LIGHTNING PHENOMENOLOGY NOTES

NOTE 8

APRIL 1983

NATURAL FREQUENCIES OF A POST ATTACHED TO A LIGHTNING RETURN STROKE

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ABSTRACT

The natural frequencies of a metallic post attached to a lightning channel are calculated. Two different lightning channel models are discussed and the asymptotic antenna technique is employed to solve the problem formulated in terms of an integro-differential equation. The results are presented by way of figures and approximate analytic formulas.

PREFACE

We wish to thank R. Aguero and J. Butman of Dikewood for performing the numerical calculation, to Dr. C.E. Baum and Lt. D. Andersh of AFWL for their interest and suggestions in this effort.

CONTENTS

I. INTRODUCTION

The objective of this effort is to calculate the natural frequencies of a metallic post attached to a lightning channel. Two different lightning channel models will be employed for the calculation. The natural frequencies calculated herefrom can be compared with measured data when the latter become available. Such a comparison will enable one to determine the parameters inherent in the models. With the models the electromagnetic characteristics, such as the fields, of a lightning channel can be calculated and simulated.

Section II describes the two different lightning channel models. Section III details the approach which eventually leads to transcendental equations required for numerical computation. Explicit approximate analytical formulas for the natural frequencies are also derived in Section III. The transcendental equations are described by Equations 18 and 19, and the explicit analytical formulas are given by Equations 20 through 22. Section IV presents the numerical results.

II. MODELS

In this section two lightning models, which may be referred to as the resistive model and the corona-sheath model, are introduced for the calculation of the natural frequencies of a post with a lightning channel attachment. The important features and the required parameters of these models are described below.

1. RESISTIVE MODEL

In Reference 1, it was demonstrated that a lightning channel and its surrounding corona could be represented by a finitely conducting rod with an effective radius. Such a representation (Fig. 1a), according to Reference 2, can be considered as a transmission line. The required parameters for this model are

- the effective radius r_e, and
- the effective resistance per unit length R'.

In terms of these two parameters, the lightning channel has a characteristic impedance given by (See Ref. 2)

$$Z_c^r(s) = \frac{Z_o}{2\pi} \psi^r(s) \tag{1}$$

where

$$\psi^r(s) \simeq \left\{ \left[\ell n(sr_e/c)\right]^2 - \frac{2\pi R' \ell n(sr_e/c)}{s\mu_o} \right\}^{1/2}$$

$s = j\omega$ is the complex frequency; Z_o, c, μ_o are, respectively, the free-space wave impedance, speed of light, permeability; and the superscript "r" is used to indicate "resistive" model quantities.

2. CORONA-SHEATH MODEL

The corona-sheath model has been suggested in Reference 3 for a lightning channel and its surrounding corona. In this model, the lightning current flows only in the perfectly conducting center channel, while all the electric

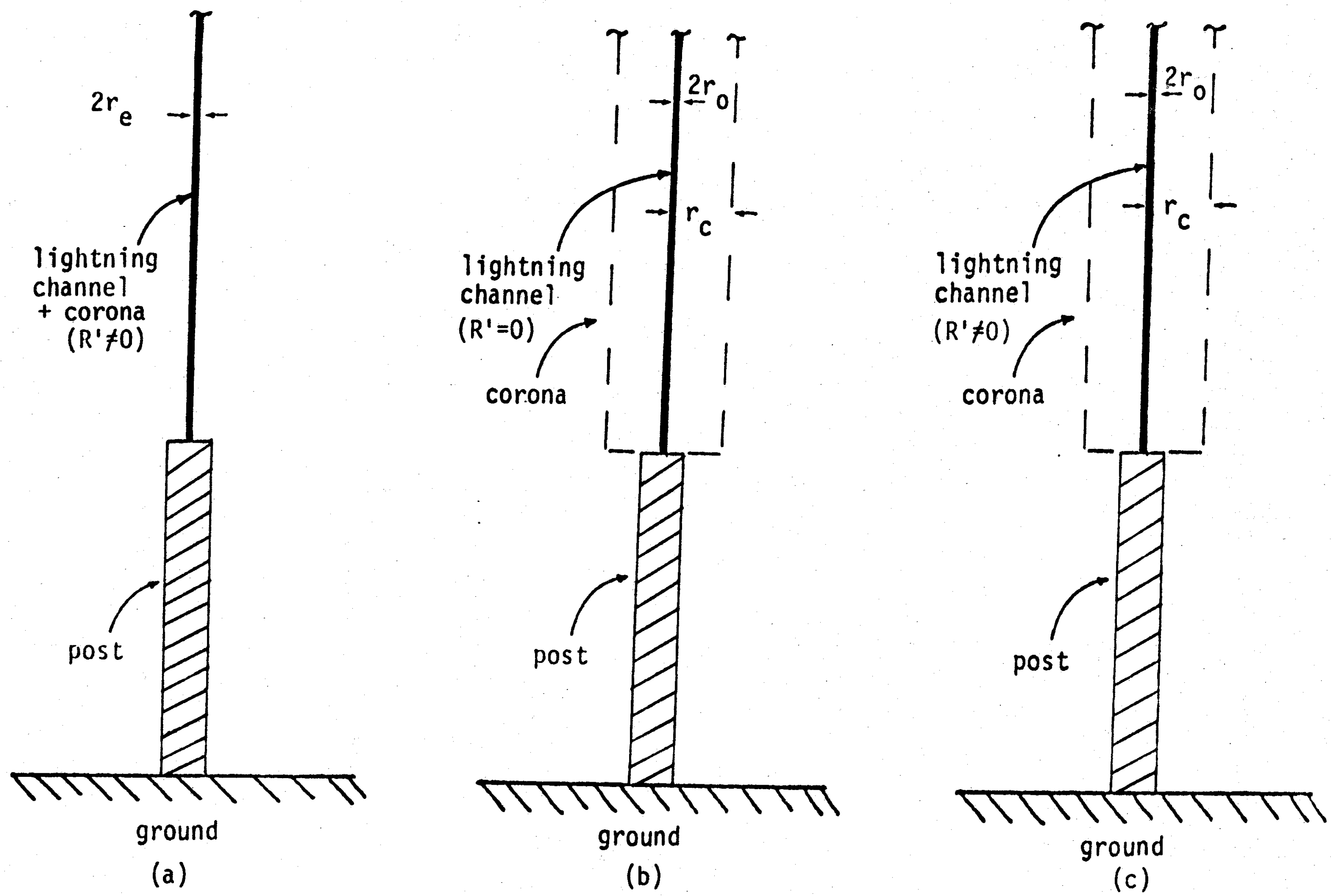

Figure 1. Various models for lightning which strikes a metallic post. (a) Resistive model, (b) corona-sheath model, and, (c) a general representation for both (a) and (b). In the figure, R' is the resistance per unit length.

charges reside on an effective corona surface. This model is shown in Figure 1b together with a post. The required parameters for the corona-sheath model are

- the effective radius of the lightning center channel r_o, and
- the effective radius of the corona surface r_c.

The effective corona radius r_c is related to the charge per unit length of the corona Q' via

$$r_c = \frac{|Q'|}{2\pi\epsilon_o E_b} \tag{2}$$

where E_b and ϵ_o are the air breakdown electric field and the permittivity of the surrounding air. This model can also be considered as a transmission line with the characteristic impedance

$$Z_c^c(s) = \frac{Z_o}{2\pi} \psi^c(s) \tag{3}$$

where

$$\psi^c(s) \simeq \left\{ \ell n \left(\frac{sr_o}{c}\right) \ell n \left(\frac{sr_c}{c}\right) \right\}^{1/2}$$

and the superscript "c" is used to indicate "corona-sheath" model quantities. Clearly, the above two lightning models are special cases of a more complicated model depicted in Figure 1c, which allows for resistivity in the center channel and, therefore, has the following corresponding characteristic impedance

$$\begin{aligned} Z_c(s) &= \frac{Z_o}{2\pi} \psi(s) \\ &\simeq \frac{Z_o}{2\pi} \left\{ \ell n \left(\frac{sr_o}{c}\right) \ell n \left(\frac{sr_c}{c}\right) - \frac{2\pi R' \, \ell n \left(sr_c/c\right)}{s\mu_o} \right\}^{1/2} \end{aligned} \tag{4}$$

The next section will describe an approach to solving this more general model. The two special models can then be obtained by taking

- $r_o = r_c = r_e$ for the resistive model, and
- $R' = 0$ for the corona-sheath model.

