Interaction Notes

Note 621

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On the Natural Oscillation Frequencies of a Straight Wire

F. M. Tesche
Holcombe Dept. of Electrical and Computer Engineering
College of Engineering & Science, 337 Fluor Daniel Building
Box 340915, Clemson, SC 29634-0915

and

D. V. Giri
Pro-Tech, 11 C Orchard Court, Alamo, CA 94507-1541
Dept. of Electrical and Computer Engineering, University of New Mexico,
Albuquerque, NM 87131

Abstract

There are a number of different methods for computing the complex natural frequencies of a straight wire antenna or scatterer. The most accurate method is that derived from a numerical solution to the integral equation for the wire current. Due to the relative difficulty in obtaining this solution, however, several different approximations to these resonant frequencies have been developed and used in the past. Most recently, a paper by Myers, et al. has described a variational approach for computing these resonant frequencies.

This note provides a brief description of each of these methods for determining the wire resonances, and then presents a comparison of the result of each method for a wire of diameter to length ratio (d/L) ranging from 0 to 0.1.

1. Introduction

The Singularity Expansion Method (SEM) was first introduced by Baum in 1971 in an US Air Force note series [1]. It was later thoroughly documented in a book chapter [2] and has been discussed in a large number of other publications. Reference [3] provides an overview of this subject and an extensive list of pertinent references.

The essential point of the SEM analysis method for antennas or scatterers is that the behavior of the induced current on the structure in the frequency domain can be represented by a sum of terms involving the natural resonances of the structure and the corresponding current modes. This is similar to the representation of the response of a conventional circuit using a pole-residue analysis.

A key element to the SEM analysis is the determination of the natural resonances of the antenna. Early work with a spherical scatterer using an analytical expression for the current distribution showed that there were a doubly infinite set of natural resonance frequencies which corresponded to the zeros of certain spherical Hankel function expressions [4]. Earlier work by Oseen [5] and Hallén [6] had studied similar resonances in a straight wire, and approximate expressions for the principal resonant frequencies of the wire were developed. In 1973 Tesche [7] showed through a numerical solution of the Pocklington integro-differential equation that the straight wire has complex natural resonances much like those in the spherical antenna. They occur in *layers* in the complex frequency plane and occur arranged with an increasing imaginary part ($j\omega$), which occur at approximately equal intervals. A similar study was conducted by Giri and his co-workers [8], using a different method for searching for the resonances of the integral equation solution.

Aside from the direct numerical evaluation of the resonances of a straight wire in [7], a number of other researchers have developed methods for approximating the complex natural frequencies. All of these methods have increasing accuracy as the wire becomes very thin, and all provide only the first layer of resonances, which are the closest to the $j\omega$ axis. Lee and Leung [9] and Weinstein [10] use a Weiner-Hopf method, while Hoorfar and Chang [11] an infinite wire solution for the current, with additional reflected traveling current waves at the ends of the wire. Bouwkamp [12] and Marin and Liu [13] both use the asymptotic antenna theory of Hallén to obtain the resonances.

More recently, a novel variational method, which is completely different from the previous approaches, but which stems from an integral equation analysis of the wire, has been developed by Meyers, Sandler and Wu [14]. We also note in passing that finite methods like the finite-difference time-domain (FDTD) [15] or finite element methods are also used to compute antenna responses. However, these are not useful, as they provide results in the time domain or on the $j\omega$ axis, and are not suitable for exploring the complex frequency plane.

With these differing methods for determining the natural frequencies of the wire, it is interesting to evaluate which is most accurate when compared with the numerically determined values of [7]. Reference [13] provides a comparison of their results with those of [7], but there do not seem to be comparisons of the other methods in the literature. The purpose of this paper, therefore, is to compare the results of the complex natural frequencies from the above authors in a consistent way, so as to illustrate which approximate method is the most accurate.

2. Survey of Methods for Determining the Wire Resonant Frequencies

In describing the literature on this subject, it is important to point out that some authors use the $e^{-i\omega t}$ time convention, while others use $e^{j\omega t}$. In this paper, we elect the use of $e^{j\omega t}$, and we have modified some of the referenced equations accordingly. Furthermore, some of the authors define the wire geometry by its half-length *h* and wire radius *a*. We will use the parameters $L = 2h$ and $d = 2a$ to represent the *length* and *diameter* of the wire, respectively. The comparisons of all of the results will be made using these parameters. In some of the discussion to follow, the wire thickness is described by the factor $\Omega = 2 \ln(2L/d)$.

The complex natural resonances of the wire are generally defined by the notation $s_{\ell,n} = \sigma_{\ell,n} + j\omega_{\ell,n}$, where ℓ denotes the resonance "layer" and *n* is the index for the resonances contained in the layer. For resonances in the first layer ($\ell = 1$), the resonances are frequently referred to simply as s_n. As in a resonant circuit, the real part $\sigma_{\ell,n}$ corresponds to a damping constant and the imaginary part $j\omega_{\ell,n}$ is an oscillation frequency. It should be noted that these resonances always occur in complex conjugate pairs, as $\left(s_{\ell,n}, s^*_{\ell,n}\right)$.

Other parameters that are used in this development are: $\gamma = 0.57721$ (Euler's constant), $\Gamma = e^{\gamma} = 1.781\ldots$, $Z_c = 120\,\pi\,(\Omega)$ (free space impedance), and $c = 3 \times 10^8$ (m/s) (speed of light).

