Interaction Notes

Note 562

22 October 2000

The Boundary-Connection Supermatrix for Uniform Isotropic Walls

Carl E. Baum

Air Force Research laboratory
Directed Energy Directorate

Abstract

A useful way to describe a wall of non-zero thickness is the boundary-connection supermatrix (BCS) relating the tangential components of the electromagnetic fields on the two surfaces of the wall. This paper develops the BCS for uniform isotropic walls. This is applied to the canonical problem of the spherical shell (shield). A delay corrected BCS is also introduced to give the wall an equivalent zero thickness.

This work was sponsored in part by the Air Force Office of Scientific Research, Arlington, VA, and in part by the Air Force Research Laboratory, Directed Energy Directorate.

TABLE OF CONTENTS

1. Introduction

In the 1970's, discussion began on a subject to become known as the boundary connection supermatrix. In this regard, I was fortunate to have colleagues Dr. Kendall F. Casey, who was applying this to composite materials [4], and Dr. (later Prof.) Korada Umashankar (deceased) who was applying this to wire cages [1]. We exchanged numerous ideas and I outlined some possible papers. In 1978, Prof. James R. Wait (deceased) suggested that I write a book on the theory of electromagnetic shielding, of which a chapter in my outline would have covered this subject. I have also suggested on various occasions that Dr. Kelvin S. H. Lee should write a book on EM shielding.

So, resurrecting my old notes and extending them, the present paper has resulted. After defining the general concept of the BCS and the delay-corrected BCS, these are computed for a uniform isotropic wall (of thickness Δ). Noting the differences for E and H waves, conditions are developed under which the BCS and delay-corrected BCS are approximately independent of the details of the electromagnetic fields outside the wall, relying ony on the tangential components of the electric and magnetic fields on both surfaces. This is then applied to the canonical problem of the spherical shell and compared to the exact solution contained in the appendices.

2. Boundary Connection Supermatrix

The basic idea of a BCS is to relate the tangential components of both electric and magnetic fields on two nearty surfaces (locally parallel) where some structure (to which we refer as a wall) is located between these two surfaces. In mathematical terms we are looking for a BCS defined by

$$\begin{pmatrix} \overleftrightarrow{1}_t \cdot \tilde{\vec{E}}^{(2)}(\vec{r}_2, s) \\ Z_2 \overleftrightarrow{1}_t \cdot \tilde{\vec{H}}^{(2)}(\vec{r}_2, s) \end{pmatrix} = (\tilde{\overleftrightarrow{B}}_{u,v}(2,1;s)) \odot \begin{pmatrix} \overleftrightarrow{1}_t \cdot \tilde{\vec{E}}^{(1)}(\vec{r}_1, s) \\ Z_1 \overleftrightarrow{1}_t \cdot \tilde{\vec{H}}^{(1)}(\vec{r}_1, s) \end{pmatrix}$$

$(\tilde{\overleftrightarrow{B}}(2,1;s)) \equiv$ boundary connection supermatrix (BCS)

$\sim$ $\equiv$ two-sided Laplace transform over time t (2.1)

$s = \Omega + j\omega$ $\equiv$ Laplace-transform variable or complex frequency

The 1 and 2 indices refer to surfaces S_1 and S_2 in Fig. 2.1 separated by a distance Δ (typically small). These surfaces are locally approximately planar, i.e., have radii of curvature large compared to Δ. If the spacing Δ is allowed to vary, it should do so *slowly* so as to retain a constant Δ as an approximate local condition.

Note that $\vec{r}_1$ (on S_1) and $\vec{r}_2$ (on S_2) are assumed to be corresponding positions separated by Δ, i.e., at points of closest approach. The two wave impedances Z_1 and Z_2 refer to the two media on opposite sides of the wall. Often they both correspond to free space with

$$Z_o = \left[\frac{\mu_0}{\varepsilon_0}\right]^{\frac{1}{2}} \equiv \text{wave impedance of free space}$$

$$c = [\mu_0 \varepsilon_0]^{-\frac{1}{2}} \equiv \text{propagation speed in free space}$$

$$\gamma_0 = \frac{s}{c} \equiv \text{propagation constant in free space}$$

$$\mu_0 \equiv \text{permeability of free space} \qquad (2.2)$$

$$\varepsilon_0 \equiv \text{permittivity of free space}$$

In the simplest case, the tangential fields are slowly varying along each S_n. This assumes that radian wavelengths $\lambda\!\!\!^{-}$ in the two media external to the wall are large compared to Δ, and that the wall is uniform, i.e., not varying in its properties in directions parallel to the S_n. (It may be layered, i.e., varying in the direction

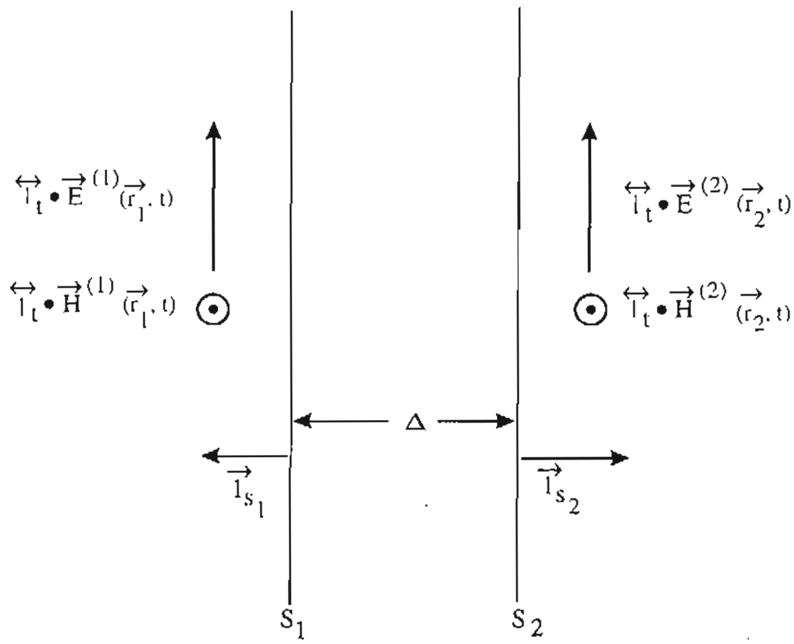

Fig. 2.1 Locally Flat, Parallel, Closely Spaced Boundary Surfaces.

perpendicular to the S_n, i.e., the unit surface normals $\vec{1}_{S_n}$.) However, some kinds of walls have significant transverse variation, e.g, wire mesh, rebars, etc. In this case, λ should be large compared to the transverse periods (two transverse directions) of the wall variation. In this case, the tangential fields (including incident, scattered, and transmitted (from other-side components) are not locally uniform. In such a case, the tangential fields need to be reinterpreted as appropriate averages.

Since we are dealing with transverse components on the S_n, we can regard the vectors as having two components. The supervectors then have four components and the BCS is 4 x 4. In this case we have the BCS inverse

$$(\tilde{\overleftrightarrow{B}}_{u,v}(2,1;s)) = (\tilde{\overleftrightarrow{B}}_{u,v}(2,1;s))^{-1} \tag{2.3}$$

In some cases one can successively dot multiply BCSs for a sequence of walls to obtain a composite BCS for the set of walls. The BCS is a kind of chain matrix (or matrizant) in these cases. However, transversely (perpendicular to $\vec{1}_{S_n}$) varying walls have nonuniform tangential fields, which can upset this chain-matrix property. Such cases require special consideration.

One can also define a delay-corrected BCS to account for the transit time (or phase shift) of an electromagnetic wave through the wall thickness Δ in the absence of the wall. This allows, as an approximation, one to refer the tangential fields to surfaces S_{1+} and S_{2-} which in the limit are the same surface S_0 somewhere between S_1 and S_2 as indicated in Fig. 2.2. This can be convenient in some computations. Formally we can construct such a delay corrected BCS as

$$(\tilde{\overleftrightarrow{B}}^{(d)}(2_-,1_+;s)) = (\tilde{\overleftrightarrow{B}}(2_-,2;s)) \odot (\tilde{\overleftrightarrow{B}}(2,1;s)) \odot (\tilde{\overleftrightarrow{B}}(1,1_+;s)) \tag{2.4}$$

That this is an approximation will be clearer when we consider the effects of angle of incidence of the wave(s) outside the wall. Note that the outer two supermatrices in the product refer to a hypothetical wall with properties on the 2 or 1 sides of the original wall.

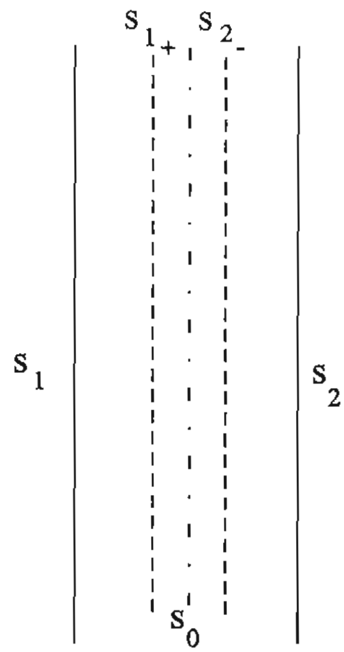

Fig. 2.2 Reference Surface for Delay-Corrected BCS.

3. Boundary Connection Supermatrix for Uniform Isotropic Wall

The simplest wall is a uniform isotropic one described by

$$Z_2 = \left[\frac{\mu_2}{\sigma + \_ s\varepsilon_2}\right]^{\frac{1}{2}} \equiv \text{wave impedance}$$

$$\gamma_2 = \left[s\mu_2\left[\sigma + s\varepsilon_2\right]\right]^{\frac{1}{2}} \equiv \text{propagation constant}$$

$$\mu_r \equiv \frac{\mu_2}{\mu_0} \equiv \text{relative permeability} \tag{3.1}$$

$$\frac{Z_2}{Z_0} = \mu_r \frac{\gamma_0}{\gamma_2}$$

As indicated in Fig. 3.1 we have a set of Cartesian coordinates for this section with z perpendicular to the wall, occupying $z_1 < z < z_2$. The plane of incidence is taken as the yz plane. This gives the usual radar (h, v) coordinates with

$$\vec{1}_h = -\vec{1}_x \quad , \quad \vec{1}_i = -\vec{1}_h \times \vec{1}_v \equiv \text{direction of incidence} \tag{3.2}$$

The angle of incidence θ_i is measured with respect to the z axis with

$$\vec{1}_i = \left[\vec{1}_y\vec{1}_y + \vec{1}_z\vec{1}_z\right] \cdot \vec{1}_i = \vec{1}_y \sin(\theta_i) + \vec{1}_z \cos(\theta_i)$$

$$0 \le \theta_i < \frac{\pi}{2} \tag{3.3}$$

$$\vec{1}_v = \left[\vec{1}_y\vec{1}_y + \vec{1}_z\vec{1}_z\right] \cdot \vec{1}_v = \vec{1}_y \cos(\theta_i) - \vec{1}_z \sin(\theta_i)$$

The wall thickness is

$$\Delta = z_1 - z_2 \tag{3.4}$$

with the choice of z_1 (or z_2) taken at our convenience.

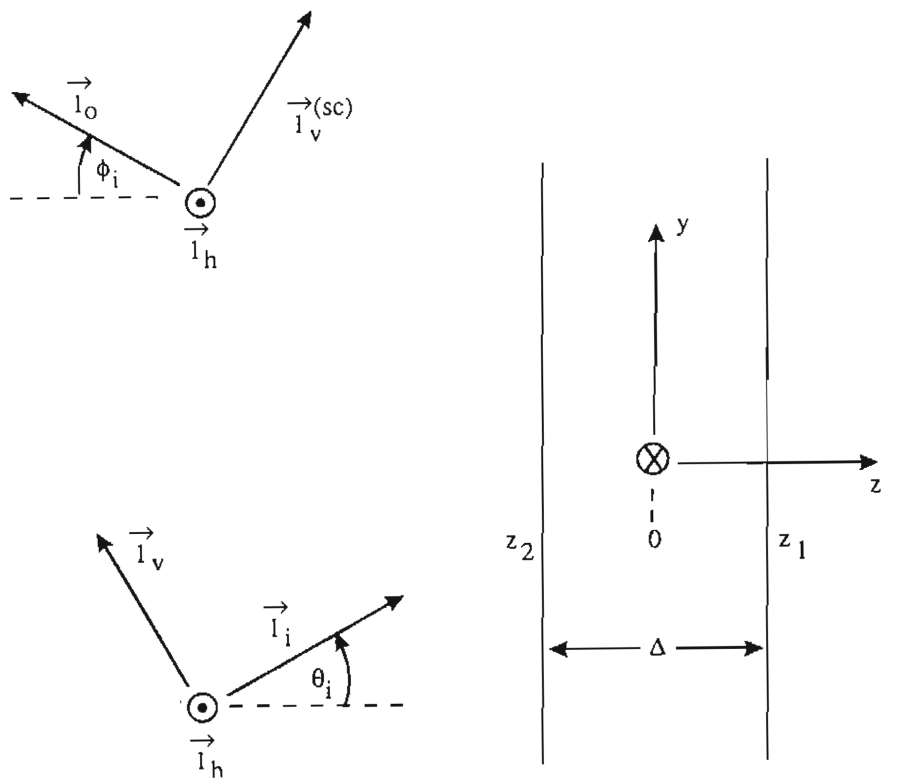

Fig. 3.1 Uniform Isotropic Wall.

3.1 H-wave

The incident wave in volume 1 is an H (or TE) wave as

$$
\begin{aligned}
\vec{\tilde{E}}^{(inc;h)}(\vec{r},s) &= \vec{1}_h E_0^{(h)} e^{-\gamma_0 \vec{1}_i \cdot \vec{r}} \\
Z_0 \vec{\tilde{H}}^{(inc;h)}(\vec{r},s) &= -\vec{1}_v E_0^{(h)} e^{-\gamma_0 \vec{1}_i \cdot \vec{r}} \quad , \quad z < z_1
\end{aligned}
\tag{3.5}
$$

The scattered wave in volume 1 is

$$
\begin{aligned}
\vec{\tilde{E}}^{(inc;h)}(\vec{r},s) &= \vec{1}_h E_1^{(h)} e^{-\gamma_0 \vec{1}_o \cdot \vec{r}} \\
Z_0 \vec{\tilde{H}}^{(sc;h)}(\vec{r},s) &= \vec{1}_v^{(sc)} E_1^{(h)} e^{-\gamma_0 \vec{1}_o \cdot \vec{r}} \\
\vec{1}_o &= \left[\vec{1}_y \vec{1}_y - \vec{1}_z \vec{1}_z\right] \cdot \vec{1}_i = \vec{1}_y \sin(\theta_i) - \vec{1}_z \cos(\theta_i) \\
\vec{1}_v^{(sc)} &= \left[\vec{1}_y \vec{1}_y - \vec{1}_z \vec{1}_z\right] \cdot \vec{1}_v = \vec{1}_y \cos(\theta_i) + \vec{1}_z \sin(\theta_i)
\end{aligned}
\tag{3.6}
$$

The tangential fields on S_1 are then

$$
\begin{aligned}
\vec{\tilde{E}}_t^{(1,h)}(s) &= \vec{1}_z \cdot \left[\vec{1}_h E_0^{(h)}(s) e^{-\gamma_0\left[y\sin(\theta_i) + z_1\cos(\theta_i)\right]} + \vec{1}_h E_1^{(h)}(s) e^{-\gamma_0\left[y\sin(\theta_i) - z_1\cos(\theta_i)\right]}\right] \\
&= -\vec{1}_x e^{-\gamma_0 y \sin(\theta_i)}\left[E_0^{(h)}(s) e^{-\gamma_0 z_1 \cos(\theta_i)} + E_1^{(h)}(s) e^{\gamma_0 z_1 \cos(\theta_i)}\right] \\
Z_0 \vec{H}_t^{(1,h)}(s) &= \vec{1}_z \cdot \left[-\vec{1}_v \tilde{E}_0^{(h)}(s) e^{-\gamma_0\left[y\sin(\theta_i) + z_1\cos(\theta_i)\right]} + \vec{1}_v^{(sc)} \tilde{E}_1^{(h)}(s) e^{-\gamma_0\left[y\sin(\theta_i) - z_1\cos(\theta_i)\right]}\right] \\
&= -\vec{1}_y \cos(\theta_i) e^{-\gamma_0 y \sin(\theta_i)}\left[E_0^{(h)}(s) e^{-\gamma_0 z_1 \cos(\theta_i)} - \tilde{E}_1^{(h)}(s) e^{\gamma_0 z_1 \cos(\theta_i)}\right] \\
\vec{1}_z &\equiv \overleftrightarrow{1} - \vec{1}_z \vec{1}_z = \vec{1}_x \vec{1}_x + \vec{1}_y \vec{1}_y \equiv \text{transverse (to } z\text{) identity}
\end{aligned}
\tag{3.7}
$$

Note the common term $e^{-\gamma_0 y \sin(\theta_i)}$ which expresses the variation transverse to z. All our subsequent fields will be constrained to have this variation to match boundary conditions on the S_n.