III. APPROACH

The original problem of Figure 1c where the post sits vertically on a perfectly conducting ground is equivalent to that of Figure 2 where the images are used to account for the ground effects. The natural modes of the imaged problem can be obtained by solving for the nontrivial solutions of the following homogeneous integro-differential equation with certain appropriate boundary conditions (see Ref. 4 and the references quoted therein):

$$\left(\frac{d^2}{dz^2} - \frac{s^2}{c^2}\right) \int_{-\ell}^{\ell} \frac{e^{-(s/c)\sqrt{a^2 + (z - z')^2}}}{4\pi\sqrt{a^2 + (z - z')^2}} I(z')dz' = 0 , \quad |z| \leq \ell \qquad (5)$$

where a and ℓ are, respectively, the radius and length of the post.

Equation 5 can first be reduced to an integral equation

$$\int_{-\ell}^{\ell} \frac{e^{-(s/c)\sqrt{a^2 + (z - z')^2}}}{\sqrt{a^2 + (z - z')^2}} I(z')dz' = B \cosh(sz/c), \quad |z| \leq \ell \qquad (6)$$

after imposing the following symmetrical condition

$$I(z) = I(-z) \qquad (7)$$

Additional conditions are required for solving Equation 6: they are the continuation of current and the equality of potential at the attachment point of the post and lightning channel, i.e., at $z = \ell$,

$$I_-(\ell) = I_+(\ell) = I(\ell)$$

$$\phi_-(\ell) = \phi_+(\ell) = \phi(\ell) \qquad (8)$$

where the subscript "-" or "+" indicates quantities at the post side or the

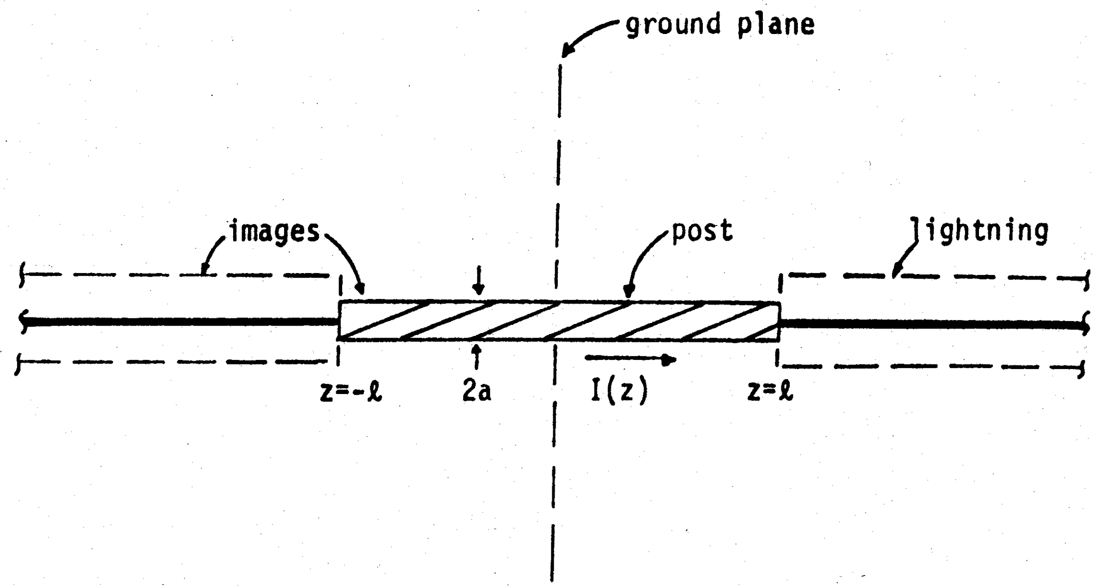

Figure 2. An equivalent representation of Figure 1c, with the ground effect being accounted for with images.

lightning side.

Since the potential at the post side is given by

$$\phi_- = -\frac{1}{4\pi s\varepsilon_0}\frac{d}{dz}\int_{-\ell}^{\ell}\frac{e^{-(s/c)\sqrt{a^2+(z-z')^2}}}{\sqrt{a^2+(z-z')^2}}I(z')dz'\Bigg|_{z=\ell} \tag{9}$$

and the potential at the lightning side is equal to the potential drop for a current I_+ flowing through $Z_c(s)$, i.e.,

$$\phi_+ = I_+Z_c(s) \tag{10}$$

Equation 8 is then equivalent to

$$\frac{d}{dz}\int_{-\ell}^{\ell}\frac{e^{-(s/c)\sqrt{a^2+(z-z')^2}}}{\sqrt{a^2+(z-z')^2}}I(z')dz'\Bigg|_{z=\ell} = -4\pi s\varepsilon_0 Z_c(s)I(\ell) \tag{11}$$

which can be combined with Equation 6 to give

$$\int_{-\ell}^{\ell}\frac{e^{-(s/c)\sqrt{a^2+(z-z')^2}}}{\sqrt{a^2+(z-z')^2}}I(z')dz' = \frac{-2\psi(s)I(\ell)}{\sinh(s\ell/c)}\cosh(sz/c) \tag{12}$$

This is the final integral equation that one needs to solve for the natural modes of the problem posed in Figure 2.

Since the radius a of the post is much smaller than its length ℓ, an asymptotic theory can be used to solve Equation 12. First, one rewrites Equation 12 in the following form

$$I(z)\int_{-\ell}^{\ell}\frac{1}{\sqrt{a^2+(z-z')^2}}\,dz' + \int_{-\ell}^{\ell}\frac{I(z')e^{-(s/c)\sqrt{a^2+(z-z')^2}} - I(z)}{\sqrt{a^2+(z-z')^2}}\,dz'$$

$$= -\frac{2\psi(s)I(\ell)}{\sinh(s\ell/c)}\cosh(sz/c) \tag{13}$$

or equivalently,

$$I(z)\Omega(z) + \frac{2\psi(s)I(\ell)}{\sinh(s\ell/c)}\cosh(sz/c)$$

$$= -\int_{-\ell}^{\ell}\frac{I(z')e^{-(s/c)\sqrt{a^2+(z-z')^2}} - I(z)}{\sqrt{a^2+(z-z')^2}}\,dz' \tag{14}$$

where

$$\Omega(z) = \sinh^{-1}\left(\frac{\ell+z}{a}\right) + \sinh^{-1}\left(\frac{\ell-z}{a}\right) \tag{15}$$

is a slowly varying function of z with $\Omega(0) = 2\sinh^{-1}(\ell/a) \simeq 2\ln(2\ell/a)$. Using the fact that $\Omega(z) \simeq \Omega(0) >> 1$ and $|\psi(s)| >> 1$, one can expand s and I(z) in the following fashion

$$s = s^{(0)} + s^{(1)} + \cdots\cdots$$

$$I = I^{(0)} + I^{(1)} + \cdots\cdots \tag{16}$$

where $s^{(m)}$, $I^{(m)}$ are $O(1/\Omega^m, 1/\psi^m)$. Substituting Equation 16 in Equation 14, one has, after a tedious but straightforward mathematical manipulation,

$$I^{(0)}(z) = A_0\,\frac{\Omega(0)}{\Omega(z)}\cosh\left(s^{(0)}z/c\right) \tag{17}$$

$$1 + \frac{2\psi(s^{(0)})}{\Omega(\ell)}\coth\left(s^{(0)}\ell/c\right) = 0 \tag{18}$$

and

$$2\,\frac{s^{(1)}\ell}{c} = \left\{ \Omega(0) + \frac{2\Omega(0)}{\Omega(\ell)}\,\frac{c}{\ell}\cosh^2\left(\frac{s^{(0)}\ell}{c}\right)\frac{d\psi\left(s^{(0)}\right)}{ds^{(0)}} \right\}^{-1}$$

$$\cdot\; e^{s^{(0)}\ell/c}\,\sinh\left(\frac{s^{(0)}\ell}{c}\right) E\left(\frac{4s^{(0)}\ell}{c}\right) \qquad (19)$$

where (see Ref. 5)

$$E(\zeta) \equiv E_1(\zeta) + \ln\zeta + \gamma$$

$$E_1(\zeta) \equiv \int_{\zeta}^{\infty} \frac{e^{-t}}{t}\,dt \equiv \text{exponential integral}$$

$$\gamma \equiv 0.577\ldots\ldots \equiv \text{Euler's constant.}$$

When $|\psi(s)| \to \infty$, Equations 16 through 19 agree, as they should, with Equations 16 and 17 of Reference 4 where a post without a lightning attachment was discussed. Equation 18 can be used to solve for $s^{(0)}$, from which $s^{(1)}$ can then be calculated via Equation 19.

