Interaction Notes

Note 565

17 November 2000

Axial Scattering from Thin Cones

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Abstract

A model is developed for axial scattering from thin cones of arbitrary cross section based on the electric- and magnetic-polarizability dyadics per unit length. This is later specialized to perfectly conducting cones for axial backscattering. For circular cones it is shown to agree well with exact and physical-optics results. Applying the model to elliptic cones the disagreement with physical-optics results is clear. In the limit this gives the nonzero results for a thin angular sector.

This work was sponsored in part by the Air Force Office of Scientific Research, and in part by the Air Force Research Laboratory, Directed Energy Directorate.

1. Introduction

A recent paper [4] considers the physical-optics (PO) approximation for early-time scattering. For axial backscattering examples of perfectly conducting bodies of revolution (BOR), including paraboloids and circular cones, seem to have good PO results as compared to exact and asymptotic results. However, the perfectly conducting angular sector is not approximated at all by PO (giving zero axial backscattering) for the case of the incident electric field parallel to the plane of the angular sector.

It would then seem useful to understand scattering from cones of various cross sections and compare the results to the PO approximation [11]. The general problem with arbitrary interior cone angle has not been solved. However, we know the exact form the solution must take [4, 10] which factors the angular dependence from the temporal/frequency dependence. This applies to various types of cones (e.g., dielectric), but our examples here are for perfectly conducting cones.

The incident field is given by

$$\vec{E}^{(inc)}(\vec{r},t) = E_0 f\left(t - \frac{\vec{1}_i \cdot \vec{r}}{c}\right)\vec{1}_e \ , \quad \tilde{\vec{E}}^{(inc)}(\vec{r},s) = E_0 \tilde{f}(s) e^{-\gamma \vec{1}_i \cdot \vec{r}}\, \vec{1}_e$$

$$\vec{H}^{(inc)}(\vec{r},t) = \frac{E_0}{Z_0} f\left(t - \frac{\vec{1}_i \cdot \vec{r}}{c}\right)\vec{1}_i \times \vec{1}_e \ , \quad \tilde{\vec{H}}^{(inc)}(\vec{r},s) = \frac{E_0}{Z_0} \tilde{f}(s) e^{-\gamma \vec{1}_i \cdot \vec{r}}\, \vec{1}_i \times \vec{1}_e$$

$$\vec{1}_e \cdot \vec{1}_i = 0$$

$$\vec{1}_i \equiv \text{direction of incidence}$$

$$\vec{1}_e \equiv \text{polarization}$$

$$f(t) \equiv \text{waveform}$$

$$c = [\mu_0 \varepsilon_0]^{-\frac{1}{2}} \equiv \text{speed of light}$$

$$Z_0 = \left[\frac{\mu_0}{\varepsilon_0}\right]^{\frac{1}{2}} \equiv \text{wave impedance}$$

$$\sim \ \equiv \text{two-sided Laplace transform over time } t \qquad (1.1)$$

$$s = \Omega + j\omega \equiv \text{Laplace-transform variable or complex frequency}$$

$$\gamma = \frac{s}{c} \equiv \text{propagation constant}$$

The scatterer (target) being a cone, the cone tip is taken as the coordinate origin $(\vec{r} = \vec{0})$.

The scattered far field is given by (general case)

$$\vec{E}_f^{(sc)}(\vec{r},t) = \frac{1}{4\pi r} \overleftrightarrow{\Lambda}(\vec{1}_0, \vec{1}_i;t) \circ \vec{E}^{(inc)}\left(\vec{0}, t-\frac{r}{c}\right)$$

$$\tilde{\vec{E}}_f^{(sc)}(\vec{r},s) = \frac{e^{-\gamma r}}{r} \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_0, \vec{1}_i;s) \cdot \vec{E}^{(inc)}(\vec{0},s)$$

$\tilde{\overleftrightarrow{\Lambda}}^{(sc)}(\vec{1}_0, \vec{1}_i;s) \equiv$ scattering dyadic

$\tilde{\overleftrightarrow{\Lambda}}^{(sc)}(\vec{1}_0, \vec{1}_i;t) \circ \equiv$ scattering-dyadic operator

$\circ \equiv$ convolution with respect to time

$\overleftrightarrow{\Lambda}(\vec{1}_0, \vec{1}_i;t) \equiv$ scattering-dyadic impulse response (1.2)

$r = |\vec{r}|$

$\vec{1}_o \equiv$ scattering direction (direction to observer)

$$\vec{E}_f^{(sc)}(\vec{r},t) \cdot \vec{1}_o = 0$$

For backscattering we have

$$\vec{1}_o = -\vec{1}_i$$
$$\overleftrightarrow{\Lambda}_b(\vec{1}_i,t) \equiv \overleftrightarrow{\Lambda}(-\vec{1}_i, \vec{1}_i;t) \tag{1.3}$$

Reciprocity implies

$$\overleftrightarrow{\Lambda}(\vec{1}_o, \vec{1}_i;t) = \overleftrightarrow{\Lambda}^T(-\vec{1}_i, -\vec{1}_o;t)$$
$$\overleftrightarrow{\Lambda}_b(\vec{1}_i,t) = \overleftrightarrow{\Lambda}_b^T(\vec{1}_i,t) \tag{1.3}$$

These relations are general and not specific to the cone. We also have the transverse dyadics

$$\begin{aligned}
\vec{1}_i &\equiv \overleftrightarrow{1} - \vec{1}_i \vec{1}_i \\
\vec{1}_o &\equiv \overleftrightarrow{1} - \vec{1}_o \vec{1}_o \\
\vec{1}_r &\equiv \overleftrightarrow{1} - \vec{1}_r \vec{1}_r \\
\vec{1}_z &\equiv \overleftrightarrow{1} - \vec{1}_z \vec{1}_z \\
\overleftrightarrow{1} &\equiv \vec{1}_x \vec{1}_x + \vec{1}_y \vec{1}_y + \vec{1}_z \vec{1}_z \equiv \text{three dimensional identity}
\end{aligned} \tag{1.4}$$

for later use.