The transmitted field in volume 2 is also an H wave of the form

$$\begin{aligned}
\tilde{\vec{E}}^{(t,h)}(\vec{r},s) &= \vec{1}_h E_2^{(h)} e^{-\gamma_0 \vec{1}_i \cdot \vec{r}} \\
Z_0 \tilde{\vec{H}}^{(t,h)}(\vec{r},s) &= -\vec{1}_v E_2^{(h)} e^{-\gamma_0 \vec{1}_i \cdot \vec{r}} \quad , \quad z > z_2
\end{aligned} \tag{3.8}$$

The tangential fields on S_2 are then

$$\begin{aligned}
\tilde{\vec{E}}_t^{(t,h)}(s) &= \overleftrightarrow{1}_z \cdot \vec{1}_h E_2^{(h)}(s) e^{-\gamma_0 \left[y\sin(\theta_i) + z_2 \cos(\theta_i) \right]} \\
&= -\vec{1}_x E_2^{(h)}(s) e^{-\gamma_0 y \sin(\theta_i)} e^{-\gamma_0 z_2 \cos(\theta_i)} \\
Z_0 \vec{H}_t^{(1,h)}(s) &= -\vec{1}_z \cdot \vec{1}_v \tilde{E}_2^{(h)}(s) e^{-\gamma_0 \left[y\sin(\theta_i) + z_2 \cos(\theta_i) \right]} \\
&= -\vec{1}_y \cos(\theta_i) \tilde{E}_2^{(h)}(s) e^{-\gamma_0 y \sin(\theta_i)} e^{-\gamma_0 z_2 \cos(\theta_i)}
\end{aligned} \tag{3.9}$$

Inside the wall we need the parameters in (3.1) in two waves (right and left propagating) with the same transverse propagation as above. The waves here are more complicated due to the complex directions (unit vectors and associated angles). We have the generalized plane waves

$$\begin{aligned}
&\tilde{\vec{E}}^{(in,h)}(\vec{r},s) = \sum_{\pm} \tilde{\vec{E}}_{\pm}^{(h)}(s) e^{-\gamma_2 \vec{1}_{\pm} \cdot \vec{r}} = \sum_{\pm} \tilde{\vec{E}}_{\pm}^{(h)}(s) e^{-\gamma_y y \mp \gamma_z z} \\
&\tilde{\vec{H}}^{(in,h)}(\vec{r},s) = \sum_{\pm} \tilde{\vec{H}}_{\pm}^{(h)}(s) e^{-\gamma_2 \vec{1}_{\pm} \cdot \vec{r}} = \sum_{\pm} \tilde{\vec{H}}_{\pm}^{(h)}(s) e^{-\gamma_y y \mp \gamma_z z} \\
&\vec{1}_{\pm} \times \tilde{\vec{E}}_{\pm}^{(h)}(s) = Z_2 \tilde{\vec{H}}_{\pm}^{(h)}(s) \quad , \quad \tilde{\vec{E}}_{\pm}^{(h)}(s) = -Z_2 \vec{1}_{\pm} \times \tilde{\vec{H}}_{\pm}^{(h)}(s) \\
&\vec{1}_{\pm} \cdot \tilde{\vec{E}}_{\pm}^{(h)}(s) = 0 = Z_2 \vec{1}_{\pm} \cdot \tilde{\vec{H}}_{\pm}^{(h)}(s) \\
&\text{upper sign} \Rightarrow \text{right-propagating wave} \\
&\text{lower sign} \Rightarrow \text{left-propagating wave}
\end{aligned} \tag{3.10}$$

Constraining the polarization (H wave) gives

$$\begin{aligned}
\tilde{\vec{E}}_{\pm}^{(h)}(s) &= \vec{1}_h \tilde{E}_{\pm}^{(h)}(s) = -\vec{1}_x \tilde{E}_{\pm}^{(h)}(s) \\
Z_2 \tilde{\vec{H}}_{\pm}^{(h)}(s) &= -\vec{1}_{\pm} \times \vec{1}_x \tilde{E}_{\pm}^{(h)}(s) = -\vec{1}_v^{(sc)} \times \left[\vec{1}_y \times \vec{1}_z \right] \tilde{E}_{\pm}^{(h)}(s)
\end{aligned}$$

$$= -\left[\vec{1}_y \left[\vec{1}_\pm \cdot \vec{1}_z \right] - \vec{1}_z \left[\vec{1}_\pm \cdot \vec{1}_z \right] \right] \tilde{E}_\pm^{(h)}(s) \tag{3.11}$$

Matching transverse propagation (y direction) to the exterior waves gives

$$\vec{1}_\pm \cdot \vec{1}_y = \vec{1}_i \cdot \vec{1}_y = \sin(\theta_i) \tag{3.12}$$

Set

$$\begin{aligned} \gamma_2 \vec{1}_\pm \cdot \vec{r} &= \gamma_2 \left[\left[\vec{1}_\pm \cdot \vec{1}_y \right] y + \left[\vec{1}_\pm \cdot \vec{1}_z \right] z \right] \\ &= \gamma_2 \left[\sin(\theta_{in}) y \pm \cos(\theta_{in}) z \right] \\ &= \gamma_y \, y \pm \gamma_z \, z \end{aligned} \tag{3.13}$$

where θ_{in} (generally complex) is the propagation angle (direction $\vec{1}_+$) in the wall. We then have

$$\begin{aligned} \gamma_y &= \gamma_2 \sin(\theta_{in}) = \gamma_0 \sin(\theta_{in}) \\ \sin(\theta_{in}) &= \frac{\gamma_y}{\gamma_2} = \frac{\gamma_0}{\gamma_2} \sin(\theta_i) \\ \cos(\theta_{in}) &= \left[1 - \sin^2(\theta_{in}) \right]^{\frac{1}{2}} = \left[1 - \left[\frac{\gamma_0}{\gamma_2} \right]^2 \sin^2(\theta_{in}) \right]^{\frac{1}{2}} \\ \gamma_z &= \gamma_0 \cos(\theta_{in}) \end{aligned} \tag{3.14}$$

Note that for large γ_2 / γ_0 (an important case) $\theta_{in} \simeq 0$ which will be useful for later approximations.

The tangential fields on S_1 are then

$$\begin{aligned} \tilde{\vec{E}}_t^{(1,h)}(s) &= \overleftrightarrow{1}_z \cdot \tilde{\vec{E}}^{(in,h)}(\vec{r}_1, s) = \sum_\pm \overleftrightarrow{1}_z \cdot \tilde{\vec{E}}_\pm^{(h)}(s) \, e^{-\gamma_y y \mp \gamma_z z_1} \\ &= -\vec{1}_x \, e^{-\gamma_y y} \sum_\pm \tilde{E}_\pm^{(h)}(s) \, e^{\mp \gamma_z z_1} \end{aligned}$$

$$Z_2 \vec{H}_t^{(1,h)}(s) = Z_2 \overleftrightarrow{1}_z \cdot \vec{H}^{(in,h)}(\vec{r}_1, s) = Z_2 \sum_{\pm} \overleftrightarrow{1}_z \cdot \vec{H}_{\pm}^{(h)}(s)\, e^{-\gamma_y y \mp \gamma_z z_1}$$
$$= -\vec{1}_y \cos(\theta_{in}) e^{-\gamma_y y} \sum_{\pm} \pm\, \tilde{E}_{\pm}^{(h)}(s)\, e^{\mp \gamma_z z_1} \tag{3.15}$$

On S_2 we have

$$\vec{E}_t^{(2,h)}(s) = \overleftrightarrow{1}_z \cdot \vec{E}^{(in,h)}(\vec{r}_2, s) = \sum_{\pm} \overleftrightarrow{1}_z \cdot \vec{E}_{\pm}^{(h)}(s)\, e^{-\gamma_y y \mp \gamma_z z_2}$$
$$= -\vec{1}_x\, e^{-\gamma_y y} \sum_{\pm} \tilde{E}_{\pm}^{(h)}(s)\, e^{\mp \gamma_z z_2}$$
$$Z_2 \vec{H}_t^{(2,h)}(s) = Z_2 \overleftrightarrow{1}_z \cdot \vec{H}^{(in,h)}(\vec{r}_2, s) = Z_2 \sum_{\pm} \overleftrightarrow{1}_z \cdot \vec{H}_{\pm}^{(h)}(s)\, e^{-\gamma_y y \mp \gamma_z z_2} \tag{3.16}$$
$$= -\vec{1}_y \cos(\theta_{in}) e^{-\gamma_y y} \sum_{\pm} \pm\, \tilde{E}_{\pm}^{(h)}(s)\, e^{\mp \gamma_z z_2}$$

These can be manipulated into matrix form as

$$\begin{pmatrix} \tilde{E}_x^{(1,h)}(s) \\ Z_0 \tilde{H}_y^{(2,h)}(s) \end{pmatrix} = -e^{-\gamma_y y} \begin{pmatrix} e^{-\gamma_z z_1} & e^{\gamma_z z_2} \\ \frac{Z_0}{Z_2}\cos(\theta_{in}) e^{-\gamma_z z_1} & -\frac{Z_0}{Z_2}\cos(\theta_{in}) e^{\gamma_z z_2} \end{pmatrix} \cdot \begin{pmatrix} \tilde{E}_+^{(h)}(s) \\ \tilde{E}_-^{(h)}(s) \end{pmatrix}$$
$$\begin{pmatrix} \tilde{E}_x^{(2,h)}(s) \\ Z_0 \tilde{H}_y^{(2,h)}(s) \end{pmatrix} = -e^{-\gamma_y y} \begin{pmatrix} e^{-\gamma_z z_2} & e^{\gamma_z z_2} \\ \frac{Z_0}{Z_2}\cos(\theta_{in}) e^{-\gamma_z z_2} & -\frac{Z_0}{Z_2}\cos(\theta_{in}) e^{\gamma_z z_2} \end{pmatrix} \cdot \begin{pmatrix} \tilde{E}_+^{(h)}(s) \\ \tilde{E}_-^{(h)}(s) \end{pmatrix} \tag{3.17}$$

Invert the first of these as

$$\begin{pmatrix} \tilde{E}_+^{(h)}(s) \\ \tilde{E}_-^{(h)}(s) \end{pmatrix} = -\frac{e^{-\gamma_y y}}{2} \begin{pmatrix} e^{\gamma_z z_1} & \frac{Z_2 e^{\gamma_z z_1}}{Z_0 \cos(\theta_{in})} \\ e^{-\gamma_z z_1} & -\frac{Z_2 e^{-\gamma_z z_1}}{Z_0 \cos(\theta_{in})} \end{pmatrix} \cdot \begin{pmatrix} \tilde{E}_x^{(1,h)}(s) \\ Z_0 \tilde{H}_y^{(1,h)}(s) \end{pmatrix} \tag{3.18}$$

Then we can construct a BCS for H waves as

$$\begin{pmatrix} \tilde{E}_x^{(2,h)}(s) \\ Z_0 \tilde{H}_y^{(2,h)}(s) \end{pmatrix} = \left(\tilde{B}_{n,m}^{(h)}(2,1;s) \right) \cdot \begin{pmatrix} \tilde{E}_x^{(1,h)} \\ Z_0 H_y^{(1,h)} \end{pmatrix}$$

$$\left(\tilde{B}_{n,m}^{(h)}(2,1;s) \right) = \begin{pmatrix} \cosh(\gamma_z \Delta) & -\frac{Z_2}{Z_0 \cos(\theta_{in})} \sinh(\gamma_z \Delta) \\ -\frac{Z_0 \cos(\theta_{in})}{Z_2} \sinh(\gamma_z \Delta) & \cosh(\gamma_z \Delta) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{Z_0 \cos(\theta_{in})}{Z_2} \end{pmatrix} \cdot \begin{pmatrix} \cosh(\gamma_z \Delta) & -\sinh(\gamma_z \Delta) \\ -\sinh(\gamma_z \Delta) & \cosh(\gamma_z \Delta) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \frac{Z_2}{Z_0 \cos(\theta_{in})} \end{pmatrix}$$

$$\det\left(\left(\tilde{B}_{n,m}^{(h)}(2,1;s) \right) \right) = 1$$

$$\left(\tilde{B}_{n,m}^{(h)}(1,2;s) \right) = \left(\tilde{B}_{n,m}^{(h)}(1,2;s) \right)^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{Z_0 \cos(\theta_{in})}{Z_2} \end{pmatrix} \cdot \begin{pmatrix} \cosh(\gamma_z \Delta) & \sinh(\gamma_z \Delta) \\ \sinh(\gamma_z \Delta) & \cosh(\gamma_z \Delta) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \frac{Z_2}{Z_0 \cos(\theta_{in})} \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(\gamma_z \Delta) & \frac{Z_2}{Z_0 \cos(\theta_{in})} \sinh(\gamma_z \Delta) \\ \frac{Z_0 \cos(\theta_{in})}{Z_2} \sinh(\gamma_z \Delta) & \cosh(\gamma_z \Delta) \end{pmatrix} \tag{3.19}$$

Ideally, we would like the BCS to be independent of θ_i which is contained in θ_{in} and γ_z. From (3.14) we have

$$\gamma_z = \gamma_2 \cos(\theta_{in})$$

$$\cos(\theta_{in}) = \left[1 - \left[\frac{\gamma_0}{\gamma_2} \right]^2 \sin^2(\theta_i) \right]^{\frac{1}{2}} = 1 - \frac{1}{2} \left[\frac{\gamma_0}{\gamma_2} \right]^2 \sin^2/(\theta_i) + O\left(\left[\frac{\gamma_0}{\gamma_2} \right]^4 \right) \quad \text{as } \frac{\gamma_0}{\gamma_2} \to 0 \tag{3.20}$$

Showing that small γ_0 / γ_2 makes $\cos(\theta_{in}) \simeq 1$ and, hence, independent of θ_i. The exponential terms are

$$\begin{aligned} e^{\pm \gamma_z \Delta} &= e^{\pm \gamma_2 \Delta} e^{\mp \frac{\gamma_2 \Delta}{2} \left[\left[\frac{\gamma_0}{\gamma_2} \right]^2 \sin^2(\theta_i) + O\left(\left[\frac{\gamma_0}{\gamma_2} \right]^4 \right) \right]} \\ &= e^{\pm \gamma_2 \Delta} \left[1 \mp \frac{\gamma_2 \Delta}{2} \left[\frac{\gamma_0}{\gamma_2} \right]^2 \sin^2(\theta_i) + O\left(\gamma_2 \Delta \left[\frac{\gamma_0}{\gamma_2} \right]^2 \right) \right] \text{as } \frac{\gamma_2 \Delta}{2} \left[\frac{\gamma_0}{\gamma_2} \right]^2 \to 0 \end{aligned} \tag{3.21}$$

This indicates that the wall need not be electrically thin provided γ_2 / γ_0 is large enough (e.g., a highly conducting wall).

With these approximations we have

$$\left(\tilde{B}_{n,m}^{(h)}(2,1;s)\right) = \begin{pmatrix} \cosh(\gamma_2\Delta) & -\frac{Z_2}{Z_0}\sinh(\gamma_2\Delta) \\ -\frac{Z_0}{Z_2}\sinh(\gamma_2\Delta) & \cosh(\gamma_2\Delta) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{Z_0}{Z_2} \end{pmatrix} \cdot \begin{pmatrix} \cosh(\gamma_2\Delta) & -\sinh(\gamma_2\Delta) \\ -\sinh(\gamma_2\Delta) & \cosh(\gamma_2\Delta) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \frac{Z_2}{Z_0} \end{pmatrix} \tag{3.22}$$

For application to more general boundary-value problems we need to keep the approximations in mind, as they may limit the range of validity of the results.

3.2 E-wave

The incident wave in volume 1 is an E (or TM) wave as

$$\begin{aligned} \vec{\tilde{E}}^{(inc,e)}(\vec{r},s) &= \vec{1}_v \, \tilde{E}_0^{(e)}(s) e^{-\gamma_0 \vec{1}_i \cdot \vec{r}} \\ Z_0 \vec{\tilde{H}}^{(inc,h)}(\vec{r},s) &= \vec{1}_h \, \tilde{E}_0^{(h)}(s) e^{-\gamma_0 \vec{1}_i \cdot \vec{r}} \, , \quad z < z_1 \end{aligned} \tag{3.23}$$

The scattered wave in volume 1 is

$$\begin{aligned} \vec{\tilde{E}}^{(sc,e)}(\vec{r},s) &= \vec{1}_v^{(sc)} \, \tilde{E}_1^{(e)}(s) e^{-\gamma_0 \vec{1}_i \cdot \vec{r}} \\ Z_0 \vec{\tilde{H}}^{(sc,e)}(\vec{r},s) &= -\vec{1}_h \, \tilde{E}_1^{(e)}(s) e^{-\gamma_0 \vec{1}_i \cdot \vec{r}} \end{aligned} \tag{3.24}$$

The tangential fields on S_1 are

$$
\begin{aligned}
\vec{E}_t^{(1,e)}(s) &= \overleftrightarrow{1}_z \cdot \left[\vec{1}_v \tilde{E}_0^{(e)}(s) e^{-\gamma_0[y\sin(\theta_i)+z_1\cos(\theta_i)]} + \vec{1}_v^{(sc)} \tilde{E}_1^{(e)}(s) e^{-\gamma_0[y\sin(\theta_i)-z_1\cos(\theta_i)]} \right] \\
&= \vec{1}_y \cos(\theta_i) e^{-\gamma_0 y \sin(\theta_i)} \left[\tilde{E}_0^{(e)}(s) e^{-\gamma_0 z_1 \cos(\theta_i)} + \tilde{E}_1^{(e)}(s) e^{\gamma_0 z_1 \cos(\theta_i)} \right] \\
Z_0 \vec{H}_t^{(1,e)} &= \overleftrightarrow{1}_z \cdot \left[\vec{1}_h \tilde{E}_0^{(h)}(s) e^{-\gamma_0[y\sin(\theta_i)+z_1\cos(\theta_i)]} - \vec{1}_h \tilde{E}_1^{(h)}(s) e^{-\gamma_0[y\sin(\theta_i)-z_1\cos(\theta_i)]} \right] \\
&= -\vec{1}_x e^{-\gamma_0 y \sin(\theta_i)} \left[\tilde{E}_0^{(h)}(s) e^{-\gamma_0 z_1 \cos(\theta_i)} - \tilde{E}_1^{(e)}(s) e^{\gamma_0 z_1 \cos(\theta_i)} \right]
\end{aligned} \tag{3.25}
$$

The transmitted field in volume 2 is an E wave of the form

$$
\begin{aligned}
\tilde{\vec{E}}^{(t,e)}(\vec{r},s) &= \vec{1}_v \tilde{E}_2^{(e)}(s) e^{-\gamma_0 \vec{1}_i \cdot \vec{r}} \\
Z_0 \tilde{\vec{H}}^{(t,e)}(\vec{r},s) &= \vec{1}_h \tilde{E}_2^{(e)}(s) e^{-\gamma_0 \vec{1}_i \cdot \vec{r}}, \; z > z_2
\end{aligned} \tag{3.26}
$$

The tangential fields on S_2 are

$$
\begin{aligned}
\tilde{\vec{E}}_t^{(t,e)}(s) &= \overleftrightarrow{1}_z \cdot \vec{1}_v \tilde{E}_2^{(e)} e^{-\gamma_0[y\sin(\theta_i)+z_2\cos(\theta_i)]} \\
&= \vec{1}_y \cos(\theta_i) \tilde{E}_2^{(e)}(s) e^{-\gamma_0 y \sin(\theta_i)} e^{-\gamma_0 z_2 \cos(\theta_i)} \\
Z_0 \tilde{\vec{H}}_t^{(t,e)}(s) &= \overleftrightarrow{1}_z \cdot \vec{1}_h \tilde{E}_2^{(e)} e^{-\gamma_0[y\sin(\theta_i)+z_2\cos(\theta_i)]} \\
&= -\vec{1}_x \tilde{E}_2^{(e)}(s) e^{-\gamma_0 y \sin(\theta_i)} e^{-\gamma_0 z_2 \cos(\theta_i)}
\end{aligned} \tag{3.27}
$$

Insite the wall we have two E waves as

$$
\begin{aligned}
\tilde{\vec{E}}^{(in,e)}(\vec{r},s) &= \sum_{\pm} \tilde{\vec{E}}_{\pm}^{(e)}(s) e^{-\gamma_2 \vec{1}_{\pm} \cdot \vec{r}} = \sum_{\pm} \tilde{\vec{E}}_{\pm}^{(e)}(s) e^{-\gamma_y y \pm \gamma_z z} \\
\tilde{\vec{H}}^{(in,e)}(\vec{r},s) &= \sum_{\pm} \tilde{\vec{H}}_{\pm}^{(e)}(s) e^{-\gamma_2 \vec{1}_{\pm} \cdot \vec{r}} = \sum_{\pm} \tilde{\vec{H}}_{\pm}^{(e)}(s) e^{-\gamma_y y \pm \gamma_z z} \\
\vec{1}_{\pm} \times \tilde{\vec{E}}_{\pm}^{(e)}(s) &= Z_0 \tilde{\vec{H}}_{\pm}^{(e)}(s) \; , \quad \tilde{\vec{E}}_{\pm}^{(e)}(s) = -Z_2 \vec{1}_{\pm} \times \tilde{\vec{H}}_{\pm}^{(e)}(s) \\
\vec{1}_{\pm} \cdot \tilde{\vec{E}}_{\pm}^{(e)}(s) &= 0 = Z_2 \vec{1}_{\pm} \cdot \tilde{\vec{H}}_{\pm}^{(e)}(s)
\end{aligned}
$$

upper sign $\Rightarrow$ right-propagating wave (3.28)

lower sign $\Rightarrow$ left-propagating wave

Constraining the polarization (E wave) gives

$$\begin{aligned}
\vec{\tilde{H}}^{(e)}_{\pm}(s) &= \vec{1}_h \tilde{H}^{(e)}_{\pm}(s) = -\vec{1}_x \tilde{H}^{(e)}_{\pm}(s) \\
\vec{\tilde{E}}^{(e)}_{\pm}(s) &= Z_2 \vec{1}_{\pm} \times \vec{1}_x \tilde{H}^{(e)}_{\pm}(s) = Z_2 \vec{1}_{\pm} \times \left[\vec{1}_y \times \vec{1}_z\right] \tilde{H}^{(e)}_{\pm}(s) \\
&= Z_2 \left[\vec{1}_y \left[\vec{1}_{\pm} \cdot \vec{1}_z\right] - \vec{1}_z \left[\vec{1}_{\pm} \cdot \vec{1}_x\right]\right] \tilde{H}^{(e)}_{\pm}(s)
\end{aligned} \tag{3.29}$$

Note that (3.12) through (3.14) carry over directly.

