Interaction Notes

Note 589

24 October 2003

High-Frequency Propagation on Nonuniform Multiconductor Transmission Lines

Carl E. Baum
Air Force Research Laboratory
Directed Energy Directorate

Abstract

By means of a symmetric renormalization of the voltage and current vectors the transmission-line equations are cast in a form which brings out a leading term which dominates the high-frequency propagation. Using the sum rule of the product integral a correction term is also exhibited.

This work was sponsored in part by the Air Force Office of Scientific Research, and in part by the Air Force Research Laboratory, Directed Energy Directorate.

1. Introduction

A nonuniform multiconductor transmission line (NMTL) is described by the telegrapher equations as

$$\begin{aligned}
\frac{d}{dz}\left(\tilde{V}_n(z,s)\right) &= -\left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(\tilde{I}_n(z,s)\right) + \left(\tilde{V}_n^{(s)'}(z,s)\right) \\
\frac{d}{dz}\left(\tilde{I}_n(z,s)\right) &= -\left(\tilde{Y}'_{n,m}(z,s)\right) \cdot \left(\tilde{V}_n(z,s)\right) + \left(\tilde{I}_n^{(s)'}(z,s)\right)
\end{aligned} \tag{1.1}$$

Here we have N-component vectors (for N conductors plus reference)

$$\begin{aligned}
\left(\tilde{V}_n(z,s)\right) &= \text{voltage vector} \\
\left(\tilde{I}_n(z,s)\right) &= \text{current vector (positive current convention in direction of increasing } z\text{)} \\
\left(\tilde{V}_n^{(s)'}(z,s)\right) &= \text{voltage source per unit length} \\
\left(\tilde{I}_n^{(s)'}(z,s)\right) &= \text{current source per unit length} \\
z\ (\text{real}) &= \text{position coordinate along the line} \\
\sim\ &\equiv \text{two-sided Laplace transform} \\
s = \Omega + j\omega\ &\equiv \text{Laplace-transform variable or complex frequency} \\
\left(\tilde{Z}'_{n,m}(z,s)\right) = \left(\tilde{Z}'_{n,m}(z,s)\right)^T &\equiv \text{per-unit-length (series) impedance matrix (N x N)} \\
\left(\tilde{Y}'_{n,m}(z,s)\right) = \left(\tilde{Y}'_{n,m}(z,s)\right)^T &\equiv \text{per-unit-length (transverse) admittance matrix (N x N)}
\end{aligned} \tag{1.2}$$

where reciprocity has been used.

There are various approaches to the solution of these equations [11]. From a computational point of view, one can divide the line into a set of sections, approximate the line as uniform in each section by some sort of section average, solve each resulting approximate section analytically, and multiply the resulting set of chain matrices to obtain an approximate solution for the entire transmission line. As discussed in [5, 6] one can improve on this by adding a small variable term to the average and use the sum rule of the product integral to obtain a correction which reduces the reflections at the section boundaries.

In this paper we adopt a different approach. While the approximations in [5, 6] work well for wavelengths of the order of the section length (or even somewhat less), the present approach is based on a high-frequency approximation (thereby being a complementary technique). Previous papers [2, 4] have explored some aspects of this with the assumption of equal modal speeds (all N modes). Here we form a similar normalization of the voltage and current vector by the square root of the characteristic impedance matrix and separate the waves into the two propagation directions with coupling between the two. This yields a WKB type of leading term plus corrections based on the sum rule of the product integral followed by a (matrizant) series expansion of the correction product integral. It should be noted that the WKB approximation is treated in references such as [8] for single-conductor (plus reference) nonuniform transmission lines. Here, however, it is generalized to multiconductor nonuniform transmission lines.

2. Symmetric Renormalized Form of the NMTL Equations

Following the procedure in [1] let us first look at the local aspects of the propagation, neglecting the z variation. We then have

$$
\begin{aligned}
\left(\tilde{\gamma}_{n,m}(z,s)\right) &= \left[\left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(\tilde{Y}'_{n,m}(z,s)\right)\right]^{1/2} \quad \text{(p.r. square root)} \\
&\equiv \text{propagation matrix} \\
\left(\tilde{Z}_{c_{n,m}}(z,s)\right) &= \left(\tilde{\gamma}_{n,m}(z,s)\right) \cdot \left(\tilde{Y}'_{n,m}(z,s)\right)^{-1} \\
&= \left(\tilde{\gamma}_{n,m}(z,s)\right)^{-1} \cdot \left(\tilde{Z}'_{n,m}(z,s)\right) \\
&= \left(\tilde{Y}_{c_{n,m}}(z,s)\right)^{-1} = \left(\tilde{Z}_{c_{n,m}}(z,s)\right)^{T} \\
&\equiv \text{characteristic impedance matrix}
\end{aligned}
\tag{2.1}
$$

In [1] the square of the propagation matrix is diagonalized to find right and left eigenmodes. Taking positive-real (p.r.) square roots of the eigenvalues gives the eigenvalues of the propagation matrix. These modes are then used to give representations of the other above matrices. However, in the present context these eigenmodes and eigenvalues are functions of z, making their use less desirable.

We can still use (2.1) to define the characteristic impedance and admittance matrices, as well as the propagation matrix. Now they are allowed to vary as functions of z. As in [2, 4] we form

$$
\begin{aligned}
\left(\tilde{z}_{c_{n,m}}(z,s)\right) &\equiv \left(\tilde{Z}_{c_{n,m}}(z,s)\right)^{1/2} \quad \text{(p.r. square root)} \\
\left(\tilde{y}_{c_{n,m}}(z,s)\right) &\equiv \left(\tilde{Y}_{c_{n,m}}(z,s)\right)^{1/2} \quad \text{(p.r. square root)} \\
&= \left(\tilde{z}_{c_{n,m}}(z,s)\right)^{-1}
\end{aligned}
\tag{2.2}
$$

We then have

$$
\begin{aligned}
&\left(\tilde{z}_{c_{n,m}}(z,s)\right) \cdot \left(\tilde{y}_{c_{n,m}}(z,s)\right) = \left(1_{n,m}\right) = \left(\tilde{y}_{c_{n,m}}(z,s)\right) \cdot \left(\tilde{z}_{c_{n,m}}(z,s)\right) \\
&\left[\frac{d}{dz}\left(\tilde{z}_{c_{n,m}}(z,s)\right)\right] \cdot \left(\tilde{y}_{c_{n,m}}(z,s)\right) + \left(\tilde{z}_{c_{n,m}}(z,s)\right) \cdot \left[\frac{d}{dz}\left(\tilde{y}_{c_{n,m}}(z,s)\right)\right] = \left(0_{n,m}\right) \\
&\left[\frac{d}{dz}\left(\tilde{y}_{c_{n,m}}(z,s)\right)\right] \cdot \left(\tilde{z}_{c_{n,m}}(z,s)\right) + \left(\tilde{y}_{c_{n,m}}(z,s)\right) \cdot \left[\frac{d}{dz}\left(\tilde{z}_{c_{n,m}}(z,s)\right)\right] = \left(0_{n,m}\right)
\end{aligned}
\tag{2.3}
$$

these last two results being the transposes of each other.