2.1 Solutions Based on the Integral Equation Solution

The frequency domain Pocklington integro-differential equation describes the current flowing on the wire, for either a local excitation source (the antenna problem), or for a distributed incident field (the scattering problem). This current may be determined numerically by using the method of moments, as described by Harrington [16]. This procedure results in a system impedance matrix equation that contains information about all of the resonances of the wire. By generalizing this integral equation solution to complex frequencies s (i.e., $j\omega \rightarrow s$), the natural frequencies $s_{\ell,n}$ are determined by searching for the zeros of the determinant of the system matrix.

2.1.1 Resonances Determined by a Newton Method Search in the Complex s-Plane

The paper by Tesche [7] describes the initial calculation of the natural resonances of the wire structure through the use of Newton's method in the complex s-plane. This amounted to an iterative search for the zeros of the system determinant in the s-plane, and for the first time, showed the existence of the multi-layer pole structure for the wire. In a separate communication [17], it was shown that the use of the Hallén integral equation provides the same natural frequencies for the wire.

This analysis also resulted in the determination of the natural current modes at each resonant frequency, and permitted the construction of both the spectral and transient responses of the wire in terms of these parameters. Details of this SEM approach are provided in [7] and are not described further here.

2.1.2 Resonances Determined by a Contour Integration Approach.

As an alternative to the Newton search method for finding the wire resonances from the integral equation solution, a contour integration method proposed by Baum [18] can be used. Giri and his co-authors in [8] used this contour integration method to determine the frequencies in the s-plane.

When the integral equation is matricized, the natural frequencies are computed from finding the zeroes of the system matrix. This determinant is an analytical function of the complex frequency. The contour integration method involves performing certain integrals on a closed contour around the locations of the natural frequencies in the complex s-plane. It is superior to the Newton or Newton-Raphson method, which require a guess or starting value and they find their way to the actual resonance. If the resonance is sharp, these methods can suffer from numerical instabilities. The contour integration method does not require a guess or starting value. All calculations are made from a safe distance from the resonance location. For this reason, the results of the contour integration method are robust and numerically stable.

2.1.3 Myers, Sandler and Wu Variational Solution

The most recent contribution to the set of complex resonant frequencies based on an integral equation for the wire is described in [14]. The method used in this paper is unlike the other methods described previously, in that it uses a *variational approach* with the Pocklington integro-differential equation, together with an assumed current distribution that representative of the current at each resonant frequency. Thus, this is an approximate technique, but since it directly uses the integral equation in the solution, it is listed in this section on integral equation methods.

This analysis method results in a characteristic equation involving a complex propagation constant k (Eq.(20) in [14]), from which the roots may be determined iteratively. When modified to reflect the wire parameters we are using (L and d) as well as the $e^{j\omega t}$ time variation, this equation becomes

$$\begin{aligned} F(k) = \left(\kappa_n^2 - k^2\right) \Bigg\{ 2L \left[\ln\left(\frac{4L}{d}\right) - \int_0^L \frac{1 - e^{-jky}\cos\left(\kappa_n y\right)}{y} dy \right] \\ - \frac{j}{k+\kappa_n}\left[e^{-j(k+\kappa_n)L} - 1\right] - \frac{j}{k-\kappa_n}\left[e^{-j(k-\kappa_n)L} - 1\right] \Bigg\} \\ - \left(\kappa_n^2 + k^2\right) \frac{2}{\kappa_n} \int_0^L \frac{e^{-jky}\sin\left(\kappa_n y\right)}{y} dy \end{aligned} \quad (1)$$

The complex resonant frequencies s_n are then found by numerically searching for the roots to the equation

$$F\left(\frac{s_n}{jc}\right)=0 \quad for\ n=1,2,3\cdots . \tag{2}$$

2.2 Approximate Solutions

Other approximate methods for determining the resonances of the wire have been developed in the literature. These are described below.

2.2.1 Lee and Leung Solution

In [9] a frequency domain expression for the wire current using a Wiener-Hopf approach is developed. This result is given in Eq.(4.8) of [9] and it yields the following explicit expression for the complex natural resonance frequencies:

$$s_n=\sigma_n+j\omega_n=\frac{jn\pi c}{L}\left(1+\frac{1-j\frac{2}{\pi}\ln\left(2n\pi\right)}{4n\ln\left(\Gamma\frac{d}{L}\frac{n\pi}{2}\right)}\right) \qquad n=1,\,2,\,3,\,\ldots \tag{3}$$

This expression is valid for $(n\pi/2)(d/L)<<1$, which implies that the solution for the resonances degrades as the wire becomes thicker. In the development in [9], only the frequency indices of n = 1, 3, 5, ⋯ are mentioned, and these correspond to even current modes on the wire. There is no mention of the odd modes for n = 2, 4, 6, ⋯, but (3) does appear to provide approximate resonant frequencies for these modes as well.

