry, Interaction Note 502, May 1994; Ch. 7, pp. 219-242, in C. E. Baum (ed.), Detection and Identification of Visually Obscured Targets, Taylor & Francis, 1998.

10. C. E. Baum, Target Symmetry and the Scattering Dyadic, Interaction Note 507, September 1994.

11. C. E. Baum, Symmetry in Electromagnetic Scattering as a Target Discriminant, Interaction Note 523, October 1996; Wideband Interferometric Sensing and Imaging Polarimetry, H. Mott and W. Boerner (eds.), Proc. SPIE, Vol. 3120, 1997, pp. 295-307.

12. C. E. baum, Symmetric Recormalization of the Nonuniform Multiconductor-Transmission-Line Equations with a Single Modal Speed for Analytically Solvable Sections, Interaction Note 537, January 1998.

13. C. E. Baum, Splitting of Degenerate Natural Modes for Buried Targets with Almost-O_2 Symmetry, Interaction Note 545, July 1998.

14. C. E. Baum, Target-Vicinity Scattering Parameters, Interaction Note 546, August 1998.

15. L. Carin, R. Kapoor, and C. E. Baum, Polarimetric SAR Imaging of Buried Landmines, IEEE Trans. Geoscience and Remote Sensing, 1985, pp. 1985-1988.

16. H. W. Kroto, Molecular Rotation Spectra, Wiley, 1975.

17. M. A. Morrison, T. L. Estle, N. F. Lane, Quantum States of Atoms, Molecules, and Solids, prentice-Hall, 1976.

18. C. E. Baum, SEM and EEM Scattering Matrices, and Time-Domain Scatterer Polarization in the Scattering Residue Matrix, Ch. I-9, pp. 427-486, in W.-M. Boerner et al (eds.), *Direct and Inverse Methods in Radar Polarimetry*, Kluwer, 1992.

19. C. E. Baum and H. N. Kritikos, Symmetry in Electromagnetics, Ch. 1, pp. 1-90, in C. E. Baum and H. N. Kritikos (eds.), *Electromagnetic Symmetry*, Taylor & Francis, 1998.