Now define normalized variables

$$
\begin{aligned}
\left(\tilde{\mathrm{v}}_n(z,s)\right) &\equiv \left(\tilde{y}_{c_{n,m}}(z,s)\right) \bullet \left(\tilde{V}_n(z,s)\right) \\
\left(\tilde{i}_n(z,s)\right) &\equiv \left(\tilde{z}_{c_{n,m}}(z,s)\right) \bullet \left(\tilde{I}_n(z,s)\right) \\
\left(\tilde{\mathrm{v}}_n^{(s)'}(z,s)\right) &\equiv \left(\tilde{y}_{c_{n,m}}(z,s)\right) \bullet \left(\tilde{V}_n^{(s)'}(z,s)\right) \\
\left(\tilde{i}_n^{(s)'}(z,s)\right) &\equiv \left(\tilde{z}_{c_{n,m}}(z,s)\right) \bullet \left(\tilde{I}_n(z,s)\right)
\end{aligned}
\tag{2.4}
$$

and note that

$$
\begin{aligned}
\left(\tilde{Z}'_{n,m}(z,s)\right) &= \left(\tilde{\gamma}_{n,m}(z,s)\right) \bullet \left(\tilde{Z}_{c_{n,m}}(z,s)\right) = \left(\tilde{\gamma}_{n,m}(z,s)\right) \bullet \left(\tilde{z}_{c_{n,m}}(z,s)\right)^2 \\
\left(\tilde{Y}'_{n,m}(z,s)\right) &= \left(\tilde{Y}_{c_{n,m}}(z,s)\right) \bullet \left(\tilde{\gamma}_{n,m}(z,s)\right) = \left(\tilde{y}_{c_{n,m}}(z,s)\right)^2 \bullet \left(\tilde{\gamma}_{n,m}(z,s)\right) \\
&= \left(\tilde{\gamma}_{n,m}(z,s)\right)^T \bullet \left(\tilde{y}_{c_{n,m}}(z,s)\right)^2
\end{aligned}
\tag{2.5}
$$

Substitute the above in the telegrapher equations as

$$
\begin{aligned}
\frac{d}{dz}\left[\left(\tilde{z}_{c_{n,m}}(z,s)\right) \bullet \left(\tilde{\mathrm{v}}_n(z,s)\right)\right] &= -\left(\tilde{\gamma}_{n,m}(z,s)\right) \bullet \left(\tilde{z}_{c_{n,m}}(z,s)\right) \bullet \left(\tilde{i}_n(z,s)\right) \\
&\quad + \left(\tilde{z}_{c_{n,m}}(z,s)\right) \bullet \left(\tilde{\mathrm{v}}_n^{(s)'}(z,s)\right) \\
\frac{d}{dz}\left[\left(\tilde{y}_{c_{n,m}}(z,s)\right) \bullet \left(\tilde{i}_n(z,s)\right)\right] &= -\left(\tilde{\gamma}_{n,m}(z,s)\right)^T \bullet \left(\tilde{y}_{c_{n,m}}(z,s)\right) \bullet \left(\tilde{\mathrm{v}}_n(z,s)\right) \\
&\quad + \left(\tilde{y}_{c_{n,m}}(z,s)\right) \bullet \left(\tilde{i}_n^{(s)'}(z,s)\right)
\end{aligned}
\tag{2.6}
$$

Expanding the derivatives and clearing terms gives

$$
\begin{aligned}
\frac{d}{dz}\left(\tilde{\mathrm{v}}_n(z,s)\right) &= -\left(\tilde{y}_{c_{n,m}}(z,s)\right) \bullet \left[\frac{d}{dz}\left(z_{c_{n,m}}(z)\right)\right] \bullet \left(\tilde{\mathrm{v}}_n(z,s)\right) \\
&\quad - \left(\tilde{y}_{c_{n,m}}(z,s)\right) \bullet \left(\tilde{\gamma}_{n,m}(z,s)\right) \bullet \left(\tilde{z}_{c_{n,m}}(z,s)\right) \bullet \left(\tilde{i}_n(z,s)\right) + \left(\tilde{\mathrm{v}}_n^{(s)'}(z,s)\right) \\
\frac{d}{dz}\left(\tilde{i}_n(z,s)\right) &= -\left(\tilde{z}_{c_{n,m}}(z,s)\right) \bullet \left[\frac{d}{dz}\left(\tilde{y}_{c_{n,m}}(z)\right)\right] \bullet \left(\tilde{i}_n(z,s)\right) \\
&\quad - \left(\tilde{z}_{c_{n,m}}(z,s)\right) \bullet \left(\tilde{\gamma}_{n,m}(z,s)\right)^T \bullet \left(\tilde{y}_{c_{n,m}}(z,s)\right) \bullet \left(\tilde{\mathrm{v}}_n(z,s)\right) + \left(\tilde{i}_n^{(s)'}(z,s)\right)
\end{aligned}
\tag{2.7}
$$

Define combined normalized voltages as

$$\left(\tilde{v}_n(z,s)\right)_{\substack{1\\2}} = \left(\tilde{v}_n(z,s)\right) \pm \left(\tilde{i}_n(z,s)\right) \tag{2.8}$$

and similarly for the sources. The upper index (1, + sign) corresponds to waves propagating in the $+z$ direction while the lower index (2, - sign) corresponds to waves propagating in the $-z$ direction. The normalized voltages and currents can, of course, be reconstructed from

$$\begin{aligned}
\left(\tilde{v}_n(z,s)\right) &= \frac{1}{2}\left[\left(\tilde{v}_n(z,s)\right)_1 + \left(\tilde{v}_n(z,s)\right)_2\right] \\
\left(\tilde{i}_n(z,s)\right) &= \frac{1}{2}\left[\left(\tilde{v}_n(z,s)\right)_1 - \left(\tilde{v}_n(z,s)\right)_2\right]
\end{aligned} \tag{2.9}$$

