Interaction Notes
Note 557

February 2000

Solutions of the Transmission Line Equations Using Product Integrals of Variable Matrices

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Abstract

Product integrals and Lie algebraic ideas are used to find useful representations of solutions of variable coefficient differential equations that have applications to transmission-line problems.

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Abstract

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1 Introduction

The general transmission-line equations describe the currents and voltages on a tube of transmission lines:

$$\begin{aligned}\frac{dV(z)}{dz} &= -Z(z)\,I(z)\\ \frac{dI(z)}{dz} &= -Y(z)\,V(z)\end{aligned} \tag{1.1}$$

where

z	is the position along the tube
$I(z)$	is the complex current vector with N components
$V(z)$	is the complex voltage vector with N components
$Y(z)$	is the N by N complex admittance matrix
$Z(z)$	is the N by N complex impedance matrix

For more details, see the notes [1] by Baum, Liu, and Tesche. In the applications to transmission lines, the vectors and matrices also depend on s, the complex variable conjugate to time under the Laplace transform. Also, if we find a complete set of analytic solutions then we can easily solve problems with inhomogeneous terms and either initial or boundary conditions, so we do not include these terms in the statement of the problem.

These transmission-line equations (1.1) can be easily written as a single second-order equation for either the currents or the voltages:

$$\frac{d^2 I}{dz^2} - Y' Y^{-1} \frac{dI}{dz} - Y Z I = 0 \quad \text{or} \quad \frac{d^2 V}{dz^2} - Z' Z^{-1} \frac{dV}{dz} - Z Y V = 0\,, \tag{1.2}$$

where $Y' = dY/dz$ and $Z' = dZ/dz$. It is also useful to write these equations as a system of $2\,N$ by $2\,N$ equations:

$$\frac{d}{dz}\begin{bmatrix} V \\ I \end{bmatrix} = -\begin{bmatrix} 0 & Z \\ Y & 0 \end{bmatrix}\begin{bmatrix} V \\ I \end{bmatrix}. \tag{1.3}$$

In fact, the matrices Y and Z cannot be arbitrary. The reciprocity condition requires that

$$Y^T = Y\,, \quad Z^T = Z\,. \tag{1.4}$$

To be clear, if the complex conjugate is given by an over-bar, then the adjoint A^* of a matrix A is given by $A^* = \overline{A}^T$ so we do not necessarily have $Y^* = Y$ or $Z^* = Z$. So in the general case we do not assume the matrices are Hermitian (self-adjoint). The condition that there are no infinite physical parameters requires that the Y and Z matrices be invertible, that is,

$$Y^{-1} \quad \text{and} \quad Z^{-1} \quad \text{exist.} \tag{1.5}$$

As noted above, the matrices Y and Z are functions of a parameter s (the Laplace-transform varible or complex frequency). For apropriate ranges of s, in particular for s purely imaginary, these matrices have positive real part, so we just look at that case and assume that for all vectors V and I

$$Re\left(V^T Y V\right) > 0 \quad \text{and} \quad Re\left(I^T Z I\right) > 0\,. \tag{1.6}$$

To summarize, we will always assume that the matrices in the (1.1) are symmetric and have strictly positive real part.

There are other special cases of interest. When the materials in the cables are uniform, then the matrices are constant in z,

$$Y(z) = Y \quad \text{and} \quad Z(z) = Z\,. \tag{1.7}$$

In the lossless cables case, the vectors and matrices are real for real s and the solutions for all s can be found from $s = \imath$ by rescaling, so we can take

$$Y(z) = \imath\, L(z) \quad \text{and} \quad Z(z) = \imath\, C(z)\,. \tag{1.8}$$

where L is the inductance matrix and C is the capacitance matrix and both L and C are symmetric and thus Hermitian (self-adjoint) and are positive. The most specific case of interest is when there is uniform permeability μ and uniform permittivity ϵ in which case

$$L = \mu F \quad \text{and} \quad C = \epsilon F^{-1} \tag{1.9}$$

where F only depends on the geometry of the cables and is real symmetric, so Hermitian and positive definite.