2.2.2 Weinstein Solutions

The natural resonance frequencies and current distributions on a thin cylinder are also discussed in [10], which uses a Weiner-Hopf technique. In this development, the expression for current distribution contains a denominator term D and this term vanishes at certain complex frequencies. This term is given in (62.56) of [10] and is

$$D(s_n)=1-\left(\Psi(L,s_n)e^{-s_nL/c}\right)^2=0 \tag{4}$$

The function $\psi(z,s)$ is provided in (62.63) of [10] and upon substituting the required parameters, it is

$$\Psi(z,s)=\frac{2\ln\left(\frac{2c}{\Gamma sd}\right)}{\ln\left(\frac{8cz}{\Gamma sd^2}\right)-E\left(-j2z\frac{s}{c}\right)e^{2zs/c}}. \tag{5}$$

The function $E(y)$ in (5) is defined by Weinstein as

$$\begin{aligned} E(y) &= Ci(y) - j\,si(y) \\ &= Ci(y) - j\left(Si(y) - \frac{\pi}{2} \right), \end{aligned} \tag{6}$$

where $Ci(y)$ and $Si(y)$ are the cosine and sine integrals with complex argument y. These integrals are defined in [19], and have the form

$$\begin{aligned} Ci(y) &= \gamma + \ln(y) + \int_0^y \frac{\cos(t) - 1}{t} dt \\ &= \gamma + \ln(y) + \sum_{n=1}^{\infty} \left[\frac{(-1)^n y^{2n}}{2n(2n)!} \right] \end{aligned} \tag{7a}$$

$$\begin{aligned} Si(y) &= \int_0^y \frac{\sin(t)}{t} dt \\ &= \sum_{n=0}^{\infty} \left[\frac{(-1)^n y^{2n+1}}{(2n+1)(2n+1)!} \right]. \end{aligned} \tag{7b}$$

The roots of (4) provide the resonant frequencies s_n for Weinstein's model and these can be obtained by a suitable numerical search procedure.

Weinstein also provides an approximate solution to the resonant frequencies in Problem 9.15 in his book [10]. This results in a simple expression for s_n without the need of a numerical search. This expression is

$$s_n \approx -\frac{c}{2L} \frac{H(2n\pi)}{\ln\left(\frac{-j2L}{\Gamma n\pi d} \right)} + j\frac{n\pi c}{L} \tag{8}$$

where

$$H(y) = \int_0^y \frac{1 - e^{-jt}}{t} dt. \tag{9}$$

2.2.3 Hoorfar and Chang Solution

Hoorfar and Chang [11] use the method of Shen, et al. [20] to represent the current on a straight wire as the sum of the current in an infinite wire, plus traveling wave components reflected from the ends of the wire. This development leads to the following resonance condition:

$$\Delta(s_n) = 1 - R^2(s_n) I_\infty^2(s_n; L) = 0 \tag{10}$$

In this expression, $I_\infty(s;z)$ is related to the current flowing along an infinite wire, and is approximated by

$$I_\infty(s;L) \approx \frac{2\pi}{Z_c} \frac{e^{-s|z|/c}}{\Gamma_1(s) + \lambda\left(\frac{2s\,|\,z\,|}{c}\right)}, \tag{11}$$

where
$$\Gamma_1(s) = -2\ln\left(\frac{sd}{2c} e^{\gamma}\right) \tag{12}$$

and
$$\lambda(y) = \gamma + \ln(y) + e^{y} E_1(y). \tag{13}$$

The term E_1 in this last expression is the exponential integral of the first kind defined in [11, eq.(5.1.1)], which is

$$E_1(y) = \gamma + \ln(y) + \sum_{n=1}^{\infty}\left[\frac{(-1)^n y^n}{n(n)!}\right]. \tag{14}$$

The term $R(s)$ in (10) is the reflection coefficient at the ends of the wire and is given in [11]

$$R(s) \approx \frac{Z_c}{2\pi} \frac{\Gamma_1(s)}{|A_o(s)|\left(1 + \mathrm{Re}(B_o(s))\right)^2} \tag{15}$$

where
$$A_o(s) = \Gamma_1(s) \frac{\left(1 + \ln(2)/\Gamma_1(s)\right)^2}{-j\pi J_0\left(-j\frac{s}{c}\frac{d}{2}\right) H_0^{(2)}\left(-j\frac{s}{c}\frac{d}{2}\right)} \tag{16}$$

and
$$\begin{aligned} B_o(s) = \frac{-j}{2\pi}&\left\{\ln\left(\Gamma_1(s) - g_o(s) + j3\pi/2\right)\right. \\ &\left. -\ln\left(\Gamma_1(s) - g_o(s) - j3\pi/2\right)\right\} - \frac{1}{\Gamma_1(s) + \ln(2)}, \end{aligned} \tag{17}$$

with
$$g_o(s) = 33.88\left(-j\frac{s}{c}\frac{d}{2}\right)^2 \exp\left(\frac{3.26}{j\frac{s}{c}\frac{d}{2}}\right). \tag{18}$$

J_o and $H_0^{(2)}$ in (16) are the usual cylindrical Hankel functions.

The natural frequencies s_n for this case are again determined from (10) through a numerical search procedure.

2.2.4 Bouwkamp Solution (from Hallén)

Bouwkamp [12] discusses the use of Hallén's asymptotic antenna theory for the determination of the resonant frequencies of a center-fed wire antenna, or a field-excited wire. Hallén's original reference [6] is in German and is difficult to obtain, so this analysis approach relies on the results of Bouwkamp's work.

This solution relies on constructing a representation of the wire current as a numerator and denominator fraction, each of which is expanded of powers of Ω^{-n}. Equation (28) of [12] provides the midpoint current on the wire for three terms in the power series expansion. It is apparent from this expression that the roots of the denominator will correspond to the resonant frequencies of the wire. Due to the fact that this analysis is based on a symmetric excitation of the wire, only the frequencies of the even modes are be provided by the equations to follow.