When $R'\ell >> Z_o$ and/or $a >> (r_o, r_c, r_e)$, one has $|\psi(s)| >> \Omega(\ell)$, and Equation 18 can be approximately solved. The approximate solution is

$$\frac{s_n^{(0)}\ell}{c} \simeq j\frac{(2n+1)\pi}{2} - \frac{\Omega(\ell)}{2\psi_n}\;, \quad n = 0,1,2,\ldots \qquad (20)$$

where $\psi_n = \psi(s)$ with $s = j(2n+1)\pi c/(2\ell)$, i.e.,

$$\psi_n \simeq \left\{ \left(\ln\left[\frac{2\ell}{(2n+1)\pi r_o}\right] - j\frac{\pi}{2}\right)\left(\ln\left[\frac{2\ell}{(2n+1)r_c\pi}\right] - j\frac{\pi}{2}\right) - \frac{R'\ell}{Z_o}\,\frac{4j}{(2n+1)}\left(\ln\left[\frac{2\ell}{(2n+1)r_c\pi}\right] - j\frac{\pi}{2}\right)\right\}^{1/2} \qquad (21)$$

Furthermore, Equation 19 can be approximated by

$$\frac{s_n^{(1)}\ell}{c} \simeq -\frac{1}{2\Omega(0)}\left\{\ell n[(2n+1)2\pi] + \gamma + j\left[\frac{\pi}{2} - \frac{1}{(2n+1)2\pi}\right]\right\} \qquad (22)$$

In equations 20 through 22, $\Omega(\ell) \simeq \ell n(4\ell/a)$, $\Omega(0) \simeq 2\ell n(2\ell/a)$, and the subscript n indicates the quantity referring to the n-th natural mode.

The sum, $s_n^{(0)} + s_n^{(1)}$, obtained from Equations 20 through 22 gives the approximate analytical formula for the natural frequencies of a post with a lightning channel attachment. The accuracy of the approximate analytical formula will be checked against the numerical results computed from Equations 18 and 19 in Section IV. It is seen from Equation 20 and 22 that the effect of a channel attachment on the post's natural frequencies is entirely acounted for by the last term of Equation 20. This term is proportional to $(R'\ell/Z_0)^{-1/2}$ when the lightning channel is highly resistive, and vanishes, as expected, when $R' \to \infty$.

IV. NUMERICAL RESULTS

Equations 18 and 19 were numerically solved on a microcomputer for $\psi(s)$ given by Equation 4. The results of $s^{(0)} + s^{(1)}$ are given in Figures 3 through 14 for post radius a = 0.05 m, and various post length ℓ and lightning parameter values. The lightning parameters are the corona radius r_c, the channel radius r_0, the combined effective radius r_e, and the effective resistance R' per unit length (see Figure 1).

Figures 3 through 9 are the natural-frequency plots for the resistive model with $r_0 = r_c = r_e$ and nonzero $R'\ell/Z_0$. Figures 3 and 4 show the effect of the channel resistance R' on the post's natural frequencies for two different values of effective lightning radius r_e. The imaginary part of the natural frequencies remains almost unchanged as $R'\ell/Z_0$ varies. On the other hand, the absolute value of the real part (i.e., the decaying constant) increases when $R'\ell/Z_0$ decreases. This is expected because a post with a lightning attachment of lower resistance is different from a post without a lightning attachment.

Figures 5 and 6 show the effect of the effective lightning radius r_e on the post's natural frequencies for $R'\ell/Z_0 \gtrsim 10^4$ and equal to 10. The natural frequencies are found to depend only weakly on r_e. This is because the dependence of the characteristic impedance Z_c of a lightning channel on r_e is logarithmic. Also, when the channel is very resistive, i.e., when $R'\ell/Z_0 \gtrsim 10^4$, the natural frequencies become independent of r_e, and are the same as those of a post without a lightning attachment.

Figures 7 through 9 show the effect of the post length ℓ on the natural frequencies. When $R'\ell/Z_0$ is large (i.e., $\gtrsim 10^4$), the decaying constant increases as ℓ decreases. The reason is that there is more radiation loss for a thicker post of a given length. When $R'\ell/Z_0$ is small ($\lesssim 10$), the dependence of the normalized natural frequency on ℓ is weak. This is attributable to the fact that the damping of the post current is mainly caused by the leakage into the lightning channel whose dependence on ℓ is logarithmic.

Figures 10 through 14 give the natural frequencies of the corona-sheath model with $R'\ell\, Z_0 = 0$. Figures 10 and 11 show the effect of the corona radius r_c whereas Figures 12 through 14 show the effect of the post length ℓ on the post's natural frequencies. The normalized natural frequencies are nearly

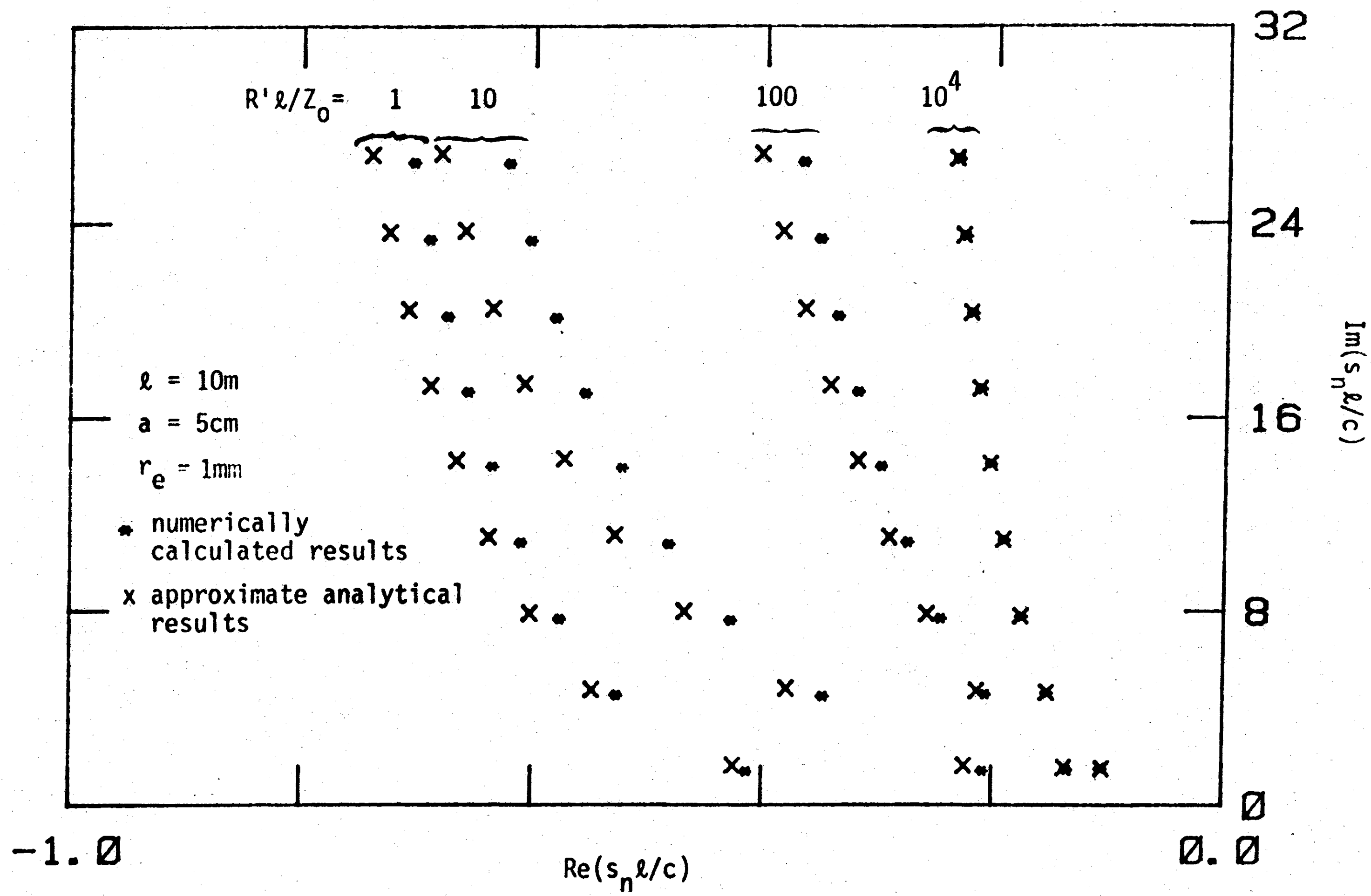

Figure 3. Normalized natural frequencies of a post of 10m long, 5 cm in radius, with a lightning attachment of r_e=1mm and various R'.

