Interaction Notes

Note 594

19 November 2004

The Forward-Scattering Theorem Applied to the Scattering Dyadic

Carl E. Baum
Air Force Research Laboratory
Directed Energy Directorate

Abstract

From the forward-scattering theorem we have relations between the absorption and scattering cross sections, and the forward scattering. The scattered fields are represented by a scattering dyadic times the incident plane wave. This allows one to reformulate the results in terms of the scattering dyadic, exhibiting some general characteristics of this dyadic.

This work was sponsored in part by the Air Force Office of Scientific Research, and in part by the Air Force Research Laboratory, Directed Energy Directorate.

1. Introduction

In [4] the forward-scattering theorem was considered in both time and complex-frequency domains. This relates the power absorbed and scattered by an object to that scattered in the forward direction. This is a classical problem; the references in [4] can be consulted in this regard. Note that various authors also call this the cross-section theorem or optical theorem.

Figure 1.1 shows a general scatterer in free space. The incident field is taken as a plane wave of the form

$$\tilde{\vec{E}}^{(inc)}(\vec{r},\vec{1}_i,\vec{1}_p;s) = E_0 \tilde{f}(s)\ \tilde{\vec{e}}^{(inc)}(\vec{r},\vec{1}_i,\vec{1}_p;s)$$

$$\tilde{\vec{e}}^{(inc)}(\vec{r},\vec{1}_i,\vec{1}_p;s) = \vec{1}_p e^{-\gamma \vec{1}_i \cdot \vec{r}}$$

$$\tilde{\vec{H}}^{(inc)}(\vec{r},\vec{1}_i,\vec{1}_p;s) = \frac{E_0}{Z_0}\tilde{f}(s)\ \vec{1}_i \times \tilde{\vec{e}}^{(inc)}(\vec{r},\vec{1}_i,\vec{1}_p;s) = \frac{E_0}{Z_0}\tilde{f}(s)\ \tilde{\vec{h}}^{(inc)}(\vec{r},\vec{1}_i,\vec{1}_p;s)$$

$\vec{1}_i \equiv$ direction of incidence

$\vec{1}_p \equiv$ polarization (assumed linear (real valued))

$\vec{1}_i \cdot \vec{1}_p = 0$ (perpendicular)

$\overleftrightarrow{1}_z = \overleftrightarrow{1} = \vec{1}_i\,\vec{1}_i \equiv$ transverse incidence dyadic

$$\overleftrightarrow{1} \equiv \vec{1}_x\vec{1}_x + \vec{1}_y\vec{1}_y + \vec{1}_z\vec{1}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \text{identity dyadic}$$

$\gamma = \frac{s}{c} \equiv$ propagation constant

$s = \Omega + j\omega =$ Laplace-transform variable or complex frequency

$c = [\mu_0 \varepsilon_0]^{1/2} \equiv$ light speed

$$Z_0 = \left[\frac{\mu_0}{\varepsilon_0}\right]^{1/2} \equiv \text{wave impedance} \tag{1.1}$$

$\tilde{\vec{e}}^{(inc)}(\vec{r},\vec{1}_i,\vec{1}_p;s) \equiv$ normalized incident field (dimensionless)

The far scattered fields (as $r \to \infty$ in direction $\vec{1}_r$) take the form

$$\tilde{\vec{E}}_f(\vec{r},\vec{1}_i,\vec{1}_p;s) = \frac{E_0 \tilde{f}(s)}{r} e^{-\gamma r}\ \tilde{\vec{v}}_f(\vec{1}_r,\vec{1}_i,\vec{1}_p;s)$$