The tangential fields on S_1 are

$$\begin{aligned}
\vec{\tilde{E}}^{(1,e)}_t(s) &= \overleftrightarrow{1}_z \cdot \vec{\tilde{E}}^{(in,e)}(\vec{r}_1, s) = \sum_{\pm} \overleftrightarrow{1}_z \cdot \vec{\tilde{E}}^{(e)}_{\pm}(s)\, e^{-\gamma_y y \mp \gamma_z z_1} \\
&= \vec{1}_y \cos(\theta_i)\, e^{-\gamma_y y} \sum_{\pm} Z_2 \tilde{H}^{(e)}_{\pm}(s)\, e^{\mp \gamma_z z_1} \\
\vec{\tilde{H}}^{(1,e)}_t(s) &= \overleftrightarrow{1}_z \cdot \vec{\tilde{H}}^{(in,e)}(\vec{r}_1, s) = \sum_{\pm} \overleftrightarrow{1}_z \cdot \vec{\tilde{H}}^{(e)}_{\pm}(s)\, e^{-\gamma_y y \mp \gamma_z z_1} \\
&= -\vec{1}_x\, e^{-\gamma_y y} \sum_{\pm} \tilde{H}^{(e)}_{\pm}(s)\, e^{\mp \gamma_z z}
\end{aligned} \tag{3.30}$$

On S_2 we have

$$\begin{aligned}
\vec{\tilde{E}}^{(2,e)}_t(s) &= \overleftrightarrow{1}_z \cdot \vec{\tilde{E}}^{(in,e)}(\vec{r}_2, s) = \sum_{\pm} \overleftrightarrow{1}_z \cdot \vec{\tilde{E}}^{(e)}_{\pm}(s)\, e^{-\gamma_y y \mp \gamma_z z_2} \\
&= \vec{1}_y \cos(\theta_i)\, e^{-\gamma_y y} \sum_{\pm} Z_2 \tilde{H}^{(e)}_{\pm}(s)\, e^{\mp \gamma_z z_2} \\
\vec{\tilde{H}}^{(1,e)}_t(s) &= \overleftrightarrow{1}_z \cdot \vec{\tilde{H}}^{(in,e)}(\vec{r}_2, s) = \sum_{\pm} \overleftrightarrow{1}_z \cdot \vec{\tilde{H}}^{(e)}_{\pm}(s)\, e^{-\gamma_y y \mp \gamma_z z_2} \\
&= -\vec{1}_x\, e^{-\gamma_y y} \sum_{\pm} \tilde{H}^{(e)}_{\pm}(s)\, e^{\mp \gamma_z z_2}
\end{aligned} \tag{3.31}$$

In matrix form we have

$$\begin{pmatrix} \tilde{E}_y^{(1,e)}(s) \\ -Z_0 \tilde{H}_-^{(1,e)}(s) \end{pmatrix} = \frac{e^{\gamma_y y}}{2} \begin{pmatrix} \cos(\theta_{in}) e^{-\gamma_z z_1} & \cos(\theta_{in}) e^{\gamma_z z_1} \\ \frac{Z_0}{Z_2} e^{-\gamma_z z_1} & -\frac{Z_0}{Z_2} e^{\gamma_z z_1} \end{pmatrix} \cdot \begin{pmatrix} Z_2 \tilde{H}_+^{(e)}(s) \\ Z_2 \tilde{H}_-^{(e)}(s) \end{pmatrix}$$

$$\begin{pmatrix} \tilde{E}_y^{(2,e)}(s) \\ -Z_0 \tilde{H}_x^{(2,e)}(s) \end{pmatrix} = e^{-\gamma_y y} \begin{pmatrix} \cos(\theta_{in}) e^{-\gamma_z z_2} & \cos(\theta_{in}) e^{\gamma_z z_2} \\ \frac{Z_0}{Z_2} e^{-\gamma_z z_2} & -\frac{Z_0}{Z_2} e^{\gamma_z z_2} \end{pmatrix} \cdot \begin{pmatrix} Z_2 \tilde{H}_+^{(e)}(s) \\ Z_2 \tilde{H}_-^{(e)}(s) \end{pmatrix} \tag{3.32}$$

Invert the first of these as

$$\begin{pmatrix} Z_2 \tilde{H}_+^{(e)}(s) \\ Z_2 \tilde{H}_-^{(e)}(s) \end{pmatrix} = \frac{e^{\gamma_y y}}{2} \begin{pmatrix} \frac{e^{\gamma_z z_2}}{\cos(\theta_{in})} & \frac{Z_2}{Z_0} e^{\gamma_z z_2} \\ \frac{e^{-\gamma_z z_2}}{\cos(\theta_{in})} & -\frac{Z_2}{Z_0} e^{\gamma_z z_2} \end{pmatrix} \cdot \begin{pmatrix} \tilde{E}_y^{(1,e)}(s) \\ -Z_0 \tilde{H}_x^{(1,e)}(s) \end{pmatrix} \tag{3.33}$$

The BCS for E waves is then

$$\begin{pmatrix} \tilde{E}_y^{(2,e)}(s) \\ -Z_0 \tilde{H}_x^{(2,e)}(s) \end{pmatrix} = \left(\tilde{B}_{n,m}^{(e)}(2,1;s)\right) \cdot \begin{pmatrix} \tilde{E}_x^{(1,e)}(s) \\ -Z_0 \tilde{H}_x^{(1,e)}(s) \end{pmatrix}$$

$$\left(\tilde{B}_{n,m}^{(e)}(2,1;s)\right) = \begin{pmatrix} \cosh(\gamma_z \Delta) & -\frac{Z_2 \cos(\theta_{in})}{Z_0} \sinh(\gamma_z \Delta) \\ -\frac{Z_0}{Z_2 \cos(\theta_{in})} \sinh(\gamma_z \Delta) & \cosh(\gamma_z \Delta) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{Z_0}{Z_2 \cos(\theta_{in})} \end{pmatrix} \cdot \begin{pmatrix} \cosh(\gamma_z \Delta) & -\sinh(\gamma_z \Delta) \\ -\sinh(\gamma_z \Delta) & \cosh(\gamma_z \Delta) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \frac{Z_2 \cos(\theta_{in})}{Z_0} \end{pmatrix}$$

$$\det\left(\left(\tilde{B}_{n,m}^{(e)}(2,1;s)\right)\right) = 1$$

$$\left(\tilde{B}_{n,m}^{(e)}(1,2;s)\right) = \left(\tilde{B}_{n,m}^{(e)}(2,1;s)\right)^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{Z_0}{Z_2 \cos(\theta_{in})} \end{pmatrix} \cdot \begin{pmatrix} \cosh(\gamma_z \Delta) & \sinh(\gamma_z \Delta) \\ \sinh(\gamma_z \Delta) & \cosh(\gamma_z \Delta) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \frac{Z_2 \cos(\theta_{in})}{Z_0} \end{pmatrix}$$

$$
= \begin{pmatrix} \cosh(\gamma_z \Delta) & -\frac{Z_2 \cosh(\theta_{in})}{Z_0} \sinh(\gamma_z \Delta) \\ \frac{Z_0}{Z_2 \cosh(\theta_{in})} \sinh(\gamma_z \Delta) & \cosh(\gamma_z \Delta) \end{pmatrix} \tag{3.34}
$$

Note the similarity to the H-wave case. In the terms (other than $\gamma_{z)} \cos(\theta_{in})$ has been replaced by $\cos^{-1}(\theta_{in})$.

As before, we would like the BCS to be independent of θ_i. Than we can treat γ_z and $\cos(\theta_{in})$ as in (3.20) and (3.21) with the same results, leading to

$$
\left(\tilde{B}^{(e)}_{n,m}(2,1;s) \right) \simeq \begin{pmatrix} \cosh(\gamma_z \Delta) & -\frac{Z_2}{Z_0} \sinh(\gamma_z \Delta) \\ -\frac{Z_0}{Z_2} \sinh(\gamma_z \Delta) & \cosh(\gamma_z \Delta) \end{pmatrix} \simeq \left(\tilde{B}^{(h)}_{n,m}(2,1;s) \right) \tag{3.35}
$$

If you were wondering why a minus sign was used on the left side of (3.32) (both equations), this should now be clear.

3.3 Combined BCS for E- and H-waves

Now we can define

$$
\begin{aligned}
\left(\tilde{B}_{n,m}(2,1;s) \right) &\equiv \begin{pmatrix} \cosh(\gamma_z \Delta) & -\frac{Z_2}{Z_0} \sinh(\gamma_z \Delta) \\ -\frac{Z_0}{Z_2} \sinh(\gamma_z \Delta) & \cosh(\gamma_z \Delta) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \frac{Z_0}{Z_2} \end{pmatrix} \cdot \begin{pmatrix} \cosh(\gamma_z \Delta) & -\sinh(\gamma_z \Delta) \\ -\sinh(\gamma_z \Delta) & \cosh(\gamma_z \Delta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{Z_2}{Z_0} \end{pmatrix} \\
&\simeq \left(\tilde{B}^{(h)}_{n,m}(2,1;s) \right) \simeq \left(\tilde{B}^{(h)}_{n,m}(2,1;s) \right)
\end{aligned} \tag{3.36}
$$

using the approximations previously discussed. The previous results can then be combined in supermatrix form as

$$\begin{pmatrix} \begin{pmatrix} \tilde{E}_x^{(2,h)}(s) \\ Z_0\tilde{H}_y^{(2,h)}(s) \end{pmatrix} \\ \begin{pmatrix} \tilde{E}_y^{(2,h)} \\ -Z_0\tilde{H}_y^{(2,h)} \end{pmatrix} \end{pmatrix} = \left(\left(\tilde{B}_{n,m}^{(h,e)}(2,1;s) \right)_{u,v} \right) \odot \begin{pmatrix} \begin{pmatrix} \tilde{E}_x^{(1,h)}(s) \\ Z_0\tilde{H}_y^{(1,h)}(s) \end{pmatrix} \\ \begin{pmatrix} \tilde{E}_y^{(1,h)} \\ -Z_0\tilde{H}_y^{(1,h)} \end{pmatrix} \end{pmatrix}$$

$$\left(\left(\tilde{B}_{n,m}^{(h,e)}(2,1;s) \right)_{u,v} \right) = \begin{pmatrix} \left(\tilde{B}_{n,m}(2,1;s)\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(\tilde{B}_{n,m}(2,1;s)\right) \end{pmatrix}$$

$$= \left(\tilde{B}_{n,m}(2,1;s)\right) \otimes \left(1_{u,v}\right)$$

$$\det\left(\left(\left(\tilde{B}_{n,m}^{(h,e)}(2,1;s) \right)_{u,v} \right) \right) = 1$$

$$\left(\left(\tilde{B}_{n,m}(1,2;s)\right)_{u,v} \right) = \left(\left(\tilde{B}_{n,m}(2,1;s)\right)_{u,v} \right)^{-1} = \begin{pmatrix} \left(\tilde{B}_{n,m}(1,2;s)\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(\tilde{B}_{n,m}(1,2;s)\right) \end{pmatrix}$$

$$= \left(\tilde{B}_{n,m}(1,2;s)\right) \otimes \left(1_{u,v}\right)$$

$$\left(\tilde{B}_{n,m}(1,2;s)\right) = \left(\tilde{B}_{n,m}(1,2;s)\right)^{-1} = \begin{pmatrix} \cosh(\gamma_2\Delta) & -\frac{Z_2}{Z_0}\sinh(\gamma_2\Delta) \\ -\frac{Z_0}{Z_2}\sinh(\gamma_2\Delta) & \cosh(\gamma_2\Delta) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{Z_0}{Z_2} \end{pmatrix} \cdot \begin{pmatrix} \cosh(\gamma_2\Delta) & \sinh(\gamma_2\Delta) \\ \sinh(\gamma_2\Delta) & \cosh(\gamma_2\Delta) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \frac{Z_2}{Z_0} \end{pmatrix} \tag{3.37}$$

This is a very convenient form since it is block diagonal.

In Section 2 the BCS is defined with a different ordering of the tangential field components, which in the present context takes the form

$$\begin{pmatrix} \vec{\tilde{E}}_t^{(2)}(s) \\ Z_0\vec{\tilde{H}}_t^{(2)}(s) \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \tilde{E}_x^{(2)}(s) \\ \tilde{E}_y^{(2)}(s) \end{pmatrix} \\ \begin{pmatrix} Z_0\tilde{H}_x^{(2)}(s) \\ Z_0\tilde{H}_y^{(2)}(s) \end{pmatrix} \end{pmatrix} = \left(\left(\tilde{B}_{n,m}(2,1;s)\right)_{u,v} \right) \odot \begin{pmatrix} \begin{pmatrix} \tilde{E}_x^{(1)}(s) \\ \tilde{E}_y^{(1)}(s) \end{pmatrix} \\ \begin{pmatrix} Z_0\tilde{H}_x^{(1)}(s) \\ Z_0\tilde{H}_y^{(1)}(s) \end{pmatrix} \end{pmatrix}$$

$$\left(\left(\tilde{B}_{n,m}(2,1;s) \right)_{u,v} \right) \equiv \begin{pmatrix} \begin{pmatrix} \cosh(\gamma_2\Delta) & 0 \\ 0 & \cosh(\gamma_2\Delta) \end{pmatrix} & \begin{pmatrix} 0 & -\frac{Z_2}{Z_0}\sinh(\gamma_2\Delta) \\ \frac{Z_2}{Z_0}\sinh(\gamma_2\Delta) & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{Z_0}{Z_2}\sinh(\gamma_2\Delta) \\ -\frac{Z_0}{Z_2}\sinh(\gamma_2\Delta) & 0 \end{pmatrix} & \begin{pmatrix} \cosh(\gamma_2\Delta) & 0 \\ 0 & \cosh(\gamma_2\Delta) \end{pmatrix} \end{pmatrix} \tag{3.38}$$

The determinant is still 1, and the inverse is found by reversing the signs on all the sinh terms, as can be verified by multiplying (dot product) the two supermatrices [5].

3.4 Delay corrected BCS

In Section 2 the concept of a delay-corrected BCS is introduced, so as to be able to reference fields in volumes 1 and 2 to a common surface S_0. For present purposes, we can take this surface as $z = z_0$ with z_0 somewhere inside the wall (Fig. 3.1). Following (2.4) we need BCSs to extend the external fields from S_1 and S_2 to S_0.

Using the results already obtained for a uniform isotropic wall we can take the special case that this medium is free space now with substitutions

$$\gamma_2 \to \gamma_0 , \quad Z_2 \to Z_0 , \quad \theta_{in} \to \theta_i , \quad \gamma_z \to \gamma_0 , \quad \omega s(\theta_i) \tag{3.39}$$

From (3.19) we then form

$$\begin{aligned} \begin{pmatrix} \tilde{E}_x^{(1,h)}(s) \\ Z_0 \tilde{H}_y^{(1,h)}(s) \end{pmatrix} &= \left(\tilde{B}_{n,m}^{(h)}(1,1_+;s) \right) \cdot \begin{pmatrix} \tilde{E}_x^{(1_+,h)}(s) \\ Z_0 \tilde{H}_y^{(1_+,h)}(s) \end{pmatrix} \\ \left(\tilde{B}_{n,m}^{(h)}(1,1_+;s) \right) &= \begin{pmatrix} \cosh\left(\gamma_0 \cos(\theta_i)[z_0 - z_1]\right) & \dfrac{\sinh\left(\gamma_0 \cos(\theta_i)[z_0 - z_1]\right)}{\cos(\theta_i)} \\ \cosh(\theta_i)\sinh\left(\gamma_0 \cos(\theta_i)[z_0 - z_1]\right) & \cosh\left(\gamma_0 \cos(\theta_i)[z_0 - z_1]\right) \end{pmatrix} \\ \begin{pmatrix} \tilde{E}_x^{(2_-,h)}(s) \\ Z_0 \tilde{H}_y^{(2_-,h)}(s) \end{pmatrix} &= \left(\tilde{B}_{n,m}^{(h)}(2_-,2_+;s) \right) \cdot \begin{pmatrix} \tilde{E}_x^{(2,h)}(s) \\ Z_0 \tilde{H}_y^{(1_+,h)}(s) \end{pmatrix} \\ \left(\tilde{B}_{n,m}^{(h)}(2_-,2;s) \right) &= \begin{pmatrix} \cosh\left(\gamma_0 \cos(\theta_i)[z_2 - z_0]\right) & \dfrac{\sinh\left(\gamma_0 \cos(\theta_i)[z_2 - z_1]\right)}{\cos(\theta_i)} \\ \cosh(\theta_i)\sinh\left(\gamma_0 \cos(\theta_i)[z_2 - z_1]\right) & \cosh\left(\gamma_0 \cos(\theta_i)[z_2 - z_1]\right) \end{pmatrix} \end{aligned} \tag{3.40}$$

For the special case of

$$z_0 - z_1 = z_2 - z_0 = \frac{\Delta}{2} \tag{3.41}$$

we have

$$\left(\tilde{B}_{n,m}^{(h)}(1,1_+;s)\right) = \left(\tilde{B}_{n,m}^{(h)}(2_-,2;s)\right) = \begin{pmatrix} \cosh\left(\gamma_0 \cos(\theta_i)\frac{\Delta}{2}\right) & \dfrac{\sinh\left(\gamma_0 \cos(\theta_i)\frac{\Delta}{2}\right)}{\cosh(\theta_i)} \\ \cosh(\theta_i)\sinh\left(\gamma_0 \cos(\theta_i)\frac{\Delta}{2}\right) & \cosh\left(\gamma_0 \cos(\theta_i)\frac{\Delta}{2}\right) \end{pmatrix} \tag{3.42}$$

Note that in the above $\Delta/2$ can be replaced by Δ_1 and Δ_2 with

$$\Delta_1 + \Delta_2 = \Delta \tag{3.43}$$

if one wishes to take S_0 as some surface other than $z = 0$.

While similar to (3.19), we have the dependence on θ_i to consider. Unlike $\cos(\theta_{in})$ in (3.12) this does not go to unity as in the wall which we can take as highly conducting. For small $\gamma_0 \cos(\theta_i)\Delta$ we can write

$$\begin{aligned} \left(\tilde{B}_{n,m}^{(h)}(1,1_+;s)\right) = \left(\tilde{B}_{n,m}^{(h)}(2_-,2;s)\right) &= \begin{pmatrix} 1 & \gamma_0 \frac{\Delta}{2} \\ \gamma_0 \cos^2(\theta_i)\frac{\Delta}{2} & 1 \end{pmatrix} + O\left(\left[\gamma_0 \cos^2(\theta_i)\frac{\Delta}{2}\right]^2\right) \\ &= \left(1_{n,m}\right) + O\left(\gamma_0 \cos^2(\theta_i)\Delta\right) \text{ as } \gamma_0 \cos(\theta_i)\Delta \to 0 \end{aligned} \tag{3.44}$$

This says that for sufficiently low frequencies $(\gamma_0\Delta \ll 1)$ we can neglect this term, the BCS and delay-corrected BCS being substantially the same. However, for $\gamma_0\Delta$ of order unity this correction can be significant, and it is unfortunately a function of θ_i.