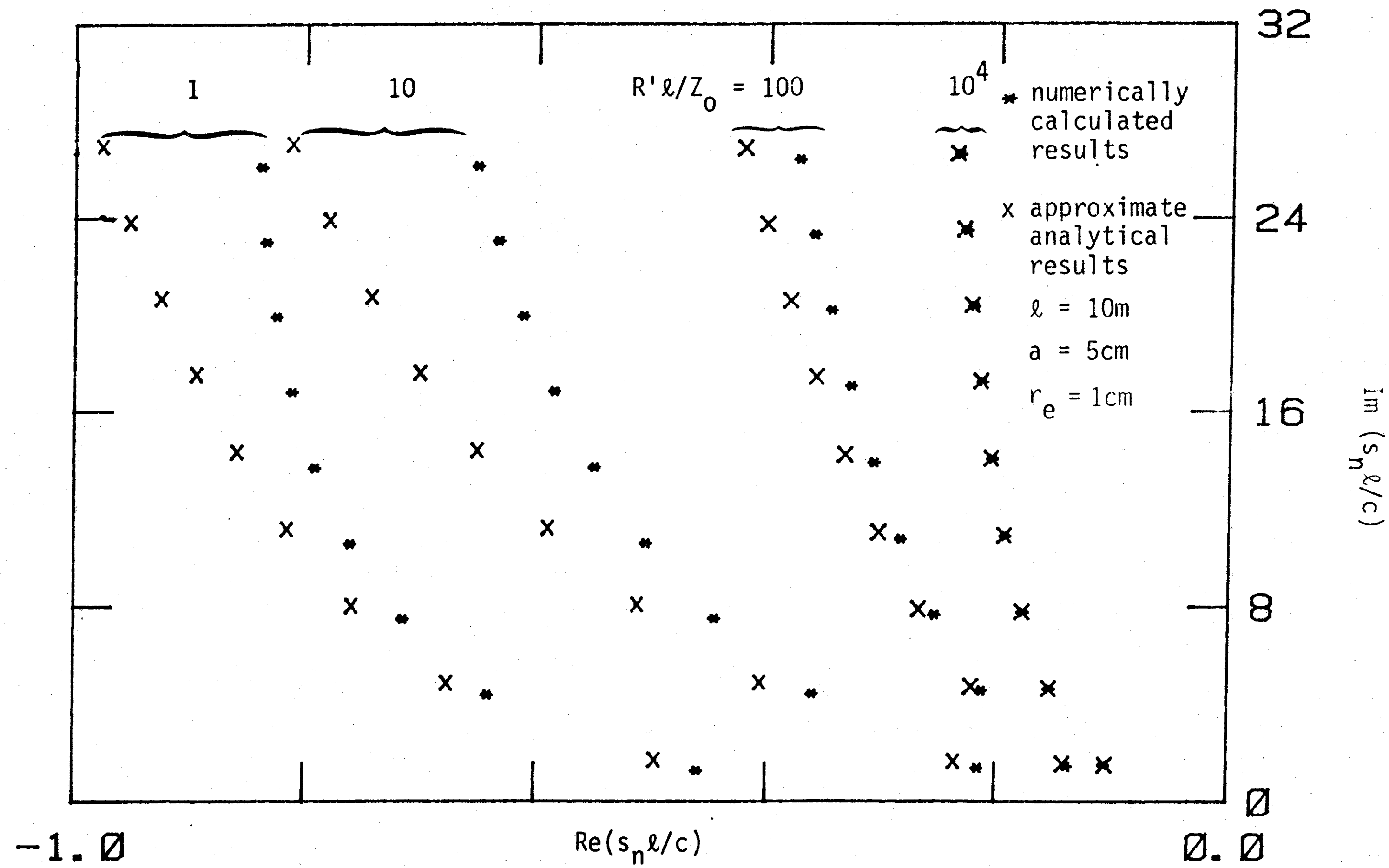

Figure 4. Normalized natural frequencies of a post of 10m long, 5cm in radius, with a lightning attachment of r_e=1cm and various R'.

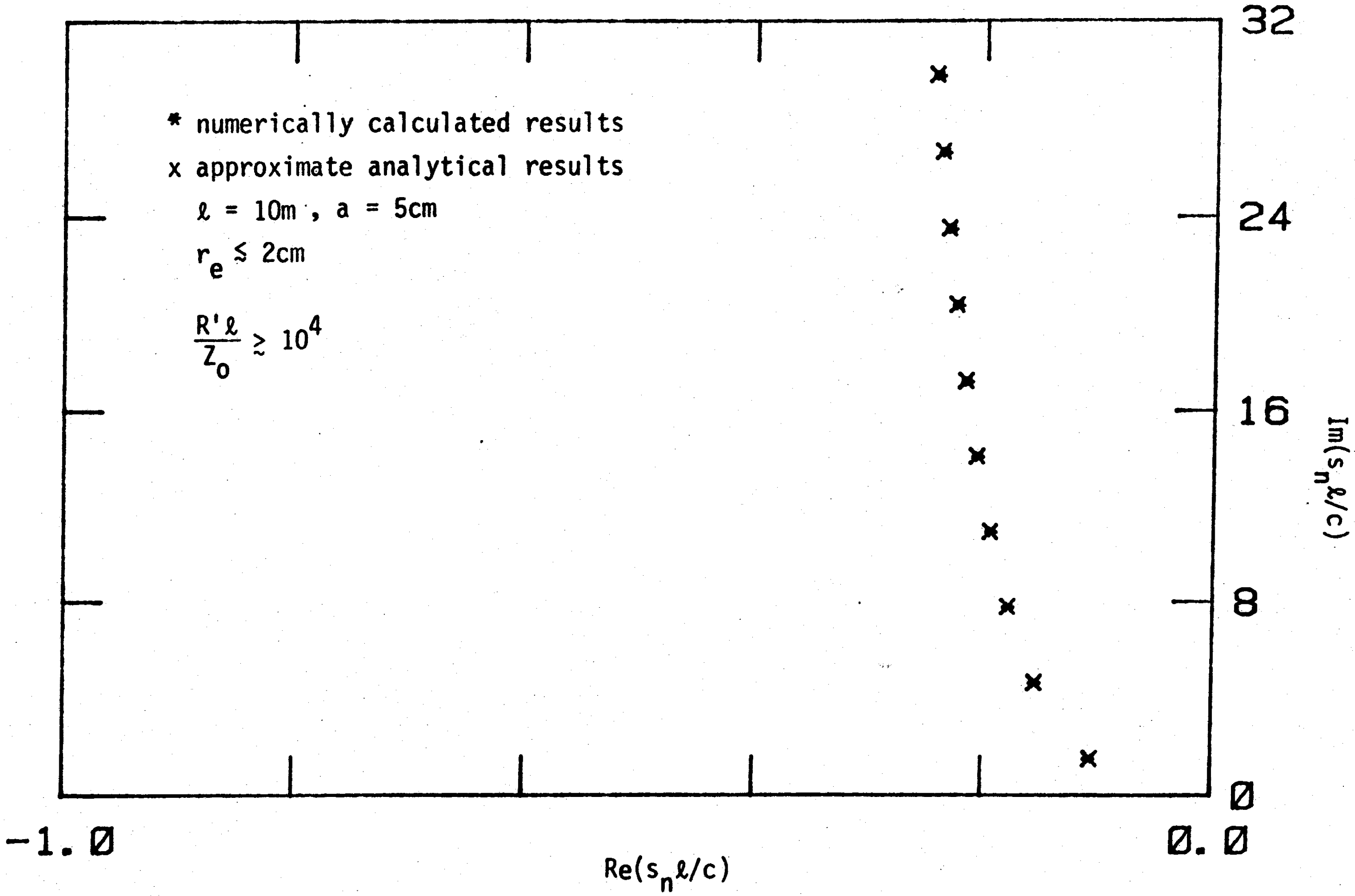

Figure 5. Normalized natural frequencies of a post of 10m long, 5cm in radius, with a lightning attachment of $r_e \lesssim 2$cm and $R'\ell/Z_o \gtrsim 10^4$.