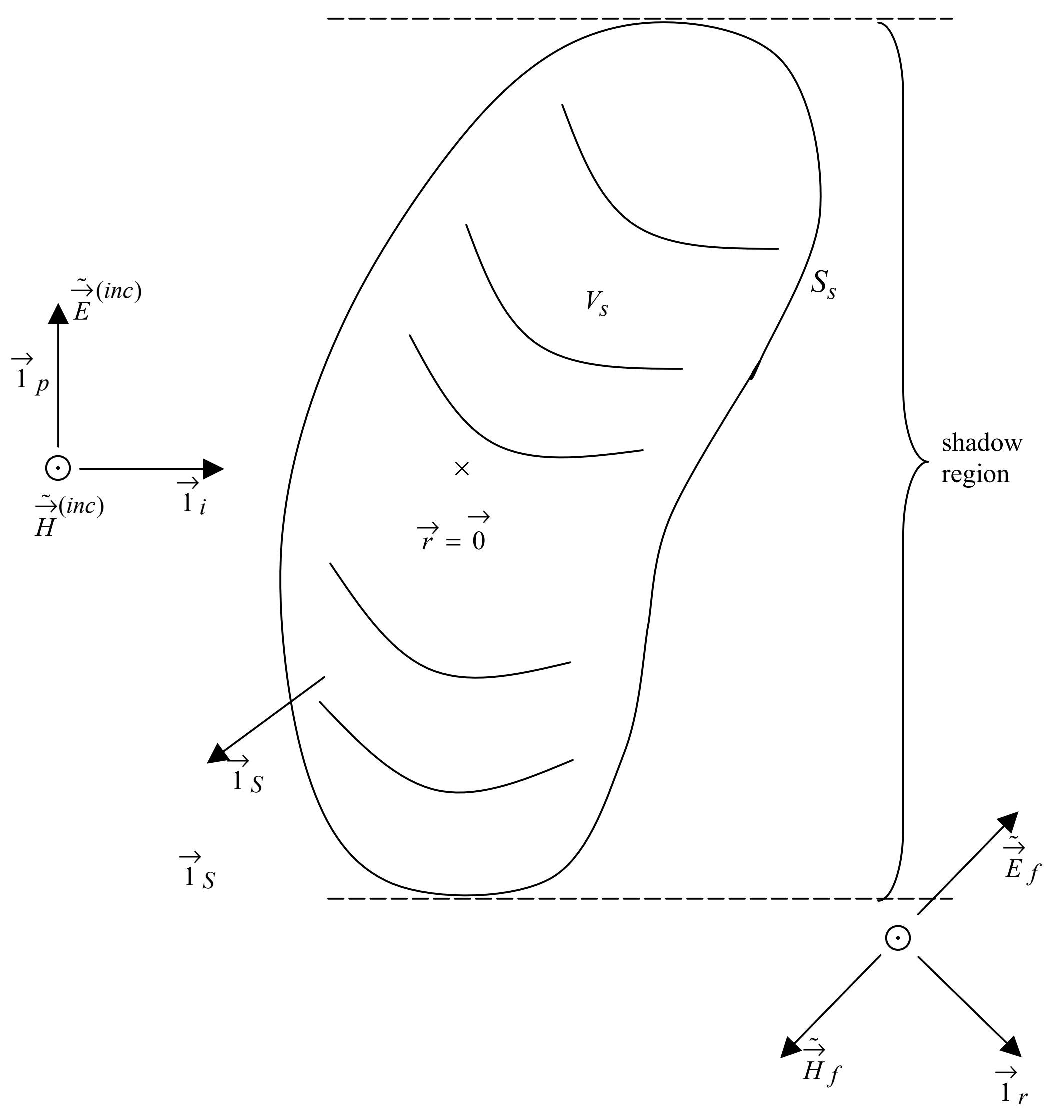

Figure 1.1. Scattering of Incident Plane Wave.

$$\tilde{\vec{v}}_f(\vec{1}_r, \vec{1}_i, \vec{1}_p; s) \equiv \text{normalized far field (dimension meters)}$$

$$\vec{1}_r \bullet \tilde{\vec{v}}_f(\vec{1}_r, \vec{1}_i, \vec{1}_p; s) = 0$$

$$\overleftrightarrow{1}_r \equiv \overleftrightarrow{1} - \vec{1}_r \vec{1}_r \equiv \text{transverse scattering dyadic}$$

$$\tilde{\vec{H}}_f(\vec{1}_r, \vec{1}_i, \vec{1}_p; s) = \frac{E_0 \tilde{f}(s)}{Z_0 r} e^{-\gamma r} \vec{1}_r \times \tilde{\vec{v}}_f(\vec{1}_r, \vec{1}_i, \vec{1}_p; s) \tag{1.2}$$

The scattering is described by a scattering dyadic as [10]

$$\tilde{\vec{E}}_f(\vec{r}, \vec{1}_i, \vec{1}_p; s) = \frac{e^{-\gamma r}}{4\pi r} \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_r, \vec{1}_i; s) \bullet \tilde{\vec{E}}^{(inc)}(\vec{0}, \vec{1}_i, \vec{1}_p; s)$$

$$\begin{aligned} \tilde{\vec{v}}_f(\vec{1}_r, \vec{1}_i, \vec{1}_p; s) &= \frac{1}{4\pi} \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_r, \vec{1}_i; s) \bullet \tilde{\vec{e}}^{(inc)}(\vec{0}, \vec{1}_i, \vec{1}_p; s) \\ &= \frac{1}{4\pi} \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_r, \vec{1}_i; s) \bullet \vec{1}_p \end{aligned}$$

$$\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_r, \vec{1}_i; s) \equiv \text{scattering dyadic (dimension meters)}$$

$$\vec{1}_r \bullet \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_r, \vec{1}_i; s) = \vec{0} = \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_r, \vec{1}_i; s) \bullet \vec{1}_p \;\; \text{(transverse property)} \tag{1.3}$$

$$\tilde{\overleftrightarrow{\Lambda}}^T(\vec{1}_r, \vec{1}_i; s) = \tilde{\overleftrightarrow{\Lambda}}(-\vec{1}_i, -\vec{1}_r; s) \;\; \text{(reciprocity)}$$

For convenience lat us also introduce another form as

$$\tilde{\overleftrightarrow{\Sigma}}(\vec{1}_r, \vec{1}_i; s) = \frac{1}{4\pi} \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_r, \vec{1}_i; s) \tag{1.4}$$

which will simplify some of the later formulae. We can refer to the two forms interchangeably as the scattering dyadic (or scattering matrix). Note that $\vec{r} = \vec{0}$ has been taken as some convenient location in the vicinity of the scatterer (say the center of the minimum circumscribing sphere).