Collect some terms as

$$\begin{aligned}
\left(\tilde{g}_{n,m}(z,s)\right) &\equiv -\left(\tilde{y}_{n,m}(z,s)\right) \bullet \left(\tilde{\gamma}_{n,m}(z,s)\right) \bullet \left(\tilde{z}_{c_{n,m}}(z,s)\right) \\
&= -\left(\tilde{z}_{c_{n,m}}(z,s)\right) \bullet \left(\tilde{Y}'_{n,m}(z,s)\right) \bullet \left(\tilde{z}_{c_{n,m}}(z,s)\right) \\
&= -\left(\tilde{z}_{c_{n,m}}(z,s)\right) \bullet \left(\tilde{\gamma}_{n,m}(z,s)\right)^T \bullet \left(\tilde{y}_{c_{n,m}}(z,s)\right) \\
&= \left(\tilde{g}_{n,m}(z,s)\right)^T \\
\left(\tilde{a}_{n,m}(z,s)\right) &= \frac{1}{2}\left[-\left(y_{c_{n,m}}(z,s)\right) \bullet \left[\frac{d}{dz}\left(z_{c_{n,m}}(z,s)\right)\right] - \left(\tilde{z}_{c_{n,m}}(z,s)\right) \bullet \left[\frac{d}{dz}\left(y_{c_{n,m}}(z,s)\right)\right]\right] \\
&= \frac{1}{2}\left[-\left(\tilde{y}_{c_{n,m}}(z,s)\right) \bullet \left[\frac{d}{dz}\left(z_{c_{n,m}}(z,s)\right)\right] + \left[\frac{d}{dz}\left(z_{c_{n,m}}(z,s)\right)\right] \bullet \left(\tilde{y}_{c_{n,m}}(z,s)\right)\right] \\
&= \frac{1}{2}\left[\left[\frac{d}{dz}\left(\tilde{y}_{c_{n,m}}(z,s)\right)\right] \bullet \left(\tilde{z}_{c_{n,m}}(z,s)\right) - \left(\tilde{z}_{c_{n,m}}(z,s)\right) \bullet \left[\frac{d}{dz}\left(\tilde{y}_{c_{n,m}}(z,s)\right)\right]\right] \\
&= \frac{1}{2}\left[\left[\frac{d}{dz}\left(\tilde{y}_{c_{n,m}}(z,s)\right)\right] \bullet \left(\tilde{z}_{c_{n,m}}(z,s)\right) + \left[\frac{d}{dz}\left(\tilde{z}_{c_{n,m}}(z,s)\right)\right] \bullet \left(\tilde{y}_{c_{n,m}}(z,s)\right)\right] \\
&= -\left(a_{n,m}(z,s)\right)^T \\
\left(\tilde{b}_{n,m}(z,s)\right) &= \frac{1}{2}\left[-\left(\tilde{y}_{c_{n,m}}(z,s)\right) \bullet \left[\frac{d}{dz}\left(\tilde{z}_{c_{n,m}}(z,s)\right)\right] + \left(\tilde{z}_{c_{n,m}}(z,s)\right) \bullet \left[\frac{d}{dz}\left(\tilde{y}_{c_{n,m}}(z,s)\right)\right]\right] \\
&= \frac{1}{2}\left[-\left(\tilde{y}_{c_{n,m}}(z,s)\right) \bullet \left[\frac{d}{dz}\left(\tilde{z}_{c_{n,m}}(z,s)\right)\right] - \left[\frac{d}{dz}\left(\tilde{z}_{c_{n,m}}(z,s)\right)\right] \bullet \left(\tilde{y}_{c_{n,m}}(z,s)\right)\right] \\
&= \frac{1}{2}\left[\left[\frac{d}{dz}\left(\tilde{y}_{c_{n,m}}(z,s)\right)\right] \bullet \left(\tilde{z}_{c_{n,m}}(z,s)\right) + \left(\tilde{z}_{c_{n,m}}(z,s)\right) \bullet \left[\frac{d}{dz}\left(\tilde{y}_{c_{n,m}}(z,s)\right)\right]\right] \\
&= \frac{1}{2}\left[\left[\frac{d}{dz}\left(\tilde{y}_{c_{n,m}}(z,s)\right)\right] \bullet \left(\tilde{z}_{c_{n,m}}(z,s)\right) - \left[\frac{d}{dz}\left(\tilde{z}_{c_{n,m}}(z,s)\right)\right] \bullet \left(\tilde{y}_{c_{n,m}}(z,s)\right)\right] \\
&= \left(\tilde{b}_{n,m}(z,s)\right)^T
\end{aligned} \tag{2.10}$$

We are now in a position to cast the new parameters in terms of a supervector/supermatrix equation by taking sums and differencfes in (2.7) as

$$
\begin{aligned}
&\frac{d}{dz}\left(\left(\tilde{\mathrm{v}}_n(z,s)\right)_\nu\right) \\
&= \left[\left(\left(\tilde{g}_{n,m}(z,s)\right)_{\nu,\nu'}\right) + \left(\left(\tilde{a}_{n,m}(z,s)\right)_{\nu,\nu'}\right) + \left(\left(\tilde{b}_{n,m}(z,s)\right)_{\nu,\nu'}\right)\right] \odot \left(\left(\tilde{\mathrm{v}}_n(z,s)\right)_\nu\right) \\
&\qquad + \left(\left(\tilde{\mathrm{v}}_n^{(s)'}(z,s)\right)_\nu\right) \\
&\left(\left(\tilde{g}_{n,m}(z,s)\right)_{\nu,\nu'}\right) = \begin{pmatrix} \left(\tilde{g}_{n,m}(z,s)\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & -\left(\tilde{g}_{n,m}(z,s)\right) \end{pmatrix} = \left(\left(\tilde{g}_{n,m}(z,s)\right)_{\nu,\nu'}\right)^T \\
&\left(\left(\tilde{a}_{n,m}(z,s)\right)_{\nu,\nu'}\right) = \begin{pmatrix} \left(\tilde{a}_{n,m}(z,s)\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(\tilde{a}_{n,m}(z,s)\right) \end{pmatrix} = -\left(\left(\tilde{a}_{n,m}(z,s)\right)_{\nu,\nu'}\right)^T \\
&\left(\left(\tilde{b}_{n,m}(z,s)\right)_{\nu,\nu'}\right) = \begin{pmatrix} \left(0_{n,m}\right) & \left(\tilde{b}_{n,m}(z,s)\right) \\ \left(\tilde{b}_{n,m}(z,s)\right) & \left(0_{n,m}\right) \end{pmatrix} = \left(\left(\tilde{b}_{n,m}(z,s)\right)_{\nu,\nu'}\right)^T
\end{aligned}
\tag{2.11}
$$

The solution of (2.11) is in terms of a matrizant solving the matrix differential equation

$$
\begin{aligned}
&\frac{d}{dz}\left(\left(\tilde{\Phi}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) \\
&= \left[\left(\left(\tilde{g}_{n,m}(z,s)\right)_{\nu,\nu'}\right) + \left(\left(\tilde{a}_{n,m}(z,s)\right)_{\nu,\nu'}\right) + \left(\left(\tilde{b}_{n,m}(z,s)\right)_{\nu,\nu'}\right)\right] \odot \left(\left(\tilde{\Phi}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) \\
&\qquad \left(\left(\tilde{\Phi}_{n,m}(z_0,z_0;s)\right)_{\nu,\nu'}\right) = \left(\left(1_{n,m}\right)_{\nu,\nu'}\right) = \begin{pmatrix} \left(1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(1_{n,m}\right) \end{pmatrix} \\
&\qquad\qquad \text{(boundary condition)}
\end{aligned}
\tag{2.12}
$$

In terms of this we have the solution of (2.11) as

$$
\begin{aligned}
\left(\left(\tilde{\mathrm{v}}_n(z,s)\right)_\nu\right) &= \left(\left(\tilde{\Phi}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{\mathrm{v}}_n(z_0,s)\right)_\nu\right) \\
&\quad + \int_{z_0}^{z} \left(\left(\tilde{\Phi}_{n,m}(z,z';s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{\mathrm{v}}_n^{(s)'}(z',s)\right)_\nu\right) dz'
\end{aligned}
\tag{2.13}
$$

If there are no sources the integral is zero.