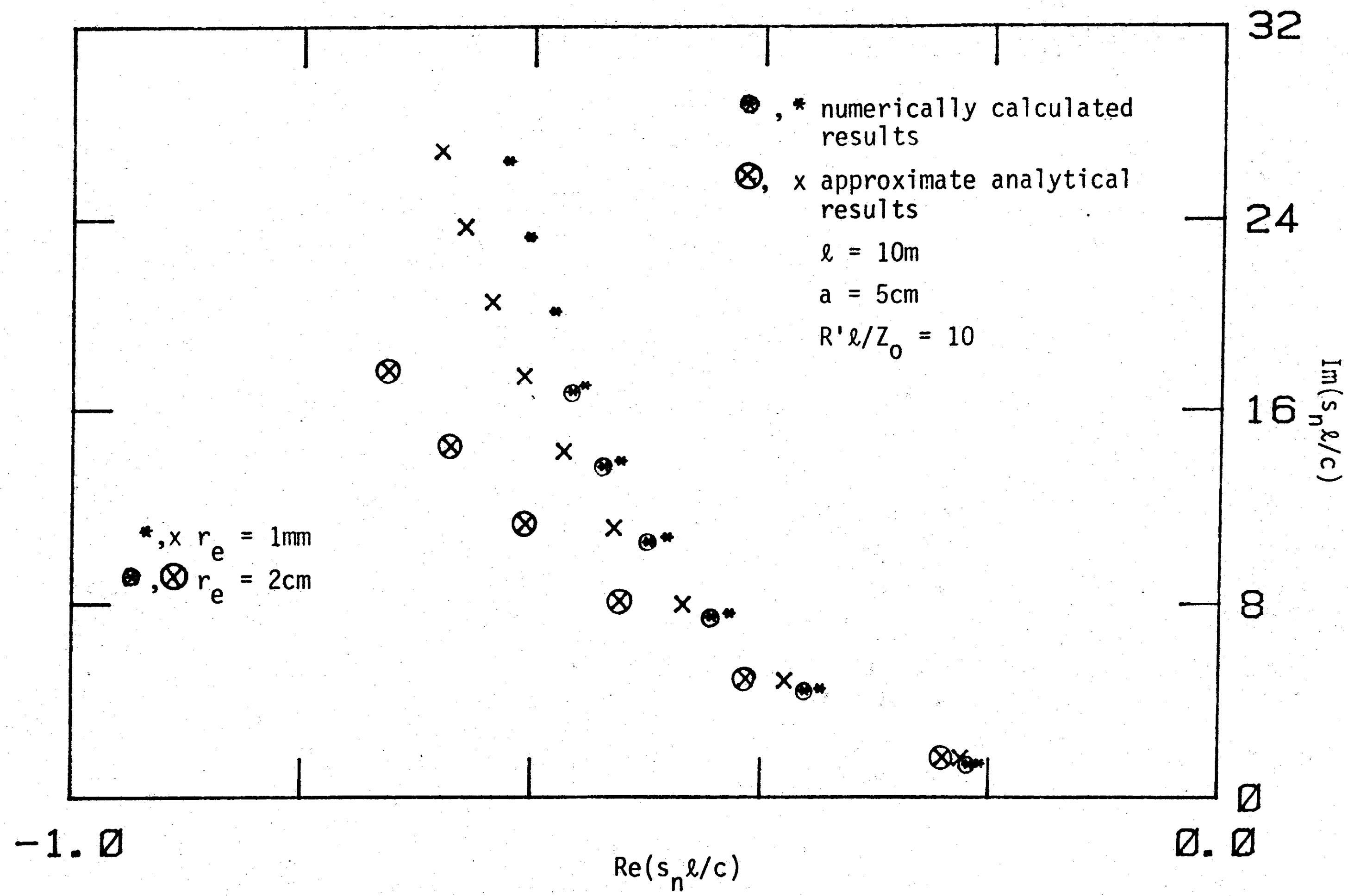

Figure 6. Normalized natural frequencies of a post of 10m long, 5cm in radius, with a lightning attachment of $R'\ell/Z_o = 10$ and various r_e.

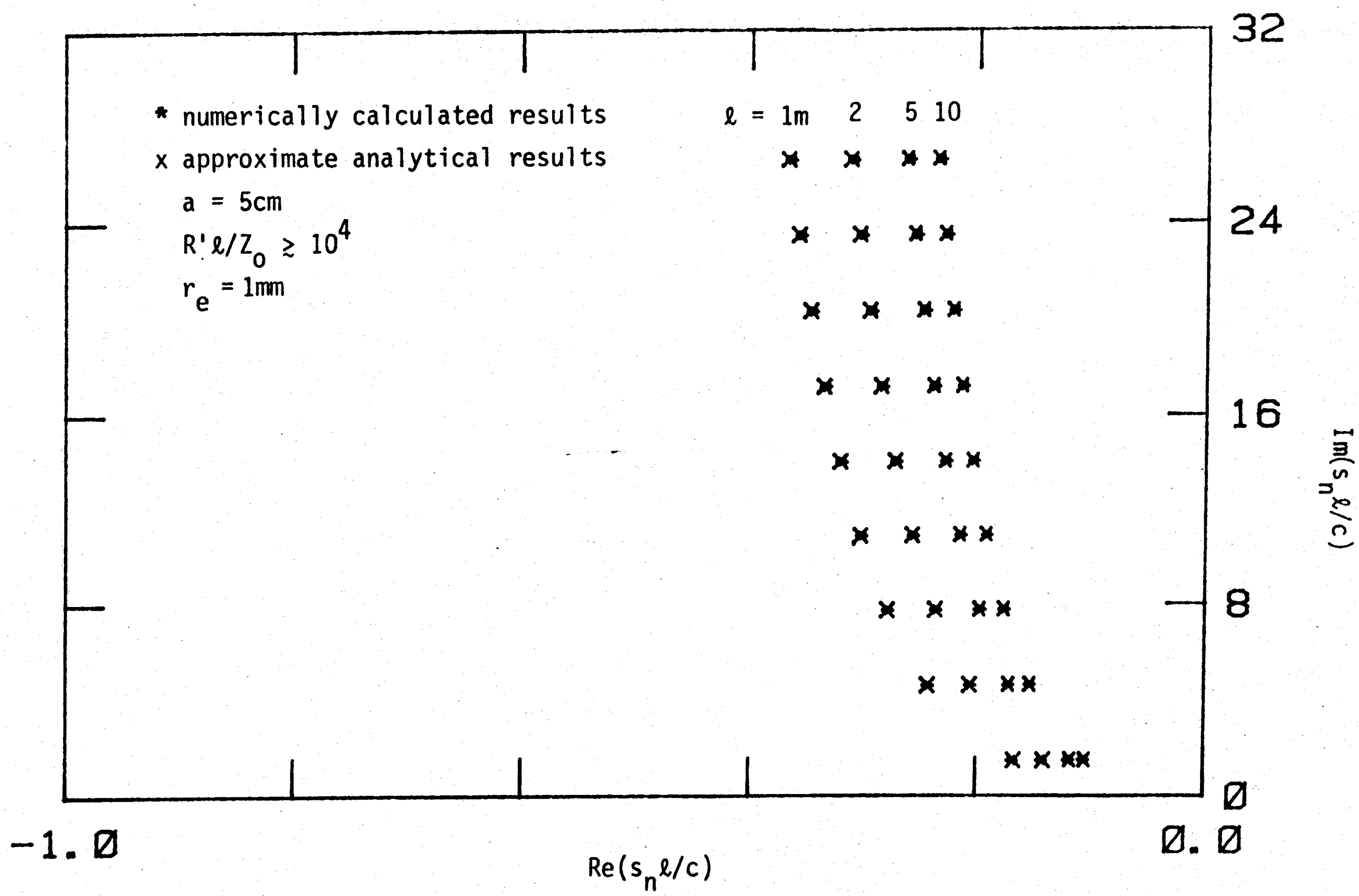

Figure 7. Normalized natural frequencies of a post of various length and 5cm in radius, with a lightning attachment of r_e=1mm and $R'\ell/Z_o \geq 10^4$.