In [4] we introduced some notation for multiple time integrals of the scattering-dyadic impulse response as

$$\begin{aligned}
&\int_{-\infty}^{t} \int_{-\infty}^{t_1} \cdots \int_{-\infty}^{t_{n-1}} \overleftrightarrow{\Lambda}(\vec{1}_o, \vec{1}_i; t_n) dt_n \, dt_{n-1} \cdots dt_1 \\
&\qquad = \int_{0_-}^{t} \int_{0_-}^{t_1} \cdots \int_{-\infty}^{t_{n-1}} \overleftrightarrow{\Lambda}(\vec{1}_o, \vec{1}_i; t_n) dt_n \, dt_{n-1} \cdots dt_1 \\
&\qquad \equiv \overleftrightarrow{\Lambda}^{(n)}(\vec{1}_o, \vec{1}_i; t) \\
&\overleftrightarrow{\Lambda}^{(0)}(\vec{1}_o, \vec{1}_i; t) \equiv \overleftrightarrow{\Lambda}(\vec{1}_o, \vec{1}_i; t) \\
&\tilde{\overleftrightarrow{\Lambda}}^{(n)}(\vec{1}_o, \vec{1}_i; s) = s^{-n} \, \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o, \vec{1}_i; s)
\end{aligned} \tag{1.5}$$

and similarly for backscattering.

For the cone we have the general form of the solution (general cone with dilation symmetry) [10]

$$\begin{aligned}
&\overleftrightarrow{\Lambda}(\vec{1}_o, \vec{1}_i; t) = c u(t) \overleftrightarrow{K}(\vec{1}_o, \vec{1}_i) \quad , \quad \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o, \vec{1}_i; s) = \frac{c}{s} \overleftrightarrow{K}(\vec{1}_o, \vec{1}_i) \\
&\overleftrightarrow{\Lambda}^{(1)}(\vec{1}_o, \vec{1}_i; t) = ct \, u(t) \overleftrightarrow{K}(\vec{1}_o, \vec{1}_i) \quad , \quad \overleftrightarrow{\Lambda}^{(2)}(\vec{1}_o, \vec{1}_i; t) = \frac{1}{2} c t^2 u(t) \overleftrightarrow{K}(\vec{1}_o, \vec{1}_i) \\
&\overleftrightarrow{K}(\vec{1}_o, \vec{1}_i) = \overleftrightarrow{K}^T(-\vec{1}_i, -\vec{1}_o) \\
&\overleftrightarrow{K}_b(\vec{1}_i) = \overleftrightarrow{K}(-\vec{1}_i, \vec{1}_i) = \overleftrightarrow{K}_b^T(\vec{1}_i)
\end{aligned} \tag{1.6}$$

In the present paper we apply this result to thin cones for the special case of axial incidence. This will give some general results and allow us to calculate the axial backscatter from a perfectly conducting thin angular sector.

2. Local Two-Dimensional Approximation for Thin Cone

Let, as in Fig. 2.1, there be a thin cone centered on the $+z$ axis with

$$\vec{1}_i = \vec{1}_z \tag{2.1}$$

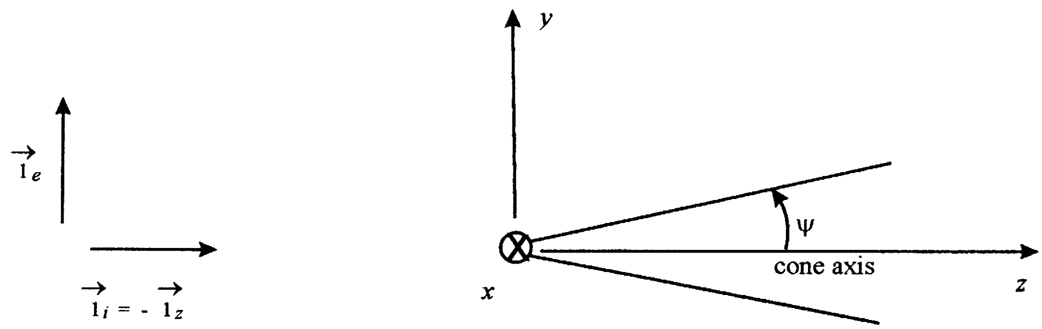
Let this cone be characterized by an interior angle ψ which is related to some characteristic dimension of $z_s \tan(\psi)$ on a cross section of constant $z = z_s$. This might be the maximum distance from the z axis or any other convenient reference.

Where then should we place the z axis? It clearly should pass through the cone tip. If the cone cross section has a symmetry plane it should lie on this. If it has a rotation axis this should be the z axis. Such cases include the circular cone, elliptic cone and angular sector (perfectly conducting) treated in the appendices.

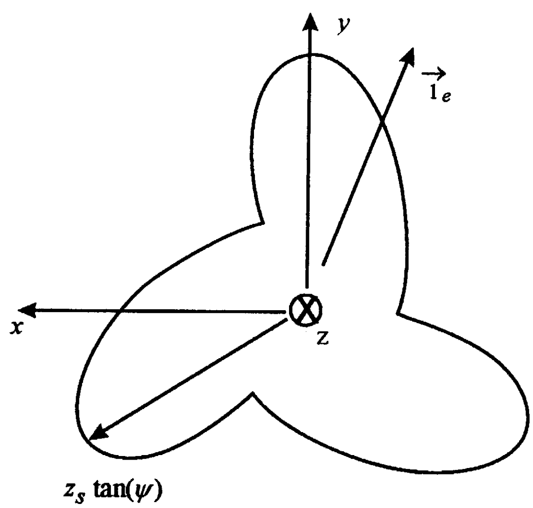
Going deeper into the matter, imagine for a moment that the direction of incidence is not parallel to this axis. Then there is a net current and a net charge per unit length (for conductors and dielectrics) and corresponding magnetic parameters if permeable magnetic materials are used. So let us choose the z axis so that incidence parallel to this produces no net current (and hence no net charge per unit length) and similarly for magnetic current (magnetic polarization current). This of course assumes that the electric and magnetic axes so defined coincide. For cross sections with rotation axes (C_N for $N \geq 2$ including both shape and constitutive parameters [9]) this happens automatically. (One can consider off-axis incidence for cases such as perfectly conducting cones with the arbitrary cross section given an effective radius for use of the thin-circular-cone results.)

With the direction of incidence along the z axis then the net charge per unit length (and similar magnetic parameter) are zero. We can then go on to the induced transverse electric- and magnetic-dipole moments per unit length. With ψ small we can locally think of the cross section in Fig. 2.1B as part of a two-dimensional structure extended in the $\pm \vec{1}_z$ directions. In the presence of the incident fields we can look at the total field distribution around and near the cone cross section as quasi static. What we need to calculate are the electric and magnetic polarizabilities per unit length so as to obtain the electric- and magnetic-dipole moments per unit length via

$$\begin{aligned} \tilde{\vec{p}}'(z_s,s) &= \varepsilon_0 \tilde{\overleftrightarrow{P}}'(z_s,s) \cdot \tilde{\vec{E}}^{(inc)}(z_s,\vec{1}_z,s) \\ \tilde{\vec{m}}'(z_s,s) &= \tilde{\overleftrightarrow{M}}'(z_s,s) \cdot \tilde{\vec{H}}^{(inc)}(z_s,\vec{1}_z,s) \end{aligned} \tag{2.2}$$



A. Side view



B. Cross-section view (plane of constant z)

Fig. 2.1 Axial Incidence on Thin Cone of General Cross Section

At this point it is useful to note certain properties (symmetries) of these polarizabilities per unit length. The very forms in (2.2) exhibit an invariance to reversal of the signs of the incident fields, equivalent to a rotation by π in the x,y plane which is C_2 symmetry [9]. This is in addition to any additional two-dimensional point symmetries of the electric and/or magnetic properties of the scatterer cross section on a constant z plane.