1.1 Uniform Cables

In the case of uniform cables, we can write the solution of the transmission-line equations (1.1) using either hyperbolic or trigonometric functions. The solution of (1.1) is given by an exponential of the coefficient matrix found in (1.3):

$$\begin{bmatrix} V(z) \\ I(z) \end{bmatrix} = e^{-t\begin{bmatrix} 0 & Z \\ Y & 0 \end{bmatrix}} \begin{bmatrix} V(0) \\ I(0) \end{bmatrix}. \tag{1.10}$$

Now if we write the exponential as a power series and collect every other term into two groups, we obtain

$$e^{-t\begin{bmatrix} 0 & Z \\ Y & 0 \end{bmatrix}} = \begin{bmatrix} \cosh(t\sqrt{ZY}) & -Y^{-1}\sqrt{ZY}\sinh(t\sqrt{ZY}) \\ -Z^{-1}\sqrt{ZY}\sinh(t\sqrt{ZY}) & \cosh(t\sqrt{ZY}) \end{bmatrix}. \tag{1.11}$$

If we write $Y = \imath L$ and $Z = \imath C$, where L is the inductance matrix and C is the capacitance matrix, and both L and C are symmetric and thus Hermitian (self-adjoint), then

$$e^{-\imath t \begin{bmatrix} 0 & C \\ L & 0 \end{bmatrix}} = \begin{bmatrix} \cos(t\sqrt{CL}) & -\imath L^{-1}\sqrt{CL}\sin(t\sqrt{CL}) \\ -\imath C^{-1}\sqrt{CL}\sin(t\sqrt{CL}) & \cos(t\sqrt{CL}) \end{bmatrix}. \tag{1.12}$$

So, in this case, the solution is given by simple waves traveling down the line.

2 Symmetrization of the Transmission Line Equations

In some cases, the Equations (1.3) can be symmetrized by a change of dependent variables. To simplify, assume the material properties are constant and then consider

$$\frac{d}{dz}\begin{bmatrix} AV \\ BI \end{bmatrix} = -\begin{bmatrix} 0 & AZB^{-1} \\ BYA^{-1} & 0 \end{bmatrix}\begin{bmatrix} AV \\ BI \end{bmatrix}. \tag{2.13}$$

where the matrices A and B are to be chosen so that the matrix in the previous equation is symmetric, that is, so that

$$AZB^{-1} = BYA^{-1}, \tag{2.14}$$

or

$$Y = B^{-1}AZB^{-1}A. \tag{2.15}$$

Setting $M = B^{-1}A$, given Y and Z we must find M so that

$$Y = MZM. \tag{2.16}$$

This is a system of N^2 quadratic equations in N^2 unknowns, so it is not clear if there even exist complex solutions or not. However, if Y and Z commute then there is a solution:

$$[Y, Z] = YZ - ZY = 0 \Rightarrow M = \sqrt{Y}\,\sqrt{Z}^{-1} = \sqrt{Z}^{-1}\sqrt{Y}. \tag{2.17}$$

So in the case of commuting matrices, the simple choice of

$$A = \sqrt{Y}, \quad B = \sqrt{Z} \tag{2.18}$$

will make the system symmetric.

2.1 Variable Coefficients and Symmetry

We now assume that for each z, the matrices in the transmission line equations (1.1) commute:

$$[Y(z), Z(z)] = Y(z)\,Z(z) - Z(z)\,Y(z) = 0. \tag{2.19}$$

If we introduce the new variables

$$\tilde{V}(z) = A(z)\,V(z), \quad \tilde{I}(z) = B(z)\,I(z), \tag{2.20}$$

then a simple computation gives

$$\begin{aligned} \frac{d\tilde{V}(z)}{dz} &= \frac{dA(z)}{dz}A^{-1}(z)\,\tilde{V} - A(z)\,Z(z)\,B^{-1}(z)\,\tilde{I}(z) \\ \frac{d\tilde{I}(z)}{dz} &= \frac{dB(z)}{dz}B^{-1}(z)\,\tilde{I} - B(z)\,Y(z)\,A^{-1}(z)\,\tilde{V}(z) \end{aligned} \tag{2.21}$$

