MATHEMATICS NOTES

NOTE 14

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TEF

A subroutine for the calculation of the Incomplete Elliptic Integrals of the First and Second Kind.

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Abstract

This note describes a computer subroutine which calculates the Incomplete Elliptic Integrals $F(\phi|m)$ and $E(\phi|m)$. The routine accepts any value both positive and negative of the amplitude $\phi$. The parameter $m$ is restricted to $0 \leq m \leq 1$. 
INTRODUCTION

Subroutine TEF is a computer code written in standard ASA FORTRAN IV for the Control Data Corporation 6600 computer at the Air Force Weapons Laboratory at Kirtland Air Force Base, New Mexico. Two completely different methods are employed to calculate the Incomplete Elliptic Integral of the First Kind $F(\phi \mid m)$ and the Second Kind $E(\phi \mid m)$ depending upon the size of the parameter $m$. Also listed is a routine to calculate the Complete Elliptic Integrals $K(m)$ and $E(m)$. 
The incomplete elliptic integrals of the First and Second Kind are defined as follows:

\[ F(\phi \mid m) = \int_0^\phi \frac{1}{\sqrt{1 - m \sin^2(\theta)}} \, d\theta \]

\[ E(\phi \mid m) = \int_0^\phi \frac{1}{\sqrt{1 - m \sin^2(\theta)}} \, d\theta \quad (1) \]

where \( \phi \) is the amplitude and \( m \) is the parameter. The complementary parameter \( m_1 \) is defined as

\[ m + m_1 = 1 \quad (2) \]

and

\[ m = \sin^2(\alpha) \quad (3) \]

where \( \alpha \) is the modular angle. It should be noted that the incomplete elliptic integrals are written in several different forms. Dependence on the parameter \( m \) is denoted by a vertical stroke preceding the parameter as written above. Dependence on the modular angle \( \alpha \) is denoted by a backward stroke preceding the modular angle as

\[ F(\phi \backslash \alpha), \quad E(\phi \backslash \alpha) \quad (4) \]

Dependence on the modulus \( k \) is denoted in one of two ways as

\[ F(\phi, k), \quad E(\phi, k) \]

or

\[ F(\phi \mid k), \quad E(\phi \mid k) \quad (5) \]
where
\[ m = k^2 \]
and
\[ m_1 = (k')^2 \]  \hspace{1cm} (6)

Several different forms are used in this note and these forms are generally dictated by the references.

For the purpose of the routine given in this note the parameter \( m \) is chosen for the input to the subroutine. The restrictions placed upon \( m \) are
\[ 0 \leq m \leq 1 \]  \hspace{1cm} (7)

The routine accepts amplitudes \( \phi \) of any magnitude using the relationship\(^1\)

\[ F(s \pi \pm \phi \mid m) = 2 s K \pm F(\phi \mid m) \]  \hspace{1cm} (8)

and
\[ E(s \pi \pm \phi \mid m) = 2 s E \pm E(\phi \mid m) \]  \hspace{1cm} (9)

where the complete elliptic integrals \( K \) and \( E \) are defined as the incomplete elliptic integrals with the amplitude equal to \( \frac{\pi}{2} \) as

\[ K(m) = K = F(\frac{\pi}{2} \mid m) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - m \sin^2 (\theta)}} \ d\theta \]  \hspace{1cm} (10)

and
\[ E(m) = E = E(\frac{\pi}{2} \mid m) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - m \sin^2 (\theta)}} \ d\theta \]  \hspace{1cm} (11)

\(^1\)Handbook of Mathematical Functions, AMS 55, M. Abramowitz and I. A. Stegun, Editors, National Bureau of Standards, 1964, eqns. 17.4.3 and 17.4.4.
The following relationships are used for negative amplitudes:

\[ F(-\phi \mid m) = -F(\phi \mid m) \]
\[ E(-\phi \mid m) = -E(\phi \mid m) \] (12)

The incomplete elliptic integral of the First Kind is difficult to calculate, especially for values of \( \phi \) near \( \pi/2 \) and \( m \) near unity. For this reason TEF calculates \( F(\phi \mid m) \) using two methods depending upon the size of the parameter \( m \). Consequently \( E(\phi \mid m) \) is also computed using two methods.