Similarly for E waves from (3.34) we have

$$
\begin{pmatrix} \tilde{E}_y^{(1,e)}(s) \\ -Z_0 \tilde{H}_x^{(1,e)}(s) \end{pmatrix} = \left(\tilde{B}_{n,m}^{(e)}(1,1_+;s) \right) \cdot \begin{pmatrix} \tilde{E}_y^{(1_+,e)}(s) \\ -Z_0 \tilde{H}_x^{(1_+,e)}(s) \end{pmatrix}
$$

$$
\left(\tilde{B}_{n,m}^{(e)}(1,1_+;s) \right) = \begin{pmatrix} \cosh\left(\gamma_0 \cos(\theta_i)[z_0 - z_1]\right) & \cos(\theta_i)\sinh\left(\gamma_0 \cos(\theta_i)[z_0 - z_1]\right) \\ \dfrac{\sinh\left(\gamma_0 \cos(\theta_i)[z_0 - z_1]\right)}{\cosh(\theta_i)} & \cosh\left(\gamma_0 \cos(\theta_i)[z_0 - z_1]\right) \end{pmatrix}
$$

$$
\begin{pmatrix} \tilde{E}_y^{(2_-,e)}(s) \\ -Z_0 \tilde{H}_x^{(2_-,e)}(s) \end{pmatrix} = \left(\tilde{B}_{n,m}^{(e)}(2_-,2;s) \right) \cdot \begin{pmatrix} \tilde{E}_y^{(2,e)}(s) \\ -Z_0 \tilde{H}_x^{(2,e)}(s) \end{pmatrix} \tag{3.45}
$$

$$
\left(\tilde{B}_{n,m}^{(e)}(2_-,2;s) \right) = \begin{pmatrix} \cosh\left(\gamma_0 \cos(\theta_i)[z_2 - z_0]\right) & \cos(\theta_i)\sinh\left(\gamma_0 \cos(\theta_i)[z_2 - z_1]\right) \\ \dfrac{\sinh\left(\gamma_0 \cos(\theta_i)[z_2 - z_1]\right)}{\cosh(\theta_i)} & \cosh\left(\gamma_0 \cos(\theta_i)[z_2 - z_1]\right) \end{pmatrix}
$$

For the special case of S_0 centered in the wall this is

$$
\begin{aligned}
\left(\tilde{B}_{n,m}^{(e)}(1,1_+;s) \right) &= \left(\tilde{B}_{n,m}^{(e)}(2_-,2;s) \right) \\
&= \begin{pmatrix} \cosh\left(\gamma_0 \cos(\theta_i) \dfrac{\Delta}{2} \right) & \cos(\theta_i)\sinh\left(\gamma_0 \cos(\theta_i) \dfrac{\Delta}{2} \right) \\ \dfrac{\sinh\left(\gamma_0 \cos(\theta_i) \dfrac{\Delta}{2} \right)}{\cosh(\theta_i)} & \cosh\left(\gamma_0 \cos(\theta_i) \dfrac{\Delta}{2} \right) \end{pmatrix}
\end{aligned} \tag{3.46}
$$

This is the transpose of the matrix in (3.42). The same small $\gamma_0 \Delta$ restrictions apply.

As in Section 3.3, these matrices can be combined to give supermatrices which can be combined with the supermatrices in (3.37) and (3.38) to give delay-corrected supermatrices. Noting that the matrices introduced here reduce to identities (under restrictions of small $\gamma_0 \Delta$ for independence from θ_i), this is only a formal correction.

4. Application of BCS to Spherical Shell

Appendix A treats the exact solution of the fields inside a uniform isotropic spherical shell due to an incident plane wave. Figure A.2 gives the dimensions (inner radius a, outer radius b, thickness Δ). The constitutive parameters of the shell are the same as those for the wall in Section 3. In the present section, let us consider the spherical shell using the BCS.

As in Appendix A we have the same incident wave as in (A.18) and Fig. A.1 expanded in spherical vector wave functions with coefficients $a^{(i)}_{n,m,p}$ and $b^{(i)}_{n,m,p}$ in (A.14). The internal fields (region 1) are as in (A.20) with coefficients $a^{(1)}_{n,m,p}$ and $b^{(1)}_{n,m,p}$ for $0 \le r < a$. The external fields (region 3) are as in (A.22) with coefficients $a^{(sc)}_{n,m,p}$ and $b^{(sc)}_{n,m,p}$ for $b < r$. In the spherical shell (region 2) we use the BCS to account (approximately) for the fields for $a < r < b$ with thickness $\Delta = b - a$.

To connect the fields in regions 1 and 3, there are various ways to use the BCS. In Section 3 the BCS is developed for the transverse Cartesian coordinates x and y, or directions $\vec{1}_x$ and $\vec{1}_y$. For the spherical-shield problem these are replaced by directions $\vec{1}_\theta$ and $\vec{1}_\phi$. In addition, the longitudinal coordinate z is replaced by r. One can match the tangentiela fields in this form, if desired.

The concept of E waves and H waves carries over to the spherical vector wave functions. E-waves have an electric field expanded in $\vec{N}$ functions with the $b_{n,m,p}$ coefficients, while H-waves have an electric field expanded in $\vec{M}$ functions with the $a_{n,m,p}$ coefficients. In Section 3 the BCS is developed for E- and H-waves including the incidence angle θ_i as a parameter. Now we have spherical vector wave functions, making this angle problematical. Fortunately, for large wall conductivity, the dependence on this angle is removed giving a common approximate BCS for both types of waves.

The spherical vector wave functions form an orthogonal set for tangential components on any constant-r sphere. The approximate BCS is independent of θ and ϕ giving complete O_3 symmetry. Each term in the expansion has the tangential components of $\vec{E}$ and $\vec{H}$ mutually perpendicular. Thus this BCS can be applied termwise. In equation form we have

$$\begin{pmatrix} \left[\tilde{E}^{(inc)}_\theta(s) + \tilde{E}^{(sc)}_\theta(s)\right]\Big|_{r=b} \\ Z_0\left[\tilde{H}^{(inc)}_\phi(s) + \tilde{H}^{(sc)}_\phi(s)\right]\Big|_{r=b} \end{pmatrix} = \left(\tilde{B}_{n,m}(b,a;s)\right) \bullet \begin{pmatrix} \tilde{E}^{(1)}_\theta(s)\Big|_{r=a} \\ Z_0\,\tilde{H}^{(1)}_\phi\Big|_{r=a} \end{pmatrix}$$

$$\begin{pmatrix} \left[\tilde{E}_\phi^{(inc)}(s)+\tilde{E}_\phi^{(sc)}(s)\right]\Big|_{r=b} \\ -Z_0\left[\tilde{H}_\theta^{(inc)}(s)+\tilde{H}_\theta^{(sc)}\right]\Big|_{r=b} \end{pmatrix} = \left(\tilde{B}_{n,m}(b,a;s)\right)\cdot\begin{pmatrix} \tilde{E}_\phi^{(1)}(s)\Big|_{r=a} \\ -Z_0\,\tilde{H}_\theta^{(1)}\Big|_{r=a} \end{pmatrix}$$

$$\left(B_{n,m}(b,a;s)\right) = \begin{pmatrix} \cosh(\gamma_2\Delta) & -\frac{Z_0}{Z_2}\sinh(\gamma_2\Delta) \\ -\frac{Z_0}{Z_2}\sinh(\gamma_2\Delta) & \cosh(\gamma_2\Delta) \end{pmatrix} \tag{4.1}$$

Applying this to the individual terms we have

$$\begin{aligned}
&\begin{pmatrix} i_n(\gamma_0 b)a_{n,m,p}^{(i)}+k_n(\gamma_0 b)a_{n,m,p}^{(sc)} \\ -\frac{\left[\gamma_0 b i_n(\gamma_0 b)\right]'}{\gamma_0 b}a_{n,m,p}^{(i)}-\frac{\left[\gamma_0 b k_n(\gamma_0 b)\right]'}{\gamma_0 b}a_{n,m,p}^{(sc)} \end{pmatrix} \\
&\quad= \frac{1}{\gamma_0 b}\begin{pmatrix} \gamma_0 b i_n(\gamma_0 b) & \gamma_0 b k_n(\gamma_0 b) \\ -\left[\gamma_0 b i_n(\gamma_0 b)\right]' & -\left[\gamma_0 b k_n(\gamma_0 b)\right]' \end{pmatrix}\cdot\begin{pmatrix} a_{n,m,p}^{(i)} \\ a_{n,m,p}^{(sc)} \end{pmatrix} \\
&\quad= \left(\tilde{B}_{n,m}(b,a;s)\right)\cdot\begin{pmatrix} i_n(\gamma_0 a)a_{n,m,p}^{(i)} \\ -\frac{\left[\gamma_0 b i_n(\gamma_0 a)\right]'}{\gamma_0 a}a_{n,m,p}^{(i)} \end{pmatrix} \\
&\quad= \left(\tilde{B}_{n,m}(b,a;s)\right)\cdot\begin{pmatrix} \gamma_0 a i_n(\gamma_0 a) \\ -\left[\gamma_0 a i_n(\gamma_0 a)\right]' \end{pmatrix}\frac{a_{n,m,p}^{(1)}}{\gamma_0 a}
\end{aligned}$$

$$\begin{aligned}
&\begin{pmatrix} \frac{\left[\gamma_0 b i_n(\gamma_0 b)\right]'}{\gamma_0 b}b_{n,m,p}^{(i)} & +\frac{\left[\gamma_0 b k_n(\gamma_0 b)\right]'}{\gamma_0 b}b_{n,m,p}^{(sc)} \\ -\gamma_0 b i_n(\gamma_0 b)b_{n,m,p}^{(i)} & -\gamma_0 b k_n(\gamma_0 b)b_{n,m,p}^{(sc)} \end{pmatrix} \\
&\quad= \frac{1}{\gamma_0 b}\begin{pmatrix} \left[\gamma_0 b i_n(\gamma_0 b)\right]' & \left[\gamma_0 b k_n(\gamma_0 b)\right]' \\ -\gamma_0 b i_n(\gamma_0 b) & -\gamma_0 b k_n(\gamma_0 b) \end{pmatrix}\cdot\begin{pmatrix} b_{n,m,p}^{(i)} \\ b_{n,m,p}^{(sc)} \end{pmatrix} \\
&\quad= \left(\tilde{B}_{n,m}(b,a;s)\right)\cdot\begin{pmatrix} \left[\gamma_0 a i_n(\gamma_0 a)\right]' b_{n,m,p}^{(1)} \\ -i_n(\gamma_0 a)b_{n,m,p}^{(1)} \end{pmatrix} \\
&\quad= \left(\tilde{B}_{n,m}(b,a;s)\right)\cdot\begin{pmatrix} \left[\gamma_0 a i_n(\gamma_0 a)\right]' \\ -\gamma_0 a i_n(\gamma_0 a) \end{pmatrix}\frac{b_{n,m,p}^{(1)}}{\gamma_0 a}
\end{aligned} \tag{4.2}$$

Using one of the Wronskians in (A.6) we have the inverse matrices

$$\begin{pmatrix} \gamma_0 b i_n(\gamma_0 b) & \gamma_0 b k_n(\gamma_0 b) \\ -[\gamma_0 b i_n(\gamma_0 b)]' & -[\gamma_0 b k_n(\gamma_0 b)]' \end{pmatrix}^{-1} = \begin{pmatrix} -[\gamma_0 b k_n(\gamma_0 b)]' & -\gamma_0 b k_n(\gamma_0 b) \\ [\gamma_0 b i_n(\gamma_0 b)]' & \gamma_0 b i_n(\gamma_0 b) \end{pmatrix} \quad (\det = 1)$$

$$\begin{pmatrix} [\gamma_0 b i_n(\gamma_0 b)]' & [\gamma_0 b k_n(\gamma_0 b)]' \\ -\gamma_0 b i_n(\gamma_0 b) & -\gamma_0 b k_n(\gamma_0 b) \end{pmatrix}^{-1} = \begin{pmatrix} \gamma_0 b k_n(\gamma_0 b) & [\gamma_0 b k_n(\gamma_0 b)]' \\ -\gamma_0 b i_n(\gamma_0 b) & -[\gamma_0 b i_n(\gamma_0 b)]' \end{pmatrix} \quad (\det = -1) \tag{4.3}$$

So we now have

$$\begin{aligned}
\begin{pmatrix} a_{n,m,p}^{(i)} \\ a_{n,m,p}^{(sc)} \end{pmatrix} &= \frac{b}{a} \begin{pmatrix} -[\gamma_0 b k_n(\gamma_0 b)]' & -\gamma_0 b k_n(\gamma_0 b) \\ [\gamma_0 b i_n(\gamma_0 b)]' & \gamma_0 b i_n(\gamma_0 b) \end{pmatrix} \cdot \left(\tilde{B}_{n,m}(b,a;s) \right) \cdot \begin{pmatrix} \gamma_0 a i_n(\gamma_0 a) \\ -[\gamma_0 a i_n(\gamma_0 a)]' \end{pmatrix} a_{n,m,p}^{(1)} \\
&\simeq \frac{b}{a} \begin{pmatrix} -[\gamma_0 b k_n(\gamma_0 b)]' & -\gamma_0 b k_n(\gamma_0 b) \\ [\gamma_0 b i_n(\gamma_0 b)]' & \gamma_0 b i_n(\gamma_0 b) \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} \gamma_0 a i_n(\gamma_0 a) \cosh(\gamma_0 \Delta) + \frac{Z_2}{Z_0} [\gamma_0 a i_n(\gamma_0 a)]' \sinh(\gamma_2 \Delta) \\ -\frac{Z_0}{Z_2} \gamma_0 a i_n(\gamma_0 a) \sinh(\gamma_2 \Delta) - [\gamma_0 a i_n(\gamma_0 a)]' \cosh(\gamma_2 \Delta) \end{pmatrix} a_{n,m,p}^{(1)} \\
\begin{pmatrix} b_{n,m,p}^{(i)} \\ b_{n,m,p}^{(sc)} \end{pmatrix} &= \frac{b}{a} \begin{pmatrix} \gamma_0 b k_n(\gamma_0 b) & [\gamma_0 b k_n(\gamma_0 b)]' \\ -\gamma_0 b i_n(\gamma_0 b) & -[\gamma_0 b i_n(\gamma_0 b)]' \end{pmatrix} \cdot \left(\tilde{B}_{n,m}(b,a;s) \right) \cdot \begin{pmatrix} [\gamma_0 a i_n(\gamma_0 a)]' \\ -\gamma_0 a i_n(\gamma_0 a) \end{pmatrix} b_{n,m,p}^{(1)} \\
&\simeq \frac{b}{a} \begin{pmatrix} \gamma_0 b k_n(\gamma_0 b) & [\gamma_0 b k_n(\gamma_0 b)]' \\ -\gamma_0 b i_n(\gamma_0 b) & -[\gamma_0 b i_n(\gamma_0 b)]' \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} [\gamma_0 a i_n(\gamma_0 a)]' \cosh(\gamma_2 \Delta) + \frac{Z_2}{Z_0} [\gamma_0 a i_n(\gamma_0 a)]' \sinh(\gamma_2 \Delta) \\ -\frac{Z_0}{Z_2} [\gamma_0 a i_n(\gamma_0 a)]' \sinh(\gamma_2 \Delta) - \gamma_0 a i_n(\gamma_0 a) \cosh(\gamma_2 \Delta) \end{pmatrix} b_{n,m,p}^{(1)}
\end{aligned} \tag{4.4}$$

We now solve for the coefficients for the internal fields as

$$\frac{a_{n,m,p}^{(1)}}{a_{n,m,p}^{(i)}}$$

$$\simeq\Bigg[\big[\gamma_0 b k_n(\gamma_0 b)\big]'\gamma_0 a i_n(\gamma_0 a)X_1+\gamma_0 b k_n(\gamma_0 b)\big[\gamma_0 a i_n(\gamma_0 a)\big]'X_2$$

$$+\frac{Z_0}{Z_2}\gamma_0 b k_n(\gamma_0 b)\gamma_0 a i_n(\gamma_0 a)X_3+\frac{Z_2}{Z_0}\big[\gamma_0 b k_n(\gamma_0 a)\big]'\big[\gamma_0 a i_n(\gamma_0 a)\big]'X_4\Bigg]^{-1}$$

$$\frac{b_{n,m,p}^{(1)}}{a_{n,m,p}^{(i)}}$$

$$\simeq\Bigg[\big[\gamma_0 b k_n(\gamma_0 b)\big]'\gamma_0 a i_n(\gamma_0 a)X_1+\gamma_0 b k_n(\gamma_0 b)\big[\gamma_0 a i_n(\gamma_0 a)\big]'X_2$$

$$+\frac{Z_2}{Z_0}\gamma_0 b k_n(\gamma_0 b)\gamma_0 a i_n(\gamma_0 a)X_3+\frac{Z_0}{Z_2}\big[\gamma_0 b k_n(\gamma_0 b)\big]'\big[\gamma_0 a i_n(\gamma_0 a)\big]'X_4\Bigg]^{-1}$$

$$X_1=-X_2=-\cosh(\gamma_2\Delta)\frac{a}{b}$$

$$X_3=-X_4=-\sinh(\gamma_2\Delta)\frac{a}{b} \tag{4.5}$$

The coefficients for the incident field are given in (A.14).

The results are expressed in this form for direct comparison to the exact results for the X_n in Appendix B ((B.1), (B.2), and (B.11)). For large $\gamma_2 a$ (allowing even not-too-small $\gamma_2\Delta$) the exact X_n are approximated in (B.4). For *a/b* near 1.0 the results in (4.5) are in close agreement. For small $\gamma_2 a$ the exact X_n are approximated in (B.8). For *a/b* near 1.0 the results for X_1 and X_2 in (4.5) ($\cosh(\gamma_2\Delta)\to 1$) are in close agreement. Similarly, the result for X_4 ($-\sinh(\gamma_2\Delta)\to-\gamma_2\Delta$) is in close agreement. However, the result for X_3 is very much in disagreement. Thus, at least for the X_n, the BCS is not a good approximation for low frequencies.

With (4.5) let us rearrange the results for the internal-field coefficients as

$$\begin{aligned}
&\frac{a_{n,m,p}^{(1)}}{a_{n,m,p}^{(i)}} \\
&\simeq \frac{b}{a}\Bigg[\Big[\gamma_0 b k_n(\gamma_0 b)\big[\gamma_0 a i_n(\gamma_0 a)\big]' - \big[\gamma_0 b k_n(\gamma_0 b)\big]' \gamma_0 a i_n(\gamma_0 a)\Big]\cosh(\gamma_2\Delta) \\
&\qquad + \left[\frac{Z_0}{Z_2}\gamma_0 b k_n(\gamma_0 b)\gamma_0 a i_n(\gamma_0 a) - \frac{Z_2}{Z_0}\big[\gamma_0 b k_n(\gamma_0 b)\big]'\big[\gamma_0 a i_n(\gamma_0 a)\big]'\right]\sinh(\gamma_2\Delta)\Bigg]^{-1} \\
&\frac{b_{n,m,p}^{(1)}}{b_{n,m,p}^{(i)}} \\
&\simeq \frac{b}{a}\Bigg[\Big[\gamma_0 b k_n(\gamma_0 b)\big[\gamma_0 a i_n(\gamma_0 a)\big]' - \big[\gamma_0 b k_n(\gamma_0 b)\big]' \gamma_0 a i_n(\gamma_0 a)\Big]\cosh(\gamma_2\Delta) \\
&\qquad + \left[\frac{Z_2}{Z_0}\gamma_0 b k_n(\gamma_0 b)\gamma_0 a i_n(\gamma_0 a) - \frac{Z_0}{Z_2}\big[\gamma_0 b k_n(\gamma_0 b)\big]'\big[\gamma_0 a i_n(\gamma_0 a)\big]'\right]\sinh(\gamma_2\Delta)\Bigg]^{-1}
\end{aligned} \tag{5.1}$$

So the results are becoming more compact.