The matrizant is given by a product integral [10]

$$\left(\left(\tilde{\Phi}_{n,m}(z,z_0;s)\right)_{v,v'}\right) = \prod_{z_0}^{z} e^{\left[\left(\left(\tilde{g}_{n,m}(z',s)\right)_{v,v'}\right) + \left(\left(\tilde{a}_{n,m}(z',s)\right)_{v,v'}\right) + \left(\left(\tilde{b}_{n,m}(z',s)\right)_{v,v'}\right)\right]dz'} \tag{2.14}$$

Let us observe here that the first term in the integrand, $((\tilde{g}_{n,m})_{v,v'})$, is, like $((\tilde{\gamma}_{n,m})_{v,v'})$, usually proportional to s. This is associated with the fact that $(\tilde{Z}'_{n,m})$ and $(\tilde{Y}'_{n,m})$ often take the form $s(L'_{n,m}(z))$ and $s(C'_{n,m}(z))$ (lossless case). Even with losses then $((g_{n,m})_{v,v'})$ tends to ∞ as $s \to \infty$ and so is large for large s. By comparison $(\tilde{a}_{n,m})$ and $(\tilde{b}_{n,m})$ may be independent of s (lossless case) or vary slowly with s. Hence our strategy is to bring out $((\tilde{g}_{n,m})_{v,v'})$ as a dominant term for high frequencies, giving a generalized type of WKB approximation. The other terms give correction terms which become more significant as the frequency is lowered.

3. Development of the Solution: Leading Term and Corrections

Define what will be the leading high-frequency term as

$$
\begin{aligned}
\left(\left(\tilde{G}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) &\equiv \prod_{z_0}^{z} e^{\left(\left(\tilde{g}_{n,m}(z',s)\right)_{\nu,\nu'}\right)dz'} \\
&= \left[\prod_{z_0}^{z} e^{\left(\tilde{g}_{n,m}(z',s)\right)dz'}\right] \oplus \left[\prod_{z_0}^{z} e^{-\left(\tilde{g}_{n,m}(z',s)\right)dz'}\right]
\end{aligned}
\tag{3.1}
$$

Letting

$$
\left(\tilde{G}_{n,m}(z,z_0;s)\right) \equiv \prod_{z_0}^{z} e^{\left(\tilde{g}_{n,m}(z',s)\right)dz'} \tag{3.2}
$$

and noting

$$
\left(\tilde{g}_{n,m}(z)\right) = \left(\tilde{g}_{n,m}(z)\right)^T \tag{3.3}
$$

we have (from (A.5))

$$
\begin{aligned}
\prod_{z_0}^{z} e^{-\left(\tilde{g}_{n,m}(z)\right)dz'} &= \left[\prod_{z_0}^{z} e^{\left(\tilde{g}_{n,m}(z')\right)dz'}\right]^{-1T} \\
&= \left(\tilde{G}_{n,m}(z,z_0;s)\right)^{-1T} \\
&= \left(\tilde{G}_{n,m}(z_0,z;s)\right)^{T}
\end{aligned}
\tag{3.4}
$$

Combining we have

$$
\begin{aligned}
\left(\left(\tilde{G}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) &= \left(\tilde{G}_{n,m}(z,z_0;s)\right) \oplus \left(\tilde{G}_{n,m}(z_0,z;s)\right)^T \\
\left(\left(\tilde{G}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right)^{-1} &= \left(\tilde{G}_{n,m}(z,z_0;s)\right)^{-1} \oplus \left(\tilde{G}_{n,m}(z_0,z;s)\right)^{-1T} \\
&= \left(\tilde{G}_{n,m}(z_0,z;s)\right) \oplus \left(\tilde{G}_{n,m}(z,z_0;s)\right)^T
\end{aligned}
\tag{3.5}
$$

Applying the sum rule the complete product integral (2.14) is

$$\begin{aligned}
&\left(\left(\Phi_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) = \left(\left(\tilde{G}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{H}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) \\
&\left(\tilde{H}_{n,m}(z,z_0;s)\right)_{\nu,\nu'} = \\
&\prod_{z_0}^{z} e^{\left(\left(\tilde{G}_{n,m}(z',z_0;s)\right)_{\nu,\nu'}\right)^{-1} \odot \left(\left(\tilde{a}_{n,m}(z',s)\right)_{\nu,\nu'}\right) + \left(\left(\tilde{b}_{n,m}(z',s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{G}_{n,m}(z',z_0;s)\right)_{\nu,\nu'}\right) dz'}
\end{aligned} \tag{3.6}$$

For treating this second, or correction, term we can use the matrizant series [2, 9]

$$\begin{aligned}
\prod_{z_0}^{z} e^{\left(A_{n,m}(z')\right)dz'} &= \sum_{n=0}^{\infty} \left(X_{n,m}(z,z_0)\right)_n \\
\left(X_{n,m}(z,z_0)\right)_0 &= \left(1_{n,m}\right) \\
\left(X_{n,m}(z,z_0)\right)_{n+1} &= \int_{z_0}^{z} \left(A_{n,m}(z')\right) \cdot \left(X_{n,m}(z',z_0)\right)_n dz'
\end{aligned} \tag{3.7}$$

Successive terms are then successive integrals of the product integrand. Gantmacher ascribes this series to G. Peano (1888).

Provided the product integrand is sufficiently small in the correction term, then it can be represented by the first few terms in the matrizant series.

4. Properties of the Leading Term

In the leading term the + (or right-going) waves are completely separated from the - (or left-going) waves. Let us write (for no distributed sources)

$$\left(\left(\tilde{\mathrm{v}}_n(z,s)\right)_\nu\right) = \left(\left(\tilde{G}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{H}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{\mathrm{v}}_n(z_0,s)\right)_\nu\right)$$
$$\left(\left(\tilde{\mathrm{v}}_n^{(0)}(z,s)\right)_\nu\right) = \left(\left(\tilde{G}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{\mathrm{v}}_n(z_0,s)\right)_{\nu,\nu'}\right)$$
$$= \text{combined voltage leading term} \tag{4.1}$$

Let us take as our boundary condition at z_0

$$\begin{pmatrix} \left(\tilde{\mathrm{v}}_n(z_0,s)\right)_1 \\ \left(\tilde{\mathrm{v}}_n(z_0,s)\right)_2 \end{pmatrix} = \begin{pmatrix} \left(\tilde{\mathrm{v}}_n(z_0,s)\right)_1 \\ (0_n) \end{pmatrix} \tag{4.2}$$