For values of \( m \) less than .75 the following infinite series are used to calculate the incomplete elliptic integral of the First and Second Kinds respectively:

\[
F(\phi \mid m) = \int_0^\phi \left[ 1 - m \sin^2(\theta) \right]^{-\frac{1}{2}} d\theta
\]

\[
= \frac{2\phi}{\pi} K - \sin(\phi) \cos(\phi) \left[ \frac{1}{2} A_2 m + \frac{1.3}{2.4} A_4 m^2 + \frac{1.3.5}{2.4.6} A_6 m^3 + \ldots \right] \] (13)

\[
E(\phi \mid m) = \int_0^\phi \left[ 1 - m \sin^2(\theta) \right]^{\frac{1}{2}} d\theta
\]

\[
= \frac{2\phi}{\pi} E + \sin(\phi) \cos(\phi) \left[ \frac{1}{2} A_2 m + \frac{1}{2.4} A_4 m^2 + \frac{1.3}{2.4.6} A_6 m^3 + \ldots \right] \] (14)

\(^2\)Reference 1, eqns. 17.4.1 and 17.4.2.

where \[ A_2 = \frac{1}{2} \]
\[ A_4 = \frac{3}{2\cdot4} + \frac{1}{4} \sin^2(\phi) \]
\[ A_6 = \frac{3\cdot5}{2\cdot4\cdot6} + \frac{5}{4\cdot6} \sin^2(\phi) + \frac{1}{6} \sin^4(\phi) \]
\[ A_8 = \frac{3\cdot5\cdot7}{2\cdot4\cdot6\cdot8} + \frac{5\cdot7}{4\cdot6\cdot8} \sin^2(\phi) + \frac{7}{6\cdot8} \sin^4(\phi) + \frac{1}{8} \sin^6(\phi) \] (15)

with \(K\) and \(E\) the complete elliptic integrals of the First and Second Kind.

This method although extremely fast requires many iterations, that is, terms (one term per iteration) of the series to produce the required degree of accuracy for \(m\) close to 1. For example, for \(\phi = 85^0\) and \(m = \sin^2(88^0)\) approximately 20,000 iterations were required to make the last term less than \(10^{-14}\). Consequently a second method is employed to produce \(F(\phi | m)\) for values of \(m\) close to unity. To calculate \(F(\phi | m)\) and \(E(\phi | m)\) the descending Landen transformation\(^4\) is used where \(\phi_1', \phi_2', \ldots, \phi_N\) are successively determined from

\[ \tan(\phi_{n+1} - \phi_n) = \frac{b_n}{a_n} \tan(\phi_n), \; n=0, 1, 2, \ldots N \] (16)

where \(\phi_0 = \phi\).

Then to the accuracy desired\(^5\)

\[ F(\phi | \alpha) = \frac{\phi_N}{2^N a_N} \] (17)

and

\[ E(\phi | \alpha) = \frac{E}{K} F(\phi | \alpha) + c_1 \sin(\phi_1) + c_2 \sin(\phi_2) + \ldots + c_N \sin(\phi_N) \] (18)

\(^4\)Reference 1, eqn. 17.6.8.

\(^5\)Reference 1, eqns. 17.6.9 and 17.6.10.
where \( a_n \) and \( b_n \) are determined by the process of the Arithmetic-Geometric Mean\(^6\). Starting with

\[
a_0 = 1, \quad b_0 = \cos (\alpha), \quad c_0 = \sin (\alpha)
\]

(19)

\( a_n, b_n, \) and \( c_n \) can then be determined as

\[
a_n = \frac{a_{n-1} + b_{n-1}}{2}
\]

\[
b_n = \left( a_{n-1} \cdot b_{n-1} \right)^{\frac{1}{2}}
\]

\[
c_n = \frac{1}{2} \left( a_{n-1} - b_{n-1} \right)
\]

where \( 0 \leq |c_n| \leq \epsilon \) to the degree of accuracy \( \epsilon \) specified.

It might be noted that \( E(\phi | m) \) can be accurately calculated using any one of several methods including Gaussian Quadrature which would essentially reproduce the tables of the incomplete elliptic integrals in reference 1. However, since most of the logic to compute \( E(\phi | m) \) is contained in the logic for \( F(\phi | m) \) the remaining logic was also added.