For the next step we note that the delay-corrected BCS changes the BCS by factors (pre and post) which are approximately identities for small $\gamma_0\Delta$ (as in (3.44)). Within this restriction the modified Bessel functions in (5.1) can be reduced to a common argument (say $\gamma_0 a$), the difference between $\gamma_0 b$ and $\gamma_0 a$ being just $\gamma_0\Delta$. The various functions are approximated as

$$f(\gamma_0 b) = f(\gamma_0 a + \gamma_0\Delta) = f(\gamma_0 a) + f'(\gamma_0 a)\gamma_0\Delta + O\left([\gamma_0\Delta]^2\right) \text{ as } \gamma_0\Delta \to 0 \tag{5.2}$$

where, as usual a prime denotes a derivative with respect to the argument. Applying this to the collection of terms multiplying $\cosh(\gamma_0\Delta)$ in (5.1) gives

$$\begin{aligned}
&\gamma_0 b k_n(\gamma_0 b)\big[\gamma_0 a i_n(\gamma_0 b)\big]' - \big[\gamma_0 b k_n(\gamma_0 b)\big]' \gamma_0 a i_n(\gamma_0 a) \\
&\quad = \gamma_0 a k_n(\gamma_0 a)\big[\gamma_0 a i_n(\gamma_0 a)\big]' - \big[\gamma_0 a k_n(\gamma_0 a)\big]' \gamma_0 a i_n(\gamma_0 a) + O(\gamma_0\Delta) \\
&\quad = 1 + O(\gamma_0 a) \text{ as } \gamma_0\Delta \to 0
\end{aligned} \tag{5.3}$$

using the Wronskian (A.6). Applying this to (5.1), making similar approximations in the other Bessel functions and noting $b/a \simeq 1$, we have

$$
\begin{aligned}
\frac{a^{(1)}_{n,m,p}}{a^{(i)}_{n,m,p}} &\simeq \Bigg[\cosh\Big(\gamma_2\Delta\Big) \\
&\quad + \left[\frac{Z_0}{Z_2}\gamma_0 a k_n(\gamma_0 a)\gamma_0 a i_n(\gamma_0 a) - \frac{Z_2}{Z_0}\left[\gamma_0 a k_n(\gamma_0 a)\right]'\left[\gamma_0 a i_n(\gamma_0 a)\right]'\right]\sinh(\gamma_2\Delta)\Bigg]^{-1} \\
\frac{b^{(1)}_{n,m,p}}{b^{(i)}_{n,m,p}} &\simeq \Bigg[\cosh\Big(\gamma_2\Delta\Big) \\
&\quad + \left[\frac{Z_0}{Z_2}\gamma_0 a k_n(\gamma_0 a)\gamma_0 a i_n(\gamma_0 a) - \frac{Z_0}{Z_2}\left[\gamma_0 a k_n(\gamma_0 a)\right]'\left[\gamma_0 a i_n(\gamma_0 a)\right]'\right]\sinh(\gamma_2\Delta)\Bigg]^{-1}
\end{aligned}
\tag{5.4}
$$

Here a can be interpreted as the inner or outer radius of the spherical shell, or something in between.

From (3.1) we have

$$
\begin{aligned}
\frac{Z_2}{Z_0} &= \mu_r \frac{\gamma_0 a}{\gamma_2 a} \\
\frac{a^{(1)}_{n,m,p}}{a^{(i)}_{n,m,p}} &\simeq \Bigg[1 + \Bigg[\frac{1}{\mu_r}\frac{\gamma_2 a}{\gamma_0 a}\gamma_0 a k_n(\gamma_0 a)\gamma_0 a i_n(\gamma_0 a) \\
&\quad - \mu_r \frac{\gamma_2 a}{\gamma_0 a}\left[\gamma_0 a k_n(\gamma_0 a)\right]'\left[\gamma_0 a i_n(\gamma_0 a)\right]'\Bigg]\tanh(\gamma_0\Delta)\Bigg]^{-1} \operatorname{sech}(\gamma_2\Delta) \\
\frac{b^{(1)}_{n,m,p}}{b^{(i)}_{n,m,p}} &\simeq \Bigg[1 + \Bigg[\mu_r \frac{\gamma_0 a}{\gamma_2 a}\gamma_0 a k_n(\gamma_0 a)\gamma_0 a i_n(\gamma_0 a) \\
&\quad - \frac{1}{\mu_r}\frac{\gamma_2 a}{\gamma_0 a}\left[\gamma_0 a k_n(\gamma_0 a)\right]'\left[\gamma_0 a i_n(\gamma_0 a)\right]'\Bigg]\tanh(\gamma_2\Delta)\Bigg]^{-1} \operatorname{sech}(\gamma_2\Delta)
\end{aligned}
\tag{5.5}
$$

In this form let us look for further approximations.

Noting that

$$
\gamma_2\Delta = \sqrt{s\mu_2\left[\sigma_2 + s\varepsilon_2\right]} \simeq \sqrt{s\mu_2\sigma}
\tag{5.6}
$$

let s be in the right half plane, including the jω axis. Then

$$
\begin{aligned}
\operatorname{Re}[\gamma_2\Delta] &\geq \left[\frac{|s|\mu_2\sigma}{2}\right]^{\frac{1}{2}} \quad \text{for} \quad \operatorname{Re}[s] \geq 0 \\
\tanh(\gamma_2\Delta) &= \frac{e^{\gamma_2\Delta} - e^{-\gamma_2\Delta}}{e^{\gamma_2\Delta} + e^{-\gamma_2\Delta}} \simeq 1 \quad \text{for large } \gamma_2\Delta \text{ in RHP}
\end{aligned}
\tag{5.7}
$$

For highly conducting spherical shells then (5.5) further simplifies. Noting that for $\gamma_0 a$ of order one (resonance region) we have

$$\left|\frac{\gamma_2 a}{\gamma_0 a}\right| >> 1 \tag{5.8}$$

then (5.5) simplifies to

$$\begin{aligned}
\frac{a^{(1)}_{n,m,p}}{a^{(i)}_{n,m,p}} &\simeq \left[1 + \frac{1}{\mu_r}\,\frac{\gamma_2 a}{\gamma_0 a}\,\gamma_0 a k_n(\gamma_0 a)\gamma_0 a i_n(\gamma_0 a)\right]^{-1} \sec h(\gamma_2 a) \\
\frac{b^{(1)}_{n,m,p}}{b^{(i)}_{n,m,p}} &\simeq \left[1 - \frac{1}{\mu_r}\,\frac{\gamma_2 a}{\gamma_0 a}\left[\gamma_0 a k_n(\gamma_0 a)\right]'\left[\gamma_0 a i_n(\gamma_0 a)\right]'\right]^{-1} \sec h(\gamma_2 \Delta) \\
\sec h(\gamma_2 \Delta) &\simeq 2\, e^{-\gamma_2 \Delta} \quad \text{(small in RHP including } j\omega \text{ axis)}
\end{aligned} \tag{5.9}$$

Here we can identify the internal resonances as the zeroes of the $i_n(\gamma_0 a)$ for H (TE) modes and of the $[\gamma_a i_n(\gamma_0 a)]'$ for E (TM) modes. The above approximations also show the amplitudes of the resonances, and can be used to estimate the small shifts of the resonances (pole locations) to the left of the jω axis.

6. Concluding Remarks

The BCS is a suitable approximation for a uniform isotropic wall under restricted conditions. Basically the propagation through the wall should be perpendicular to its surfaces. This applies under conditions of high conductivity and/or permeability provided that the frequency is not too low. Under static conditions, for example, the magnetic field passes through a conducting spherical shell of permeability μ_0 as though it were not there. In such a case there is a large radial (longitudinal) component of the magnetic field, comparable to the transverse components. For such low frequencies other kinds of approximations, such as an equivalent sheet impedance, can be more useful [2].

One can go a step further and construct a delay-corrected BCS if the frequency is not too high. Basically the radian wavelengths outside the wall need to be large compared to the wall thickness. This gives the wall an equivalent zero thickness for convenience in analysis.

Appendix A. The Spherical Shell

The fields penetrating into a spherical shell from an incident plane wave is a classical electromagnetic boundary-value problem [3]. Here we revisit this problem and carry the results further.

A.1 Electromagnetic fields in spherical coordinates

Summarizing from various previous papers, such as [2], we have solutions expressed in terms of the complex-frequency or two-sided Laplace-transform variable $s = \Omega + j\omega$. We have spherical harmonics

$$
\begin{aligned}
Y_{n,m,{e \atop o}}(\theta,\phi) &= P_n^{(m)}(\cos(\theta))\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix} \\
\vec{P}_{n,m,p}(\theta,\phi) &= Y_{n,m,p}(\theta,\phi)\,\vec{1}_r \\
\vec{Q}_{n,m,p}(\theta,\phi) &= \nabla_s Y_{n,m,p}(\theta,\phi)\,\vec{1}_r \times \vec{R}_{n,m,p}(\theta,\phi) \\
\vec{Q}_{n,m,{e \atop o}}(\theta,\phi) &= \vec{1}_\theta \frac{dP_n^{(m)}(\cos(\theta))}{d\theta}\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix} + \vec{1}_\phi \frac{dP_n^{(m)}(\cos(\theta))}{\sin\theta} m \begin{Bmatrix}-\sin(m\phi)\\ \cos(m\phi)\end{Bmatrix} \\
\vec{R}_{n,m,p}(\theta,\phi) &= \nabla_s \times \vec{P}_{n,m,p}(\theta,\phi) = -\vec{1}_r \times \vec{Q}_{n,m,p}(\theta,\phi) \\
\vec{R}_{n,m,{e \atop o}}(\theta,\phi) &= \vec{1}_\theta \frac{P_n^{(m)}(\cos(\theta))}{\sin(\theta)} m \begin{Bmatrix}-\sin(m\phi)\\ \cos(m\phi)\end{Bmatrix} - \vec{1}_\phi \frac{dP_n^{(m)}(\cos(\theta))}{d\theta} m \begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}
\end{aligned}
\tag{A.1}
$$

with the Legendre functions given by

$$
\begin{aligned}
P_n^{(m)}(\xi) &\equiv [-1]^m \left(1-\xi^2\right)^{m/2} \frac{d^m}{d\xi^m} P_n(\xi) \\
P_n(\xi) &\equiv P_n^{(0)}(\xi) \equiv \frac{1}{2^n n!}\frac{d^n}{d\xi^n}\left[\left[\xi^2-1\right]^n\right]
\end{aligned}
\tag{A.2}
$$

These are used in constructing the spherical wave functions

$$
\begin{aligned}
\Xi_{n,m,p}^{(\ell)}(\gamma\vec{r}) &= f_n^{(\ell)}(\gamma r) P_n^{(m)}(\theta,\phi) \quad , \quad r = |\vec{r}| \\
\vec{L}_{n,m,p}^{(\ell)}(\gamma\vec{r}) &= \frac{1}{\gamma}\nabla\Xi_{n,m,p}^{(\ell)}(\gamma\vec{r}) = f_n^{(\ell)\prime}(\gamma r)\,\vec{P}_{n,m,p}(\theta,\phi) + \frac{f_n^{(\ell)}(\gamma r)}{\gamma r}\vec{Q}_{n,m,p}(\theta,\phi) \\
\vec{M}_{n,m,p}^{(\ell)}(\gamma\vec{r}) &= \nabla \times \left[\vec{r}\,\Xi_{n,m,p}^{(\ell)}(\gamma\vec{r})\right] = -\gamma r \times \vec{L}_{n,m,p}^{(\ell)}(\gamma\vec{r}) = -\frac{1}{\gamma}\nabla\times\vec{N}_{n,m,p}^{(\ell)}(\gamma\vec{r}) \\
&= f_n^{(\ell)}(\gamma r)\,\vec{R}_{n,m,p}(\theta,\phi)
\end{aligned}
$$

$$
\begin{aligned}
\vec{N}^{(\ell)}_{n,m,p}(\gamma \vec{r}) &= \frac{1}{\gamma}\nabla \times \vec{M}^{(\ell)}_{n,m,p}(\gamma \vec{r}) \\
&= n(n+1)\frac{f_n^{(\ell)}(\gamma r)}{\gamma r}\vec{P}_{n,m,p}(\theta,\phi) + \frac{\left[\gamma r f_n^{(\ell)}(\gamma r)\right]'}{\gamma r}\vec{Q}_{n,m,p}(\theta,\phi)
\end{aligned}
\tag{A.3}
$$

The spherical Bessel functions (modified) are denoted by

$$
f_n^{(1)}(\gamma r) = i_n(\gamma r) \quad , \quad f_n^{(2)}(\gamma r) = k_n(\gamma r) \tag{A.4}
$$

with

$$
\begin{aligned}
i_n(\zeta) &= \frac{e^{\zeta}}{2\zeta}\sum_{\beta=0}^{n}\frac{[n+B]!}{B![n-\beta]!}[-2\zeta]^{-\beta} + (-1)^{n+1}\frac{e^{-\zeta}}{2\zeta}\sum_{\beta=0}^{n}\frac{[n+B]!}{B![n-\beta]!}[2\zeta]^{-\beta} \\
k_n(\zeta) &= \frac{e^{-\zeta}}{\zeta}\sum_{\beta=0}^{n}\frac{[n+B]!}{B![n-\beta]!}(2\zeta)^{-\beta} \\
i_n(\zeta) &= \frac{1}{2}\left[[-1]^{n+1}k_n(\zeta) \quad - k_n(-\zeta)\right]
\end{aligned}
\tag{A.5}
$$

A prime is used to indicate a derivative with respect to the argument of a Bessel function. We have the Wronskian relations

$$
\begin{aligned}
W\left(i_n(\zeta), k_n(\zeta)\right) &= i_n(\zeta)k_n'(\zeta) - i_n'(\zeta)k_n(\zeta) = -\zeta^{-2} \\
W\left(\zeta i_n(\zeta), \zeta k_n(\zeta)\right) &= \zeta i_n(\zeta)\left[\zeta k_n(\zeta)\right]' - \left[\zeta i_n(\zeta)\right]'\zeta k_n(\zeta) = -1
\end{aligned}
\tag{A.6}
$$

The propagation constant is

$$
\begin{aligned}
\gamma &= \left[s\mu(\sigma + s\varepsilon)\right]^{1/2} \\
\mu &\equiv \text{permeability} \\
\sigma &\equiv \text{conductivity} \\
\varepsilon &\equiv \text{permittivity}
\end{aligned}
\tag{A.7}
$$

For later use we have particular Bessel-function representations

$$
\begin{aligned}
i_1(\zeta) &= \frac{1}{2}\left[e^{\zeta}\left[\zeta^{-1} - \zeta^{-2}\right] + e^{-\zeta}\left[\zeta^{-1} + \zeta^{-2}\right]\right] \\
&= \zeta^{-1}\cosh(\zeta) - \zeta^{-2}\sinh(\zeta) \\
&= \frac{\zeta}{3}\left[1 + O\left(\zeta^2\right)\right] \text{ as } \zeta \to 0 \\
k_1(\zeta) &= e^{-\zeta}\left[\zeta^{-1} + \zeta^{-2}\right] \\
&= \zeta^{-2}\left[1 + O\left(\zeta^2\right)\right] \text{ as } \zeta \to 0 \\
\left[\zeta i_1(\zeta)\right]' &= \frac{1}{2}\left[e^{\zeta}\left[1 - \zeta^{-1} + \zeta^{-2}\right] - e^{-\zeta}\left[1 + \zeta^{-1} + \zeta^{-2}\right]\right] \\
&= \left[1 + \zeta^{-2}\right]\sinh(\zeta) - \zeta^{-1}\cosh(\zeta) \\
&= \frac{2}{3}\zeta\left[1 + O\left(\zeta^2\right)\right] \text{ as } \to 0 \\
\left[\zeta k_1(\zeta)\right]' &= -e^{-\zeta}\left[1 + \zeta^{-1} + \zeta^{-2}\right] \\
&= -\zeta^{-2}\left[1 + O\left(\zeta^2\right)\right] \text{ as } \to 0
\end{aligned}
\tag{A.8}
$$

Associated particular Legendre-function representations are

$$
\begin{aligned}
P_1^{(0)}(\xi) &= \xi \quad , \quad P_1^{(0)}(\cos(\theta)) = \cos(\theta) \\
P_1^{(1)}(\xi) &= -\left[1 - \xi^2\right]^{1/2} \quad , \quad P_1^{(1)}(\cos(\theta)) = -\sin(\theta)
\end{aligned}
\tag{A.9}
$$

Particular spherical harmonics are

$$
\begin{aligned}
\vec{Q}_{1,0,{e \atop o}} &= -\vec{1}_\theta \sin(\theta)\begin{Bmatrix}1\\0\end{Bmatrix} \\
\vec{Q}_{1,1,{e \atop o}} &= -\vec{1}_\theta \sin(\theta)\begin{Bmatrix}\cos(\phi)\\ \sin(\phi)\end{Bmatrix} - \vec{1}_\phi\begin{Bmatrix}-\sin(\phi)\\ \cos(\phi)\end{Bmatrix} \\
\vec{R}_{1,0,{e \atop o}} &= -\vec{1}_\phi \sin(\theta)\begin{Bmatrix}1\\0\end{Bmatrix} \\
\vec{R}_{1,1,{e \atop o}} &= -\vec{1}_\theta \sin(\theta)\begin{Bmatrix}-\sin(\phi)\\ \cos(\phi)\end{Bmatrix} + \vec{1}_\phi \cos(\theta)\begin{Bmatrix}\cos(\phi)\\ \sin(\phi)\end{Bmatrix}
\end{aligned}
\tag{A.10}
$$

A.2. Plane Waves in Spherical Coordinates

Define a set of orthogonal (right-handed) unit vectors by

$$\begin{aligned}
\overrightarrow{1}_1 &= \sin(\theta_1)\cos(\phi_1)\,\overrightarrow{1}_x + \sin(\theta_1)\sin(\phi_1)\,\overrightarrow{1}_y + \cos(\theta_1)\,\overrightarrow{1}_z \\
\overrightarrow{1}_2 &= -\cos(\theta_1)\cos(\phi_1)\,\overrightarrow{1}_x - \cos(\theta_1)\sin(\phi_1)\,\overrightarrow{1}_y + \sin(\theta_1)\,\overrightarrow{1}_z \\
\overrightarrow{1}_3 &= \sin(\phi_1)\,\overrightarrow{1}_x - \cos(\phi_1)\,\overrightarrow{1}_y
\end{aligned} \tag{A.11}$$

Here $\overrightarrow{1}_1$ is the direction of propagation and $\overrightarrow{1}_2$ and $\overrightarrow{1}_3$ are mutually orthogonal unit vectors, each orthogonal to $\overrightarrow{1}_1$ to indicate the polarization of the electromagnetic fields in the incident plane wave. As indicated in Fig. A.1, θ_1 is the angle of $\overrightarrow{1}_1$ with respect to the z axis and ϕ_1 is the angle of its projection on the x, y plane with respect to the x axis. For convenience $\overrightarrow{1}_2$ is chosen in a plane parallel to $\overrightarrow{1}_1$ and the z axis (E or TM polarization if the electric field is parallel to $\overrightarrow{1}_2$) while $\overrightarrow{1}_3$ is then parallel to the x, y plane (H or TE polarization if the electric field is parallel to $\overrightarrow{1}_3$). In (A.11) we can use the relations between Cartesian and spherical coordinates

$$\begin{aligned}
x &= r\,\sin(\theta)\cos(\phi) \\
y &= r\,\sin(\theta)\sin(\phi) \\
z &= r\,\cos(\theta) \\
\overrightarrow{1}_x &= \sin(\theta)\cos(\phi)\,\overrightarrow{1}_r + \cos(\theta)\cos(\phi)\,\overrightarrow{1}_\theta - \sin(\phi)\,\overrightarrow{1}_\theta \\
\overrightarrow{1}_y &= \sin(\theta)\sin(\phi)\,\overrightarrow{1}_r + \cos(\theta)\sin(\phi)\,\overrightarrow{1}_\theta + \cos(\phi)\,\overrightarrow{1}_\phi \\
\overrightarrow{1}_z &= \cos(\theta)\,\overrightarrow{1}_r - \sin(\theta)\,\overrightarrow{1}_\theta
\end{aligned} \tag{A.12}$$

to express the incident-wave unit vectors in terms of (θ_1,ϕ_1) and (θ,ϕ).