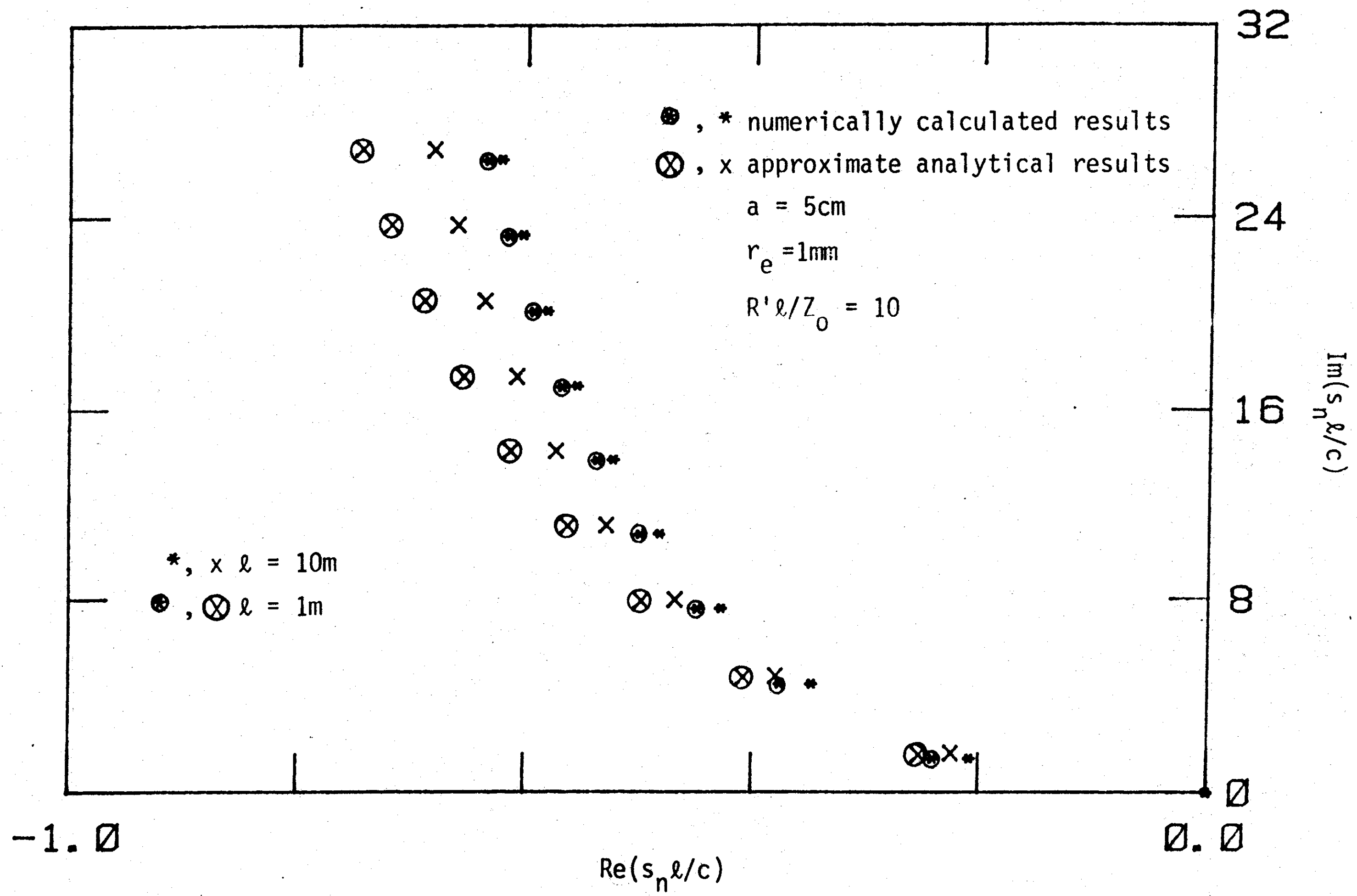

Figure 8. Normalized natural frequencies of a post of various length and 5cm in radius, with a lightning attachment of r_e=1mm and $R'\ell/Z_o$=10.

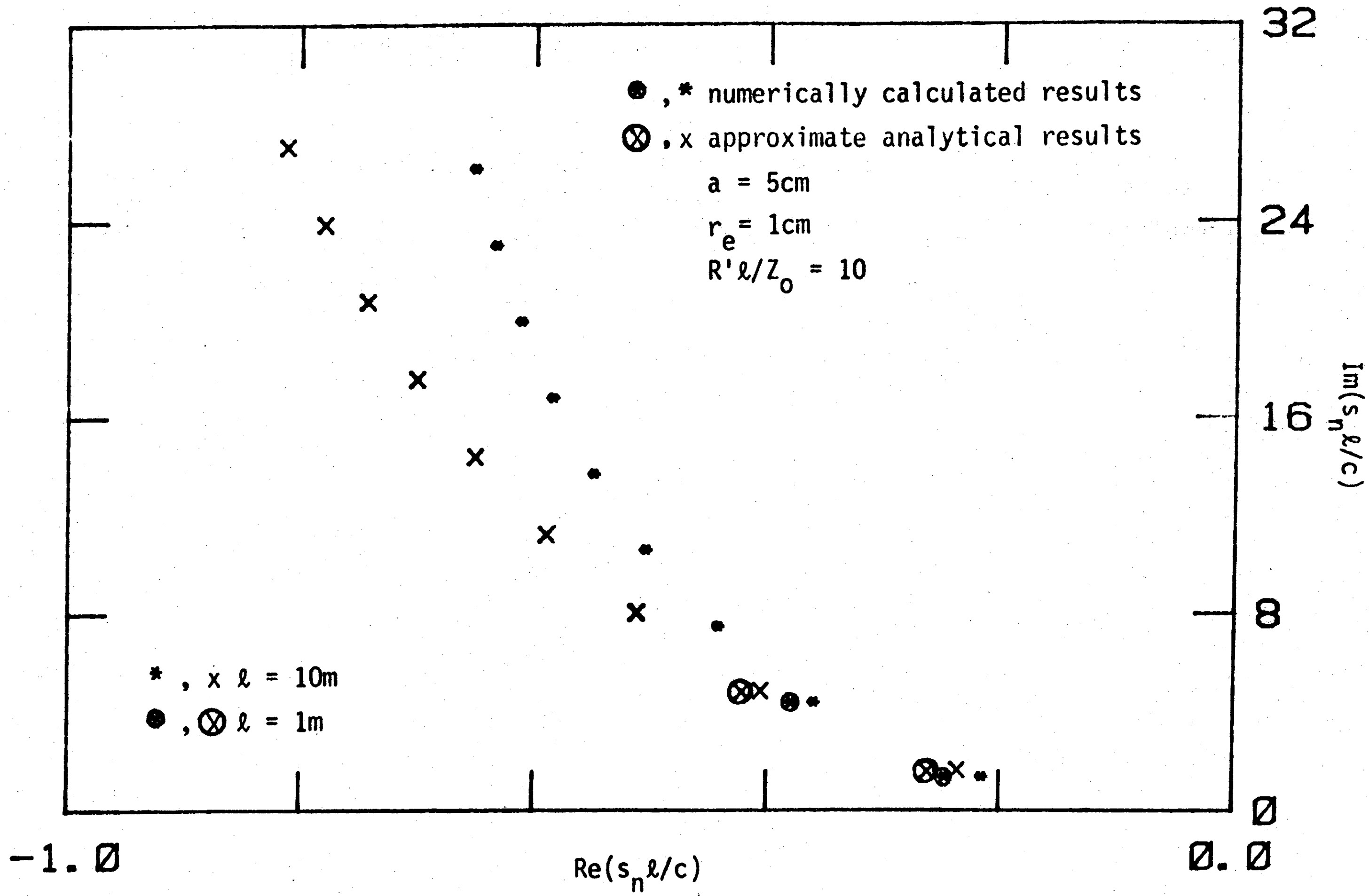

Figure 9. Normalized natural frequencies of a post of various length and 5 cm in radius, with a lightning attachment of r_e=1cm and $R'\ell/Z_o$=10.

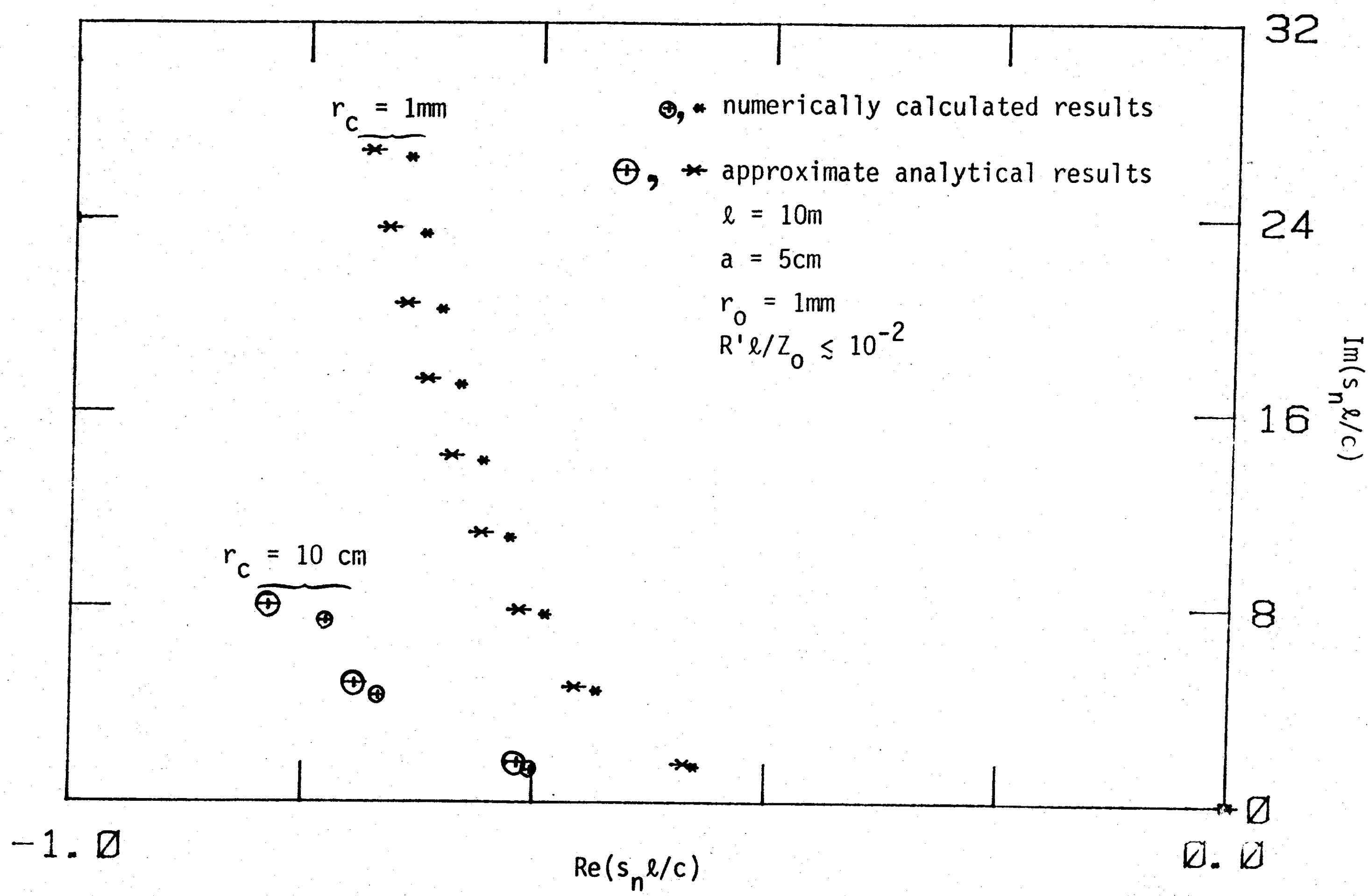

Figure 10. Normalized natural frequencies of a post of 10m long and 5 cm in radius with a lightning attachment of $R'\ell/Z_o \lesssim 10^{-2}$, r_o = 1mm and various r_c.