For perfectly conducting scatterers one can solve for various cross-section geometries by conformal transformation (as in the appendices). There is also a symmetry on interchange of the transverse electric and magnetic fields (similar to self duality [9]). Considering a cross section as in Fig. 2.1B and noting that both fields are derived from the same complex potential we note that an incident electric field in the x direction has the same effect as an incident magnetic field in the y direction except for a change in sign. (The two-dimensional perfectly conducting body concentrates electric flux (positive polarizability) and excludes magnetic flux (negative polarizability.) In two-dimensional form we define

$$\overleftrightarrow{\tau}_z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad \overleftrightarrow{\tau}_z^{-1} = -\overleftrightarrow{\tau}_z = \overleftrightarrow{\tau}_z^{T} \tag{2.3}$$

so that $\overleftrightarrow{\tau}_z \cdot$ is equivalent to $\vec{1}_z \times$, a positive $\pi/2$ rotation in the x, y plane. In three dimensions one can use

$$\overleftrightarrow{\tau}_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.4}$$

(Note also that $\cdot \overleftrightarrow{\tau}_z$ is equivalent to $\times \vec{1}_z \cdot$)

So we have

$$\begin{aligned} \overleftrightarrow{M}' &= \overleftrightarrow{\tau}_z \cdot \overleftrightarrow{P}' \cdot \overleftrightarrow{\tau}_z = \vec{1}_z \times \overleftrightarrow{P}' \times \vec{1}_z \\ \overleftrightarrow{P}' &= \overleftrightarrow{\tau}_z \cdot \overleftrightarrow{M}' \cdot \overleftrightarrow{\tau}_z = \vec{1}_z \times \overleftrightarrow{M}' \times \vec{1}_z \end{aligned} \tag{2.5}$$

whichh for bodies of resolution or others with polarizabilities proportional to the transverse identity (due to special two-dimensional symmetries) gives

$$
\begin{aligned}
\overleftrightarrow{P} &= P\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = P\,\overleftrightarrow{1}_z \\
\overleftrightarrow{M} &= M\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = M\,\overleftrightarrow{1}_z \\
M &= -P
\end{aligned}
\tag{2.6}
$$

for perfectly conducting two-dimensional scatterers.

For present purposes we will assume that the polarizabilities per unit length $\overleftrightarrow{P}'$ and $\overleftrightarrow{M}'$ are independent of frequency. (Special frequency dependences are allowed within the constraint of dilation symmetry [10].) Given that the cross-section area is proportional to z_s^2 the polarizabilities per unit length can be written as

$$
\begin{aligned}
\overleftrightarrow{P}' &= \overleftrightarrow{P}'_0\, z_s^2 \tan^2(\psi) \\
\overleftrightarrow{M}' &= \overleftrightarrow{M}'_0\, z_s^2 \tan^2(\psi)
\end{aligned}
\tag{2.7}
$$

where $\overleftrightarrow{P}'_0 = \overleftrightarrow{M}'_0$ are dimensionless. It is these quantities which we will need for any given cross section to include in our general scattering results. The appendices consider these for circular, strip and elliptic cross sections.

3. Scattering Dyadic for Axial Incidence on Thin Cones

Now consider the scattering from elementary dipoles on the z axis (coordinate z_s). We have

$$\vec{R}(z_s) = \vec{r} - z_s \vec{1}_z$$

$$z_s \vec{1}_z \equiv \text{source position}$$

$$\vec{r} = r\vec{1}_o \equiv \text{observer position}$$

$$\begin{aligned} R(z_s) &= |\vec{R}(z_s)| = r\left|\vec{1}_o - \frac{z_s}{r}\vec{1}_z\right| \\ &= r - \vec{1}_o \cdot \vec{1}_z z_s + O(r^{-1}) \text{ as } r \to \infty \\ e^{-\gamma R} &= e^{-\gamma r} e^{\gamma \vec{1}_o \cdot \vec{1}_z z_s}\left[1 + O(r^{-1})\right] \text{ as } r \to \infty \end{aligned} \tag{3.1}$$

$$\vec{1}_o \cdot \vec{1}_z \equiv \cos(\theta)$$

which can be inserted in the formulae for elementary dipoles at the source position. For electric dipoles we have the far fields [1, 3]

$$\begin{aligned} \tilde{\vec{E}}_f^{(p)}(\vec{r},s) &= -e^{-\gamma r} e^{\gamma\cos(\theta) z_s} \frac{\mu_0 s^2}{4\pi r} \vec{1}_o \cdot \tilde{\vec{p}}(z_s,s) \\ \tilde{\vec{H}}_f^{(p)}(\vec{r},s) &= -e^{-\gamma r} e^{\gamma\cos(\theta) z_s} \frac{s^2}{4\pi r}\frac{1}{c^2} \vec{1}_o \times \tilde{\vec{p}}(z_s,s) \end{aligned} \tag{3.2}$$

For magnetic dipoles we similarly have

$$\begin{aligned} \tilde{\vec{E}}_f^{(m)}(\vec{r},s) &= -e^{-\gamma r} e^{\gamma\cos(\theta) z_s} \frac{\mu_0 s^2}{4\pi r}\frac{1}{c} \vec{1}_o \times \tilde{\vec{m}}(z_s,s) \\ \tilde{\vec{H}}_f^{(m)}(\vec{r},s) &= -e^{-\gamma r} e^{\gamma\cos(\theta) z_s} \frac{s^2}{4\pi r}\frac{1}{c^2} \vec{1}_o \cdot \tilde{\vec{m}}(z_s,s) \end{aligned} \tag{3.3}$$