Next we have the result for a dyadic plane wave [2]

$$\begin{aligned}
\overleftrightarrow{1}\, e^{-\gamma_0 \overrightarrow{1}_1 \cdot \overrightarrow{r}} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{p=e,0} \left[2 - 1_{0,m}\right]^n \left[-1\right]^n \left[2n+1\right] \frac{[n-m]!}{[n+m]!} \\
\Bigg[-\overrightarrow{P}_{n,m,p}(\theta_1,\phi_1)\, \overrightarrow{L}^{(1)}_{n,m,p}(\gamma_0 \overrightarrow{r}) \\
+ \frac{1}{n(n+1)} \left[\overrightarrow{R}_{n,m,p}(\theta_1,\phi_1)\, \overrightarrow{M}^{(1)}_{n,m,\ell}(\gamma_0 \overrightarrow{r}) - \overrightarrow{Q}_{n,m,p}(\theta_1,\phi_1)\, \overrightarrow{N}^{(1)}_{n,m,\ell}(\gamma_0 \overrightarrow{r}) \right] \Bigg]
\end{aligned} \tag{A.13}$$

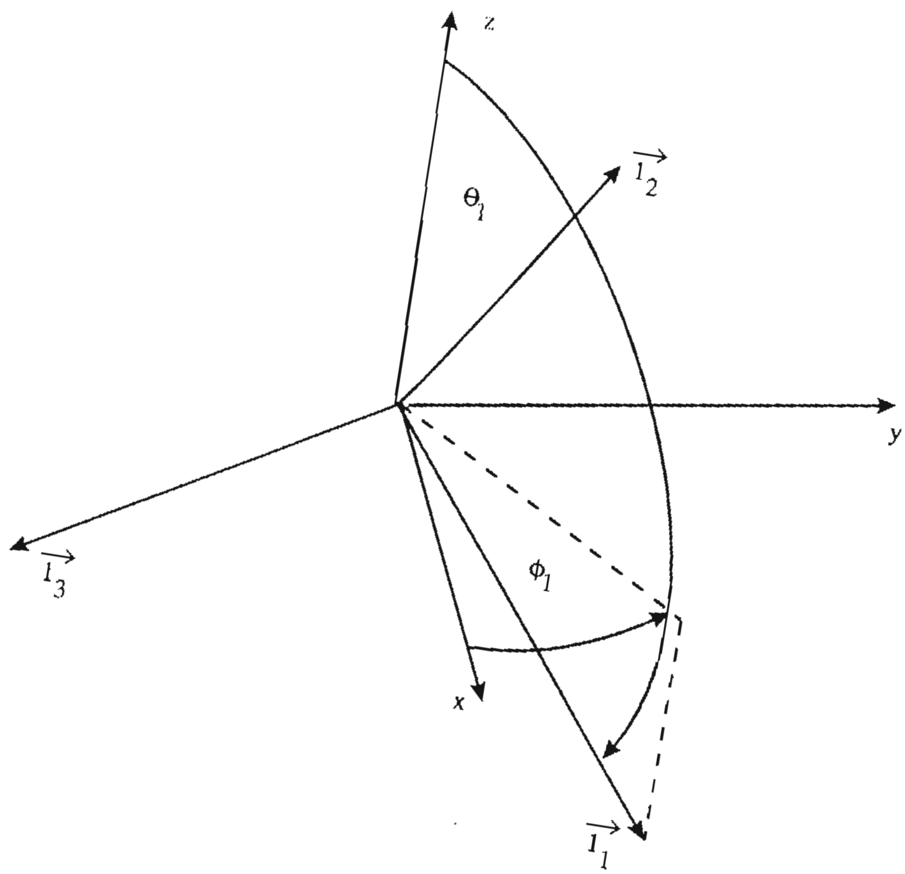

Fig. A.1. Unit Vectors for Plane Wave.

where for $n = 0$ the summation does not extend over the identically zero $\vec{Q}$, $\vec{R}$, $\vec{M}$, and $\vec{N}$ functions. For vector plane waves we have the set of orthogonal unit vectors $\{\vec{1}_1, \vec{1}_2, \vec{1}_3\}$. In free space, electromagnetic plane waves have both electric and magnetic fields orthogonal to $\vec{1}_1$ (as well as to each other). Thus only $\vec{1}_2$ and $\vec{1}_3$ are of concern. This removes the presence of the $\vec{L}$ functions in the expansion (since plane waves have zero-divergence fields). Taking dot products of $\vec{1}_2$ and $\vec{1}_3$ with (A.12) gives

$$\vec{1}_2 e^{-\gamma_0 \vec{1}_1 \cdot \vec{r}} = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{p=e,0} \left[a_{n,m,p}^{(i)} \vec{M}_{n,m,p}^{(1)}(\gamma_0 \vec{r}) + b_{n,m,p}^{(i)} \vec{N}_{n,m,p}^{(1)}(\gamma_0 \vec{r}) \right]$$

$$\vec{1}_3 e^{-\gamma_0 \vec{1}_1 \cdot \vec{r}} = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{p=e,0} \left[b_{n,m,p}^{(i)} \vec{M}_{n,m,p}^{(1)}(\gamma_0 \vec{r}) - a_{n,m,p}^{(i)} \vec{N}_{n,m,p}^{(1)}(\gamma_0 \vec{r}) \right]$$

$$a_{n,m,{e \atop o}}^{(i)} = \left[2 - 1_{0,m}\right]\left[-1\right]^{n+1} \frac{2n+1}{n[n+1]} \frac{[n-m]!}{[n+m]!} m \frac{P_n^{(m)}(\cos(\theta_1))}{\sin(\theta_1)} \begin{Bmatrix} -\sin(m\phi_1) \\ \cos(m\phi_1) \end{Bmatrix}$$

$$b_{n,m,{e \atop o}}^{(i)} = \left[2 - 1_{0,m}\right]\left[-1\right]^{n} \frac{2n+1}{n[n+1]} \frac{[n-m]!}{[n+m]!} \frac{dP_n^{(m)}(\cos(\theta_1))}{d\theta_1} \begin{Bmatrix} \cos(m\phi_1) \\ \sin(m\phi_1) \end{Bmatrix} \tag{A.14}$$

The superscript i here refers to the incident wave. Particular coefficients of interest are

$$a'^{(i)}_{1,0,{e \atop o}} = 0$$

$$a'^{(i)}_{1,1,{e \atop o}} = \frac{3}{2} \frac{P_1^{(1)}(\cos(\theta_1))}{\sin(\theta_1)} \begin{Bmatrix} -\sin(\phi_1) \\ \cos(\phi_1) \end{Bmatrix} = -\frac{3}{2} \begin{Bmatrix} -\sin(\phi_1) \\ \cos(\phi_1) \end{Bmatrix}$$

$$b'^{(i)}_{1,0,{e \atop o}} = -\frac{3}{2} \frac{dP_1^{(0)}(\cos(\theta_1))}{d\theta_1} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \frac{3}{2} \sin(\theta_1) \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$b'^{(i)}_{1,1,{e \atop o}} = -\frac{3}{2} \frac{dP_1^{(1)}(\cos(\theta_1))}{d\theta_1} \begin{Bmatrix} \cos(\phi_1) \\ \sin(\phi_1) \end{Bmatrix} = \frac{3}{2} \cos(\theta_1) \begin{Bmatrix} \cos(\phi_1) \\ \sin(\phi_1) \end{Bmatrix} \tag{A.15}$$

A.3. Boundary-Value Problem

The geometry is shown in Fig. A.2. There are three regions. Regions 1 and 3 are free space with

$$\gamma_0 = s\left[\mu_0 \varepsilon_0\right]^{-1} = \frac{s}{c} \equiv \text{propagation constant}$$

$$Z_0 = \left[\frac{\mu_0}{\varepsilon_0}\right] \equiv \text{wave impedance} \tag{A.16}$$

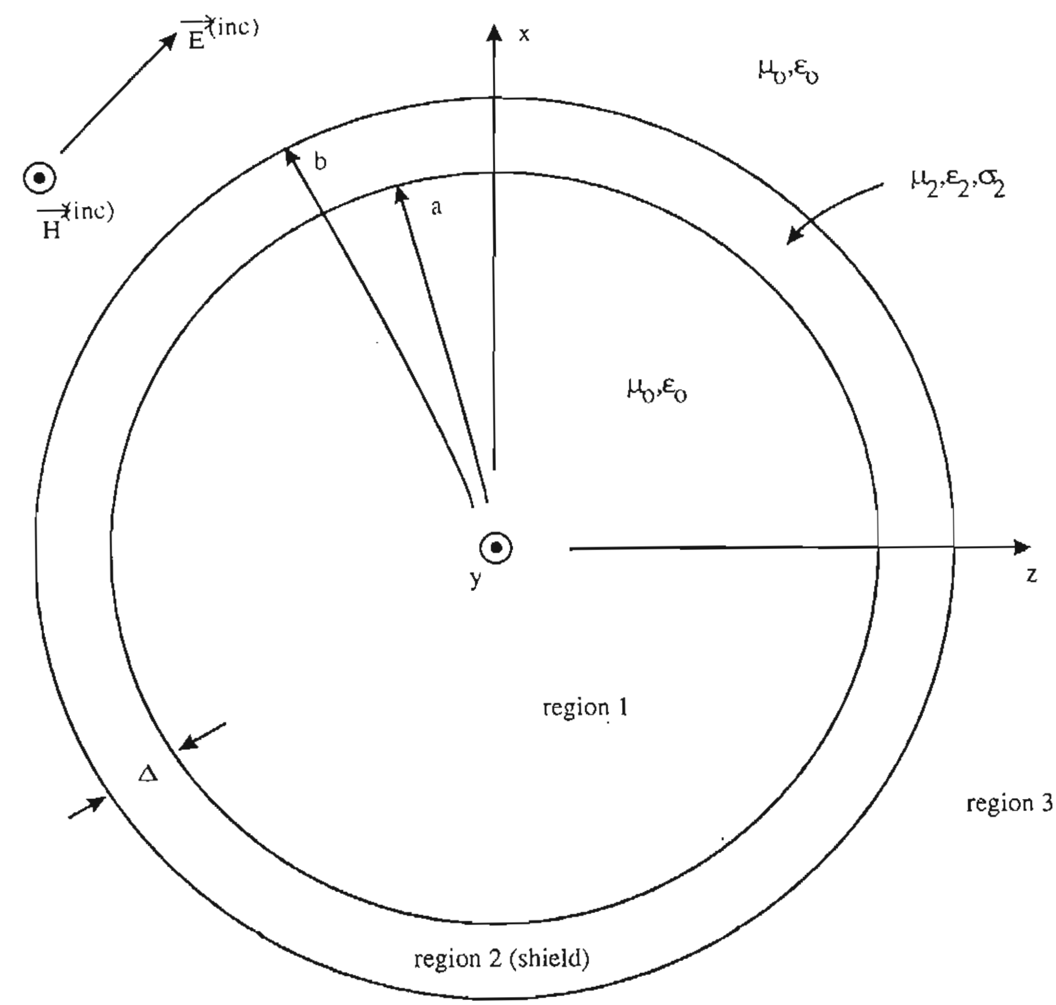

Fig. A.2 The Spherical Shell

Region 2 is the spherical homogeneous, isotropic shell with

$$\gamma_2 = \left[s\mu_2 \left[\sigma_2 + s\varepsilon_2 \right] \right]^{1/2}$$
$$Z_2 = \left[\frac{s\mu_2}{\sigma_2 + s_2} \right]^{1/2} \tag{A.17}$$

For convenience we can choose our incident plane wave as an E (or TM) wave, giving

$$\vec{E}^{(inc)}(\vec{r}, s) = E_0 \vec{1}_2 e^{-\gamma_0 \vec{1}_1 \bullet \vec{r}}$$
$$\vec{H}^{(inc)}(\vec{r}, s) = \frac{E_0}{Z_0} \vec{1}_3 e^{-\gamma_0 \vec{1}_1 \bullet \vec{r}} \tag{A.18}$$

Due to the symmetry of the problem (O_3, all rotations and reflections in three dimensions) the above applies to H (or TE) waves of well by a redefinition of the coordinates. Furthermore, one can choose polarization and direction of incidence at our convenience, e.g.,

$$\vec{1}_1 = \vec{1}_z \quad , \quad \theta_1 = 0$$
$$\vec{1}_2 = \vec{1}_x \quad , \quad \phi_1 = \pi \qquad \text{A.19)}$$
$$\vec{1}_3 = \vec{1}_y$$

Having the incident wave we need to similarly expand the rest of the fields. The internal shielded region (region 1, $r < a$) has

$$\vec{E}^{(1)}(\vec{r}, s) = E_0 \sum_{n=1}^{\infty} \sum_{m=1}^{n} \sum_{p=e,0} \left[a_{n,m,p}^{(1)} \vec{M}_{n,m,p}^{(1)}(\gamma_0 \vec{r}) + b_{n,m,p}^{(1)} \vec{N}_{n,m,p}^{(1)}(\gamma_0 \vec{r}) \right]$$
$$\vec{H}^{(1)}(\vec{r}, s) = \frac{E_0}{Z_0} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \sum_{p=e,0} \left[b_{n,m,p}^{(1)} \vec{M}_{n,m,p}^{(1)}(\gamma_0 \vec{r}) - a_{n,m,p}^{(1)} \vec{N}_{n,m,p}^{(1)}(\gamma_0 \vec{r}) \right] \tag{A.20}$$

Note the use of only $\ell = 1$ wave functions for non-singular fields at the origin ($\vec{r} = \vec{0}$). Within the shell or shield wall itself (region 2, $a < r < b$) we need both types of wave functions giving

$$
\begin{aligned}
\vec{E}^{(2)}(\vec{r},s) &= E_0 \sum_{n=1}^{\infty} \sum_{m=1}^{n} \sum_{p=e,0} \Big[a^{(2)}_{n,m,p} \vec{M}^{(1)}_{n,m,p}(\gamma_2 \vec{r}) + a^{(2')}_{n,m,p} \vec{M}^{(2)}_{n,m,p}(\gamma_2 \vec{r}) \\
&\qquad + b^{(2)}_{n,m,p} \vec{N}^{(1)}_{n,m,p}(\gamma_2 \vec{r}) + b^{(2')}_{n,m,p} \vec{N}^{(2)}_{n,m,p}(\gamma_2 \vec{r}) \Big] \\
\vec{H}^{(2)}(\vec{r},s) &= \frac{E_0}{Z_2} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \sum_{p=e,0} \Big[b^{(2)}_{n,m,p} \vec{M}^{(1)}_{n,m,p}(\gamma_2 \vec{r}) + b^{(2')}_{n,m,p} \vec{M}^{(2)}_{n,m,p}(\gamma_2 \vec{r}) \\
&\qquad - a^{(2)}_{n,m,p} \vec{N}^{(1)}_{n,m,p}(\gamma_2 \vec{r}) - a^{(2')}_{n,m,p} \vec{N}^{(2)}_{n,m,p}(\gamma_2 \vec{r}) \Big]
\end{aligned} \tag{A.21}
$$

In region 3 we have the incident field, but also need the scattered field (superscript *sc*) which we expand in terms of only outgoing ($\ell = 2$) wave functions as

$$
\begin{aligned}
\vec{E}^{(sc)}(\vec{r},s) &= E_0 \sum_{n=1}^{\infty} \sum_{m=1}^{n} \sum_{p=e,0} \Big[a^{(sc)}_{n,m,p} \vec{M}^{(2)}_{n,m,p}(\gamma_0 \vec{r}) + b^{(sc)}_{n,m,p} \vec{N}^{(2)}_{n,m,p}(\gamma_0 \vec{r}) \Big] \\
\vec{H}^{(sc)}(\vec{r},s) &= \frac{E_0}{Z_0} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \sum_{p=e,0} \Big[b^{(sc)}_{n,m,p} \vec{M}^{(2)}_{n,m,p}(\gamma_0 \vec{r}) - a^{(sc)}_{n,m,p} \vec{N}^{(2)}(\gamma_0 \vec{r}) \Big]
\end{aligned} \tag{A.22}
$$

Matching tangential electric and magnetic fields across the r = a boundary gives

$$
\begin{aligned}
a^{(1)}_{n,m,p}\, i_n(\gamma_0 a) &= a^{(2)}_{n,m,p}\, i_n(\gamma_2 a) + a^{(2')}_{n,m,p}\, k_n(\gamma_2 a) \\
a^{(1)}_{n,m,p} \frac{\left[\gamma_0 a i_n(\gamma_0 a)\right]'}{\gamma_0 a} &= \frac{Z_0}{Z_2} \left[a^{(2)}_{n,m,p} \frac{\left[\gamma_2 a i_n(\gamma_2 a)\right]'}{\gamma_2 a} + a^{(2')}_{n,m,p} \frac{\left[\gamma_2 a k_n(\gamma_2 a)\right]'}{\gamma_2 a} \right] \\
b^{(1)}_{n,m,p} \frac{\left[\gamma_0 a i_n(\gamma_0 a)\right]'}{\gamma_0 a} &= b^{(2)}_{n,m,p} \frac{\left[\gamma_2 a i_n(\gamma_2 a)\right]'}{\gamma_2 a} + b^{(2')}_{n,m,p} \frac{\left[\gamma_2 a k_n(\gamma_2 a)\right]'}{\gamma_2 a} \\
b^{(1)}_{n,m,p} i_n(\gamma_0 a) &= \frac{Z_0}{Z_2} \left[b^{(2)}_{n,m,p} i_n(\gamma_2 a) + b^{(2')}_{n,m,p} k_n(\gamma_0 a) \right]
\end{aligned} \tag{A.23}
$$

Similarly matching the tangential fields across the $r = b$ boundary gives

$$
\begin{aligned}
&a_{n,m,p}^{(2)} i_n(\gamma_2 b) + a_{n,m,p}^{(2')} k_n(\gamma_2 b) = a_{n,m,p}^{(i)} i_n(\gamma_0 b) + a^{(sc)} k_n(\gamma_0 b) \\
&a_{n,m,p}^{(2)} \frac{[\gamma_2 b i_n(\gamma_2 b)]'}{\gamma_2 b} + a_{n,m,p}^{(2')} \frac{[\gamma_2 b k_n(\gamma_2 b)]'}{\gamma_2 b} \\
&\qquad = \frac{Z_2}{Z_0}\left[a_{n,m,p}^{(i)} \frac{[\gamma_0 b i_n(\gamma_0 b)]'}{\gamma_0 b} + a_{n,m,p}^{(sc)} \frac{[\gamma_0 b k_n(\gamma_0 b)]'}{\gamma_0 b} \right] \\
&b_{n,m,p}^{(2)} \frac{[\gamma_2 b i_n(\gamma_2 b)]'}{\gamma_2 b} + b_{n,m,p}^{(2')} \frac{[\gamma_2 b k_n(\gamma_2 b)]'}{\gamma_2 b} \\
&\qquad = b_{n,m,p}^{(i)} \frac{[\gamma_0 b i_n(\gamma_0 b)]'}{\gamma_0 b} + b_{n,m,p}^{(sc)} \frac{[\gamma_0 b k_n(\gamma_0 b)]'}{\gamma_0 b} \\
&b_{n,m,p}^{(2)} i_n(\gamma_2 b) + b_{n,m,p}^{(2')} k_n(\gamma_2 b) \\
&\qquad = \frac{Z_2}{Z_0}\left[b_{n,m,p}^{(i)} i_n(\gamma_0 b) + b_{n,m,p}^{(sc)} k_n(\gamma_0 b) \right]
\end{aligned}
\tag{A.24}
$$

There are five $a_{n,m,p}$ coefficients with one known (the incident one) and four equations to solve for the four unknown ones, and similarly for the $b_{n,m,p}$ coefficients.