These equations can be written as a system

$$\frac{d}{dz}\begin{bmatrix}\tilde{V}\\ \tilde{I}\end{bmatrix}=\begin{bmatrix}A'(z)\,A^{-1}(z) & A(z)\,Z(z)\,B^{-1}(z)\\ B(z)\,Y(z)\,A^{-1}(z) & B'(z)\,B^{-1}(z)\end{bmatrix}\begin{bmatrix}\tilde{V}\\ \tilde{I}\end{bmatrix}. \tag{2.22}$$

The choice of

$$A(z)=\sqrt{Y(z)}\,,\quad B(z)=\sqrt{Z(z)} \tag{2.23}$$

makes the off-diagonal part of the system symmetric, and the diagonal parts are product derivatives (also called product derivatives, but the order of the factors is important).

2.2 Problems With Non-Commuting Matrices

Here we indicate why the dealing with matrix functions of z that do not commute is difficult. So, if a matrix function $M(z)$ has the property that the matrices for different values of z commute, that is,

$$[M(z_1),M(z_2)]=0 \tag{2.24}$$

for all z_1 and z_2, and $f(x)$ is any differentiable function, then

$$\frac{d}{dz}f(M(z))=\frac{df}{dx}(M(z))\,\frac{dM}{dz}(z)\,, \tag{2.25}$$

that is, the chain rule works. Without the commuting condition, the chain rule does not work in general, as elementary examples will show.

There is one important exception for the function $1/x$. If we differentiate $M(z)\,M^{-1}(z)=$ Identity, the we get

$$\frac{d}{dz}M^{-1}(z)=-M^{-1}(z)\left(\frac{d}{dz}M(z)\right)M^{-1}(z)\,. \tag{2.26}$$

One important problem is that the derivative of a square root of a matrix cannot be found in closed form, but we can find a simple equation for the square root. To see this, we follow the calculus derivation of the derivative. We have that

$$M(z)=\sqrt{Y(z)}\Rightarrow M^2(z)=Y(z)\,. \tag{2.27}$$

and then differentiation gives

$$M(z)\,\frac{dM(z)}{dz}+\frac{dM(z)}{dz}M(z)=Y'(z)\,. \tag{2.28}$$

This doesn't lead to a simple formula for the derivative of the square root. It does give a $N\times N$ system of nonlinear equations to find the derivative of the the square root, but again it is difficult to know when this system is solvable.

We would also like to have a simple form for the product derivatives

$$\frac{dA(z)}{dz}A^{-1}, \quad \frac{dB(z)}{dz}B^{-1} \tag{2.29}$$

that appear in the transformed equations. However, these are not the derivatives of $\ln(A(z))$ and $\ln(B(z))$ unless the commuting condition (2.24) holds. If we set

$$M(z) = \ln(A(z)), \tag{2.30}$$

then

$$e^{M(z)} = A(z). \tag{2.31}$$

This last expression can be differentiated using Formula 6.196 in [4], but this produces a formula difficult to solve for dM/dz.