For the first-order estimation of the resonances, only the Ω^{-1} term in the denominator will be considered. This yields the following determinant, for which the roots s_n can be determined for n = 1, 3, 5, ..:

$$\Delta(s_n) = \cos\left(\frac{s_n L}{j2c}\right) + \frac{\alpha_1\left(\frac{s_n L}{j2c}\right)}{\Omega} = 0\,. \tag{19}$$

The parameter α_1 in this equation is given by

$$\alpha_1(y) = \tfrac{1}{2} e^{jy} E_B(4y) - \cos(y) E_B(2y) \tag{20}$$

where E_B is defined by Bouwkamp as

$$\begin{aligned} E_B(y) &= \int_0^y \frac{1-\cos(t)}{t}\,dt + j\int_0^y \frac{\sin(t)}{t}\,dt \\ &= -\sum_{n=1}^{\infty}\left[\frac{(-1)^n y^{2n}}{2n(2n)!}\right] + j\sum_{n=0}^{\infty}\left[\frac{(-1)^n y^{2n+1}}{(2n+1)(2n+1)!}\right]. \end{aligned} \tag{21}$$

As in the previous cases, the natural frequencies for the even modes can be determined numerically from (19).

2.2.5 Marin and Liu Solution (from Oseen)

Marin and Liu [13] discuss the use of Hallén's asymptotic antenna theory to approximate the wire resonances using an approach attributed to Oseen [5]. Unfortunately, this latter reference is difficult to find, so we rely on [13] for this solution.

This reference provides a simple first order approximation for the resonances as

$$s_n \approx -\frac{c}{\Omega L}\left(\gamma + \ln\left(2n\pi\right) - Ci\left(2n\pi\right) + j\,Si\left(2n\pi\right)\right) + j\frac{c}{L}n\pi \ . \qquad (22)$$

Reference [13] also states that there is a higher order approximation for the wire resonances, and shows a plot comparing both the first and second-order Hallén resonances and the integral equation-determined resonances. However, details of this second order solution are not provided, and consequently, they are not compared here.

3. Numerical Results and Comparisons

3.1 Wire Resonances from the Integral Equation Solution

In this section, we provide a comparison of the complex resonant frequencies for the straight wire provided by the preceding methods. To start, we illustrate the locations of the wire resonances (in the upper left portion of the s-plane) that are provided by the integral equation solution of [7]. Figure 1 shows a shaded plot of the moment method determinant surface for the case of d/L = 0.01 (Ω = 11.98), which is a relatively thin wire. Note that the frequency axes in this plot are normalized as $j\omega L/c$ and $-\sigma L/c$.

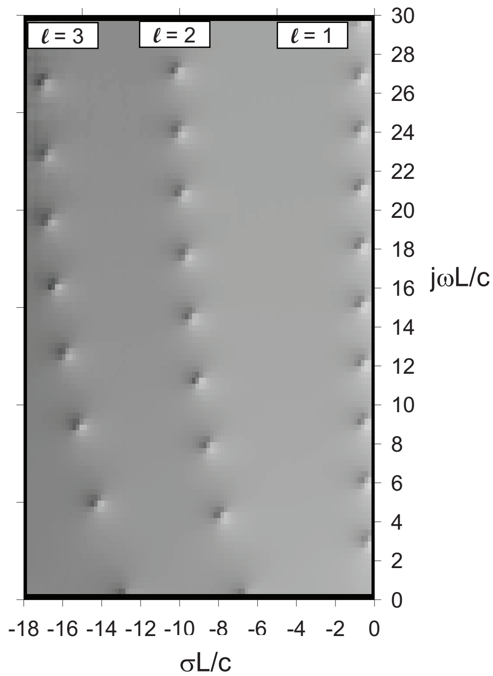


Figure 1. Locations of the natural resonant frequencies for a wire with *d/L* = 0.01, as computed by an integral equation solution. (Only frequencies in the upper left complex s-plane are shown.)

In Figure 1, the fact that the natural frequencies of the wire occur in layers, each having a higher value of damping ($-\sigma L/c$) is clearly evident. Figure 2 examines the behavior of the first layer of resonances (ℓ = 1) as the d/L ratio varies from 0 to 0.1. For the case of the wire with a vanishing diameter, the values of the imaginary part of the complex frequency is simply $j\omega_n L/c$ = $n\pi$ for n = 1, 2, 3, .. and $\sigma_n L/c$ = 0. As the wire diameter becomes larger, the resonances migrate off into the left part of the s-plane and the damping constants of the resonances increase.