First reduce (A.23) by eliminating the coefficients with 2' superscripts, giving

$$
\begin{aligned}
&a_{n,m,p}^{(1)} \left[i_n(\gamma_0 a) \frac{[\gamma_2 a k_n(\gamma_2 a)]'}{\gamma_2 a} - \frac{Z_2}{Z_0} \frac{[\gamma_0 b i_n(\gamma_0 a)]'}{\gamma_0 a} k_n(\gamma_2 a) \right] \\
&= a_{n,m,p}^{(2)} \left[i_n(\gamma_2 a) \frac{[\gamma_2 a k_n(\gamma_2 a)]'}{\gamma_2 a} - \frac{[\gamma_2 a i_n(\gamma_2 a)]'}{\gamma_2 a} k_n(\gamma_2 a) \right] \\
&= a_{n,m,p}^{(2)} \frac{-1}{[\gamma_2 a]^2} \\
&b_{n,m,p}^{(1)} \left[\frac{[\gamma_0 a i_n(\gamma_0 a)]'}{\gamma_0 a} k_n(\gamma_2 a) - \frac{Z_2}{Z_0} i_n(\gamma_0 a) \frac{[\gamma_2 a k_n(\gamma_2 a)]'}{\gamma_2 a} \right] \\
&= b_{n,m,p}^{(2)} \left[\frac{[\gamma_0 a i_n(\gamma_0 a)]'}{\gamma_0 a} k_n(\gamma_2 a) - i_n(\gamma_0 a) \frac{[\gamma_2 a k_n(\gamma_2 a)]'}{\gamma_2 a} \right] \\
&= b_{n,m,p}^{(2)} \frac{1}{[\gamma_2 a]^2}
\end{aligned}
\tag{A.25}
$$

where the Wronskian (A.6) is used to remove half of the Bessel functions. Next in (A.23) remove the coefficients with 2 superscripts giving

$$
\begin{aligned}
&a_{n,m,p}^{(1)}\left[i_n(\gamma_0 a)\frac{\left[\gamma_2 a i_n(\gamma_2 a)\right]'}{\gamma_2 a}-\frac{Z_2}{Z_0}\frac{\left[\gamma_0 a i_n(\gamma_0 a)\right]'}{\gamma_0 a}i_n(\gamma_2 a)\right]\\
&=a_{n,m,p}^{(2')}\left[k_n(\gamma_2 a)\frac{\left[\gamma_2 a i_n(\gamma_2 a)\right]'}{\gamma_2 a}-\frac{\left[\gamma_2 a k_n(\gamma_2 a)\right]'}{\gamma_2 a}i_n(\gamma_2 a)\right]\\
&=a_{n,m,p}^{(2')}\frac{1}{[\gamma_2 a]^2}\\
&b_{n,m,p}^{(1)}\left[\frac{\left[\gamma_0 a i_n(\gamma_0 a)\right]'}{\gamma_0 a}i_n(\gamma_2 a)-\frac{Z_2}{Z_0}i_n(\gamma_0 a)\frac{\left[\gamma_2 a i_n(\gamma_2 a)\right]'}{\gamma_2 a}\right]\\
&=b_{n,m,p}^{(2')}\left[\frac{\left[\gamma_2 a k_n(\gamma_2 a)\right]'}{\gamma_2 a}i_n(\gamma_2 a)-k_n(\gamma_2 a)\frac{\left[\gamma_2 a i_n(\gamma_2 a)\right]'}{\gamma_2 a}\right]\\
&=b_{n,m,p}^{(2')}\frac{-1}{[\gamma_2 a]^2}
\end{aligned}
\tag{A.26}
$$

Turning now to (A.24) let us eliminate the coefficients with *sc* superscripts giving

$$
\begin{aligned}
&a_{n,m,p}^{(2)}\left[i_n(\gamma_2 b)\frac{\left[\gamma_0 b k_n(\gamma_0 b)\right]'}{\gamma_0 b}-\frac{Z_0}{Z_2}\frac{\left[\gamma_2 b i_n(\gamma_2 b)\right]'}{\gamma_2 a}k_n(\gamma_0 b)\right]\\
&\quad+a_{n,m,p}^{(2')}\left[k_n(\gamma_2 b)\frac{\left[\gamma_0 b k_n(\gamma_0 b)\right]'}{\gamma_0 b}-\frac{Z_0}{Z_2}\frac{\left[\gamma_2 b k_n(\gamma_2 b)\right]'}{\gamma_2 b}k_n(\gamma_0 b)\right]\\
&\quad=a_{n,m,p}^{(i)}\left[i_n(\gamma_0 b)\frac{\left[\gamma_0 b k_n(\gamma_0 b)\right]'}{\gamma_0 b}-\frac{\left[(\gamma_0 b) i_n(\gamma_0 b)\right]'}{\gamma_0 b}k_n(\gamma_0 b)\right]\\
&\quad=a_{n,m,p}^{(i)}\frac{-1}{[\gamma_0 b]^2}\\
&b_{n,m,p}^{(2)}\left[\frac{\left[\gamma_2 b i_n(\gamma_2 b)\right]'}{\gamma_2 a}k_n(\gamma_0 b)-\frac{Z_0}{Z_2}i_n(\gamma_2 b)\frac{\left[\gamma_0 b k_n(\gamma_0 b)\right]'}{\gamma_0 b}\right]\\
&\quad+b_{n,m,p}^{(2')}\left[\frac{\left[\gamma_2 b k_n(\gamma_2 b)\right]'}{\gamma_2 b}k_n(\gamma_0 b)-\frac{Z_0}{Z_2}k_n(\gamma_2 b)\frac{\left[\gamma_0 b k_n(\gamma_0 b)\right]'}{\gamma_0 b}\right]\\
&\quad=b_{n,m,p}^{(i)}\left[k_n(\gamma_0 b)\frac{\left[\gamma_0 b i_n(\gamma_0 b)\right]'}{\gamma_0 b}-\frac{\left[(\gamma_0 b) k_n(\gamma_0 b)\right]'}{\gamma_0 b}i_n(\gamma_0 b)\right]\\
&\quad=b_{n,m,p}^{(i)}\frac{-1}{[\gamma_0 b]^2}
\end{aligned}
\tag{A.27}
$$

We now can solve for the 1 superscripted coefficients by substituting from (A.25) and (A.26) into (A.27) to give

$$\frac{a^{(1)}_{n,m,p}}{a^{(i)}_{n,m,p}} = [\gamma_2 a]^{-2}[\gamma_0 b]^{-2}\Bigg[\left[i_n(\gamma_2 b)\frac{[\gamma_0 b k_n(\gamma_0 b)]'}{\gamma_2 b} - \frac{Z_0}{Z_2}\frac{[\gamma_2 b i_n(\gamma_2 b)]'}{\gamma_2 b}k_n(\gamma_0 b)\right]$$

$$\left[i_n(\gamma_0 a)\frac{[\gamma_2 a k_n(\gamma_2 a)]'}{\gamma_2 a} - \frac{Z_2}{Z_0}\frac{[\gamma_0 a i_n(\gamma_0 a)]'}{\gamma_0 a}k_n(\gamma_2 a)\right]$$

$$-\left[k_n(\gamma_2 b)\frac{[\gamma_0 b k_n(\gamma_0 b)]'}{\gamma_0 b} - \frac{Z_0}{Z_2}\frac{[\gamma_2 b k_n(\gamma_2 b)]'}{\gamma_2 b}k_n(\gamma_0 b)\right]$$

$$\left[i_n(\gamma_0 a)\frac{[\gamma_2 a i_n(\gamma_2 a)]'}{\gamma_2 a} - \frac{Z_2}{Z_0}\frac{[\gamma_0 a i_n(\gamma_0 a)]'}{\gamma_0 a}i_n(\gamma_2 a)\right]\Bigg]^{-1}$$

$$\frac{b^{(1)}_{n,m,p}}{b^{(i)}_{n,m,p}} = [\gamma_2 a]^{-2}[\gamma_0 b]^{-2}\Bigg[\left[\frac{[\gamma_2 b i_n(\gamma_0 a)]'}{\gamma_2 b}k_n(\gamma_0 b) - \frac{Z_0}{Z_2}i_n(\gamma_2 b)\frac{[\gamma_0 b k_n(\gamma_0 b)]'}{\gamma_0 b}\right]$$

$$\left[\frac{[\gamma_0 a i_n(\gamma_0 a)]'}{\gamma_0 a}k_n(\gamma_2 a) - \frac{Z_2}{Z_0}i_n(\gamma_0 a)\frac{[\gamma_2 a k_n(\gamma_2 a)]'}{\gamma_2 a}\right] \tag{A.28}$$

$$-\left[\frac{[\gamma_2 b k_n(\gamma_2 b)]'}{\gamma_2 b}k_n(\gamma_0 b) - \frac{Z_0}{Z_2}k_n(\gamma_2 b)\frac{[\gamma_0 b k_n(\gamma_0 b)]'}{\gamma_0 b}\right]$$

$$\left[\frac{[\gamma_0 a i_n(\gamma_0 a)]'}{\gamma_0 a}i_n(\gamma_2 a) - \frac{Z_2}{Z_0}i_n(\gamma_0 a)\frac{[\gamma_0 a i_n(\gamma_2 a)]'}{\gamma_2 a}\right]\Bigg]^{-1}$$

These results for the region–1 field coefficients are basically the same as [3] except that the numerator Bessel functions have been replaced by $[\gamma_2 a]^{-2}[\gamma_0 b]^{-2}$, a significant simplification. The Bessel functions in the denominator involve mixtures with arguments involving all of $\gamma_0 a$, $\gamma_0 b$, $\gamma_2 a$, and $\gamma_2 b$. An alternate form of (A.28) is

$$\frac{a^{(1)}_{n,m,p}}{a^{(i)}_{n,m,p}}$$

$$= \Bigg[\Big[\gamma_2 b i_n(\gamma_2 b)[\gamma_0 b k_n(\gamma_0 b)]' - \frac{Z_0}{Z_2}[\gamma_2 b i_n(\gamma_0 a)]' \gamma_0 b k_n(\gamma_0 b)\Big]$$

$$\Big[\gamma_0 i_n(\gamma_0 a)[\gamma_2 a k_n(\gamma_2 a)]' - \frac{Z_2}{Z_0}[\gamma_0 a i_n(\gamma_0 a)]' \gamma_2 a k_n(\gamma_2 a)\Big]$$

$$- \Big[\gamma_0 b k_n(\gamma_2 b)[\gamma_0 b k_n(\gamma_0 b)]' - \frac{Z_0}{Z_2}[\gamma_2 b k_n(\gamma_2 b)]' \gamma_2 b k_n(\gamma_0 b)\Big]$$

$$\Big[\gamma_0 a i_n(\gamma_0 a)[\gamma_2 a i_n(\gamma_2 a)]' - \frac{Z_2}{Z_0}[\gamma_0 a i_n(\gamma_0 a)]' \gamma_2 a i_n(\gamma_2 a)\Big]\Bigg]^{-1}$$

$$\frac{b^{(1)}_{n,m,p}}{b^{(i)}_{n,m,p}}$$

$$= \Bigg[\Big[[\gamma_2 b i_n(\gamma_2 b)]' \gamma_0 b k_n(\gamma_0 b) - \frac{Z_0}{Z_2}\gamma_2 b i_n(\gamma_2 b)[\gamma_0 b k_n(\gamma_0 b)]'\Big]$$

$$\Big[[\gamma_0 a i_n(\gamma_0 a)]' \gamma_2 a k_n(\gamma_2 a) - \frac{Z_2}{Z_0}\gamma_0 a i_n(\gamma_0 a)[\gamma_2 a k_n(\gamma_2 a)]'\Big]$$

$$- \Big[[\gamma_2 b k_n(\gamma_2 b)]' \gamma_0 b k_n(\gamma_0 b) - \frac{Z_0}{Z_2}\gamma_2 b k_n(\gamma_2 b)[\gamma_0 b k_n(\gamma_0 b)]'\Big]$$

$$\Big[[\gamma_0 a i_n(\gamma_0 a)]' \gamma_2 a i_n(\gamma_2 a) - \frac{Z_2}{Z_0}\gamma_0 a i_n(\gamma_0 a)[\gamma_2 a i_n(\gamma_2 a)]'\Big]\Bigg]^{-1} \quad \text{(A.29)}$$

In this form the Bessel functions are all mutiplied by their arguments so we need only consider functions $\zeta i_n(\zeta)$ and $\zeta k_n(\zeta)$, these being called Riccati-Bessel functions [7].

At this point we can note that only n = 1 terms contribute to fields at the center of region 1, the coordinate origin $\vec{r} = \vec{0}$. In this case the only nonzero coefficients in (A.15) for the special incidence and polarization conditions in (A.19) are

$$a^{(i)}_{1,1,o} = \frac{3}{2} \quad , \quad b^{(i)}_{1,1,e} = \frac{3}{2} \quad \text{(A.30)}$$

Considering the description of the incident fields in (A.2) we find from (A.3) that it is the $\vec{N}$ functions for N=1 which are nonzero at the origin. The field coefficients for the origin are

$$\frac{\vec{E}^{(1)}(\vec{0},s)}{E_0} = \frac{b_{1,1,e}^{(1)}}{b_{1,1,e}^{(i)}} \vec{1}_x$$

$$\frac{Z_o \vec{H}^{(1)}(\vec{0},s)}{E_0} = \frac{a_{1,1,e}^{(1)}}{a_{1,1,e}^{(i)}} \vec{1}_y \tag{A.31}$$

Note, however, that as one goes away from the origin other terms become significant. In particular, the electric field has a term which circulates around the y axis (maximum near the spherical shell at low frequencies) and is related to the magnetic field in (A.31), i.e., to $a_{1,1,e}^{(1)} \vec{M}^{(1)}_{n,m,p}(\gamma_0 \vec{r})$. It is this term that dominates the low-frequency electric field [6].

Appendix B: Approximation of the Terms Involving $\gamma_2 a$ and $\gamma_2 b$

The basic result in (A.29) can be further manipulated to obtain more insight into the form that the shielding takes.

B.1 $a_{n,m,p}^{(1)}$ coefficients

The first part of (A.29) can be manipulated into the form

$$\frac{a_{n,m,p}^{(1)}}{a_{n,m,p}^{(i)}}$$

$$= \left[\left[\gamma_0 b k_n(\gamma_0 b)\right]' \gamma_0 a i_n(\gamma_0 a) X_1 + \gamma_0 b k_n(\gamma_0 b)\left[\gamma_0 a i_n(\gamma_0 a)\right]' X_2 \right. \tag{B.1}$$

$$\left. + \frac{Z_0}{Z_2}\gamma_0 b k_n(\gamma_0 b)\gamma_0 a i_n(\gamma_0 a) X_3 + \frac{Z_2}{Z_0}\left[\gamma_0 b k_n(\gamma_0 b)\right]'\left[\gamma_0 a i_n(\gamma_0 a)\right]' X_4\right]^{-1}$$

where the X_n contain all terms involving Bessel functions of $\gamma_2 a$ and $\gamma_2 b$ (only), noting that γ_2 involves the shell parameters with $|\gamma_2| >> |\gamma_0|$. Writing out these terms and writing the i_n functions in terms of k_n functions (A.5) gives

$$X_1 = \gamma_2 b i_n(\gamma_2 b)\left[\gamma_2 a k_n(\gamma_2 a)\right]' - \gamma_2 b k_n(\gamma_2 b)\left[\gamma_2 a i_n(\gamma_2 a)\right]'$$

$$= \frac{1}{2}\left[\left[-1\right]^{n+1}\gamma_2 b k_n(\gamma_2 b) - \gamma_2 b k_n(-\gamma_2 a)\right]\left[\gamma_2 a k_n(\gamma_2 a)\right]'$$

$$- \gamma_2 b k_n(\gamma_2 b)\frac{1}{2}\left[\left[-1\right]^{n+1}\gamma_2 a k_n(\gamma_2 a) - \left[-\gamma_2 a k_n(-\gamma_2 a)\right]'\right]$$

$$= \frac{1}{2}\left[\left[-\gamma_2 b k_n(-\gamma_2 b)\right]\left[\gamma_2 a k_n(\gamma_2 a)\right]' + \left[\gamma_2 b k_n(\gamma_2 b)\right]\left[-\gamma_2 a k_n(-\gamma_2 a)\right]'\right]$$

$$X_2 = \left[\gamma_2 b k_n(\gamma_2 b)\right]'\gamma_2 a k_n(\gamma_2 a) - \left[\gamma_2 b k_n(\gamma_2 b)\right]'\gamma_2 a i_n(\gamma_2 a)$$

$$= \frac{1}{2}\left[\left[-1\right]^{n+1}\left[\gamma_2 b k_n(\gamma_2 b)\right]' - \left[-\gamma_2 b k_n(-\gamma_2 b)\right]'\right]\gamma_2 a k_n(\gamma_2 a)$$

$$- \left[\gamma_2 b k_n(\gamma_2 b)\right]'\frac{1}{2}\left[\left[-1\right]^{n+1}\gamma_2 a k_n(\gamma_2 a) + \left[-\gamma_2 a k_n(-\gamma_2 a)\right]\right]$$

$$= -\frac{1}{2}\left[\left[-\gamma_2 b k_n(-\gamma_2 b)\right]'\left[\gamma_2 a k_n(\gamma_2 a)\right] + \left[\gamma_2 b k_n(\gamma_2 b)\right]'\left[-\gamma_2 a k_n(-\gamma_2 a)\right]\right]$$

$$
\begin{aligned}
X_3 &= \left[\gamma_2 b k_n(\gamma_2 b)\right]' \left[\gamma_2 a i_n(\gamma_2 a)\right]' - \left[\gamma_2 b i_n(\gamma_2 b)\right]' \left[\gamma_2 a k_n(\gamma_2 a)\right]' \\
&= \left[\gamma_2 b k_n(\gamma_2 b)\right]' \frac{1}{2}\left[\left[[-1]^{n+1}\left[\gamma_2 a k_n(\gamma_2 a)\right] - \left[-\gamma_2 a k_n(-\gamma_2 a)\right]'\right]\right] \\
&\quad - \frac{1}{2}\left[[-1]^{n+1}\left[\gamma_2 b k_n(\gamma_2 b)\right]' - \left[-\gamma_2 b k_n(-\gamma_2 b)\right]'\right]\left[\gamma_2 a k_n(\gamma_2 a)\right]' \\
&= \frac{1}{2}\left[\left[-\gamma_2 b k_n(-\gamma_2 b)\right]'\left[\gamma_2 a k_n(\gamma_2 a)\right]' + \left[\gamma_2 b k_n(\gamma_2 b)\right]' \left[-\gamma_2 a k_n(-\gamma_2 a)\right]'\right] \\
X_4 &= \gamma_2 b k_n(\gamma_2 b)\gamma_2 a i_n(\gamma_2 a) - \gamma_2 b i_n(\gamma_2 b)\gamma_2 a k_n(\gamma_2 a) \\
&= \gamma_2 b k_n(\gamma_2 b)\frac{1}{2}\left[[-1]^{n+1}\gamma_2 a k_n(\gamma_2 a) - \gamma_2 a k_n(-\gamma_2 a)\right] \\
&\quad - \frac{1}{2}\left[[-1]^{n+1}\gamma_2 b k_n(\gamma_2 b) - \gamma_2 b k_n(-\gamma_2 a)\right] \\
&= -\frac{1}{2}\left[\left[-\gamma_2 b k_n(-\gamma_2 b)\right]\left[\gamma_2 a k_n(\gamma_2 a)\right] - \left[\gamma_2 b k_n(\gamma_2 b)\right]\left[-\gamma_2 a k_n(-\gamma_2 a)\right]\right]
\end{aligned}
\tag{B.2}
$$