2.3 Product Integrals and Derivatives

The theory of product integrals and product derivatives was invented to deal with just the situation described in the previous section. Using the definition of the product integral (see Section 5 of Baum and Steinberg [4] for definitions and basic properties) applied to the exponential of $M(z)$, we define

$$G(z) = \prod_{0}^{z} e^{ds\, M(s)}, \tag{2.32}$$

and then we get (see [4], Equation (5.155))

$$\frac{dG}{dz}(z) = M(z)\, G(z). \tag{2.33}$$

Also (see [4], Equation (5.158)) the product derivative is defined as

$$D_z G(z) = G'(z)\, G^{-1}(z), \tag{2.34}$$

so that

$$D_z G(z) = G'(z)\, G^{-1}(z) = M(z). \tag{2.35}$$

We also have that (see [4], Equation (5.160))

$$G(z) = \prod_{0}^{z} e^{D_s G(s)\, ds}. \tag{2.36}$$

One of the more interesting formulas for product integrals is the analog of the formula $\exp(a+b) = \exp(a)\exp(b)$ ((see [4], Equation (5.163)): if

$$G(z) = \prod_{0}^{z} e^{A(s)\, ds}, \tag{2.37}$$

then

$$\prod_{0}^{t} e^{(A(s)+B(s))\, ds} = G(z) \prod_{0}^{z} e^{G^{-1}(s)\, B(s)\, G(s)\, ds}. \tag{2.38}$$

We will use this formula in the next section.

2.4 Product Integral Reduction of the Transmission Line Equations

Here we show that the sum rule (2.38) can be used to check that we have correctly transformed the transmission line equations or, conversely, that the sum rule is correct. We rewrite Equation (2.22) as

$$\frac{d}{dz}\begin{bmatrix} \tilde{V} \\ \tilde{I} \end{bmatrix} = \left(\begin{bmatrix} A'A^{-1} & 0 \\ 0 & B'B^{-1} \end{bmatrix} + \begin{bmatrix} 0 & AZB^{-1} \\ BYA^{-1} & 0 \end{bmatrix} \right) \begin{bmatrix} \tilde{V} \\ \tilde{I} \end{bmatrix}, \tag{2.39}$$

so and then apply the sum rule (2.38) with

$$\begin{aligned} A &\rightarrow \begin{bmatrix} A'A^{-1} & 0 \\ 0 & B'B^{-1} \end{bmatrix}, \\ B &\rightarrow \begin{bmatrix} 0 & AZB^{-1} \\ BYA^{-1} & 0 \end{bmatrix}, \end{aligned} \tag{2.40}$$

so that

$$\begin{aligned} G &\rightarrow \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \\ G^{-1}BG &\rightarrow \begin{bmatrix} 0 & Z \\ Y & 0 \end{bmatrix}, \end{aligned} \tag{2.41}$$

and then

$$\prod_0^t e^{\begin{bmatrix} A'(s)A^{-1}(s) & A(s)Z(s)B^{-1}(s) \\ B(s)Y(s)A^{-1}(s) & B'(s)B^{-1}(s) \end{bmatrix} ds} = \begin{bmatrix} A(z) & 0 \\ 0 & B(z) \end{bmatrix} \prod_0^t e^{\begin{bmatrix} 0 & Z(s) \\ Y(s) & 0 \end{bmatrix} ds}, \tag{2.42}$$

which agrees with the definition (2.20) which we can write as

$$\begin{bmatrix} \tilde{V}(z) \\ \tilde{I}(z) \end{bmatrix} = \begin{bmatrix} A(z) & 0 \\ 0 & B(z) \end{bmatrix} \begin{bmatrix} V(z) \\ I(z) \end{bmatrix}. \tag{2.43}$$

3 Diagonalizing Symmetric Transmission Line Equations

In this section we assume the commuting property (2.19)

$$[Y(z), Z(z)] = Y(z)\,Z(z) - Z(z)\,Y(z) = 0\,. \tag{3.44}$$

and that one of the matrices $Y(z)$ or $Z(z)$ can be diagonalized. Then, because of the commuting property, both of the matrices can be diagonalized and have a complete common set of eigen-vectors $\phi_i(z)$ for all z. Define the S matrix to be the inverse of the matrix with columns the eigenvectors:

$$S(z) = [\phi_1(z), \cdots \phi_N(z)]^{-1}\,, \tag{3.45}$$

and then we have

$$\Lambda_Y(z) = S(z)\,Y(z)\,S^{-1}(z)\,, \quad \Lambda_Z(z) = S(z)\,Z(z)\,S^{-1}(z)\,, \tag{3.46}$$