Integrating over z_s we have

$$d\tilde{\vec{p}}(z_s,s) = \tilde{\vec{p}}'(z_s,s)\,d\,z_s$$

$$d\tilde{\vec{m}}(z_s,s) = \tilde{\vec{m}}'(z_s,s)\,d\,z_s \tag{3.4}$$

$$\tilde{\vec{E}}_f^{(sc)}(\vec{r},s) = -\frac{e^{-\gamma r}}{4\pi r}\mu_0 s^2 \int_0^\infty e^{\gamma\cos(\theta) z_s}\left[\overleftrightarrow{1}_0 \cdot \tilde{\vec{p}}'(z_s,s) - \frac{1}{c}\vec{1}_0 \times \tilde{\vec{m}}'(z_s,s)\right] dz_s$$

From (2.2) and (1.1) we have

$$\tilde{\vec{p}}'(z_s,s) = \varepsilon_o\, \tilde{\overleftrightarrow{P}}'(z_s,s) \cdot \tilde{\vec{E}}^{(inc)}(z_s \vec{1}_z,s)$$

$$= \varepsilon_o\, E_0 \tilde{f}(s)\, e^{-\gamma z_s}\, \tilde{\overleftrightarrow{P}}'(z_s,s) \cdot \vec{1}_e$$

$$\tilde{\vec{m}}'(z_s,s) = \tilde{\overleftrightarrow{M}}'(z_s,s) \cdot \tilde{\vec{H}}^{(inc)}(z_s \vec{1}_z,s) \tag{3.5}$$

$$= \frac{1}{Z_0} E_0 \tilde{f}(s)\, e^{-\gamma z_s}\, \tilde{\overleftrightarrow{M}}'(z_s,s) \cdot \left[\vec{1}_z \times \vec{1}_e\right]$$

for axial incidence. The far scattered field then is

$$\tilde{\vec{E}}_f^{(sc)}(\vec{r},s)$$

$$= \frac{-e^{-\gamma r}}{4\pi r} E_o \tilde{f}(s) \gamma^2 \int_0^\infty e^{-[1-\cos(\theta)]\gamma z_s}\left[\overleftrightarrow{1}_o \cdot \tilde{\overleftrightarrow{P}}'(z_s,s) \cdot \vec{1}_e - \vec{1}_o \times \tilde{\overleftrightarrow{M}}'(z_s,s) \cdot \left[\vec{1}_z \times \vec{1}_e\right]\right] dz_s \tag{3.6}$$

Using the identity [7]

$$\tilde{\overleftrightarrow{M}}'(z_s,s) \cdot \left[\vec{1}_z \times \vec{1}_e\right] = \left[\tilde{\overleftrightarrow{M}}' \times \vec{1}_z\right] \cdot \vec{1}_e \tag{3.7}$$

we have the scattering dyadic

$$\tilde{\overleftrightarrow{\Lambda}}^{(sc)}(\vec{1}_o, \vec{1}_z;s) = \tilde{\overleftrightarrow{\Lambda}}^{(sc)T}(-\vec{1}_z, -\vec{1}_o;s)$$

$$= -\gamma^2 \int_0^\infty e^{-[1-\cos(\theta)]\gamma z_s}\left[\overleftrightarrow{1}_o \cdot \tilde{\overleftrightarrow{P}}'(z_s,s) - \vec{1}_o \times \tilde{\overleftrightarrow{M}}'(z_s,s) \times \vec{1}_z\right] dz_s \tag{3.8}$$

Note that the integration over z_s from 0 to ∞ is, in time domain, over 0 to ct/[1-cos(θ)] (including any singularity at this point) based on the causal properties of the polarizabilities. Note that θ hould not be too near the cone ($\theta >> \psi$). The polarizability dyadics have non-zero elements only for the transverse (x, y) coordinates, being zero for anything to do with the z coordinate (5 elements).

At this point let us note that (3.8) is quite general, allowing various forms of polarizabilities. These can be then converted to time domain via inverse Laplace transform, or via convolution with $\overset{\leftrightarrow}{P}(z_s,t)$ and $\overset{\leftrightarrow}{M}(z_s,t)$.

4. Scattering Dyadic for Axial Incidence on Perfectly Conducting Cones

For perfectly conducting cones we have

$$
\begin{aligned}
\overleftrightarrow{P}' &= \overleftrightarrow{P}'_0 \tan^2(\psi) z_s^2 \\
\overleftrightarrow{M}' &= \overleftrightarrow{M}'_0 \tan^2(\psi) z_s^2 \\
\overleftrightarrow{M}'_0 &= \overleftrightarrow{\tau}_z \cdot \overleftrightarrow{P}'_0 \cdot \overleftrightarrow{\tau}_z = \vec{1}_z \times \overleftrightarrow{P}'_0 \times \vec{1}_z \\
\overleftrightarrow{M}'_0 \times \vec{1}_z &= -\vec{1}_z \times \overleftrightarrow{P}'_0 = -\overleftrightarrow{\tau}_z \cdot \overleftrightarrow{P}'_0
\end{aligned}
\tag{4.1}
$$

The scattering dyadic is then

$$
\begin{aligned}
&\tilde{\overleftrightarrow{\Lambda}}^{(sc)}(\vec{1}_o, \vec{1}_z; s) \\
&= -\gamma^2 \tan^2(\psi) \left[\overleftrightarrow{1}_o \cdot \overleftrightarrow{P}'_0 - \vec{1}_o \times \overleftrightarrow{M}'_0 \times \vec{1}_z \right] \int_0^\infty e^{-[1-\cos(\theta)]\gamma z_s} z_s^2 dz_s \\
&= -\gamma^2 \tan^2(\psi) \left[\overleftrightarrow{1}_o + \vec{1}_o \times \overleftrightarrow{\tau}_z \right] \cdot \overleftrightarrow{P}'_0 \int_0^\infty e^{-[1-\cos(\theta)]\gamma z_s} z_s^2 dz_s \\
&= -\frac{2}{\gamma} \frac{\tan^2(\psi)}{[1-\cos(\theta)]^3} \left[\overleftrightarrow{1}_o + \vec{1}_o \times \overleftrightarrow{\tau}_z \right] \cdot \overleftrightarrow{P}'_0
\end{aligned}
\tag{4.2}
$$

Noting that the perfectly conducting cone has an exact form for the solution as [9]

$$
\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o, \vec{1}_i; s) = \gamma^{-1} \overleftrightarrow{K}(\vec{1}_o, \vec{1}_i)
\tag{4.3}
$$

we then have

$$
\overleftrightarrow{K}(\vec{1}_o, \vec{1}_z) = -2 \frac{\tan^2(\psi)}{[1-\cos(\theta)]^3} \left[\overleftrightarrow{1}_o + \vec{1}_o \times \overleftrightarrow{\tau}_z \right] \cdot \overleftrightarrow{P}'_0
\tag{4.4}
$$