The launches only a $+z$ propagating wave when considering only the leading term, due to the block-diagonal form of the product integral. This implies that

$$\begin{gathered}\left(\tilde{\mathrm{v}}_n^{(0)}(z,s)\right)_2 = \left(\tilde{\mathrm{v}}_n^{(0)}(z,s)\right) - \left(\tilde{i}_n^{(0)}(z,s)\right) = (0_n) \\ \left(\tilde{\mathrm{v}}_n^{(0)}(z,s)\right)_1 = \left(\tilde{\mathrm{v}}_n^{(0)}(z,s)\right) + \left(\tilde{i}_n^{(0)}(z,s)\right) = 2\left(\tilde{\mathrm{v}}_n^{(0)}(z,s)\right)\end{gathered} \tag{4.3}$$

Hence, we can write for the normalized voltage, without distributed source,

$$\left(\tilde{\mathrm{v}}_n^{(0)}(z,s)\right) = \left(\tilde{G}_{n,m}(z,z_0;s)\right) \cdot \left(\tilde{\mathrm{v}}_n^{(0)}(z_0,s)\right) \tag{4.4}$$

giving for the voltage vector

$$\left(\tilde{V}_n^{(0)}(z,s)\right) = \left(\tilde{z}_{c_{n,m}}(z,s)\right) \cdot \left(\tilde{G}_{n,m}(z,z_0;s)\right) \cdot \left(\tilde{y}_{c_{n,m}}(z_0,s)\right) \cdot \left(\tilde{V}_n(z_0,s)\right) \tag{4.5}$$

This can be considered as a generalization of the WKB approximation for NMTLs.

So, now look at the properties of $(\tilde{G}_{n,m})$, which is written as

$$\left(\tilde{G}_{n,m}(z,z_0;s)\right) = \prod_{z_0}^{z} e^{\left(\tilde{g}_{n,m}(z',s)\right)}$$

$$= \prod_{z_0}^{z} e^{-\left(\tilde{y}_{c_{n,m}}(z',s)\right) \cdot \left(\tilde{\gamma}_{n,m}(z',s)\right) \cdot \left(\tilde{z}_{c_{n,m}}(z',s)\right)dz'} \tag{4.6}$$

At this point the leading term is reduced to an N x N (instead of 2N x 2N) product integral. Various assumptions concerning the terms in $(\tilde{g}_{n,m})$ can now be made to further simplify the problem.

One assumption one can make is that the various N x N matrices commute with each other, including for all pairs of z' for $z_0 \leq z' \leq z$. This is the form discussed in [3]. In this case

$$\left(\tilde{G}_{n,m}(z,z_0;s)\right) = \exp\left(\int_{z_0}^{z} -\left(\tilde{y}_{c_{n,m}}(z',s)\right) \cdot \left(\tilde{\gamma}_{n,m}(z',s)\right) \cdot \left(\tilde{z}_{c_{n,m}}(z',s)\right)dz'\right) \tag{4.7}$$

reducing the problem to the usual sum integral. There are various equivalent ways to state this commutativity. In particularly one only needs that $(\tilde{Z}_{n,m})$ and $(\tilde{Y}'_{n,m})$ commute with each other for every pair of z' in $z_0 \leq z' \leq z$. Then there is a common set of eigenvectors for all of these matrices (from (2.1) and (2.2)). The above integral reduces to the form

$$\left(\tilde{G}_{n,m}(z,z_0;s)\right) = \sum_{\beta=1}^{N} e^{\int_{z_0}^{z} \tilde{g}'_{\beta}(z',s)dz'} \left(\tilde{x}_n(s)\right)_{\beta} \left(\tilde{x}_n(s)\right)_{\beta}$$

$\left(\tilde{x}_n(s)\right)_{\beta}$ = position independent orthonormal eigenvectors (symmetric matrix)

$\tilde{g}_{\beta}(z,s)$ = position dependent eigenvalues of $(\tilde{g}_{n,m})$ (4.8)

This gives us N separate scalar problems in which the N eigenwaves do not couple to each other.

Another interesting assumption concerns equal modal speeds, such as encountered with N perfectly conducting wires (plus reference) in a uniform isotropic dielectric medium (uniform at each z, allowing variation with z). The wires may vary their diameter (or even cross-section shape) and their positions relative to each other. In this case we have

$$
\begin{aligned}
\left(\tilde{\gamma}_{n,m}(z,s)\right) &= \tilde{\gamma}(z,s)\left(1_{n,m}\right) \quad , \quad \left(\tilde{g}_{n,m}(z,s)\right) = -\tilde{\gamma}(z,s)\left(1_{n,m}\right) \\
\left(\tilde{G}_{n,m}(z,s)\right) &= \tilde{G}(z,z_0;s)\left(1_{n,m}\right) \\
\tilde{G}(z,z_0;s) &= e^{-\int_{z_0}^{z} \tilde{\gamma}(z',s)dz'}
\end{aligned}
\tag{4.9}
$$

Note that this still allows variation of $(\tilde{Z}'_{n,m})$ and $(\tilde{Y}'_{n,m})$ as long as each is equal to the inverse of the other times a scalar (at each z separately), this being consistent with the foregoing assumption. Stated another way $(\tilde{Z}_{c_{n,m}})$ and $(\tilde{z}_{c_{n,m}})$ are allowed to vary as functions of z. Applying this to the $+z$ propagating wave in (4.4) we have

$$
\left(\tilde{\mathrm{v}}_n^{(0)}(z,s)\right) = \tilde{G}(z,z_0;s)\left(\tilde{\mathrm{v}}_n(z_0,s)\right) \tag{4.10}
$$

again without distributed sources. Converting back to unnormalized variables we have for the voltage vector

$$
\boxed{\left(\tilde{V}_n^{(0)}(z,s)\right) = \tilde{G}(z,z_0;s)\left(\tilde{z}_{c_{n,m}}(z,s)\right) \cdot \left(\tilde{y}_{c_{n,m}}(z_0,s)\right) \cdot \left(\tilde{V}_n(z_0,s)\right)} \tag{4.11}
$$

This is a remarkably simple result. It extends the result in [4], giving a closed form (involving a sum integral) for arbitrary N.

5. Properties of the Correction Term

From the correction term, using the matrizant series, we have

$$\left(\left(\tilde{H}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) =$$

$$\prod_{z_0}^{z} e^{\left(\left(\tilde{G}_{n,m}(z',z_0;s)\right)_{\nu,\nu'}\right)^{-1} \odot \left[\left(\left(\tilde{a}_{n,m}(z',s)\right)_{\nu,\nu'}\right) + \left(\left(\tilde{b}_{n,m}(z',s)\right)_{\nu,\nu'}\right)\right] \odot \left(\left(\tilde{G}_{n,m}(z',z_0;s)\right)_{\nu,\nu'}\right) dz'}$$

$$= \left(\left(1_{n,m}\right)_{\nu,\nu'}\right) + \tag{5.1}$$

$$\int_{z_0}^{z} \left(\left(\tilde{G}_{n,m}(z',z_0;s)\right)_{\nu,\nu'}\right)^{-1} \odot \left[\left(\left(\tilde{a}_{n,m}(z',s)\right)_{\nu,\nu'}\right) + \left(\left(\tilde{b}_{n,m}(z',s)\right)_{\nu,\nu'}\right)\right] \odot \left(\left(\tilde{G}_{n,m}(z',z_0;s)\right)_{\nu,\nu'}\right) dz'$$

$$+ \cdots$$

We can note that, for small eigenvalues (all 2N) of $((\tilde{a}_{n,m})_{\nu,\nu'}) + ((\tilde{b}_{n,m})_{\nu,\nu'})$ times $z - z_0$, the second term in the series is small. Note that the similarity transformation does not change eigenvalues. Higher order series terms are then even smaller. Looking at these second term, it is the properties of $((\tilde{a}_{n,m})_{\nu,\nu'})$ and $((\tilde{b}_{n,m})_{\nu,\nu'})$ that are important.