where Λ_Y and Λ_Z are diagonal. If we now introduce the variables

$$\tilde{I}(z) = S(z)I(z)\,, \quad \tilde{V}(z) = S(z)V(z)\,, \tag{3.47}$$

then the transmission line equations can be written as in (2.22):

$$\frac{d}{dz}\begin{bmatrix} \tilde{V} \\ \tilde{I} \end{bmatrix} = \begin{bmatrix} S'(z)\,S^{-1}(z) & S(z)\,Z(z)\,S^{-1}(z) \\ S(z)\,Y(z)\,S^{-1}(z) & S'(z)\,S^{-1}(z) \end{bmatrix} \begin{bmatrix} \tilde{V} \\ \tilde{I} \end{bmatrix}. \tag{3.48}$$

The eigen-decomposition (3.46) then gives

$$\frac{d}{dz}\begin{bmatrix} \tilde{V} \\ \tilde{I} \end{bmatrix} = \begin{bmatrix} S'(z)\,S^{-1}(z) & \Lambda_Z(z) \\ \Lambda_Y(z) & S'(z)\,S^{-1}(z) \end{bmatrix} \begin{bmatrix} \tilde{V} \\ \tilde{I} \end{bmatrix}. \tag{3.49}$$

If the diagonal product derivative terms are zero, then the system is uncoupled:

$$\frac{d}{dz}\begin{bmatrix} \tilde{V} \\ \tilde{I} \end{bmatrix} = \begin{bmatrix} 0 & \Lambda_Z(z) \\ \Lambda_Y(z) & 0 \end{bmatrix} \begin{bmatrix} \tilde{V} \\ \tilde{I} \end{bmatrix}. \tag{3.50}$$

These terms will be zero if the eigenvectors are constant. If they are not zero, then the system doesn't uncouple and the standard solution techniques cannot be applied.

3.1 Circulant Matrices

Circulant matrices give an ideal situation in which to apply the diagonalization techniques because they have constant eigenvectors. Circulant matrices, which are constant along "diagonals", are discussed in detail in Nitsch, Baum and Sturm [3] and Davis [2]. Here we adopt a definition in terms a basis which is convenient for Lie algebraic Computations. In particular, we will define circulant matrices in terms of the shift operation.

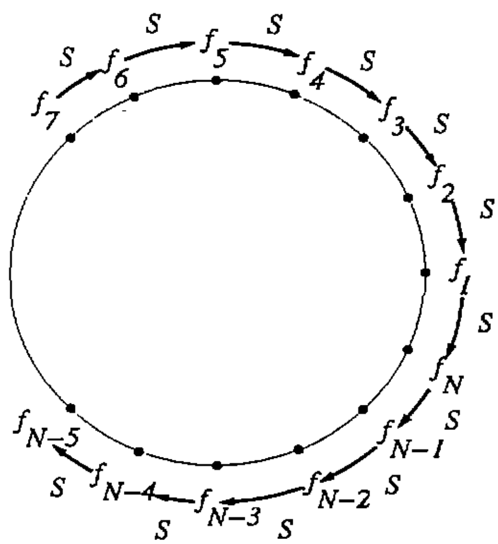

Figure 3.1: *Shift on a Circle*

As shown in Figure 3.1, put N evenly distributed points labeled by $1, 2, \cdots N-1, N$ on the unit circle and then assigning the values $f = \{f_1, f_2, \cdots f_N\}$ to these points. The shift $S = S(N)$ in the clockwise direction moves the value at the point 1 to the point $N-1$, the value at the point 2 to the point 1 and so forth. So if $g = Sf$, then

$$\begin{aligned} g_1 &= (Sf)_1 = f_2\,, \\ g_2 &= (Sf)_2 = f_3\,, \end{aligned} \tag{3.51}$$

$$g_N = (Sf)_N = f_1\,. \tag{3.52}$$

This operation can be represented by a matrix:

$$\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{N-1} \\ g_N \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix} \tag{3.53}$$

and so

$$S = S(N) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} . \tag{3.54}$$

This matrix plays a fundamental role in what follows.