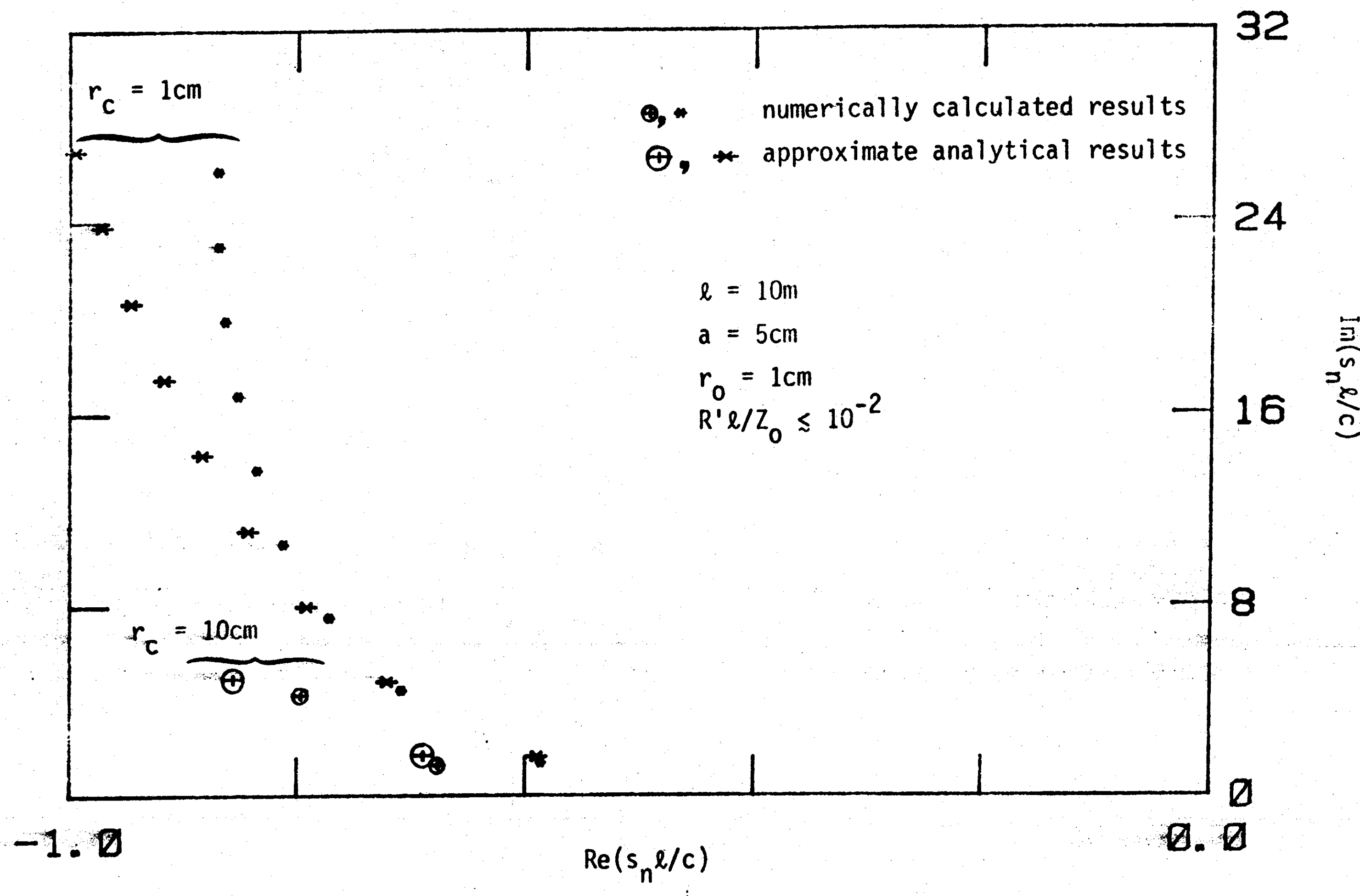

Figure 11. Normalized natural frequencies of a post of 10m long and 5cm in radius, with a lightning attachment of small $R'\ell/Z_o$, r_o = 1cm and various r_c.

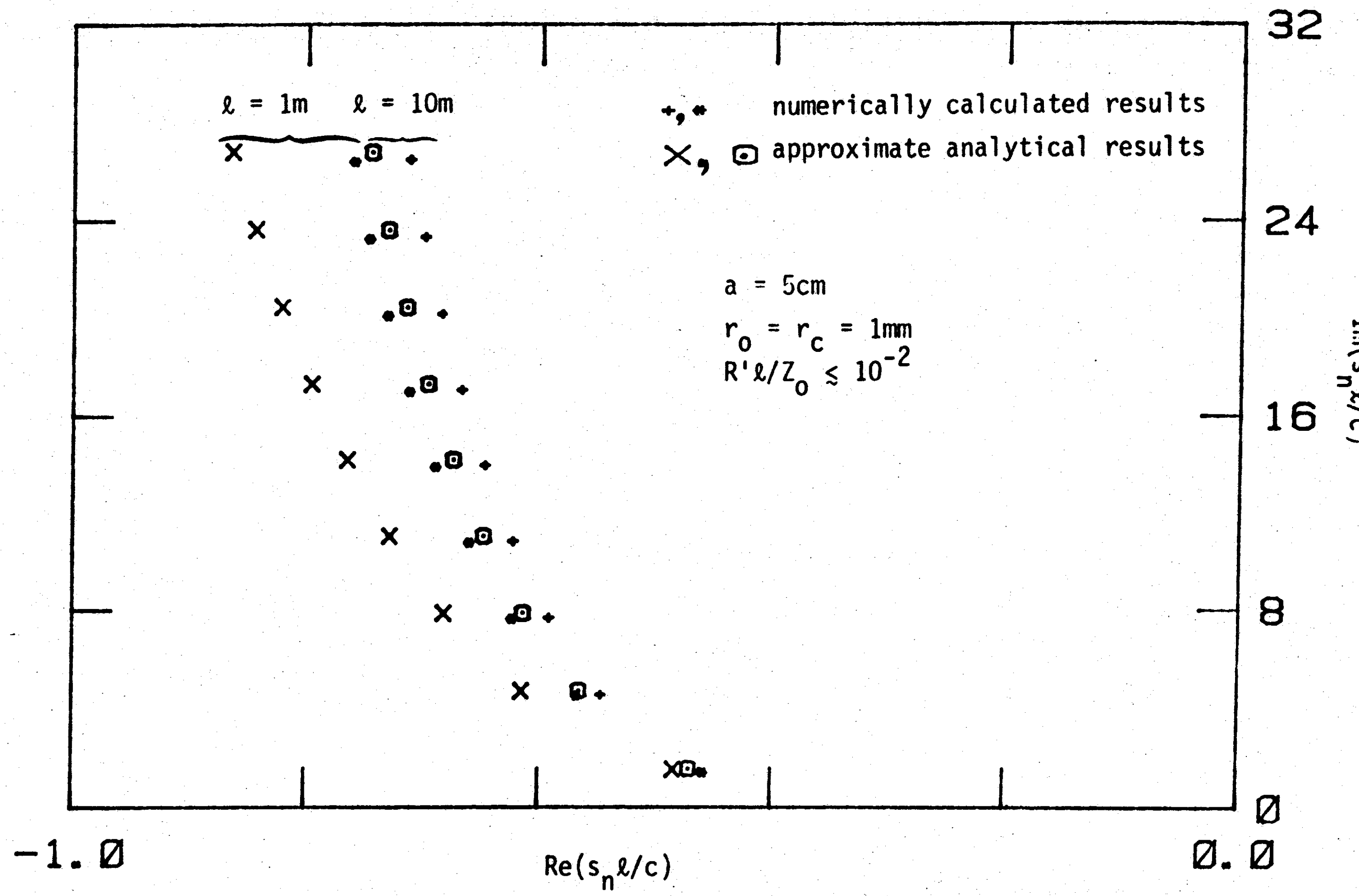

Figure 12. Normalized natural frequencies of a post of various length and 5cm in radius, with a lightning attachment of small $R'\ell/Z_o$ and $r_o=r_c=1mm$.