Note that in general, for practical NMTLs, all the $(\tilde{a}_{n,m})$ and $(\tilde{b}_{n,m})$ blocks are slowly varying with frequency as compared to $(\tilde{g}_{n,m})$ and $(\tilde{G}_{n,m})$. Typically $(\tilde{g}_{n,m})$ is proportional to frequency and $(\tilde{G}_{n,m})$ is correspondingly exponential in frequency (or for $s = j\omega$ of approximately constant magnitude). Practical cases are often approximated by perfect conductos in lossless, dispersionless dielectrics for which

$$\left.\begin{aligned} (\tilde{a}_{n,m}(z,s)) &= (a_{n,m}(z)) \\ (\tilde{b}_{n,m}(z,s)) &= (b_{n,m}(z)) \end{aligned}\right\} \text{ independent of } s \tag{5.2}$$

This shows the significance of bringing out the leading term as in Section 2. For high frequencies it is dominant.

Consider first the properties of the off-diagonal blocks $(\tilde{b}_{n,m})$. These represent the coupling between the waves propagating in opposite directions ($+z$ propagating to $-z$ propagating, and conversely). Suppose one would like there to be *no such coupling*. Then we require from (2.10)

$$
\begin{aligned}
\left(\tilde{b}_{n,m}(z,s)\right) &= \left(0_{n,m}\right) \\
&= \frac{1}{2}\left[-\left(\tilde{y}_{c_{n,m}}(z,s)\right) \bullet \left[\frac{d}{dz}\left(\tilde{z}_{c_{n,m}}(z,s)\right)\right] - \left[\frac{d}{dz}\left(\tilde{z}_{c_{n,m}}(z,s)\right)\right] \bullet \left(\tilde{y}_{c_{n,m}}(z,s)\right)\right]
\end{aligned} \tag{5.3}
$$

Clearing common factors and dot multiplying by $(\tilde{z}_{c_{n,m}})$ on both sides, we find

$$
\begin{aligned}
\left(0_{n,m}\right) &= \left[\frac{d}{dz}\left(\tilde{z}_{c_{n,m}}(z,s)\right)\right] \bullet \left(\tilde{z}_{c_{n,m}}(z,s)\right) + \left(\tilde{z}_{c_{n,m}}(z,s)\right) \bullet \left[\frac{d}{dz}\left(\tilde{z}_{c_{n,m}}(z,s)\right)\right] \\
&= \frac{d}{dz}\left[\left(\tilde{z}_{c_{n,m}}(z,s)\right)^2\right] = \frac{d}{dz}\left(\tilde{z}_{c_{n,m}}(z,s)\right) \\
\left(\tilde{Z}_{c_{n,m}}(z,s)\right) &= \left(\tilde{Z}_{c_{n,m}}(s)\right) = \left(\tilde{Y}_{c_{n,m}}(s)\right)^{-1} = \text{independent of } z
\end{aligned} \tag{5.4}
$$

which should not be surprising. (This serves as a check.) In this case there are no reflections on the line and one set of modes, independent of z, describes the propagation.

Consider second the properties of the diagonal blocks $(\tilde{a}_{n,m})$. These may represent mode conversion along the NMTL for $+z$ propagating and $-z$ propagating waves separately. If we desire no such coupling from one mode to another we might like to have

$$
\left(\tilde{a}_{n,m}(z,s)\right) = \tilde{a}(z,s)\left(1_{n,m}\right) = \left(0_{n,m}\right) \tag{5.5}
$$

so that for the $+z$ propagating waves

$$
\left(\left(\tilde{G}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right)^{-1} \bullet \tilde{a}(z,s)\left(1_{n,m}\right) \bullet \left(\left(\tilde{G}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) = \tilde{a}(z,s)\left(1_{n,m}\right) \tag{5.6}
$$

and similarly for the $-z$ propagating waves. Then including the first correction, neglecting the $(\tilde{b}_{n,m})$ terms (+ - wave coupling) we have

$$
\left(\tilde{H}_{n,m}(z,z_0;s)\right)_{1,1} = \left[1 + \tilde{a}(z,s)\right]\left(1_{n,m}\right) \tag{5.7}
$$

giving only a scalar multiplier to the leading term. However, 2.10) shows that $(\tilde{a}_{n,m})$ is skew symmetric which forces

$$\tilde{a}(z,s) = 0 \quad , \quad \left(\tilde{a}_{n,m}(z,s)\right) = \left(0_{n,m}\right) \tag{5.8}$$

so that there is no first order correction to the diagonal blocks. This is turn requires

$$\begin{gathered}\left[\frac{d}{dz}\left(\tilde{z}_{c_{n,m}}(z,s)\right)\right] \cdot \left(\tilde{y}_{c_{n,m}}(z,s)\right) = \left(\tilde{y}_{c_{n,m}}(z,s)\right) \cdot \left[\frac{d}{dz}\left(\tilde{z}_{c_{n,m}}(z,s)\right)\right] \\ \left(\tilde{z}_{c_{n,m}}(z,s)\right) \cdot \left[\frac{d}{dz}\left(\tilde{z}_{c_{n,m}}(z,s)\right)\right] = \left[\frac{d}{dz}\left(\tilde{z}_{c_{n,m}}(z,s)\right)\right] \cdot \left(\tilde{z}_{c_{n,m}}(z,s)\right)\end{gathered} \tag{5.9}$$

so that $(\tilde{z}_{c_{n,m}})$ commutes with its derivative. Consider two values of z close to each other, z and $z + \Delta z$. This implies that

$$\left(\tilde{z}_{c_{n,m}}(z,s)\right) \cdot \left(\tilde{z}_{c_{n,m}}(z+\Delta z,s)\right) = \left(\tilde{z}_{c_{n,m}}(z+\Delta z,s)\right) \cdot \left(\tilde{z}_{c_{n,m}}(z,s)\right) \tag{5.10}$$

i.e., the two matrices for two close positions on the line commute with each other (to order $[\Delta z]^2$). Extending this to the entire internal z_0 to z and letting $\Delta z \to 0$ we find that $(\tilde{z}_{c_{n,m}})$ (and hence $(\tilde{Z}_{c_{n,m}})$) must commute with the same at ever pair of points in the interval. This is the same as one of the assumptions in Section 4 leading to (4.7) and (4.8). The only coupling between modes is that from + to – waves, one + wave mode coupling with exactly one – wave mode.