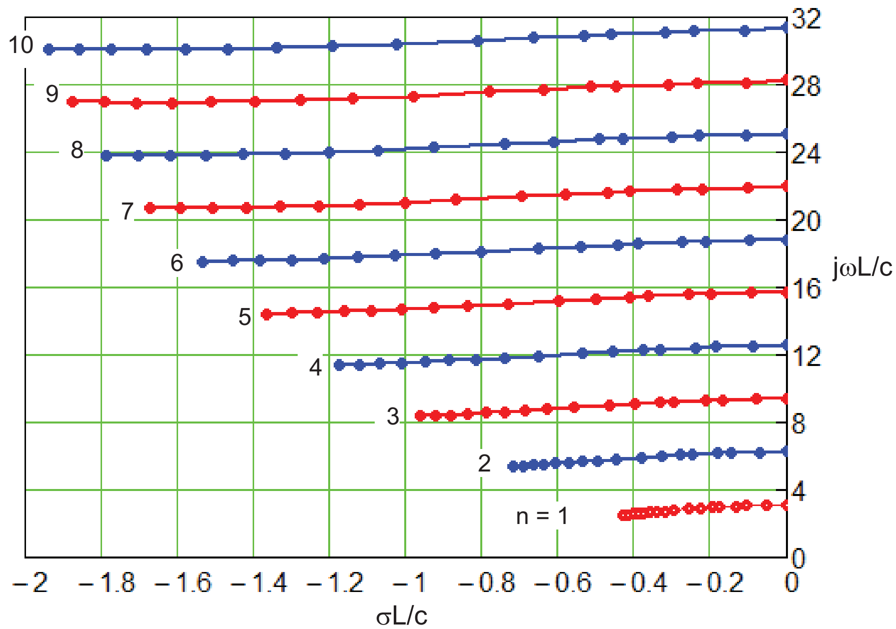


Figure 2. Plots of the lowest 10 complex resonant frequency trajectories in the $\ell = 1$ layer for 16 different values of *d*/*L*. (The 17 dots correspond to *d*/*L* = 0, 10^{-10}, 10^{-5}, 10^{-4}, 0.001, 0.002, 0.005, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09 and 0.1, starting from the jω axis.

The reader is cautioned that the integral equation analysis assumes that the wire is electrically thin. This implies that for the resonances with high values of the index n, the wire diameter should not exceed, say, 10% of the effective wavelength $\lambda_n \approx 2\pi c / \omega_n$. As an example, for the case of the n = 10 resonance and for the wire with d/L = 0.1, we see that $d \approx \lambda_n / 2$, which is too large for the integral equation solution to be valid. Thus, high order resonances for thick wire should be used with caution.

For possible future reference, the numerical values of the normalized complex resonances of the wire in layer 1, computed with the integral equation solution and shown in Figure 2, are summarized in Table 1.

Table 1. The first 10 normalized complex resonant frequencies (σ + jω)*L*/*c* of a wire, as determined from an integral equation solution.

d/L	n = 1		n = 2		n = 3		n = 4		n = 5	
	σL/c	ωL/c	σL/c	ωL/c	σL/c	ωL/c	σL/c	ωL/c	σL/c	ωL/c
0	0.000	3.142	0.000	6.283	0.000	9.425	0.000	12.566	0.000	15.708
1.0E-10	-0.053	3.108	-0.070	6.245	-0.080	9.384	-0.087	12.519	-0.093	15.651
1.0E-05	-0.107	3.065	-0.145	6.195	-0.168	9.331	-0.186	12.463	-0.200	15.595
1.0E-04	-0.134	3.040	-0.183	6.167	-0.215	9.296	-0.239	12.425	-0.258	15.554
0.001	-0.177	2.992	-0.249	6.104	-0.296	9.224	-0.333	12.346	-0.363	15.469
0.002	-0.196	2.968	-0.278	6.070	-0.334	9.183	-0.377	12.302	-0.413	15.419
0.005	-0.227	2.923	-0.329	6.004	-0.399	9.104	-0.455	12.208	-0.502	15.315
0.01	-0.257	2.873	-0.380	5.931	-0.467	9.010	-0.538	12.101	-0.598	15.196
0.02	-0.295	2.803	-0.446	5.825	-0.559	8.878	-0.651	11.947	-0.732	15.023
0.03	-0.321	2.750	-0.496	5.743	-0.628	8.778	-0.740	11.831	-0.837	14.894
0.04	-0.343	2.705	-0.538	5.677	-0.688	8.693	-0.816	11.737	-0.929	14.791
0.05	-0.361	2.666	-0.574	5.617	-0.741	8.621	-0.885	11.655	-1.012	14.706
0.06	-0.377	2.631	-0.607	5.564	-0.791	8.558	-0.949	11.583	-1.091	14.634
0.07	-0.391	2.600	-0.638	5.517	-0.837	8.501	-1.010	11.523	-1.163	14.574
0.08	-0.405	2.570	-0.667	5.473	-0.881	8.448	-1.068	11.467	-1.234	14.520
0.09	-0.417	2.546	-0.693	5.438	-0.921	8.407	-1.122	11.429	-1.300	14.486
0.1	-0.428	2.520	-0.720	5.400	-0.962	8.363	-1.177	11.385	-1.366	14.448

Table 1. (concluded)

d/L	n = 6		n = 7		n = 8		n = 9		n = 10	
	σL/c	ωL/c	σL/c	ωL/c	σL/c	ωL/c	σL/c	ωL/c	σL/c	ωL/c
0.000	0.000	18.850	0.000	21.991	0.000	25.133	0.000	28.274	0.000	31.416
1.0E-10	-0.098	18.780	-0.102	21.903	-0.105	25.023	-0.108	28.133	-0.110	31.237
1.0E-05	-0.211	18.721	-0.221	21.843	-0.229	24.960	-0.236	28.070	-0.243	31.171
1.0E-04	-0.274	18.680	-0.288	21.800	-0.300	24.916	-0.310	28.023	-0.319	31.124
0.001	-0.389	18.589	-0.411	21.702	-0.430	24.812	-0.448	27.916	-0.463	31.014
0.002	-0.444	18.532	-0.471	21.646	-0.494	24.753	-0.515	27.853	-0.534	30.948
0.005	-0.543	18.422	-0.579	21.526	-0.611	24.624	-0.640	27.718	-0.666	30.803
0.010	-0.651	18.290	-0.698	21.382	-0.740	24.473	-0.779	27.555	-0.813	30.637
0.020	-0.803	18.102	-0.868	21.181	-0.926	24.263	-0.979	27.338	-1.026	30.411
0.030	-0.925	17.964	-1.003	21.039	-1.073	24.115	-1.138	27.190	-1.195	30.263
0.040	-1.030	17.857	-1.121	20.932	-1.203	24.008	-1.275	27.090	-1.339	30.169
0.050	-1.127	17.772	-1.228	20.848	-1.319	23.930	-1.398	27.018	-1.466	30.106
0.060	-1.216	17.700	-1.328	20.782	-1.426	23.873	-1.510	26.971	-1.580	30.071
0.070	-1.300	17.643	-1.420	20.735	-1.525	23.832	-1.613	26.942	-1.682	30.062
0.080	-1.382	17.596	-1.510	20.694	-1.618	23.810	-1.707	26.933	-1.776	30.065
0.090	-1.456	17.574	-1.591	20.684	-1.703	23.813	-1.793	26.955	-1.860	30.106
0.100	-1.532	17.543	-1.671	20.669	-1.786	23.813	-1.876	26.974	-1.938	30.150