It can be shown that the summation given in equations 13 and 14 above and by the ratio test\(^7\) that

\[
|R_n| \leq \left| \frac{a_n^2}{a_n - a_{n+1}} \right|
\]

(20)

This test is used to determine the number of terms of the summation that are to be added to achieve the desired degree of accuracy.

\(^6\)Reference 1, eqns. 17.6.1 and 17.6.2.

\(^7\)Wilfred Kaplan, Advanced Calculus, Addison-Wesley Publishing Company, Reading, Massachusetts, 1952.
The discrepancies between the values of \( F(\phi | m) \) and \( E(\phi | m) \) given in reference 1 and the values returned from subroutine TEF are given in Tables 1 and 2. All of the values differ by no more than \( 10^{-8} \). The values of \( F(\phi | m) \) and \( E(\phi | m) \) were specially calculated by both methods in subroutine TEF as a test of the numerical methods involved. The series method and the method using the Landen transformation returned the same values of \( F(\phi | m) \) and \( E(\phi | m) \) when run in double precision. This fact leads one to doubt the accuracy of the tables\(^8\) of the incomplete elliptic integrals as given in the Handbook of Mathematical Functions. To avoid any error which might have been introduced by the routine that returns the complete elliptic integrals, any value of \( K(m) \) or \( E(m) \) that did not agree to \( 10^{-15} \) with the table in the AMS 55 was explicitly entered.

Subroutine TEF tests for special values of the parameter \( m \) and the amplitude \( \phi \) to avoid needless calculations. These special values and the method\(^9\) by which they are computed are as follows:

\[
F(\phi \mid 1) = \ln \left[ \tan \left( \frac{\pi}{4} + \frac{\phi}{2} \right) \right], \quad E(\phi \mid 1) = \sin (\phi) \tag{21}
\]
\[
F(\phi \mid 0) = \phi, \quad E(\phi \mid 0) = \phi \tag{22}
\]
\[
F(\frac{\pi}{2} \mid m) = K(m), \quad E(\frac{\pi}{2} \mid m) = E(m) \tag{23}
\]
\[
F(\frac{\pi}{2} \mid 1) = 10^{75} (\infty), \quad E(\frac{\pi}{2} \mid 1) = 1 \tag{24}
\]
\[
F(0 \mid 0) = 0, \quad E(0 \mid 0) = 0 \tag{25}
\]

The subroutine TEK is called by TEF to compute the complete elliptic integrals \( E(m) \) and \( K(m) \) accurate to about \( 10^{-12} \). Two methods

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\(^8\) Reference 1, pp. 613 - 618, Tables 17.5 and 17.6.

\(^9\) Reference 1, p. 594, eqns. 17.4.19, 17.4.21, 17.4.23 and 17.4.25.
Table 1.
Comparison of values of $F(\phi \setminus \alpha)$ as given in the Handbook of Mathematical Functions and values returned from subroutine TEF