The result of (4.2) can also be derived directly in time domain as

$$\overleftrightarrow{\Lambda}^{(sc)}(\vec{1}_o, \vec{1}_z; t)$$

$$= -\frac{\tan^2(\psi)}{c^2}\left[\overleftrightarrow{1}_o + \vec{1}_o \times \overleftrightarrow{\tau}_z\right] \bullet \overleftrightarrow{P}'_0 \frac{\partial^2}{\partial t^2}\int_0^\infty \delta\left(t - \frac{1-\cos(\theta)}{c} z_s\right) z_s^2 dz_s$$

$$= -\frac{\tan^2(\psi)}{c^2}\left[\overleftrightarrow{1}_o + \vec{1}_o \times \overleftrightarrow{\tau}_z\right] \bullet \overleftrightarrow{P}'_0 \left[\frac{c}{1-\cos(\theta)}\right]^3 \frac{\partial^2}{\partial t^2}\int_0^\infty \delta\left(\frac{1-\cos(\theta)}{c} z_s - t\right)\left[\frac{1-\cos(\theta)}{c} z_s\right]^2 d\left[\frac{1-\cos(\theta)}{c} z_s\right]$$

$$= -\frac{\tan^2(\psi) c}{\left[1-\cos(\theta)\right]^3}\left[\overleftrightarrow{1}_o + \vec{1}_o \times \overleftrightarrow{\tau}_z\right] \bullet \overleftrightarrow{P}'_0 \frac{\partial^2}{\partial t^2}\left[t^2 u(t)\right] \tag{4.5}$$

$$= -2c\frac{\tan^2(\psi)}{\left[1-\cos(\theta)\right]^3}\left[\overleftrightarrow{1}_o + \vec{1}_o \times \overleftrightarrow{\tau}_z\right] \bullet \overleftrightarrow{P}'_0 u(t)$$

Specializing to axial backscattering we have

$$\vec{1}_o = -\vec{1}_z \ , \ \overleftrightarrow{1}_o = \overleftrightarrow{1}_z \ , \ \theta = \pi$$

$$\left[\overleftrightarrow{1}_z - \overleftrightarrow{1}_z \times \overleftrightarrow{\tau}_z\right] \bullet \overleftrightarrow{P}'_0 = \left[\overleftrightarrow{1}_z - \overleftrightarrow{\tau}_z^{\,2}\right] \bullet \overleftrightarrow{P}'_0 = 2\,\overleftrightarrow{1}_z \bullet \overleftrightarrow{P}'_0 = 2\overleftrightarrow{P}'_0 \tag{4.6}$$

$$\overleftrightarrow{K}_b(\vec{1}_z) = \overleftrightarrow{K}(-\vec{1}_z, \vec{1}_z) = -\frac{1}{2}\tan^2(\psi)\overleftrightarrow{P}'_0$$

which is a rather simple result.

Appendix A considers the polarizability of a perfectly conducting circular cylinder of radius a, giving

$$\overleftrightarrow{P}' = 2\pi a^2 \overleftrightarrow{1}_z \tag{4.7}$$

Applying this to the thin perfectly conducting circular cone we have

$$a(z_s) = z_s \tan(\psi)$$

$$\overleftrightarrow{P}'(z_s) = \overleftrightarrow{P}'_0 z_s^2 \tan^2(\psi)$$

$$\overleftrightarrow{P}'_0 = 2\pi \overleftrightarrow{1}_z \tag{4.8}$$

$$\overleftrightarrow{K}_b = -\pi \tan^2(\psi)\overleftrightarrow{1}_z \simeq -\pi\psi^2\overleftrightarrow{1}_z$$

which is in agreement with both the physical-optics and small-ψ-asymptotic solution for axial backscattering [4, 8].

With this agreement for the circular cone, let us turn to the strip of 2a in Appendix B. In this case, letting the plane of the strip lie on the $y = 0$ plane so that only $P'_{0_{x,x}}$ is non zero as

$$\overleftrightarrow{P}'_0 = \pi a^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.9}$$

in 2 x 2 form. This then gives

$$\begin{aligned} a &= z_s \tan(\psi) \\ \overleftrightarrow{K}_b &= -\frac{\pi}{2}\tan^2(\psi)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \approx -\frac{\pi}{2}\psi^2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{4.10}$$

So now we have a solution for axial backscattering from a thin angular sector. As pointed out in [4] this is in disagreement with PO, which gives zero since such a scatterer has zero cross-section area. Note, furthermore, that the nonzero element has exactly half the value of each element for the circular cone with the angular sector and circular cone both having the same value of $a = z_s \tan(\psi)$ describing the maximum extent from the z axis.

For more insight into the deviation of these results for thin cones from the physical-optics approximation consider the perfectly conducting elliptic cone with semiaxes a_x and a_y in Appendix B. This has

$$\overleftrightarrow{P}'_o = \pi\left[a_x + a_y\right]\begin{pmatrix} a_x & 0 \\ 0 & a_y \end{pmatrix} \tag{4.11}$$

With a as the larger of a_x and a_y we have

$$\begin{aligned} a &= z_s \tan(\psi) \\ \overleftrightarrow{K}_b(\vec{1}_z) &= -\frac{\pi}{2}\tan^2(\psi)\left[\frac{a_x}{a} + \frac{a_y}{a}\right]\begin{pmatrix} \frac{a_x}{a} & 0 \\ 0 & \frac{a_y}{a} \end{pmatrix} \end{aligned} \tag{4.12}$$

As $a_y \to 0$ this has both elements bounded by those for the minimum circumscribing circular cone. As $a_y \to 0$ this goes smoothly to the result for the angular sector. As $a_y \to a_x$ it smoothly goes to the result for the circular cone.

The cross-section area of the circular cone is

$$A(z_s) = \pi z_s^2 \tan^2(\psi)\frac{a_x}{a}\frac{a_y}{a} \tag{4.13}$$

The physical-optics approximation then gives

$$\overset{\leftrightarrow}{K}_b^{(po)}(\vec{1}_z) = -\pi \tan^2(\psi)\frac{a_x}{a}\frac{a_y}{a}\vec{1}_z \tag{4.14}$$

As we can see this goes to zero smoothly for a strip in either orientation. Normalizing the result in (4.12) to the physical optics case gives

$$\left[-\pi \tan^2(\psi)\frac{a_x}{a}\frac{a_y}{a}\right]^{-1}\overset{\leftrightarrow}{K}_b(\vec{1}_z) = \frac{1}{2}\left[\frac{a}{a_x} + \frac{a}{a_y}\right]\begin{pmatrix}\frac{a}{a_y} & 0 \\ 0 & \frac{a}{a_x}\end{pmatrix} \tag{4.15}$$

showing the discrepancy. The PO result is smaller for both polarizations. The ratio of the matrix elements is not 1, but rather a_x / a_y showing the polarization dependence.