By integer division we can represent any integer n as $n = q\,N + r$ where $0 \le r < N$ and then the *mod* function is defined by $\mathrm{mod}(n, N) = r$. So the matrix for the shift is given by

$$S_{i,j} = \begin{cases} 1 & \text{if } \mathrm{mod}(i - j + 1, N) = 0 \\ 0 & \text{if } \mathrm{mod}(i - j + 1, N) \neq 0 \end{cases} \tag{3.55}$$

In fact, we will define a family of shift operators $\mathrm{Sh}(k) = \mathrm{Sh}(k, N)$:

$$\mathrm{Sh}(k)_{i,j} = \begin{cases} 1 & \text{if } \mathrm{mod}(i - j + k, N) = 0 \\ 0 & \text{if } \mathrm{mod}(i - j + k, N) \neq 0 \end{cases} \tag{3.56}$$

The geometric interpretation of $\mathrm{Sh}(k)$ is that the values at the points on the unit circle are shifted clockwise k points if $k.geq0$ and counter clockwise $-k$ points if $k \le 0$. The matrices $\mathrm{Sh}(k, N)$ are distinct for $0 \le k < N$ and $\mathrm{Sh}(k + m\,N, N) = \mathrm{Sh}(k, N)$ for all m. Also $\mathrm{Sh}(0, N)$ is the identity matrix and $\mathrm{Sh}(1, N) = \mathrm{Sh}(1) = S(N) = S$.

For $N = 5$ the distinct shift operators are:

$$\mathrm{Sh}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{Sh}(1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.57}$$

$$\mathrm{Sh}(2) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{Sh}(3) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \tag{3.58}$$

$$\mathrm{Sh}(4) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} . \tag{3.59}$$

From the geometric interpretation of the shift operator, it is clear that

$$\mathrm{Sh}(k_1, N)\,\mathrm{Sh}(k_2, N) = \mathrm{Sh}(k_1 + k_2, N) . \tag{3.60}$$

An algebraic proof of this requires only a little work with the mod function. This immediately implies that

$$\mathrm{Sh}(k, N) = S^k(N)\,, \tag{3.61}$$

for all integers k.

A *circulant* matrix is a matrix that can be written as a linear combination of shift matrices:

$$\mathrm{Ci}(c_0, c_1, \cdots c_{N-1}) = \sum_{k=0}^{N-1} c_k S^k\,. \tag{3.62}$$

Such matrices are constant on along the "diagonals". It is show in [2] that other common definitions of circulant are equivalent to this definition. In any case, all five dimensional circulant matrices are given by

$$\mathrm{Ci}(c_0, c_1, c_2, c_3, c_4) = c_0\,\mathrm{Sh}(0) + c_1\,\mathrm{Sh}(1) + c_2\,\mathrm{Sh}(2) + c_3\,\mathrm{Sh}(3) + c_4\,\mathrm{Sh}(4) \tag{3.63}$$

$$= \begin{bmatrix} c_0 & c_1 & c_2 & c_3 & c_4 \\ c_4 & c_0 & c_1 & c_2 & c_3 \\ c_3 & c_4 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_4 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_4 & c_0 \end{bmatrix}. \tag{3.64}$$

3.2 Diagonalizing Circulant Matrices

Because circulant matrices can be written as a polynomial in the shift S (3.62), they all commute under multiplication and consequently can be diagonalized simultaneously. If fact, if we find a basis that diagonalizes S, then the same basis diagonalizes all powers of S, and consequently all circulant matrices. It is well known that the discrete Fourier transform diagonalizes all translation invariant operators, that is, all operators the commute with the translation S, so lets look at the discrete Fourier transform.