The case of equal modal speeds is also discussed in Section 4 for the leading term. In ths case we have

$$\begin{aligned}\left(\left(\tilde{H}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) &= \left(\left(1_{n,m}\right)_{\nu,\nu'}\right) \\ &+\int_{z_0}^{z}\begin{pmatrix}\left(a_{n,m}(z',s)\right) & \left(\tilde{b}_{n,m}(z',s)\right)e^{-2\int_{z_0}^{z'}\tilde{g}(z'',s)dz''} \\ \left(\tilde{b}_{n,m}(z',s)\right)e^{2\int_{z_0}^{z'}\tilde{g}(z'',s)dz''} & \left(a_{n,m}(z',s)\right)\end{pmatrix}\end{aligned} \tag{5.11}$$

due to the particularly simple forms that $(\tilde{G}_{n,m})$ and its inverse take from (4.9). Here we see that $(\tilde{a}_{n,m})$ gives the mode mixing in the + wave (and similarly for the – wave) to the extent that it differs from a scalar times the identity. As discussed previously this implies that it is the zero matrix (5.8) and requires commuting matrices between the various points on the interval. This is not consistent with the simple result (4.11) for the leading term which does not require such commutativity. Thus use of the first order correction significantly complicates the result. The off-diagonal terms give the reflection between + and – waves and also involve mode mixing in general, except in the commutativity case, an extra complication.

6. The case of N = 1

The special case of a single conductor plus reference simplifies the foregoing formulae considerably. In this case we have for the leading term

$$\begin{aligned}
\left(\tilde{G}_{\nu,\nu'}(z,z_0;s)\right) &= \begin{pmatrix} \tilde{G}(z,z_0;s) & 0 \\ 0 & \tilde{G}^{-1}(z,z_0;s) \end{pmatrix} \\
\tilde{G}(z,z_0;s) &= \prod_{z_0}^{z} e^{\tilde{g}(z',s)dz'} = e^{\int_{z_0}^{z} \tilde{g}(z',s)dz'} \\
\tilde{g}(z,s) &= -\tilde{\gamma}(z,s) = -\left[\tilde{Z}'(z,s)\tilde{Y}'(z,s)\right]^{1/2} \\
\tilde{Z}_c(z,s) &= \tilde{Y}_c^{-1}(z,s) = \left[\tilde{Z}'(z,s)\tilde{Y}'^{-1}(z,s)\right]^{1/2} \\
\tilde{z}_c(z,s) &= \tilde{y}_c^{-1}(z,s) = \tilde{Z}_c^{1/2}(z,s) \\
\tilde{\mathrm{v}}^{(0)}(z,s) &= \tilde{G}(z,z_0;s)\ \tilde{\mathrm{v}}(z_0,s) \\
\tilde{V}^{(0)}(z,s) &= \tilde{G}(z,z_0;s)\,\tilde{z}_c(z,s)\,\tilde{y}_c(z_0,s)\,\tilde{V}(z_0,s) \\
&\qquad \text{(without distributed sources)}
\end{aligned} \tag{6.1}$$

The correction term simplifies to

$$\begin{aligned}
\tilde{a}(z,s) &= 0 \\
\tilde{b}(z,s) &= -\tilde{y}_c(z,s)\frac{d\tilde{z}_c(z,s)}{dz} = -\frac{d\ell n\left(\tilde{z}_c(z,s)\right)}{dz} \\
\left(\tilde{H}_{\nu,\nu'}(z,z_0;s)\right) &= \left(1_{\nu,\nu'}\right) \\
&+ \int_{z_0}^{z} \begin{pmatrix} 0 & \tilde{b}(z',s)\tilde{G}^{-2}(z',z_0;s) \\ \tilde{b}(z',s)\tilde{G}^{2}(z',z_0;s) & 0 \end{pmatrix} dz' + \cdots \\
&= \left(1_{\nu,\nu'}\right) \\
&+ \begin{pmatrix} 0 & -\displaystyle\int_{z_0}^{z} \frac{d\ell n\left(\tilde{z}_c(z',s)\right)}{dz} e^{2\int_{z_0}^{z'} \tilde{\gamma}(z'',s)dz''} dz' \\ -\displaystyle\int_{z_0}^{z} \frac{d\ell n\left(\tilde{z}_c(z',s)\right)}{dz} e^{-2\int_{z_0}^{z'} \tilde{\gamma}(z'',s)dz''} dz' & 0 \end{pmatrix} \\
&+ \cdots
\end{aligned} \tag{6.2}$$

This has no diagonal terms to modify $\tilde{V}^{(0)}$. The off-diagonal reflection terms give some conversion of the + wave into a – wave (and conversely), small at high frequencies. Given some specified forms for $\tilde{z}_c$ and $\tilde{\gamma}$. the integrals can be evaluated. For $s = j\omega$ the off-diagonal terms involve an integral over a slowly varying $d[\ell n(z_c)]/dz$ times an exponential of a constant times $j\omega$ (lossless case). As $\omega \to \infty$ the resulting integral tends to zero making the first correction term go to zero.

7. Concluding Remarks

Now we have formulae appropriate for high-frequency propagation on NMTLs. To how low in frequency the leading term extends is more complicated. One can estimate the error in the leading term from the correction term provided it deviates from the identity by a relatively small amount.

What we have here is an extension of the WKB approximation to N-conductor (plus reference) transmission-line systems. This generalizes some previous results [7]. I would like to thank E. Heyman and J. Nitsch for discussions concerning this paper.

Appendix A. Some Properties of the Product Integral

Various properties of the product integral, including supermatrices, are tabulated in [4 (appendices)]. Some additional properties are considered here. We begin with the definition as the limit of a product

$$\Delta z = \frac{z - z_0}{\mathrm{L}} \quad , \quad z_p = z_0 + p\Delta z$$

$$\begin{aligned} \prod_{z_0}^{z} e^{\left(h_{n,m}(z')\right)dz'} &= \prod_{z_0}^{z} \left[\left(1_{n,m}\right) + \left(h_{n,m}(z')\right)dz'\right] \\ &= \lim_{\substack{L\to\infty \\ (\Delta z\to 0)}} e^{\left(h_{n,m}(z_{\mathrm{L}})\right)\Delta z} \bullet \cdots \bullet e^{\left(h_{n,m}(z_2)\right)\Delta z'} \bullet e^{\left(h_{n,m}(z_1)\right)\Delta z} \\ &= \lim_{L\to\infty} \mathop{\odot}_{p=1}^{\mathrm{L}} e^{\left(h_{n,m}(z_p)\right)\Delta z} \end{aligned} \tag{A.1}$$

with continued dot multiplication taken to the left.

A first simple rule concerns complex conjugation

$$\left[\prod_{z_0}^{z} e^{\left(h_{n,m}(z')\right)dz'}\right]^* = \prod_{z_0}^{z} e^{\left(h_{n,m}(z')\right)^* dz'} \tag{A.2}$$

since the conjugate of a product is the product of the conjugates.