5. Concluding Remarks

Now we have a procedure for calculating the scattering from narrow cones for the case of axial incidence (or scattering by reciprocity). A few examples have been considered here for comparison to the PO approximation for axial backscattering. While the PO approximation seems adequate for perfectly conducting bodies of revolution (as, for example, the paraboloid and circular cone [4]), it clearly breaks down for the case of the elliptic cone. The elliptic cone is important in that a shadow boundary does not appear (until one truncates the cone). The limit of a circular cone is an angular sector, and the scattering dyadic for the elliptic cone smoothly goes to that of the angular sector (for which the PO approximation gives zero).

Noting that the present model is appropriate only for thin cones (small ψ), we still need accurate calculations for fat cones (large ψ, but $< \pi/2$) for further comparison to the PO approximation [11]. In its own right, however, this small-ψ model can now be applied to various thin structures, including ones that are not conical.

Appendix A. Polarizability Per Unit Length of Perfectly Conducting Circular Cylinder\

For computing the fields in the vicinity of a perfectly conducting object one can conveniently use conformal transformations in the complex-coordinate plane [6]. As developed in [2] we have

$$\begin{aligned}
\zeta &= x + jy \equiv \text{ complex coordinate} \\
w(\zeta) &= u(\zeta) + jv(\zeta) \equiv \text{ normalized complex potential} \\
e_0(\zeta) &= e_{0_x}(\zeta) - je_{0_y}(\zeta) = \frac{dw(\zeta)}{d\zeta} = \text{normalized complex electric field} \qquad \text{(A.1)} \\
\vec{E}(x,y) &= -\nabla\Phi(x,y) = E_0\left[e_{0_x}(\zeta)\vec{1}_x + e_{0_y}(\zeta)\vec{1}_y\right] = \text{electric field} \\
E_0 &= \text{constant (V/m)}
\end{aligned}$$

Here we deal with the electric field, but as is well known the results apply to the magnetic field as well.

As in Fig. A.1 we map a strip on $u = 0$ in the w plane to a circle of radius a in the ζ plane via

$$\frac{w}{a} = \frac{\zeta}{a} - \frac{a}{\zeta} \qquad \text{(A.3)}$$

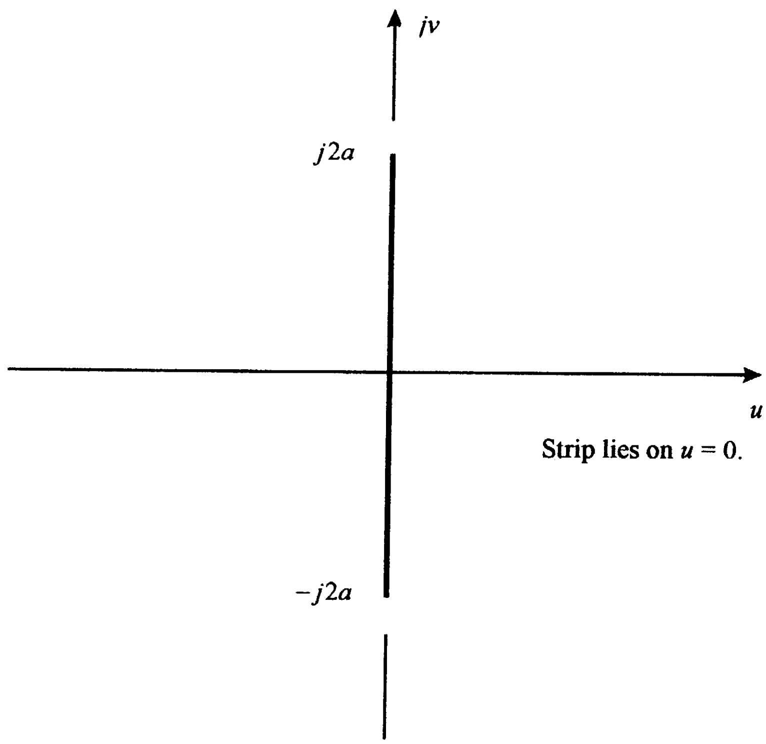
It is convenient to use cylindrical (Ψ,ϕ) coordinates via

$$\begin{aligned}
\zeta &= \Psi e^{j\phi} \\
\frac{w}{a} &= \frac{\Psi}{a}e^{j\phi} - \frac{a}{\Psi}e^{-j\phi}
\end{aligned} \qquad \text{(A.3)}$$

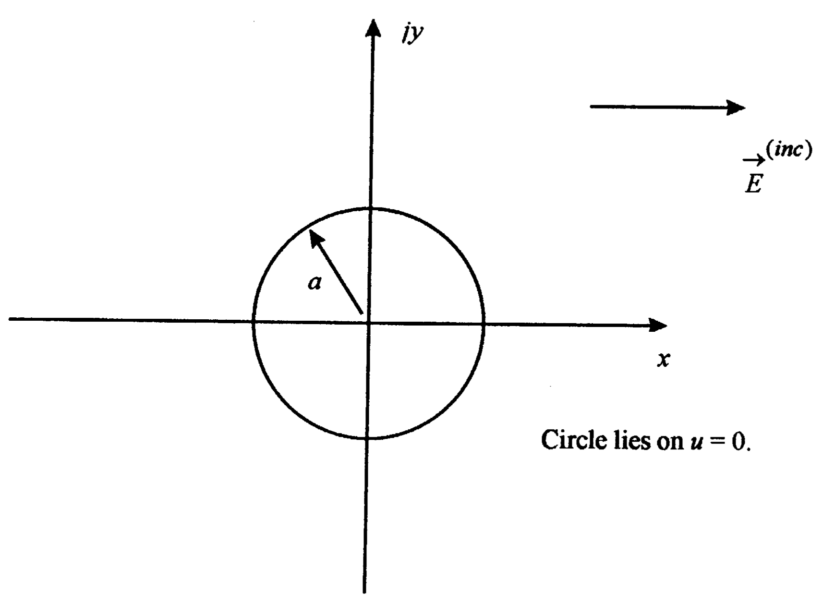
For large $|\zeta|$ we have

$$\vec{E}(x,y) \to E_o \vec{1}_x \text{ as } |\zeta| \to \infty \qquad \text{(A.4)}$$

The surface charge density on the cylinder is



A. Complex-potential plane



B. Complex-coordinate plane

Fig. A.1. Conformal Transformation for Circular Cylinder.