Here we have used the radar convention in which the scattering dyadic is related to an incident plane wave. This is also sometimes considered as a scattering amplitude (as in amplitude or magnitude, and phase, not very appropriate to a dyadic- or matrix-valued function). Some authors (e.g., [8]) instead use this term (scattering matrix) to refer the scattering to some kind of incoming wave scattering into an outgoing wave. For present purposes, let us call this form a reversal matrix (or dyadic or operator) in which form we can also state the results of this paper.

2. Forward-Scattering Theorem

Following [4] we have various cross sections. The “mixed” term gives a term involving forward scattering as

$$
\begin{aligned}
&\frac{1}{2}\left[\tilde{A}_\infty^{(mix)}(\vec{1}_i,\vec{1}_p;s) + \tilde{A}_\infty^{(mix)}(\vec{1}_i,\vec{1}_p;-s)\right] \\
&\quad = \frac{2\pi}{\gamma}\left[\vec{1}_p \cdot \vec{\mathrm{v}}_f(\vec{1}_i,\vec{1}_i,\vec{1}_p;s) - \vec{1}_p \cdot \vec{\mathrm{v}}_f(\vec{1}_i,\vec{1}_i,\vec{1}_p;-s)\right] \qquad (2.1)\\
&\quad \equiv -\tilde{A}_f(\vec{1}_i,\vec{1}_p;s)\\
&\quad = \text{even function of } s
\end{aligned}
$$

While this is derived by taking an asymptotic form for $r \to \infty$ of $\tilde{A}_\infty^{(mix)}$, the combination above avoids the divergence as $r \to \infty$. This is also the analytic continuation out into the s plane from the $j\omega$ axis. The scattered far field gives

$$
\begin{aligned}
\tilde{A}_\infty^{(sc)}(\vec{1}_i,\vec{1}_p;s) &= \int_{S_1} \tilde{\vec{\mathrm{v}}}_f(\vec{1}_r,\vec{1}_i;-s) \cdot \tilde{\vec{\mathrm{v}}}_f(\vec{1}_r,\vec{1}_i,\vec{1}_p;s)\,dS\\
&= \text{scattering cross section}\\
&= \text{even function of } s
\end{aligned}
$$

$$S_1 = \text{surface of unit sphere (integration over all } \vec{1}_r \text{ (real))} \qquad (2.2)$$

$$\tilde{A}_\infty^{(sc)}(\vec{1}_i,\vec{1}_p;j\omega) \geq 0$$

The absorption in the scatterer is

$$
\begin{aligned}
\tilde{A}_{ab}(\vec{1}_i,\vec{1}_p;s) &= \frac{1}{2}\left[\tilde{A}_s(\vec{1}_i,\vec{1}_p;s) + \tilde{A}_s(\vec{1}_i,\vec{1}_p;-s)\right]\\
&= \text{even function of } s\\
&= \text{absorption cross section} \qquad (2.3)
\end{aligned}
$$

$$\tilde{A}_{ab}(\vec{1}_i,\vec{1}_p;j\omega) \geq 0$$

$$\tilde{A}_s(\vec{1}_i,\vec{1}_p;s) = \int_{S_s}\left[\tilde{\vec{e}}(\vec{r},\vec{1}_i,\vec{1}_p;s) \times \tilde{\vec{h}}(\vec{r},\vec{1}_i,\vec{1}_p;-s)\right] \cdot \vec{1}_S\,dS$$

Where the total fields, normalized as before to give dimensionless parameters, are used here. For a lossless scatterer this is zero for $s = j\omega$ and by analytic continuation throughout the s plane.

The forward scattering theorem then states

$$\tilde{A}_{ab}(\overrightarrow{1}_i, \overrightarrow{1}_p; s) + \tilde{A}_{\infty}^{(sc)}(\overrightarrow{1}_i, \overrightarrow{1}_p; s) = \tilde{A}_f(\overrightarrow{1}_i, \overrightarrow{1}_p; s) \tag{2.4}$$

from which we conclude

$$\tilde{A}_f(\overrightarrow{1}_i, \overrightarrow{1}_p; j\omega) \geq 0 \tag{2.5}$$

The left side of (2.4) is often called the *extinction* cross section.

3. High-Frequency Forward Scattering

Referring back to Fig. 1.1, one can define a shadow area or cross-section area as

$$A_c(\overrightarrow{1}_i) = A_c(-\overrightarrow{1}_i) = \text{shadow region projected on a plane } \perp \overrightarrow{1}_i \tag{3.1}$$

Then following [9], we can evaluate the forward-scattering cross section at high frequencies for an "opaque" scatterer (a perfectly conducting scatterer being a special case).