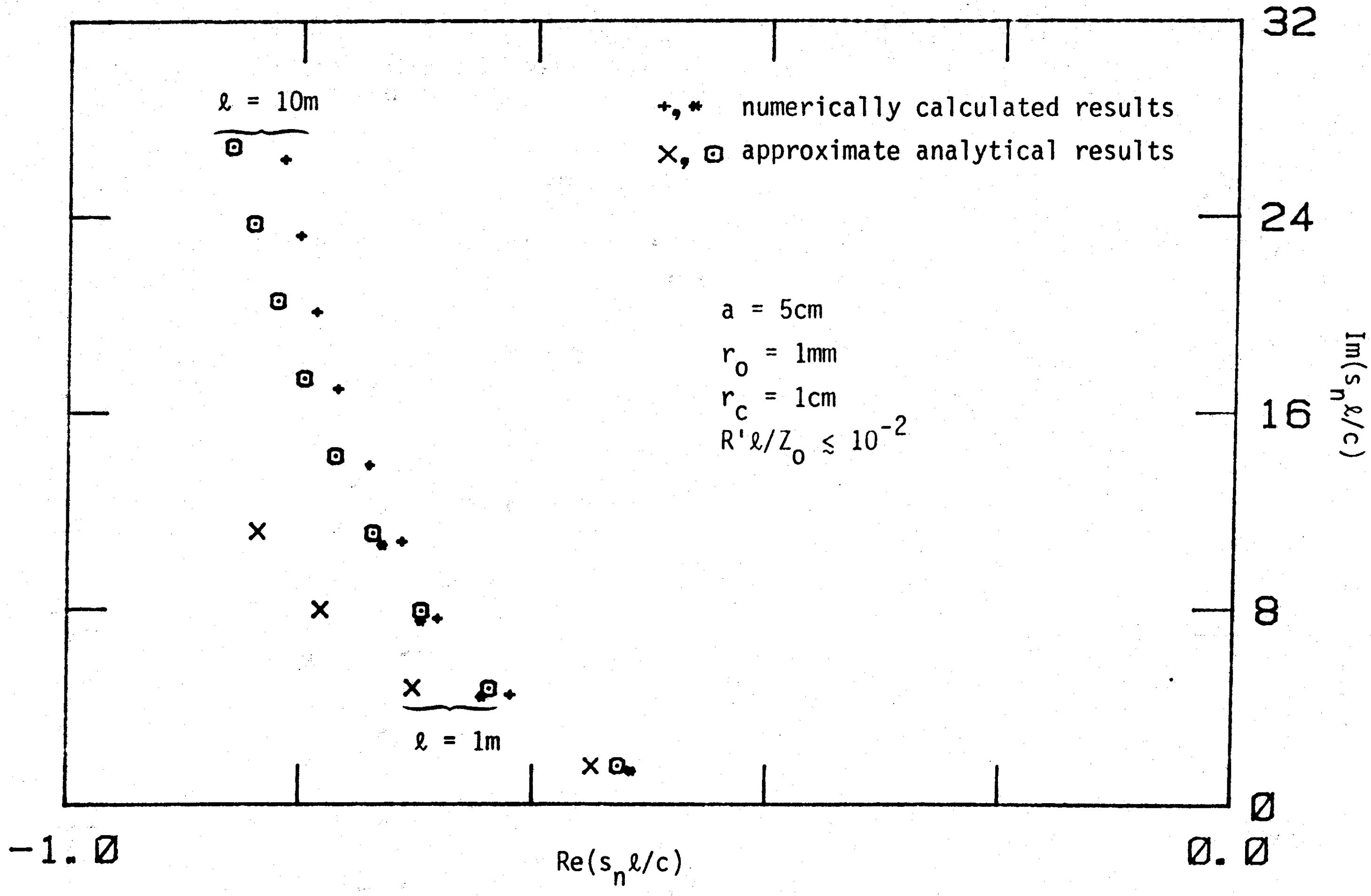

Figure 13. Normalized natural frequencies of a post of various length and 5cm in radius, with a lightning attachment of small $R'\ell/Z_o$, r_o=1mm, and r_c=1cm.

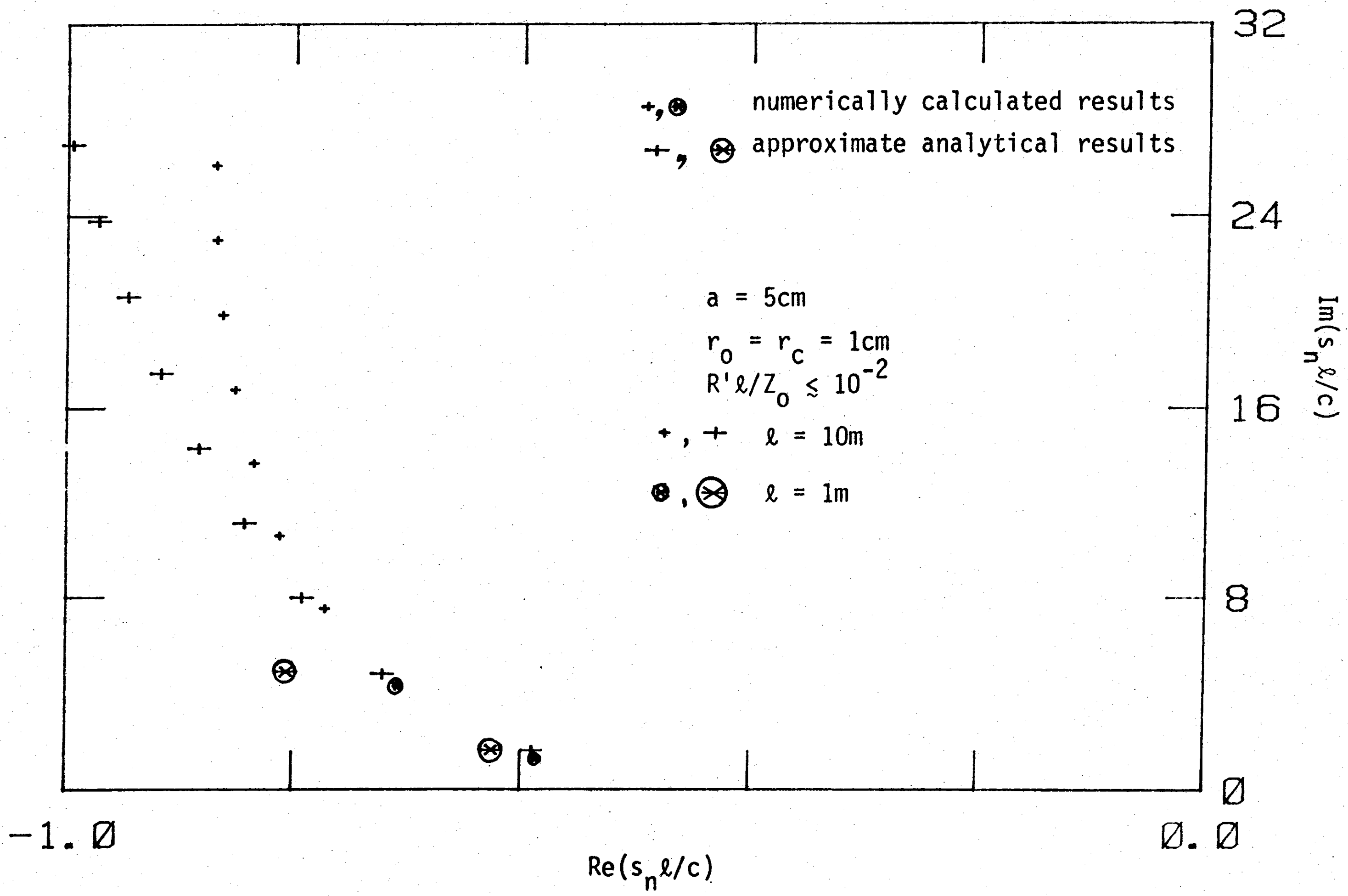

Figure 14. Normalized natural frequencies of a post of various length and 5cm in radius, with a lightning attachment of small $R'\ell/Z_o$ and $r_o=r_c=1cm$.

independent of ℓ.

The approximate analytical results given by Equations 20 and 22 are also plotted in Figures 3 through 14 for comparison with the numerical results. The agreement is generally excellent for the lower-order modes.

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