For the specific case of a wire with $d/L = 0.01$, Table 2 shows a comparison of the first 10 normalized complex resonant frequencies $(\sigma + j\omega)L/c$ in layers 1, 2 and 3, as computed by the methods of Singaraju, et al. [8], Tesche [7] and Myers, et al. [14]. Because the results of [7] and [8] are both based on the same integral equation solution (but with a different method of searching) the resonant frequencies are very close. It is noted that the contour integration approach of [8] seems to be more robust for resonances in the $\ell = 3$ layer, where the Newton method suffers from convergence problems.

Table 2. Summary of the first 10 normalized complex resonant frequencies $(\sigma + j\omega)L/c$ in layers 1, 2 and 3, as computed by the methods of Singaraju, et al. [8], Tesche [7] and Myers, et al. [14], for the wire with $d/L = 0.01$.

Resonance	Reference	Layer $\ell = 1$		Layer $\ell = 2$		Layer $\ell = 3$	
		σL/c	ωL/c	σL/c	ωL/c	σL/c	ωL/c
n = 1	Singaraju, et al.	-0.260	2.906	-6.813	0.000	-12.878	0.000
	Tesche	-0.257	2.873	-6.733	0.000	-12.800	0.000
	Myers et al.	-0.255	2.905	***	***	***	***
n = 2	Singaraju, et al.	-0.381	6.006	-7.854	4.187	-14.182	4.706
	Tesche	-0.380	5.931	-7.814	4.160	-14.134	4.689
	Myers et al.	-0.367	5.986	***	***	***	***
n = 3	Singaraju, et al.	-0.468	9.059	-8.590	7.753	-15.169	8.631
	Tesche	-0.467	9.011	-8.545	7.698	-15.112	8.598
	Myers et al.	-0.443	9.089	***	***	***	***
n = 4	Singaraju, et al.	-0.538	12.171	-9.156	11.101	-15.926	12.219
	Tesche	-0.538	12.100	-9.094	11.060	-15.866	12.380
	Myers et al.	-0.503	12.201	***	***	***	***
n = 5	Singaraju, et al.	-0.600	15.248	-9.567	14.375	-16.604	15.730
	Tesche	-0.598	15.190	-9.532	14.330	-16.580	15.725
	Myers et al.	-0.552	15.317	***	***	***	***
n = 6	Singaraju, et al.	-0.653	18.364	-9.940	17.623	-17.168	19.104
	Tesche	-0.651	18.290	-9.896	17.570	***	***
	Myers et al.	-0.595	18.437	***	***	***	***
n = 7	Singaraju, et al.	-0.704	21.453	-10.260	20.804	-17.680	22.455
	Tesche	-0.698	21.380	-10.200	20.770	***	***
	Myers et al.	-0.633	21.559	***	***	***	***
n = 8	Singaraju, et al.	-0.749	24.571	-10.544	24.003	-18.133	25.730
	Tesche	-0.741	24.470	-10.470	23.950	***	***
	Myers et al.	-0.667	24.682	***	***	***	***
n = 9	Singaraju, et al.	-0.792	27.667	-10.800	27.164	-18.550	29.013
	Tesche	-0.779	27.560	-10.700	27.110	***	***
	Myers et al.	-0.698	27.807	***	***	***	***
n = 10	Singaraju, et al.	-0.832	30.788	-11.030	30.334	***	***
	Tesche	-0.814	30.640	-10.900	30.260	***	***
	Myers et al.	-0.727	30.934	***	***	***	***

It is interesting to note in Table 2 that the Myers, et al. solution [14] provides only the resonances for the first layer ($\ell = 1$). This is likely due to the use of only the first layer current distributions in their variational calculation. It is speculated that if the higher layer modal distributions were to be used in the calculation, this method might provide the ℓ = 2 and 3 resonances, as well.

Figure 3 plots the resonance data of Table 2 for a visual summary of these results.