For this purpose we can use some previous results derived for impulse-radiating antennas (IRAs) [1, 2]. In this case a uniform electric field on a planar aperture is taken as the negative of the incident electric field giving

$$\begin{aligned}
\tilde{\overrightarrow{E}}_f(\overrightarrow{r}, \overrightarrow{1}_i, \overrightarrow{1}_p; s) &= -\frac{sE_0\tilde{f}(s)}{2\pi c r} e^{-\gamma r} A_c(\overrightarrow{1}_i)\overrightarrow{1}_p \\
\tilde{\overrightarrow{\mathrm{v}}}_f(\overrightarrow{1}_i, \overrightarrow{1}_i, \overrightarrow{1}_p; s) &= -\frac{sA_c(\overrightarrow{1}_i)}{2\pi c}\overrightarrow{1}_p = -\frac{\gamma}{2\pi}A_c(\overrightarrow{1}_i)\overrightarrow{1}_p \\
&= \frac{1}{4\pi}\tilde{\overleftrightarrow{\Lambda}}(\overrightarrow{1}_i, \overrightarrow{1}_i; s) \cdot \overrightarrow{1}_p \\
\overleftrightarrow{\Lambda}(\overrightarrow{1}_i, \overrightarrow{1}_i; s) &= -2\gamma A_c(\overrightarrow{1}_i)\overleftrightarrow{1}_i \\
&\text{as } \mathrm{Re}(s) \to +\infty
\end{aligned} \tag{3.2}$$

This is slightly more general in that causality allows this to be applied to dielectric scatterers which are not opaque, but merely delay the wave propagating through them. For $s = j\omega$ we still need an opaque scatterer.

From (2.1) we then have the forward-scattering cross section as

$$A_f(\overrightarrow{1}_i, \overrightarrow{1}_p; j\omega) = 2A_c(\overrightarrow{1}_i) \text{ as } \omega \to \infty \tag{3.3}$$

This being nonnegative as required by (2.5). Note the limitation to the $j\omega$ axis here due to the functions of s and $-s$ in (2.1).

4. Low-Frequency Scattering

From [6] we have the low-frequency behavior of the scattering dyadic. This is dominated by the induced electric and magnetic dipoles giving

$$\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_r, \vec{1}_i; s) = \gamma^2 \left[-\overleftrightarrow{1}_r \cdot \tilde{\overleftrightarrow{P}}(s) \cdot \vec{1}_i + \vec{1}_r \times \tilde{\overleftrightarrow{M}}(s) \times \vec{1}_i \right] \text{ as } s \to 0$$

$$\tilde{\overleftrightarrow{P}}(s) = \tilde{\overleftrightarrow{P}}^T(s) \equiv \text{electric polarizability dyadic}$$

$$\tilde{\overleftrightarrow{M}}(s) = \tilde{\overleftrightarrow{M}}^T(s) \equiv \text{magnetic polarizability dyadic} \tag{4.1}$$

In general we have

$$\tilde{\overleftrightarrow{P}}(0) \neq \overleftrightarrow{0} \quad \text{for conducting and dielectric targets}$$

$$\tilde{\overleftrightarrow{M}}(0) \begin{cases} \neq \overleftrightarrow{0} & \text{for perfectly conducting targets} \\ \neq \overleftrightarrow{0} & \text{for permeable targets} \\ = \overleftrightarrow{0} & \text{for dielectric and imperfectly conducting targets} \end{cases} \tag{4.2}$$

We can see from (4.1) that, at least for low frequencies, the scattering dyadic is an even function of s going to zero at $s = 0$.

5. Application of the Forward Scattering Theorem to the Scattering Dyadic

Now we can write the various cross sections in terms of the scattering dyadic. The forward scattering takes the form

$$\begin{aligned}\tilde{A}_f(\vec{1}_i,\vec{1}_p;s) &= \frac{2\pi}{\gamma}\vec{1}_p\cdot\left[\tilde{\overleftrightarrow{\Sigma}}(\vec{1}_i,\vec{1}_i;-s) - \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_i,\vec{1}_i;s)\right]\cdot\vec{1}_p \\ &= \frac{2\pi}{\gamma}\vec{1}_p\cdot\left[\tilde{\overleftrightarrow{\Sigma}}^T(\vec{1}_i,\vec{1}_i;-s) - \tilde{\overleftrightarrow{\Sigma}}^T(\vec{1}_i,\vec{1}_i;s)\right]\cdot\vec{1}_p\end{aligned} \tag{5.1}$$

This has various alternate forms from the transpose, such as

$$\begin{aligned}\tilde{A}_f(\vec{1}_i,\vec{1}_p;s) &= \frac{2\pi}{\gamma}\vec{1}_p\cdot\left[\tilde{\overleftrightarrow{\Sigma}}(-\vec{1}_i,-\vec{1}_i;s) - \tilde{\overleftrightarrow{\Sigma}}^T(-\vec{1}_i,-\vec{1}_i;-s)\right]\cdot\vec{1}_p \\ &= \tilde{A}_f(-\vec{1}_i,\vec{1}_p;s) \\ &= \text{even function of } s \\ &\geq 0 \text{ for } s = j\omega\end{aligned} \tag{5.2}$$

showing symmetry on reversal of the direction of incidence. This can also be stated from general symmetry conditions in [10] as

$$\tilde{\overleftrightarrow{\Sigma}}(-\vec{1}_i,-\vec{1}_i;s) = \tilde{\overleftrightarrow{\Sigma}}^T(\vec{1}_i,\vec{1}_i;-s) \tag{5.3}$$