Consider next the transpose

$$\begin{aligned} \left[\prod_{z_0}^{z} e^{\left(h_{n,m}(z')\right)dz'}\right]^T &= \left[\prod_{z_0}^{z} \left[\left(1_{n,m}\right) + \left(h_{n,m}(z')\right)dz'\right]\right]^T \\ &= \lim_{\mathrm{L}\to\infty} \left[\left[\left(1_{n,m}\right) + \left(h_{n,m}(z_{\mathrm{L}})\right)\Delta z\right] \bullet \cdots \bullet \left[\left(1_{n,m}\right) + \left(h_{n,m}(z_1)\right)\Delta z\right]\right]^T \\ &= \lim_{\mathrm{L}\to\infty} \left[\left(1_{n,m}\right) + \left(h_{n,m}(z_1)\right)^T \Delta z\right] \bullet \cdots \bullet \left[\left(1_{n,m}\right) + \left(h_{n,m}(z_{\mathrm{L}})\right)^T \Delta z\right] \\ &= \prod_{z}^{z_0} e^{\left(h_{n,m}(z')\right)^T [-dz']} = \prod_{z}^{z_0} e^{-\left(h_{n,m}(z')\right)^T dz'} \\ &= \left[\prod_{z_0}^{z} e^{-\left(h_{n,m}(z')\right)^T dz'}\right]^{-1} \end{aligned} \tag{A.3}$$

In another form we now have, replacing $(h_{n,m})$ by $(h_{n,m})^T$,

$$\begin{aligned}\prod_{z_0}^{z} e^{-\left(h_{n,m}(z')\right)^T dz'} &= \left[\prod_{z_0}^{z} e^{\left(h_{n,m}(z')\right)^T dz'}\right]^{T^{-1}} \\ &= \left[\prod_{z_0}^{z} e^{\left(h_{n,m}(z')\right)^T dz'}\right]^{-1T} \\ &= \left[\prod_{z}^{z_0} e^{\left(h_{n,m}(z')\right)^T dz'}\right]^{T}\end{aligned} \tag{A.4}$$

This leads to special cases.

1. Symmetric

$$\left(h_{n,m}(z')\right) = \left(h_{n,m}(z')\right)^T$$

$$\begin{aligned}\prod_{z_0}^{z} e^{\left(h_{n,m}(z')\right)dz'} &= \left[\prod_{z_0}^{z} e^{-\left(h_{n,m}(z')\right)dz'}\right]^{-1T} \\ &= \left[\prod_{z}^{z_0} e^{-\left(h_{n,m}(z')\right)dz'}\right]^{T}\end{aligned} \tag{A.6}$$

2. Skew symmetric

$$\left(h_{n,m}(z')\right) = -\left(h_{n,m}(z')\right)^T$$

$$\begin{aligned}\prod_{z_0}^{z} e^{\left(h_{n,m}(z')\right)dz'} &= \left[\prod_{z_0}^{z} e^{-\left(h_{n,m}(z')\right)dz'}\right]^{-1T} \\ &= \left[\prod_{z}^{z_0} e^{\left(h_{n,m}(z')\right)dz'}\right]^{T}\end{aligned} \tag{A.6}$$

(Transpose equals inverse.)

3. Hermitian

$$\left(h_{n,m}(z')\right) = \left(h_{n,m}(z')\right)^{\dagger}, \quad \dagger = \mathrm{T}* \equiv \text{adjoint} \tag{A.7}$$

Replace T by † in (A.5)

4. Anti Hermitian

$$\left(h_{n,m}(z')\right) = -\left(h_{n,m}(z')\right)^{\dagger} \tag{A.8}$$

Replace T by † in (A.6)

In [4 (Appendix B)] product integrals of direct-product matrices are considered. For completeness we note the simple rule for direct sums

$$\left(\left(h_{n,m}(z')\right)_{\nu,\nu'}\right) = \left(h_{n,m}^{(1)}(z')\right)\oplus\cdots\oplus\left(h_{n,m}^{(M)}(z')\right)$$

$$= \begin{pmatrix} \left(h_{n,m}^{(1)}(z')\right) & & \bigcirc \\ & \ddots & \\ \bigcirc & & \left(h_{n,m}^{(M)}(z')\right) \end{pmatrix}$$

(M square blocks on the diagonal) (A.9)

$$\prod_{z_0}^{z} e^{\left(\left(h_{n,m}(z')\right)_{\nu,\nu'}\right)dz'}$$

$$= \left[\prod_{z_0}^{z} e^{-\left(h_{n,m}^{(1)}(z')\right)dz'}\right]\oplus\cdots\oplus\left[\prod_{z_0}^{z} e^{-\left(h_{n,m}^{(M)}(z')\right)dz'}\right]$$

References

1. C. E. Baum, T. K. Liu, and F. M. Tesche, "On The Analysis of General Multiconductor Transmission-Line Networks", Interaction Note 350, November 1978; also in, C. E. Baum, "Electromagnetic Topology for the Analysis and Design of Complex Electromagnetic Systems", pp. 467-547, in J. E. Thompson and L. H. Luessen, *Fast Electrical and Optical Measurements*, Martiuus Nijhoff, Dordrecht, 1986.

2. C. E. Baum, "High-Frequency Propagation on Nonuniform Multiconductor Transmission Lines in Uniform Media", Interaction Note 463, March 1988; Int'l J. Numerical Modelling: Electronic Networks, Devices, and Fields, 1988, pp. 175-188.

3. J. Nitsch, C. E. Baum, and R. Sturm, "The Treatment of Commuting Nonuniform Tubes in Multiconductor-Transmission-Line Theory", Interaction Note 481, May 1990; "Analytical Treatment of Circulant Nonuniform Multiconductor Transmission Lines", IEEE Trans. EMC, 1992, pp. 28-38.

4. C. E. Baum, "Symmetric Renormalization of the Nonuniform Multiconductor-Transmission-Line Equations with a Single Modal Speed for Analytically Soluable Sections", Interaction Note 537, January 1998.

5. C. E. Baum and T. Steinmetz, "An Interpolation Technique for Analyzing Sections of Nonuniform Multiconductor Transmission Lines", Interaction Note 571, July 2001; 13th Zurich EMC Symposium, 2003, pp. 593-596.

6. C. E. Baum and T. Steinmetz, "An Interpolation Technique for Analyzing Sections of Nonuniform Multconductor Transmission Lines, Part 2", Interaction Note 578, May 2002.

7. E. Heyman, private communication, course notes, 2003.

8. A Maffucci and G. Miano, "Analysis of Nonuniform Transmission Lines by Means of the WKB Method", Proc. EMC Symposium, Zurich, 2001, pp. 241-246.

9. F. R. Gantmacher, *The Theory of Matrices*, Chelsea Publishing Company, 1959.

10. J. D. Dollard and C. N. Friedman, *Product Integration*, Addison-Wesley, 1979.

11. C. E. Baum, J. B. Nitshe, and R. J. Sturm, "Analytical Solution for Uniform and Nonuniform Multiconductor Transmission Lines with Sources", pp. 433-464, in W. R. Stone (ed.), *Review of Radio Science 1993-1996*, Oxford U. Press, 1996.