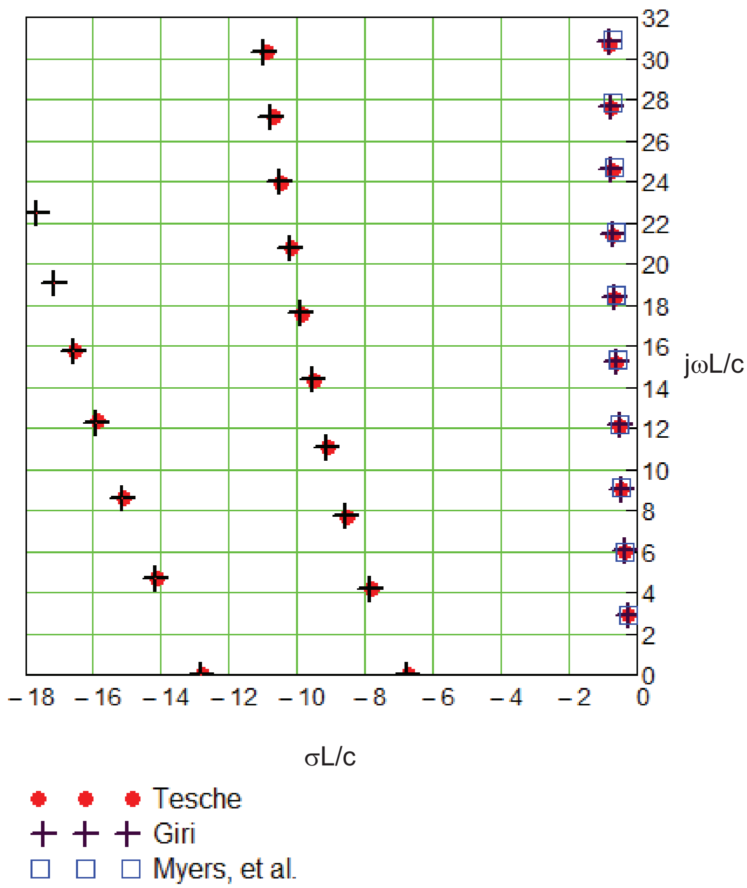


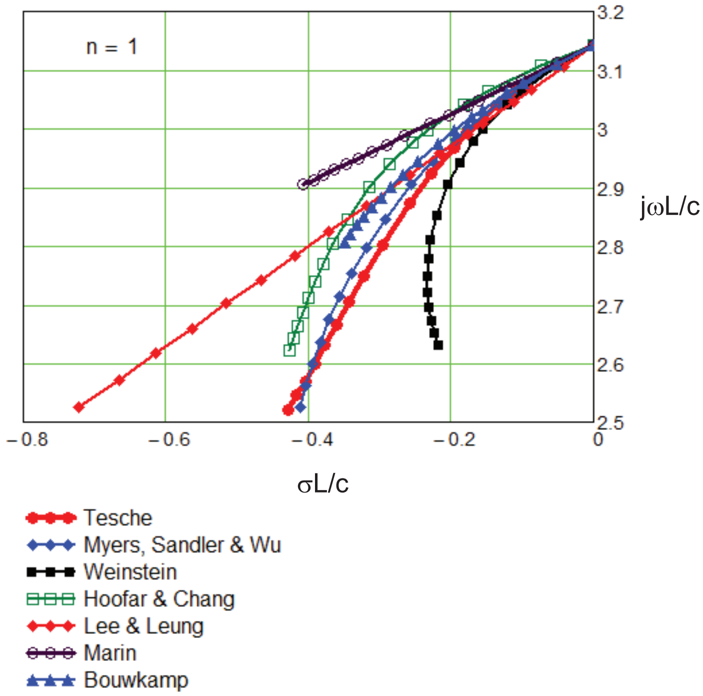
Figure 3. Overlay plots of the normalized resonant frequencies of Table 2.

3.2 Comparison of the Integral Equation Results with the Approximate Methods

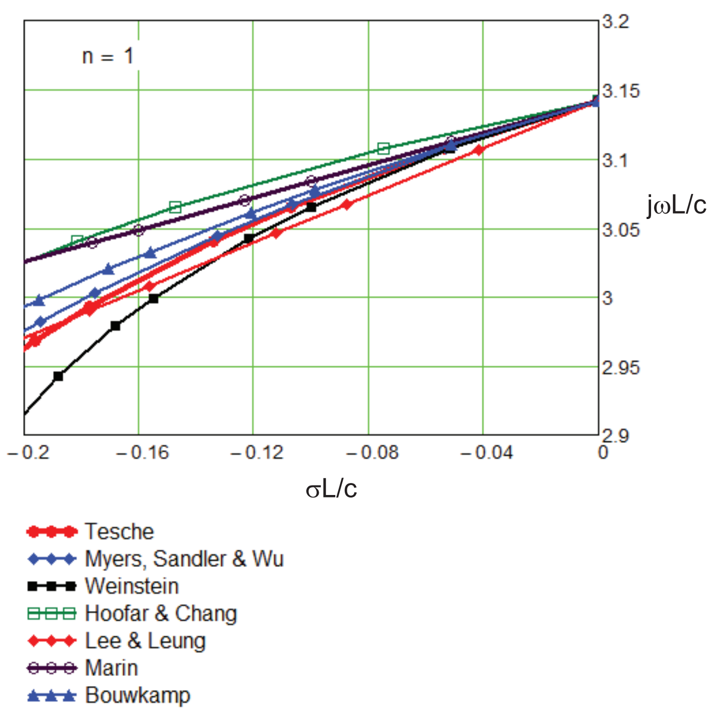
In this section we compare the previously tabulated integral equation values for the wire resonances in Table 1 with those calculated from the other models described in Section 2. We will present these comparisons as trajectory plots in the upper portion of the normalized complex frequency plane (sL/c), as done in Figure 2, for the first three principal resonances: n = 1, 2 and 3. Note that in these plots, the results of Singaraju [8] are considered to be identical to those in Table 1, and hence, are not plotted explicitly.