The far scattering gives

$$\begin{aligned}\tilde{A}_\infty^{(sc)}(\vec{1}_i,\vec{1}_p;s) &= \vec{1}_p\cdot\tilde{\overleftrightarrow{\Xi}}(\vec{1}_i,s)\cdot\vec{1}_p \\ \tilde{\overleftrightarrow{\Xi}}(\vec{1}_i,s) &= \int_{S_1}\tilde{\overleftrightarrow{\Sigma}}^T(\vec{1}_r,\vec{1}_i;-s)\cdot\tilde{\overleftrightarrow{\Sigma}}(\vec{1}_r,\vec{1}_i;s)\,dS \\ &= \int_{S_1}\tilde{\overleftrightarrow{\Sigma}}(-\vec{1}_i,-\vec{1}_r;-s)\cdot\tilde{\overleftrightarrow{\Sigma}}(\vec{1}_r,\vec{1}_i;s)\,dS\end{aligned}$$

$$\tilde{\overleftrightarrow{\Xi}}^T(\overrightarrow{1}_i,s) = \int_{S_1} \tilde{\overleftrightarrow{\Sigma}}(-\overrightarrow{1}_i,-\overrightarrow{1}_r;s) \cdot \tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_r,\overrightarrow{1}_i;-s)\, dS$$

$$= \tilde{\overleftrightarrow{\Xi}}(\overrightarrow{1}_i,-s) \tag{5.4}$$

$$\tilde{\overleftrightarrow{\Xi}}^\dagger(\overrightarrow{1}_i,j\omega) = \tilde{\overleftrightarrow{\Xi}}(\overrightarrow{1}_i,j\omega) \quad \text{(Hermitian)}$$

$$\dagger \equiv T\,* \equiv \text{adjoint}$$

The above has various alternate forms. Recall that

$$\tilde{A}_\infty^{(sc)}(\overrightarrow{1}_i,\overrightarrow{1}_p;s) = \text{even function of s}$$

$$\geq 0 \text{ for } s = j\omega \tag{5.5}$$

If the scatterer is lossless, then (2.4) gives

$$\tilde{A}_\infty^{(sc)}(\overrightarrow{1}_i,\overrightarrow{1}_p;s) = \tilde{A}_f(\overrightarrow{1}_i,\overrightarrow{1}_p;s)$$

$$\overrightarrow{1}_p \cdot \tilde{\overleftrightarrow{\Xi}}(\overrightarrow{1}_i,s) \cdot \overrightarrow{1}_p = \frac{2\pi}{\gamma}\overrightarrow{1}_p \cdot \left[\tilde{\overleftrightarrow{\Sigma}}^T(\overrightarrow{1}_i,\overrightarrow{1}_i;-s) - \tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_i,\overrightarrow{1}_i;s)\right] \cdot \overrightarrow{1}_p \tag{5.6}$$

$$= \frac{1}{2\gamma}\overrightarrow{1}_p \cdot \left[\tilde{\overleftrightarrow{\Lambda}}^T(\overrightarrow{1}_i,\overrightarrow{1}_i;-s) - \tilde{\overleftrightarrow{\Lambda}}(\overrightarrow{1}_i,\overrightarrow{1}_i;s)\right] \cdot \overrightarrow{1}_p$$

$$= \text{even function of } s$$

For $s = j\omega$ a lossless scatterer has

$$\overrightarrow{1}_p \cdot \tilde{\overleftrightarrow{\Xi}}(\overrightarrow{1}_i,j\omega) \cdot \overrightarrow{1}_p = -\frac{jc}{2\omega}\overrightarrow{1}_p \cdot \left[\tilde{\overleftrightarrow{\Lambda}}^T(\overrightarrow{1}_i,\overrightarrow{1}_i;-j\omega) - \tilde{\overleftrightarrow{\Lambda}}(\overrightarrow{1}_i,\overrightarrow{1}_i;j\omega)\right] \cdot \overrightarrow{1}_p$$

$$= -\frac{c}{\omega}\overrightarrow{1}_p \cdot \text{Im}\left(\tilde{\overleftrightarrow{\Lambda}}(\overrightarrow{1}_i,\overrightarrow{1}_i;j\omega)\right) \cdot \overrightarrow{1}_p \tag{5.7}$$

$$\geq 0$$

Letting q = 1, 2 denote the diagonal elements (transverse to $\overrightarrow{1}_i$), we then have

$$\tilde{\Xi}_{q,q}(\overrightarrow{1}_i,j\omega) = -\frac{c}{\omega}\,\text{Im}\left(\tilde{\Lambda}_{q,q}(\overrightarrow{1}_i,\overrightarrow{1}_i;j\omega)\right) \geq 0 \tag{5.8}$$

For positive ω then the imaginary part of the diagonal elements of the forward scattering dyadic are negative. The off-diagonal elements are treated later.

Going to high frequencies in the RHP then (3.2) also implies for a lossless opaque scatterer

$$\tilde{\Xi}_{q,q}(\vec{1}_i, j\omega) = 2 A_c(\vec{1}_i) \text{ as } \omega \to \infty \tag{5.9}$$

The off-diagonal elements are treated later.