<table>
<thead>
<tr>
<th>$F(\phi \setminus \alpha)$</th>
<th>value listed</th>
<th>computed value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F(5^\circ \setminus 48^\circ)$</td>
<td>0.08732765</td>
<td>0.08732766</td>
</tr>
<tr>
<td>$F(10^\circ \setminus 58^\circ)$</td>
<td>0.17517260</td>
<td>0.17517259</td>
</tr>
<tr>
<td>$F(10^\circ \setminus 62^\circ)$</td>
<td>0.17522690</td>
<td>0.17522691</td>
</tr>
<tr>
<td>$F(10^\circ \setminus 86^\circ)$</td>
<td>0.17542143</td>
<td>0.17542142</td>
</tr>
<tr>
<td>$F(15^\circ \setminus 44^\circ)$</td>
<td>0.26324404</td>
<td>0.26324403</td>
</tr>
<tr>
<td>$F(15^\circ \setminus 46^\circ)$</td>
<td>0.26335019</td>
<td>0.26335020</td>
</tr>
<tr>
<td>$F(20^\circ \setminus 70^\circ)$</td>
<td>0.35547959</td>
<td>0.35547958</td>
</tr>
<tr>
<td>$F(20^\circ \setminus 82^\circ)$</td>
<td>0.35622881</td>
<td>0.35622880</td>
</tr>
<tr>
<td>$F(25^\circ \setminus 28^\circ)$</td>
<td>0.43932365</td>
<td>0.43932364</td>
</tr>
<tr>
<td>$F(25^\circ \setminus 48^\circ)$</td>
<td>0.44404397</td>
<td>0.44404396</td>
</tr>
<tr>
<td>$F(25^\circ \setminus 74^\circ)$</td>
<td>0.44967538</td>
<td>0.44967539</td>
</tr>
<tr>
<td>$F(30^\circ \setminus 80^\circ)$</td>
<td>0.54842535</td>
<td>0.54842534</td>
</tr>
<tr>
<td>$F(35^\circ \setminus 50^\circ)$</td>
<td>0.63363947</td>
<td>0.63363946</td>
</tr>
<tr>
<td>$F(35^\circ \setminus 52^\circ)$</td>
<td>0.63511150</td>
<td>0.63511149</td>
</tr>
<tr>
<td>$F(35^\circ \setminus 64^\circ)$</td>
<td>0.64351521</td>
<td>0.64351520</td>
</tr>
<tr>
<td>$F(35^\circ \setminus 78^\circ)$</td>
<td>0.65067415</td>
<td>0.65067414</td>
</tr>
<tr>
<td>$F(35^\circ \setminus 84^\circ)$</td>
<td>0.65228622</td>
<td>0.65228621</td>
</tr>
<tr>
<td>$F(50^\circ \setminus 72^\circ)$</td>
<td>0.99163507</td>
<td>0.99163506</td>
</tr>
<tr>
<td>$F(55^\circ \setminus 86^\circ)$</td>
<td>1.15261652</td>
<td>1.15261651</td>
</tr>
<tr>
<td>$F(60^\circ \setminus 50^\circ)$</td>
<td>1.16431637</td>
<td>1.16431636</td>
</tr>
<tr>
<td>$F(60^\circ \setminus 56^\circ)$</td>
<td>1.19275650</td>
<td>1.19275649</td>
</tr>
<tr>
<td>$F(60^\circ \setminus 60^\circ)$</td>
<td>1.21259661</td>
<td>1.21259662</td>
</tr>
<tr>
<td>$F(60^\circ \setminus 84^\circ)$</td>
<td>1.31117166</td>
<td>1.31117165</td>
</tr>
<tr>
<td>$F(70^\circ \setminus 56^\circ)$</td>
<td>1.45726935</td>
<td>1.45726934</td>
</tr>
<tr>
<td>$F(75^\circ \setminus 46^\circ)$</td>
<td>1.49668437</td>
<td>1.49668438</td>
</tr>
<tr>
<td>$F(75^\circ \setminus 82^\circ)$</td>
<td>1.97316666</td>
<td>1.97316665</td>
</tr>
<tr>
<td>$F(80^\circ \setminus 82^\circ)$</td>
<td>2.31643897</td>
<td>2.31642896</td>
</tr>
<tr>
<td>$F(85^\circ \setminus 56^\circ)$</td>
<td>1.90143591</td>
<td>1.90143590</td>
</tr>
<tr>
<td>$F(85^\circ \setminus 66^\circ)$</td>
<td>2.13070052</td>
<td>2.13070051</td>
</tr>
</tbody>
</table>
Table 2.

Comparison of values of $E(\phi \backslash \alpha)$ as given in the Handbook of Mathematical Functions and values returned from subroutine TEF