Let ω be a ***primitive*** N-th root of unity. Typically we take

$$\omega = e^{\pm\frac{2\pi i}{N}}\,, \tag{3.65}$$

but for many computations it is only important that $\omega \neq 1$. Then we have that

$$0 = \omega^N - 1 = (\omega - 1)\sum_{k=0}^{N-1} \omega^k\,, \tag{3.66}$$

which gives us the important identity

$$\sum_{k=0}^{N-1} \omega^k = 0\,. \tag{3.67}$$

In general, we need to know which factors of $\omega^N - 1$ the primitive N-th root satisfies. We take the Fourier, or more precisely the discrete Fourier transform matrix to be

$$F_{i,j} = F_{i,j}(N) = \frac{1}{\sqrt{N}}\omega^{-(i-1)(j-1)}\,, \quad 1 \le i \le N\,, 1 \le j \le N\,. \tag{3.68}$$

Note that the adjoint (complex conjugate transpose) $F^\dagger$ of F is given by

$$F^\dagger_{i,j} = F^\dagger(N)_{i,j} = \frac{1}{\sqrt{N}}\omega^{+(i-1)(j-1)}\,. \tag{3.69}$$

There are many choices possible for the discrete Fourier transform matrix. Our choice gives

$$F^\dagger S\, F = \mathrm{diag}(1, w^{-1}, w^{-2}, w^{-3}, \cdots, w^{1-N})\,, \tag{3.70}$$

that is, the k-th column of F is an eigenvector of S with eigenvalue ω^{1-k}. Additionally, F is a unitary matrix.

When we solve the transmission-line equations using Lie techniques, a critical ingredient is the exponential of each basis element. For circulant matrices, a basis is S^k, $0 \le k \le N$. We have diagonalized S, so

$$F^\dagger S^k\, F = \mathrm{diag}(1, w^{-k}, w^{-2k}, w^{-3k}, \cdots, w^{(1-N)k})\,, \tag{3.71}$$

and then we see that

$$e^{t\,S^k} = F\,\mathrm{diag}(1, e^{t\,w^{-k}}, e^{t\,w^{-2k}}, e^{t\,w^{-3k}}, \cdots, e^{t\,w^{(1-N)k}})F^\dagger\,. \tag{3.72}$$

To apply this to the transmission-line equations (1.3),

$$\frac{d}{dz}\begin{bmatrix} V \\ I \end{bmatrix} = -\begin{bmatrix} 0 & Z \\ Y & 0 \end{bmatrix}\begin{bmatrix} V \\ I \end{bmatrix}, \tag{3.73}$$

we need to write Z and Y in terms of S:

$$Z(z) = \sum_{k=0}^{N-1} b_k(z)\, S^k\,, \quad Y(z) = \sum_{k=0}^{N-1} a_k(z)\, S^k\,. \tag{3.74}$$

In this case the system of equations can be rewritten as

$$\frac{d}{dz}\begin{bmatrix} V \\ I \end{bmatrix} = \sum_{i,j=0}^{N-1} -a_i(z)\, b_j(z)\begin{bmatrix} 0 & S^j \\ S^i & 0 \end{bmatrix}\begin{bmatrix} V \\ I \end{bmatrix}, \tag{3.75}$$

so we only need to know how to exponentiate the matrices

$$\begin{bmatrix} 0 & S^j \\ S^i & 0 \end{bmatrix}, \quad 0 \le i, j \le N-1\,. \tag{3.76}$$

However,

$$\begin{bmatrix} F^\dagger & 0 \\ 0 & F^\dagger \end{bmatrix}\begin{bmatrix} 0 & S^j \\ S^i & 0 \end{bmatrix}\begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix} = \begin{bmatrix} 0 & (F^\dagger S F)^j \\ (F^\dagger S F)^i & 0 \end{bmatrix}, \tag{3.77}$$

which gives a "skew diagonalization" of the matrix involved in the exponential.

References

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