6. Extension of the Forward-Scattering Theorem

Following the procedure in [8] the scattering is cast into incoming and outgoing waves and takes the form

$$
\begin{aligned}
&\tilde{\vec{E}}'_f(\vec{r},\vec{1}_i,\vec{1}_p;s) = \frac{2\pi}{\gamma}\,\delta(\vec{1}_i+\vec{1}_r)\,\vec{1}_p\,\frac{e^{\gamma r}}{r}\ \text{(incoming wave)}\\
&+\left[\tilde{\overleftrightarrow{\Sigma}}(\vec{1}_r,\vec{1}_i;s)\cdot\vec{1}_p-\frac{2\pi}{\gamma}\,\delta(\vec{1}_i-\vec{1}_r)\,\vec{1}_p\right]\frac{e^{-\gamma r}}{r}\ \text{(outgoing wave)}
\end{aligned}
\tag{6.1}
$$

While this is not intuitively obvious, this is found by extension of the procedure in [7] in which an incident plane wave is expanded in terms of spherical wave functions and the appropriate limit is taken as $r\to\infty$. This gives a special interpretation to $\tilde{\vec{E}}'_f$. (Such expansions are also given in [3 (Appendix B)].) As we shall see this gives results consistent with the foregoing.

The outgoing wave can be related to the incoming wave by a dyadic as

$$
\begin{aligned}
&\tilde{\overleftrightarrow{\Sigma}}(\vec{1}_r,\vec{1}_i;s)-\frac{2\pi}{\gamma}\delta(\vec{1}_i-\vec{1}_r)\,\vec{1}_i = -\tilde{\overleftrightarrow{R}}(\vec{1}_r,\vec{1}_i;s)\frac{2\pi}{\gamma}\\
&\vec{1}_r\cdot\tilde{\overleftrightarrow{R}}(\vec{1}_r,\vec{1}_i;s) = \vec{0} = \tilde{\overleftrightarrow{R}}(\vec{1}_r,\vec{1}_i;s)\cdot\vec{1}_i\\
&\overleftrightarrow{R}^T(\vec{1}_r,\vec{1}_i;s) = \overleftrightarrow{R}(-\vec{1}_i,-\vec{1}_r;s)\ \text{(reciprocity)}\\
&\tilde{\overleftrightarrow{R}}(\vec{1}_r,\vec{1}_i;s) \equiv \text{wave reversal dyadic}
\end{aligned}
\tag{6.2}
$$

Where $\delta(\vec{1}_r+\vec{1}_i)$ has been absorbed into the definition of $\tilde{\overleftrightarrow{R}}$ (by integration over S_1). We then have

$$
\tilde{\overleftrightarrow{R}}(\vec{1}_r,\vec{1}_i;s) = \vec{1}_i\,\delta(\vec{1}_i-\vec{1}_r)-\frac{\gamma}{2\pi}\tilde{\overleftrightarrow{\Sigma}}(\vec{1}_r,\vec{1}_i;s)
\tag{6.3}
$$

As in [8] the sign is chosen so that $\tilde{\overleftrightarrow{R}}$ reduces to an identity when there is no scattering.

Next we form

$$
\begin{aligned}
&\int_{S_1} \tilde{\overleftrightarrow{R}}^T(\vec{1}_r, \vec{1}_i; -s) \cdot \tilde{\overleftrightarrow{R}}(\vec{1}_r, \vec{1}_i\,'; s)\, dS_1 \\
&= \int_{S_1} \left[\overleftrightarrow{1}_i\, \delta(\vec{1}_i - \vec{1}_r) + \frac{\gamma}{2\pi} \tilde{\overleftrightarrow{\Sigma}}^T(\vec{1}_r, \vec{1}_i; -s) \right] \\
&\qquad \cdot \left[\overleftrightarrow{1}_i\,'\, \delta(\vec{1}_i\,' - \vec{1}_r) - \frac{\gamma}{2\pi} \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_r, \vec{1}_i; s) \right] dS \\
&= \overleftrightarrow{1}_i\, \delta(\vec{1}_i - \vec{1}_i\,') + \frac{\gamma}{2\pi} \tilde{\overleftrightarrow{\Sigma}}^T(\vec{1}_i\,', \vec{1}_i; -s) \\
&\qquad - \frac{\gamma}{2\pi} \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_i, \vec{1}_i\,'; s) - \frac{\gamma^2}{4\pi^2} \int_{S_1} \tilde{\overleftrightarrow{\Sigma}}^T(\vec{1}_r, \vec{1}_i; -s) \cdot \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_r, \vec{1}_i\,'; s)\, dS
\end{aligned}
\tag{6.4}
$$

For the case of a lossless scatterer [8] shows that

$$
\int_{S_1} \tilde{\overleftrightarrow{R}}^T(\vec{1}_r, \vec{1}_i; -s) \cdot \tilde{\overleftrightarrow{R}}(\vec{1}_r, \vec{1}_i\,'; -s)\, dS = \overleftrightarrow{1}_i\, \delta(\vec{1}_i - \vec{1}_r\,')
\tag{6.5}
$$