<table>
<thead>
<tr>
<th>$E(\phi \backslash \alpha)$</th>
<th>value listed</th>
<th>computed value</th>
</tr>
</thead>
<tbody>
<tr>
<td>E(100 \ 700)</td>
<td>0.17375210</td>
<td>0.17375209</td>
</tr>
<tr>
<td>E(150 \ 680)</td>
<td>0.25924104</td>
<td>0.25924103</td>
</tr>
<tr>
<td>E(150 \ 480)</td>
<td>0.26016110</td>
<td>0.26016109</td>
</tr>
<tr>
<td>E(200 \ 740)</td>
<td>0.34256478</td>
<td>0.34256479</td>
</tr>
<tr>
<td>E(250 \ 740)</td>
<td>0.42368913</td>
<td>0.42368914</td>
</tr>
<tr>
<td>E(300 \ 840)</td>
<td>0.50026923</td>
<td>0.50026922</td>
</tr>
<tr>
<td>E(300 \ 740)</td>
<td>0.50186633</td>
<td>0.50186634</td>
</tr>
<tr>
<td>E(350 \ 720)</td>
<td>0.57733641</td>
<td>0.57733640</td>
</tr>
<tr>
<td>E(350 \ 380)</td>
<td>0.59723431</td>
<td>0.59723432</td>
</tr>
<tr>
<td>E(400 \ 200)</td>
<td>0.69206954</td>
<td>0.69206953</td>
</tr>
<tr>
<td>E(450 \ 480)</td>
<td>0.74409773</td>
<td>0.74409772</td>
</tr>
<tr>
<td>E(500 \ 540)</td>
<td>0.80601230</td>
<td>0.80601229</td>
</tr>
<tr>
<td>E(550 \ 460)</td>
<td>0.89246858</td>
<td>0.89246857</td>
</tr>
<tr>
<td>E(600 \ 640)</td>
<td>0.90689460</td>
<td>0.90689461</td>
</tr>
<tr>
<td>E(700 \ 580)</td>
<td>1.03614663</td>
<td>1.03614664</td>
</tr>
<tr>
<td>E(750 \ 820)</td>
<td>0.97598331</td>
<td>0.97598330</td>
</tr>
<tr>
<td>E(750 \ 760)</td>
<td>0.99517606</td>
<td>0.99517605</td>
</tr>
<tr>
<td>E(750 \ 700)</td>
<td>1.02171634</td>
<td>1.02171633</td>
</tr>
<tr>
<td>E(800 \ 300)</td>
<td>1.31605841</td>
<td>1.31605840</td>
</tr>
<tr>
<td>E(850 \ 720)</td>
<td>1.07377505</td>
<td>1.07377504</td>
</tr>
<tr>
<td>E(850 \ 80)</td>
<td>1.47970717</td>
<td>1.47970716</td>
</tr>
</tbody>
</table>
are used in the subroutine depending on the size of m. If \(0 \leq m \leq (1 - 10^{-5})\) the following sequence \(^{10, 11}\) is generated

\[
\begin{align*}
k_0 &= k \\
k_{n+1} &= \frac{1 - k'_n}{1 + k'_n}
\end{align*}
\] (26)

where the complement modulus \(k'\) is given by

\[
k'_n = (1 - k_n^2)^{\frac{1}{2}} \quad n = 0, 1, 2, \ldots r
\] (27)

with \(r\) being determined from the relation

\[
k_r < 10^{-15}
\]

Since \(k_n \to 0\) as \(n \to \infty\) \(K(m)\) and \(E(m)\) can be obtained from

\[
\begin{align*}
K_n &= \frac{\pi}{2} \\
E_n &= \frac{\pi}{2} \\
K_{n-1} &= \frac{2K_n}{1 + k'_n} \\
E_{n-1} &= (1 + k'_{n-1})E_n - \frac{2k'_{n-1}}{1 + k'_{n-1}}K_n
\end{align*}
\]

\[
\vdots
\]

\[
K = K_0 \quad \text{and} \quad E = E_0
\] (28)

Subroutine TEK uses the series \(^{12}\)

\(^{10}\) J. H. Flinchum and D. E. Amos, AFWL Library Package MATH 13.


\(^{12}\) Reference 3, eqns. 773, 3 and 774, 3.
\[ K = \ln \left[ 4 (m_1)^{\frac{1}{2}} \right] + \frac{1}{2} \left\{ \ln[4 (m_1)^{\frac{1}{2}}] - \frac{2}{1 \cdot 2} \right\} m_1^2 + \frac{1}{2^2 \cdot 3^2} \left\{ \ln \left[ 4 (m_1)^{\frac{1}{2}} \right] - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4} \right\} m_1^3 + \ldots \] (29)

and

\[ E = 1 + \frac{1}{2} \left\{ \ln[4 (m_1)^{\frac{1}{2}}] - \frac{1}{1 \cdot 2} \right\} m_1 + \frac{1}{2^2 \cdot 3} \left\{ \ln \left[ 4 (m_1)^{\frac{1}{2}} \right] - \frac{2}{1 \cdot 2} - \frac{1}{3 \cdot 4} \right\} m_1^2 + \frac{1}{2^2 \cdot 3^2 \cdot 5} \left\{ \ln \left[ 4 (m_1)^{\frac{1}{2}} \right] - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4} - \frac{1}{5 \cdot 6} \right\} m_1^3 + \ldots \] (30)