$$\rho_s = \varepsilon_0 E_\Psi = \varepsilon_0 \vec{E}(x,y) \cdot \vec{1}_\Psi \Big|_{|\zeta|=a}$$

$$= \varepsilon_0 E_0 \left[e_{0_x} \cos(\phi) + e_{0_y} \sin(\phi) \right] \Big|_{|\zeta|=a} \tag{A.5}$$

$$= \varepsilon_0 E_0 \left[\mathrm{Re}\left[\frac{dw}{d\zeta} \right] \cos(\phi) - \mathrm{Im}\left[\frac{dw}{d\zeta} \right] \sin(\phi) \right] \Big|_{|\zeta|=a}$$

Using, on the cylinder,

$$\frac{w}{a} = e^{j\phi} - e^{-j\phi} = j2\sin(\phi)$$

$$d\zeta = ad\left[e^{j\phi}\right] = jae^{j\phi}d\phi \tag{A.6}$$

$$\frac{dw}{d\zeta} = 2e^{-j\phi}\cos(\phi)$$

we have

$$\rho_s(\phi) = 2\cos(\phi)\varepsilon_0 E_0 \tag{A.7}$$

This exhibits the well-known enhancement of the electric field by a factor of two on the circular cylinder.

We can now calculate the induced electric-dipole moment per unit length as

$$\vec{p} = \vec{1}_x \int_0^{2\pi} \rho_s(\phi)\; x\, d[a\phi]$$

$$= \vec{1}_x 2a^2 \varepsilon_0 E_0 \int_0^{2\pi} \cos^2(\phi) d\phi \tag{A.8}$$

$$= \vec{1}_x 2\pi a^2 \varepsilon_0 E_0$$

$$= \varepsilon_0 \overset{\leftrightarrow}{P}{}' \cdot \vec{1}_x E_0$$

Noting the rotation symmetry we have

$$\overset{\leftrightarrow}{P}{}' = 2\pi a^2 \vec{1}_z = 2\pi a^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.9}$$

or in two-dimensional form

$$\overset{\leftrightarrow}{P}' = 2\pi a^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.10}$$

for the electric-polarizability dyadic.

Since we are dealing with a perfectly conducting two-dimensional structure we also have

$$\overset{\leftrightarrow}{M}' = -\overset{\leftrightarrow}{P}' = -2\pi a^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.11}$$

consistent with (2.6).

Appendix B. Polarizability Per Unit Length of Perfectly Conducting Strip

Following the same procedure as in Appendix A we construct an appropriate conformal transformation in two steps. As indicated in Fig. B.1 we first transform a strip of width $2a$ on the $u = 0$ axis to a circle of radius $a/2$ with intermediate complex coordinates ζ_1 as

$$\frac{2w}{a} = \frac{2\zeta_1}{a} - \frac{a}{2\zeta_1} \tag{B.1}$$

In turn this is transformed to a strip of width $2a$ on the x axis of the ζ plane as

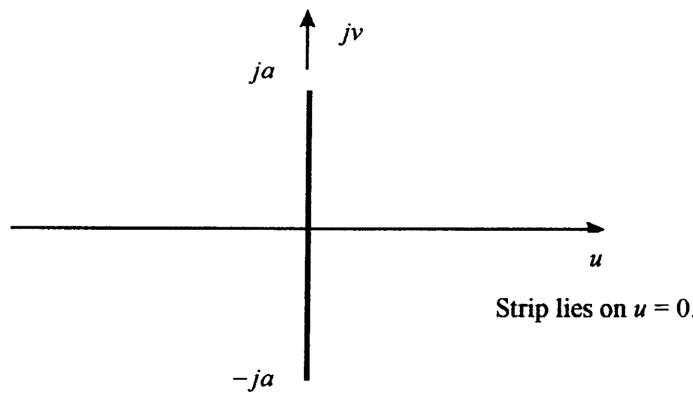
$$\frac{2\zeta}{a} = \frac{2\zeta_1}{a} + \frac{a}{2\zeta_1} \tag{B.2}$$

Combining these transformations we have

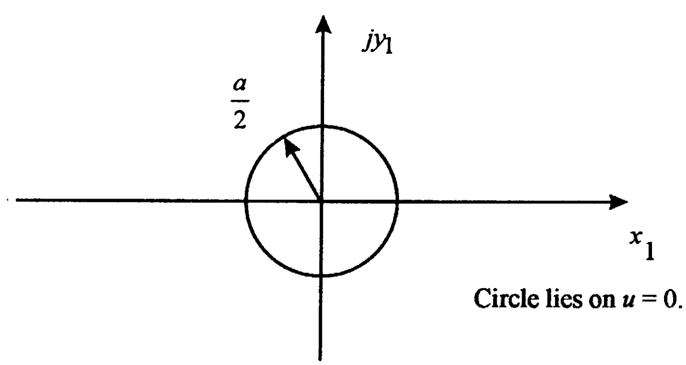
$$\begin{aligned}
\frac{2\zeta_1}{a} &= \frac{\zeta}{a} + \left[\left[\frac{\zeta}{a}\right]^2 - 1\right]^{\frac{1}{2}} \\
\frac{2w}{a} &= \frac{\zeta}{a} + \left[\left[\frac{\zeta}{a}\right]^2 - 1\right]^{\frac{1}{2}} - \left[\frac{\zeta}{a} + \left[\left[\frac{\zeta}{a}\right]^2 - 1\right]^{\frac{1}{2}}\right]^{-1} \\
&= \frac{\zeta}{a} + \left[\left[\frac{\zeta}{a}\right]^2 - 1\right]^{\frac{1}{2}} - \frac{\zeta}{a} + \left[\left[\frac{\zeta}{a}\right]^2 - 1\right]^{\frac{1}{2}} \\
&= 2\left[\left[\frac{\zeta}{a}\right]^2 - 1\right]^{\frac{1}{2}}
\end{aligned} \tag{B.3}$$

The branch has been chosen ∋ for ζ / a real and >1 the square root is positive. On the strip (the branch cut) on the top side ($y = 0_+$) we have

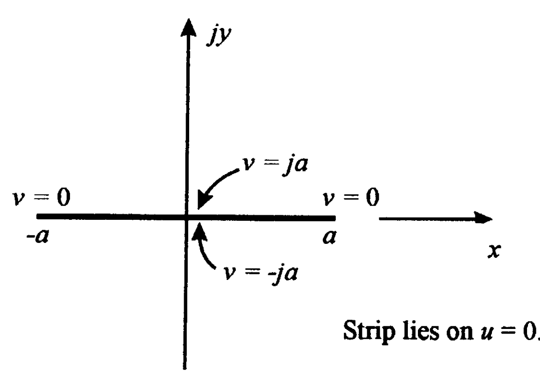
$$\frac{w}{a} = j\frac{v}{a} = j\left[1 - \left[\frac{x}{a}\right]^2\right]^{\frac{1}{2}} \tag{B.4}$$



A. Complex-potential plane



B. Intermediate transformation



C. Complex-coordinate Plane

Fig. B.1 Conformal Transformation for Strip with opposite sign on the bottom side ($y = 0_-$). We can account for both sides by multiplying the expressions by two and integrating on the top side.