Hence, for the lossless case we have

$$
\begin{aligned}
&\int_{S_1} \tilde{\overleftrightarrow{\Sigma}}^T(\vec{1}_r, \vec{1}_i; -s) \cdot \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_r, \vec{1}_i\,'; s)\, dS = \int_{S_1} \tilde{\overleftrightarrow{\Sigma}}(-\vec{1}_i, -\vec{1}_r; -s) \cdot \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_r, \vec{1}_i\,'; s)\, dS \\
&= \frac{2\pi}{\gamma} \left[\tilde{\overleftrightarrow{\Sigma}}^T(\vec{1}_i\,', \vec{1}_i; -s) - \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_i, \vec{1}_i\,'; s) \right] \\
&= \frac{2\pi}{\gamma} \left[\tilde{\overleftrightarrow{\Sigma}}(-\vec{1}_i, -\vec{1}_i\,'; -s) - \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_i, \vec{1}_i\,'; s) \right]
\end{aligned}
\tag{6.6}
$$

which is now a dyadic rather than a scalar equality. Note that the above have been derived for $s = j\omega$ so that frequency-domain energy conservation can be applied. The above form can be regarded as the analytic continuation away from the $j\omega$ axis into the s plane. Also observe that it combines two directions, $\vec{1}_i$ and $\vec{1}_i\,'$, for the incident field.

Specializing to the case of $\vec{1}'_i = \vec{1}_i$ we have for lossless scatterers

$$\tilde{\overleftrightarrow{\Xi}}(\vec{1}_i, s) = \frac{2\pi}{\gamma}\left[\tilde{\overleftrightarrow{\Sigma}}^T(\vec{1}_i, \vec{1}_i; -s) - \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_i, \vec{1}_i; s)\right]$$

$$\tilde{\overleftrightarrow{\Xi}}(\vec{1}_i, s) \equiv \int_{S_1} \tilde{\overleftrightarrow{\Sigma}}^T(\vec{1}_r, \vec{1}_i; -s) \cdot \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_r, \vec{1}_i; s)\, dS \tag{6.7}$$

$$= \int_{S_1} \tilde{\overleftrightarrow{\Sigma}}^T(-\vec{1}_i, \vec{1}_r; -s) \cdot \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_r, \vec{1}_i; s)\, dS$$

These are the same dyadic functions encountered in Section 5. Now we have relations for the off-diagonal elements as well. This is a generalized form of a Hermitian dyadic, analytically continuing the true Hermitian form for $s = j\omega$ out into the s-plane.

If, on the other hand, we look at backscattering, we have

$$\vec{1}'_i = -\vec{1}_i$$

$$\int_{S_1} \tilde{\overleftrightarrow{\Sigma}}^T(\vec{1}_r, \vec{1}_i; -s) \cdot \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_r, -\vec{1}_i; s)\, dS$$

$$= \frac{2\pi}{\gamma}\left[\tilde{\overleftrightarrow{\Sigma}}^T(-\vec{1}_i, \vec{1}_i; -s) - \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_i, -\vec{1}_i; s)\right] \tag{6.8}$$

$$\int_{S_1} \tilde{\overleftrightarrow{\Sigma}}(-\vec{1}_i, -\vec{1}_r; -s) \cdot \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_r, -\vec{1}_i; s)\, dS$$

$$= \frac{2\pi}{\gamma}\left[\tilde{\overleftrightarrow{\Sigma}}(-\vec{1}_i, \vec{1}_i; -s) - \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_i, -\vec{1}_i; s)\right]$$

for lossless scatterers. I suppose that this could be called a backscattering theorem.

7. High-Frequency Extended Forward Scattering Theorem

Recall from (3.2)

$$\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_i, \vec{1}_i; s) = -2\gamma A_c(\vec{1}_i) \overleftrightarrow{1}_i \quad \text{as} \quad \mathrm{Re}(s) \to +\infty \tag{7.1}$$

Then (6.7) gives

$$\frac{2\pi}{\gamma} \tilde{\overleftrightarrow{\Sigma}}^T(\vec{1}_i, \vec{1}_i; -s) + A_c(\vec{1}_i) \overleftrightarrow{1}_i = \tilde{\overleftrightarrow{\Xi}}(\vec{1}_i, s) \quad \text{as} \quad \mathrm{Re}(s) \to +\infty \tag{7.2}$$