Consider first large $\gamma_2 a$ and $\gamma_2 b$. For this purpose we have

$$
\begin{aligned}
\zeta k_n(\zeta) &=^{-\zeta} \sum_{\beta=0}^{n} \frac{[n+\beta]!}{\beta![n-\beta]!}[2\zeta]^{-\beta} \\
&= e^{-\zeta}\left[1+\frac{n[n+1]}{2}\zeta^{-1}+\mathrm{O}\left(\zeta^{-2}\right)\right] \quad \text{as } \zeta \to \infty \\
\left[\zeta k_n(\zeta)\right]' &= -\zeta k_n(\zeta) - 2\beta e^{-\zeta}\sum_{\beta=1}^{n}\frac{[n+\beta]!}{\beta![n-\beta]!}[2\zeta]^{-\beta-1} \\
&= e^{-\zeta}\left[-1-\sum_{\beta=1}^{n}\frac{[n+\beta]!}{\beta![n-\beta]!}\left[[2\zeta]^{-\beta}+2\beta[2\zeta]^{-\beta-1}\right]\right] \\
&= -e^{-\zeta}\left[1+\frac{n[n+1]}{2}\zeta^{-1}+\mathrm{O}\left(\zeta^{-2}\right)\right] \quad \text{as } \zeta \to \infty
\end{aligned}
\tag{B.3}
$$

Substituting in the X_n we have

$$
\begin{aligned}
X_1 &= \frac{1}{2}\left[-e^{\gamma_2\Delta}\left[1-\frac{n[n+1]}{2\gamma_2 b} + \mathrm{O}\left([\gamma_2 b]^{-2}\right)\right]\left[1+\frac{n[n+1]}{2\gamma_2 a} + \mathrm{O}\left([\gamma_2 b]^{-2}\right)\right]\right. \\
&\quad \left. -e^{-\gamma_2\Delta}\left[1+\frac{n[n+1]}{2\gamma_2 b} + \mathrm{O}\left([\gamma_2 b]^{-2}\right)\right]\left[1-\frac{n[n+1]}{2\gamma_2 a} + \mathrm{O}\left([\gamma_2 a]^{-2}\right)\right]\right] \\
&= -\cosh(\gamma_2\Delta) - \sinh(\gamma_2\Delta)\frac{n[n+1]}{2}\left[-[\gamma_2 a]^{-1}+[\gamma_2 b]^{-1}\right] + \mathrm{O}(f(\gamma_2 a,\gamma_2 b)) \\
&= -\cosh(\gamma_2\Delta) - \sinh(\gamma_2\Delta)\frac{n[n+1]}{2}\frac{\Delta}{\gamma_2 a\, b} + \mathrm{O}(f(\gamma_2 a,\gamma_2 b))
\end{aligned}
$$

$$
\begin{aligned}
X_2 &= \frac{1}{2}\left[e^{\gamma_2\Delta}\left[1-\frac{n[n+1]}{2\gamma_2 b} + \mathrm{O}\left([\gamma_2 b]^{-2}\right)\right]\left[1+\frac{n[n+1]}{2\gamma_2 a} + \mathrm{O}\left([\gamma_2 b]^{2}\right)\right]\right. \\
&\quad \left. + e^{-\gamma_2\Delta}\left[1+\frac{n[n+1]}{2\gamma_2 b} + \mathrm{O}\left([\gamma_2 b]^{2}\right)\right]\left[1-\frac{n[n+1]}{2\gamma_2 a} + \mathrm{O}\left([\gamma_2 b]^{-2}\right)\right]\right] \\
&= \cosh(\gamma_2\Delta) - \sinh(\gamma_2\Delta)\frac{n[n+1]}{2}\left[[\gamma_2 a]^{-1} - [\gamma_2 b]^{-1}\right] + \mathrm{O}(f(\gamma_2 a, \gamma_2 b)) \\
&= \cosh(\gamma_2\Delta) + \sinh(\gamma_2\Delta)\frac{n[n+1]}{2}\frac{\Delta}{\gamma_2 a\, b} + \mathrm{O}(f(\gamma_2 a, \gamma_2 b)) \\
&= -X_1 + \mathrm{O}(f(\gamma_2 a, \gamma_2 b)) \\
X_3 &= \frac{1}{2}\left[e^{\gamma_2\Delta}\left[1-\frac{n[n+1]}{2\gamma_2 b} + \mathrm{O}\left([\gamma_2 b]^{-2}\right)\right]\left[1+\frac{n[n+1]}{2\gamma_2 a} + \mathrm{O}\left([\gamma_2 a]^{2}\right)\right]\right. \\
&\quad \left. - e^{-\gamma_2\Delta}\left[1+\frac{n[n+1]}{2\gamma_2 b} + \mathrm{O}\left([\gamma_2 b]^{-2}\right)\right]\left[1-\frac{n[n+1]}{2\gamma_2 a} + \mathrm{O}\left([\gamma_2 b]^{-2}\right)\right]\right] \\
&= \sinh(\gamma_2\Delta) + \cosh(\gamma_2\Delta)\frac{n[n+1]}{2}\left[[\gamma_2 a]^{-1} - [\gamma_2 b]^{-1}\right] + \mathrm{O}(f(\gamma_2 a, \gamma_2 b)) \\
&= \sinh(\gamma_2\Delta) + \cosh(\gamma_2\Delta)\frac{n[n+1]}{2}\frac{\Delta}{\gamma_2 a\, b} + \mathrm{O}(f(\gamma_2 a, \gamma_2 b)) \\
X_4 &= \frac{1}{2}\left[-e^{\gamma_2\Delta}\left[1-\frac{n[n+1]}{2\gamma_2 b} + \mathrm{O}\left([\gamma_2 b]^{-2}\right)\right]\left[1+\frac{n[n+1]}{2\gamma_2 a} + \mathrm{O}\left([\gamma_2 b]^{2}\right)\right]\right. \\
&\quad \left. + e^{-\gamma_2\Delta}\left[1+\frac{n[n+1]}{2\gamma_2 b} + \mathrm{O}\left([\gamma_2 b]^{-2}\right)\right]\left[1-\frac{n[n+1]}{2\gamma_2 a} + \mathrm{O}\left([\gamma_2 a]^{-2}\right)\right]\right] \\
&= -\sinh(\gamma_2\Delta) - \cosh(\gamma_2\Delta)\frac{n[n+1]}{2}\left[[\gamma_2 a]^{-1} - [\gamma_2 b]^{-1}\right] + \mathrm{O}(f(\gamma_2 a, \gamma_2 b)) \qquad \text{(B.4)} \\
&= -\sinh(\gamma_2\Delta) - \cosh(\gamma_2\Delta)\frac{n[n+1]}{2}\frac{\Delta}{\gamma_2 a\, b} + \mathrm{O}(f(\gamma_2 a, \gamma_2 b)) \\
&= -X_3 + \mathrm{O}(f(\gamma_2 a, \gamma_2 b))
\end{aligned}
$$

The order symbol for large $\gamma_2 a$ (and hence large $\gamma_2 b$) can be understood as

$$
\begin{aligned}
\mathrm{O}(f(\gamma_2 a, \gamma_2 b)) &= \left[\mathrm{O}\left([\gamma_2 a]^{-2}\right) + \mathrm{O}\left([\gamma_2 b]^{-2}\right)\right]\ \left[\text{larger of } \left|e^{\gamma_2\Delta}\right|_s \left|e^{-\gamma_2\Delta}\right|\right] \\
&= \mathrm{O}\left([\gamma_2 a]^{-2}\, e^{|\mathrm{Re}(\gamma_2\Delta)|}\right) \text{ as } \gamma_2 a \to \infty
\end{aligned}
\qquad \text{(B.5)}
$$

In each of the four above cases this order is dominated by both cosh and sinh terms. Note that $|\gamma_2\Delta| << |\gamma_2 a|$ by hypothesis, so the arguments of cosh and sinh may be small while the approximation is still valid. Note for X_1 and X_2 the cosh term dominates the sinh term, except that as n becomes large, this breaks down (nonuniformly asymptotic with respect to n). At the center of the sphere only the $n = 1$ term is present and near the center only the first few n are significant. For X_3 and X_4 the sinh term dominates for large $\gamma_2 a$ with the same caution regarding n.

For small arguments we use (from (A.5))

$$\begin{aligned}
\zeta k_n(\zeta) &= \left[1-\zeta+O\left(\zeta^2\right)\right]\left[\frac{[2n]!}{n!}[2\zeta]^{-n}+\frac{[2n-1]!}{[n-1]!}[2\zeta]^{-n+1}+O\left(\zeta^{-n+2}\right)\right] \\
&= \frac{[2n]!}{n!}[2\zeta]^{-n}\left[1-\zeta+O\left(\zeta^2\right)\right]\left[1+\zeta+O\left(\zeta^2\right)\right] \\
&= [2n-1]!!\zeta^{-n}\left[1+O\left(\zeta^2\right)\right] \quad \text{as } \zeta\to 0 \\
\left[\zeta k_n(\zeta)\right]' &= -n[2n-1]!!\zeta^{-n-1}\left[1+O\left(\zeta^2\right)\right] \quad \text{as } \zeta\to 0
\end{aligned} \tag{B.6}$$

and from [7] we have

$$\begin{aligned}
\zeta i_n(\zeta) &= \frac{\zeta^{n+1}}{[2n+1]}\left[1+O\left(\zeta^2\right)\right] \quad \text{as } \zeta\to\infty \\
\left[\zeta i_n(\zeta)\right]' &= \frac{n+1}{[2n+1]!!}\zeta^n\left[1+O\left(\zeta^2\right)\right] \quad \text{as } \zeta\to\infty
\end{aligned} \tag{B.7}$$

Applying these to (B.2) we have

$$\begin{aligned}
X_1 &= \left[-\frac{[\gamma_2 b]^{n+1}}{[2n+1]!!}n[2n-1]!![\gamma_2 b]^{-n-1}-[2n-1]!![\gamma_2 b]^{-n}\frac{n+1}{[2n+1]!!}[\gamma_2 a]^n\right]\left[1+O\left([\gamma_2 b]^2\right)\right] \\
&= -\frac{1}{2n+1}\left[n\left[\frac{b}{a}\right]^{n+1}+[n+1]\left[\frac{a}{b}\right]^n\right]\left[1+O\left([\gamma_2 b]^2\right)\right] \\
&= -\frac{1}{2n+1}\left[n\left[1+\frac{\Delta}{a}\right]^{n+1}+[n+1]\left[1+\frac{\Delta}{a}\right]^{-n}\right]\left[1+O\left([\gamma_2 b]^2\right)\right] \\
&= -\left[1+O\left(\left[\frac{\Delta}{a}\right]^2\right)\right]\left[1+O\left([\gamma_2 b]^2\right)\right] \quad \text{as } \gamma_2 b,\frac{\Delta}{a}\to 0
\end{aligned}$$

$$X_2 = \left[\frac{n+1}{[2n+1]!!}[\gamma_2 b]^n [2n-1]!![\gamma_2 a]^{-n} + n[2n+1]!![\gamma_2 b]^{-n-1}\frac{[\gamma_2 a]}{[2n+1]!!}\right]\left[1+O\left([\gamma_2 b]^2\right)\right]$$

$$= \frac{1}{2n+1}\left[[n+1]\left[\frac{b}{a}\right]^n + n\left[\frac{a}{b}\right]^{n+1}\right]\left[1+O\left([\gamma_2 b]^2\right)\right]$$

$$= \frac{1}{2n+1}\left[[n+1]\left[1+\frac{\Delta}{a}\right]^n + n\left[1+\frac{\Delta}{a}\right]^{-n-1}\right]\left[1+O\left([\gamma_2 b]^2\right)\right]$$

$$= \left[1+O\left(\left[\frac{\Delta}{a}\right]^2\right)\right]\left[1+O\left([\gamma_2 b]^2\right)\right] \text{ as } \gamma_2 b, \frac{\Delta}{a} \to 0$$

$$X_3 = \left[-n[2n-1]!![\gamma_2 b]^{-n-1}\frac{n+1}{[2n+1]!!}[\gamma_2 a]^n + \frac{n+1}{[2n+1]!!}[\gamma_2 b]^n n[2n-1]!![\gamma_2 a]^{-n-1}\right]\left[1+O\left([\gamma_2 b]^2\right)\right]$$

$$= -\frac{n[n+1]}{2n+1}\frac{1}{\gamma_2 a}\left[-\left[\frac{a}{b}\right]^{n+1} + \left[\frac{b}{a}\right]^n\right]\left[1+O\left([\gamma_2 b]^2\right)\right]$$

$$= \frac{n[n+1]}{2n+1}\frac{1}{\gamma_2 a}\left[-\left[1+\frac{\Delta}{a}\right]^{-n-1} + \left[1+\frac{\Delta}{a}\right]^n\right]\left[1+O\left([\gamma_2 b]^2\right)\right]$$

$$= n[n+1]\frac{\Delta}{\gamma_2 a^2}\left[1+O\left(\left[\frac{\Delta}{a}\right]^2\right)\right]\left[1+O\left([\gamma_2 b]^2\right)\right] \text{ as } \gamma_2 b, \frac{\Delta}{a} \to 0$$

$$X_4 = \left[[2n-1]!![\gamma_2 b]^{-n}\frac{[\gamma_2 a]^{n+1}}{[2n+1]!!} - \frac{[\gamma_2 b]^{n+1}}{[2n+1]!!}[2n-1]!![\gamma_2 a]^{-n}\right]\left[1+O\left([\gamma_2 b]^2\right)\right]$$

$$= \frac{\gamma_2 a}{2n+1}\left[\left[\frac{a}{b}\right]^n - \left[\frac{b}{a}\right]^{n+1}\right]\left[1+O\left([\gamma_2 b]^2\right)\right] \tag{B.8}$$

$$= \frac{\gamma_2 a}{2n+1}\left[\left[1+\frac{\Delta}{a}\right]^{-n} - \left[1+\frac{\Delta}{a}\right]^{n+1}\right]\left[1+O\left([\gamma_2 b]^2\right)\right]$$

$$= -\gamma_2 \Delta\left[1+O\left(\frac{\Delta}{a}\right)\right]\left[1+O\left([\gamma_2 b]^2\right)\right]$$

Reviewing the above we see that the high frequency representation for X_1 (and $-X_2$) goes to the low frequency representation (with even $\gamma_2 \Delta$ small). From (B.4) we have

$$X_1 = -1 + \frac{n[n+1]}{2}\frac{\Delta^2}{ab} \quad \text{as } \gamma_2 \Delta \to 0 \tag{B.9}$$

which is close to -1 as in (B.10). So we might use

$$X_1 \simeq -X_2 \simeq -\cosh(\gamma_2\Delta) + \frac{\sinh(\gamma_{2\Delta})}{\gamma_2\Delta}\frac{n[n+1]}{2}\left[\frac{\Delta}{a}\right]^2 \tag{B.10}$$

for both high and low frequencies for n and Δ / a both small. For X_3 the high-frequency approximation has a factor $n[n+1]/2$ which disagrees with the factor $n[n+1]$ in the low-frequency approximation. For X_4, as the high-frequency approximation is taken to low frequency it is proportional to $[\gamma_2\Delta]^{-1}$, as compared to the low-frequency approximation being proportional to $\gamma_2\Delta$, which is quite a difference.

B.2 $b^{(1)}_{n,m,p}$ coefficients

The second part of (A.29) can be manipulated into the form

$$\frac{b^{(1)}_{n,m,p}}{b^{(i)}_{n,m,p}} = \left[\left[\gamma_0 b k_n(\gamma_0 b)\right]' \gamma_0 a i_n(\gamma_0 a) X_1 + \gamma_0 b k_n(\gamma_0 b)\left[\gamma_0 a i_n(\gamma_0 a)\right]' X_2 \right. \tag{B.11}$$

$$\left. + \frac{Z_2}{Z_0}\gamma_0 b k_n(\gamma_0 b)\gamma_0 a i_n(\gamma_0 a) X_3 + \frac{Z_0}{Z_2}\left[\gamma_0 b k_n(\gamma_0 b)\right]'\left[\gamma_0 a i_n(\gamma_0 a)\right]' X_4\right]^{-1}$$

Here we note that the same terms appear here as for the $a^{(1)}_{n,m,p}$ coefficients in (B.1), albeit rearranged. The first two terms above correspond exactly, but the third and fourth terms have the roles of Z_0 / Z_2 and Z_2 / Z_0 interchanged in their combinations with the Bessel functions. Hence, we need not compute the terms a second time, but just use the terms previously computed and combine them in the new way. The high- and low-frequency forms of the X_n have already been treated.

References

1. K. R. Umashankar and C. E. Baum, "Equivalent Electromagnetic Properties of a Concentric Wire Cage as Compared to a Circular Cylinder", Sensor and Simulation Note 252, March 1979.

2. C. E. Baum, "Idealized Electric- and Magnetic-Field Sensors Based on Spherical Sheet Impedances", Sensor and Simulation Note 283, March 1983; Electromagnetics, 1989, pp. 113-146.

3. C. W. Harrison, Jr. And C. H. Papas, "On the Attenuation of Transient Fields by Imperfectly Conducting Spherical Shells", Interaction Note 34, July 1964; IEEE Trans. Antennas and Propagation, 1965, pp. 960-966.

4. K. F. Casey, "Electromagnetic Shielding by Advanced Composite Materials", Interaction Note 341, Jun 1977.

5. C. E. Baum, "On the Use of Electromagnetic Topology for the Decomposition of Scattering Matrices for Complex Physical Structures", Interaction Note 454, July 1985.

6. C. E. Baum, "Monitor for Integrity of Seams in a Shield Enclosure", Measurement Note 32, April 1989; IEEE Trans. EMC, 1988, pp. 276-281.

7. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, U. S. Gov't Printing Office, AMS-55, 1964.