The surface charge density, accounting for both sides, is

$$\begin{aligned} \rho_s &= 2\varepsilon_0 \, E_y \quad , \quad y = 0_+ \\ E_y &= -E_0 \frac{dv}{dx} \end{aligned} \tag{B.5}$$

The electric-dipole moment per unit length is

$$\vec{p}\,' = \vec{1}_x \int_{-a}^{a} \rho_s(x) x dx = -\vec{1}_x 2\varepsilon_o E_o \int_{-a}^{a} x \frac{dv}{dx} dx \tag{B.6}$$

Integrating by parts gives

$$\begin{aligned} \vec{p}\,' &= -\vec{1}_x 2\varepsilon_0 E_0 \left[xv \Big|_{-a}^{a} - \int_{-a}^{a} v \, dx \right] \\ &= \vec{1}_x 2\varepsilon_0 E_0 \int_{-a}^{a} v \, dx = \vec{1}_x 2\varepsilon_0 E_0 a \int_{-a}^{a} \left[1 - \left[\frac{x}{a} \right]^2 \right]^{\frac{1}{2}} dx \\ \frac{x}{a} &\equiv \sin(\xi) \quad , \quad dx = a\cos(\xi) d\xi \\ \vec{p}\,' &= \vec{1}_x 2a^2 \varepsilon_0 E_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\xi) d\xi \\ &= \vec{1}_x \pi a^2 \varepsilon_0 E_0 \end{aligned} \tag{B.7}$$

This is for an incident electric field in the x direction. An incident electric field in the y direction, being perpendicular to the flat strip produces no induced electric-dipole moment per unit length. The electric polarizability per unit length is then

$$\overleftrightarrow{P}'_o = \pi a^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{B.8}$$

in two-dimensional form.

The magnetic polarizability per unit length from (2.5) is

$$\overleftrightarrow{M}' = \overleftrightarrow{\tau}_z \cdot \overleftrightarrow{P}' \cdot \overleftrightarrow{\tau}_z = \pi a^2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\pi a^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{B.9}$$

This then scatters a y-directed magnetic field (only) as we expect).

Appendix C. Polarizability Per Unit Length of Perfectly Conducting Elliptic Cylinder

Electric and magnetic polarizabilities of various perfectly conducting objects including elliptic cylinders are tabulated in [5, 12]. With two symmetry planes the axis is well defined. The semiaxes are a_x and a_y for the subscripted coordinate directions. While in Fig. C.1 we have shown $a_x > a_y$, it can be $a_x < a_y$ as well. The results are

$$\overset{\leftrightarrow}{P}' = \pi\left[a_x + a_y\right]\begin{pmatrix} a_x & 0 \\ 0 & a_y \end{pmatrix} \;,\; \overset{\leftrightarrow}{M}' = -\pi\left[a_x + a_y\right]\begin{pmatrix} a_y & 0 \\ 0 & a_x \end{pmatrix} \qquad \text{/C.1)}$$

Note that for $a_x = a_y = a$ this gives the result for the circular cylinder in Appendix A. For $a_x = a$ and $a_y = 0$, this gives the result for the strip in Appendix B.

The ratio of the two elements is just a_x / a_y. For comparison to the case of the circular cylinder define

$$a \equiv \text{larger of } a_x \,,\, a_y \qquad \text{(C.2)}$$

so that the elliptic cylinder has its maximum cross-section dimension equal to that of the circular cylinder. Further define

$$\begin{gathered} \alpha \equiv \frac{a_x}{a} \;,\; \beta \equiv \frac{a_y}{a} \\ \alpha \le 1 \;,\; \beta \le 1 \;,\; \frac{\alpha+\beta}{2} \le 1 \end{gathered} \qquad \text{(C.3)}$$

giving

$$\overset{\leftrightarrow}{P}' = 2\pi a^2 \begin{pmatrix} \alpha\frac{\alpha+\beta}{2} & 0 \\ 0 & \beta\frac{\alpha+\beta}{2} \end{pmatrix} \qquad \text{(C.4)}$$

The matrix elements above are each ≤ 1, showing the comparison to that for the minimum circumscribing circular cylinder.

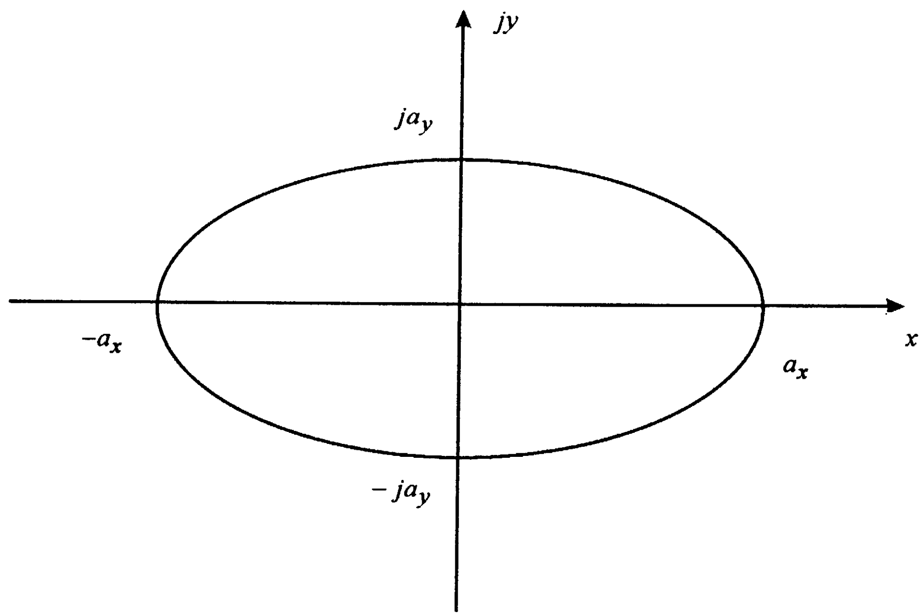


Fig. C.1 Elliptic Cylinder

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