8. Application to Poles in the Singularity Expansion Method

The scattering dyadic has the form [5]

$$\tilde{\overleftrightarrow{\Lambda}}(\overrightarrow{1}_r, \overrightarrow{1}_i; s) = \sum_{\alpha} \frac{e^{-[s-s_\alpha]t_i}}{s-s_\alpha} \overrightarrow{c}_\alpha(-\overrightarrow{1}_r)\overrightarrow{c}_\alpha(\overrightarrow{1}_i)$$

$$+ \text{ entire function of s}$$

$$\overrightarrow{c}_\alpha(\overrightarrow{1}_i) \equiv \text{ scattering natural mode} \tag{8.1}$$

$$t_i \equiv \text{ turn on time } (= 0 \text{ for simplicity})$$

Place this in (6.6) and take the limit as $s \to s_\alpha$ for some chosen s_α. This gives (for lossless scatterers)

$$\overrightarrow{c}_\alpha(\overrightarrow{1}_i)\int_{S_1} \overrightarrow{c}_\alpha(-\overrightarrow{1}_r) \bullet \tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_r, \overrightarrow{1}'_i; -s_\alpha)\, dS = -\frac{2\pi}{\gamma_\alpha}\overrightarrow{c}_\alpha(\overrightarrow{1}_i)\overrightarrow{c}_\alpha(-\overrightarrow{1}'_i)$$

$$\gamma_\alpha = \frac{s_\alpha}{c} \tag{8.2}$$

Removing $\overrightarrow{c}_\alpha(\overrightarrow{1}_i)$ (common factor) gives

$$\int_{S_1} \overrightarrow{c}_\alpha(-\overrightarrow{1}_r) \bullet \tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_r, \overrightarrow{1}'_i; -s_\alpha)\, dS = -\frac{2\pi}{\gamma_\alpha}\overrightarrow{c}_\alpha(-\overrightarrow{1}'_i) \tag{8.3}$$

relating natural modes in the left halfplane (LHP) to the scattering dyadic in the RHP. A high-frequency asymptotic expansion of the scattering dyadic in the RHP can be used to evaluate thia integral equation for specific scatterer shapes.

This can be cast in other forms as

$$
\begin{aligned}
-\frac{2\pi}{\gamma_\alpha}\vec{c}_\alpha(\vec{1}_i)
&= \int_{S_1} \tilde{\overleftrightarrow{\Sigma}}^T(\vec{1}_r, -\vec{1}_i; -s_\alpha) \cdot \vec{c}_\alpha(-\vec{1}_r)\, dS \\
&= \int_{S_1} \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_i, -\vec{1}_r; -s_\alpha) \cdot \vec{c}_\alpha(-\vec{1}_r)\, dS \\
&= \int_{S_1} \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_i, \vec{1}_r; -s_\alpha) \cdot \vec{c}_\alpha(\vec{1}_r)\, dS
\end{aligned}
\tag{8.4}
$$

where changes of variables have been applied. Note that the $\vec{c}_\alpha$ have an arbitrary scaling constant in this integral equation.

For special cases the natural modes can be degenerate (say doubly degenerate for bodies of revolution). In such cases similar results hold, except now (8.4) has multiple solutions given by the order of the degeneracy. Also, while the poles in (8.1) have been taken as first order, higher order poles give the same result.

For the case of $\mathrm{Re}(s_\alpha) \to -\infty$ (far LHP) the scattering dyadic in (8.4) can be evaluated by high-frequency asymptotic methods (such as the geometrical theory of diffraction (GTD), etc.), valid in the RHP ($\mathrm{Re}(s) \to +\infty$). This gives an integral equation for the $\vec{c}_\alpha$ and relates LHP and RHP behavior.

9. Concluding Remarks

The forward-scattering theorem and its extensions give various interesting results. This is particularly useful for the case of lossless scatterers, since we do not need to solve the scattering problem for the absorption by the scatterer.

One of the interesting extensions is that into the s plane away from the $j\omega$ axis. This gives some analytic properties of the scattering dyadic. Besides relating LHP and RHP behavior, there may be some additional properties to be discovered.

References

1. C. E. Baum, “Focused Aperture Antennas”, Sensor and Simulation Note 306, May 1987.

2. C. E. Baum, “Radiation of Impulse-Like Transient Fields”, Sensor and Simulation Note 321, November 1989.

3. C. E. Baum, “On the Singularity Expansion Method for the Solution of Electromagnetic Interaction Problems”, Interaction Note 88, December 1971.

4. C. E. Baum, “Scattering of Transient Plane Waves”, Interaction Note 473, August 1980.

5. C. E. Baum, “Representation of Surface Current Density and Far Scattering in EEM and SEM with Entire Functions”, Interaction Note 486, February 1992; ch. 13, pp. 273-316 in P. P. Delsanto and A. W. Saenz (eds.), *New Perspectives on Problems in Classical and Quantum Physics, Part II, Acoustic Propagation and Scattering, Electromagnetic Scattering*, Gordon and Breach, 1998.

6. C. E. Baum, J. E. Mooney, and L. S. Riggs, “Use of Modified Pole Series in the Singularity Expansion Method”, Interaction Note 525, April 1997.

7. E. Gerjuoy and D. S. Saxon, “Variational Principles for the Acoustic Field”, Phys. Rev., 2nd Series, Vol. 94, 1954, pp. 1445-1458.

8. D. S. Saxon, “Tensor Scattering Matrix for the Electromagnetic Field”, Phys. Rev., Vol. 100, 1955, pp. 1771-1775.

9. M. Born and E. Wolf, *Principles of Optics, 3rd Ed*., Pergamon Press, 1964.

10. C. E. Baum, “Target Symmetry and the Scattering Dyadic”, ch. 4, pp. 204-236, in D. H. Werner and R. Mittra, *Frontiers in Electromagnetics*, IEEE Press, 2000.