Figure 4 plots the trajectories of the principal normalized resonant frequency (n = 1, ℓ = 1) for the wire for values of d/L = 0, 10^{-10}, 10^{-5}, 10^{-4}, 0.001, 0.002, 0.005, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09 and 0.1, for the different models. Part (a) of this figure shows the resonance trajectories for the complete set of wire thicknesses, while part (b) shows the trajectories for very thin wires on an expanded scale in the s-plane.

Figure 5 and Figure 6 present the second and third resonance trajectories (for the (n = 2, ℓ = 1) and (n = 3, ℓ = 1) resonances), respectively, for the same collection of wire diameters as in Figure 4. Notice that the Bouwkamp results are missing from the n = 2 plot, for the reason mentioned earlier in Section 2.2.4.



(a) Normal scale



(b) Expanded scale for very thin wires

Figure 4. Plots of the trajectories of the $n = 1$ normalized complex resonant frequencies for the various analysis methods of Section 2.

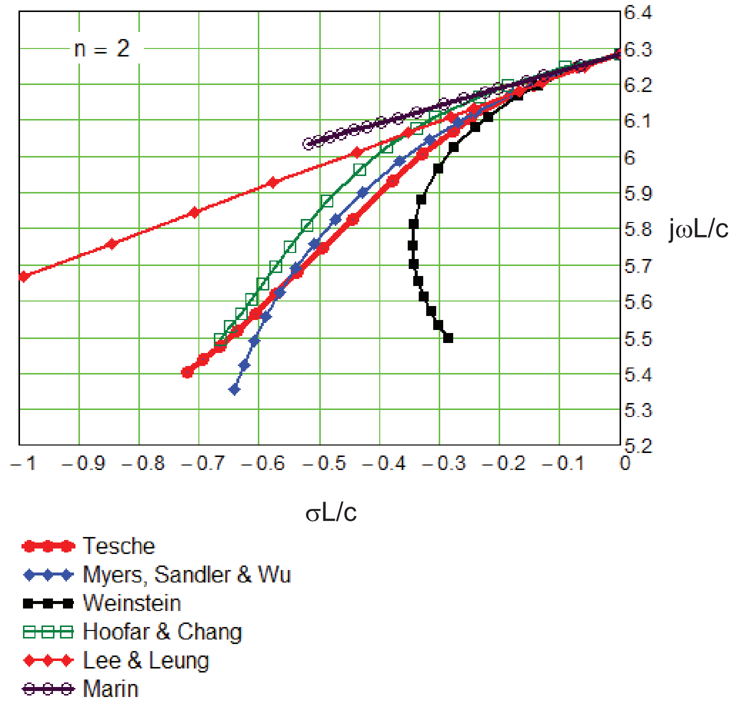


Figure 5. Plots of the trajectories of the *n* = 2 normalized complex resonant frequencies for the various analysis methods of Section 2.

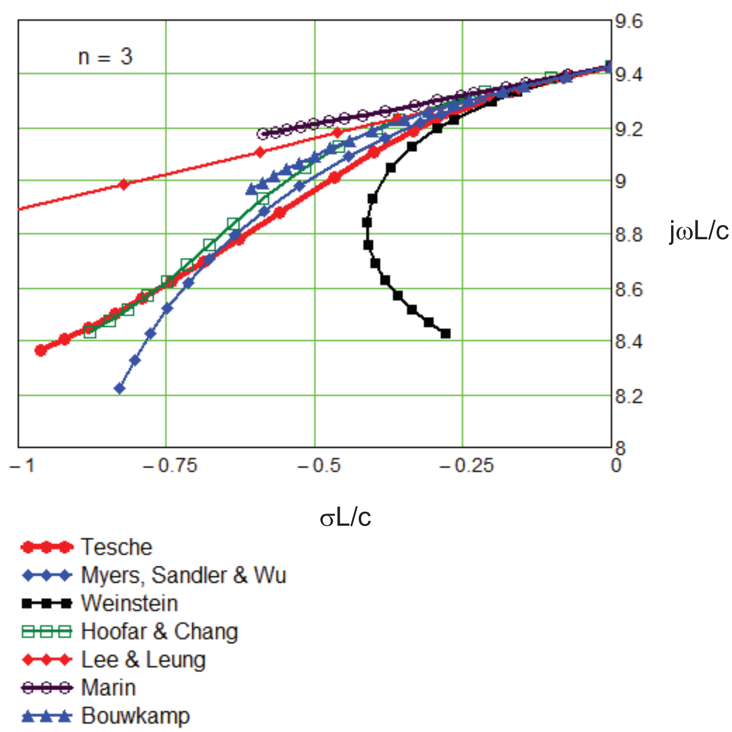


Figure 6. Plots of the trajectories of the *n* = 3 normalized complex resonant frequencies for the various analysis methods of Section 2.

4. Comments and Conclusions

From this comparison of the different methods of computing the complex resonant frequencies of a thin wire, we note that the numerical search and the contour integration methods applied to the method of moments system matrix provide consistent, and presumably, the most accurate solutions. The next most accurate approach is that of Myers et al. [14]. While it is based on the integral equation for the wire current, it is able to predict only the first layer poles. The resulting resonances agree reasonably well with those from the numerical solution of the integral equation.

For extremely thin wires, all methods provide resonant frequencies that are similar to the integral equation results. However, the accuracy of all of the approximate methods deteriorate at the wire becomes thicker. The results from Lee and Leung and from Weinstein appear to have the largest deviations for the thicker wires with $d/L > .005$.

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