for \((1 - 10^{-5}) < m < 1\). However, when \(m\) is close to one the number of significant bits lost in the subtraction (to obtain \(m_1\) as \(m_1 = 1 - m\)) can reach the word length of the computer. Subsequently, subroutine TEK has been written with the option of supplying \(m_1\) directly to the routine thereby avoiding this loss of significant bits. This procedure works provided of course \(m_1\) can be calculated in the calling routine other than by subtraction of \(m\) from 1. To exercise this option the variable ID is set non-zero and \(m_1\) is supplied to TEK instead of \(m\) through the variable RM. The parameters \(E_k\) and \(E\) return the value of \(K(m)\) and \(E(m)\) as always. A listing of subroutine TEK is given in Appendix B.

-12-
To use subroutine TEF the calling routine must furnish
the standard FORTRAN statement

CALL TEF (PHI, RM, CRIT, F, E)

The first three parameters PHI, RM and CRIT are supplied
to TEF and the last two are returned to the calling routine. The
parameters and their uses are listed below.

1. PHI TYPE REAL. This variable corresponds
to the amplitude \( \phi \).

2. RM TYPE REAL. This variable represents
the parameter \( m \).

3. CRIT TYPE REAL. This is the error criteria
for the calculations of \( F(\phi|m) \) and \( E(\phi|m) \).
If \( m \) is less than 0.75 then CRIT determines
the number of terms in the series such that
the remainder \( R_n \) of the series is less than
the criteria. If \( m \) is greater than 0.75 the
criteria is used to determine the accuracy
to which the arithmetic-geometric mean is
carried.

4. F TYPE REAL. This variable contains the
value of \( F(\phi|m) \) that is returned to the
calling routine.

5. E TYPE REAL. This variable contains the
value of \( E(\phi|m) \) that is returned to the
calling routine.
Subroutine TEK is called from subroutine TEF as follows

CALL TEK (ID, RM, EK, E)

The first two parameters ID and RM are supplied to subroutine TEK and the last two are returned to the calling routine. The use of these parameters is described below.

1. ID
   TYPE INTEGER. This determines which variable the subroutine expects to receive. If ID = 0 RM ← m. If ID ≠ 0 RM ← m₁.

2. RM
   REAL. This variable represents the parameter m or its complement m₁.

3. EK
   TYPE REAL. This variable contains the value of K(m) that is returned to the calling routine.

4. E
   TYPE REAL. This variable contains the value E(m) that is returned to the calling routine.

An output file is required by subroutine TEF and subroutine TEK for the printed error message from the check on the size of m.

Core storage requirements for the two subroutines TEF and TEK are approximately 1250₈ and 420₈ respectively.

The time to calculate $F(\phi \mid m)$ and $E(\phi \mid m)$ varies according to the value of $\phi$ and $m$. However, as a guide to the time involved, the tables of $F(\phi \mid \alpha)$ and $E(\phi \mid \alpha)$ in the Handbook of Mathematical Functions AMS 55 can essentially be reproduced in approximately 6 seconds.
SUMMARY

Subroutine TEF is a general purpose routine for computing $F(\phi|m)$ and $E(\phi|m)$ with a high degree of accuracy for a wide range of $\phi$ with the parameter $m$ in the range $0 \leq m \leq 1$. Although not a long routine there was an attempt to increase speed and accuracy at the expense of space in the general trade off between core storage and central processor time. Subroutine TEK is used in the computation of $F(\phi|m)$ and $E(\phi|m)$ but can be used by itself as a general purpose routine.
ACKNOWLEDGEMENTS

The suggestions of Dr. Carl E. Baum of AFWL concerning this routine are gratefully acknowledged as are the many helpful discussions with A1C Robert N. Marks.
Appendix A:

Listing of subroutine TEF
SUBROUTINE TEF (PH1, RM, SIG, TF, TE)
DATA PI04/785398163397448/; TPI/6.28318530717959/
DATA PI1, PI2, PI3, PI4, PI5, PI6, PI7, PI8, PI9, PI10
DIMENSION AA(50), BB(50), CC(50), PSAV(50)
IF (ABS(RM-9*5)-5) 15, 15, 5
PRINT 10, RM
FORMAT (5X, 9H*********, 3X, 13HLOOK OUT M = )
RETURN
IF (PH1) 20, 25, 25
W=1.
PH=PH1
GO TO 30
W=1.
PH=PH1
GO TO 30
RK=SORT(RM)
N=PH/TPI
A=PH-FLOAT(N)*TPI
B=A/PI02
K=B
NQ=K+1
GO TO (35, 40, 45, 50), NQ
NK=4*N
SIGNEM=1.
AP=A
GO TO 55
NK=4*N+2
SIGNEM=-1.
AP=PI-A
GO TO 55
NK=4*N+2
SIGNEM=1.
AP=A-PI
GO TO 55
NK=4*N+2
SIGNEM=-1.
AP=PI-A
CNK=NK
PH1=AP
CALL TEK (0, RM, EK, EE)
PLUS=(NK*EK
PLUS1, CNK*EE
IT=0
IF (ABS(PHI-PI02)-1*E-10) 60, 60, 65
IT=1
IF (ABS(RK-1*E0)-1*E-10) 70, 85, 85
IT=IT+1
GO TO (75, 80), IT
75 W=(PLUS+SIGNEM*ALOG(TAN(PI04+PHI*5))
TE=W*(PLUS1+SIGNEM*SIN(PHI))
RETURN
80 W=1.E75
TE=W*(PLUS1+SIGNEM)
RETURN
IF (ABS(RK)-1*E-15) 90, 95, 95
TF=W*(PLUS+SIGNEM*PHI)
TE*W*(PLUS1*SIGNEM*PHI)
RETURN
IF=IT+1
GO TO (105,100), IT
CALL TEK (0*RM*EK*EE)
TF=W*(PLUS-SIGNEM*EK)
TE=W*(PLUS1-SIGNEM*EE)
RETURN
IF (ABS(PHI)-1*E-50) 110,115,115
TF=W*PLUS
TE=W*PLUS1
RETURN
IF (RM-.75) 120,140,140
CALL TEK (0*RM*EK*EE)
S=SIN(PHI)
C=COS(PHI)
SK=RM
CE=2.*PHI/PI
TZ=CE*EK
TL=CE*EE
A=5*EO
T=5*EO*A*SK
R=T
SS=S*S
PS=1*EO
H=.5
F=5*EO
PK=SK
UI=10
DO 130 I=2,20000
J=I*2
D=FLOAT(J-1)
G=FLOAT(J-3)
E=1./FLOAT(J)
PS=SS*PS
A=E*(DA+PS)
F=D*E+F
H=H*E+H
PK=PK*SK
U=F*A*PK
IF (UI*U1/(UI-U)-SIG) 135,135,125
125 UI=U
T=U+T
130 R=H*A*PK+R
135 TF=W*((TZ-S*C*T)*SIGNEM+PLUS)
TE=W*((TI+S*C*R)*SIGNEM+PLUS1)
RETURN
ALPHAR=ASIN(RK)
AA(1)=1.
BB(1)=COS(ALPHAR)
DO 145 I=2,50
II=I-1
AA(1)*.5*(AA(I)+BB(I))
BB(I)=SQRT(AA(I)+BB(I))
CC(I)=.5*(AA(I)-BB(I))
145  IF (ABS(CC(1))-SIG) 150,145,145
150  CONTINUE
155  ISTOP=50
160  GO TO 155
165  ISTOP=1
170  P=PH1
175  P2=1.*
180  NQ=1
185  IOS=1
190  M2P=0
195  I4=0
200  ORELER=1.0E25
205  OR=1.0E25
210  DO 215 I=1,ISTOP
215  PSAV(I)=P
220  P2=P2*2.*
225  BD=TAN(P)*BB(1)/AA(1)
230  BF=ATAN(BD)
235  INS=SIGN(1.*BF)
240  IF (IOS*INS) 165,170,170
245  NQ=NG+1
250  IF (NQ=EQ.5) NQ=1
255  GO TO (175,190,195), NQ
260  IF (I4) 180,185,180
265  I4=0
270  M2P=M2P+1
275  BE=BF+FLOAT(M2P)*TPI
280  GO TO 200
285  BE=BF+P1+FLOAT(M2P)*TPI
290  GO TO 200
295  BE=BF+P1+FLOAT(M2P)*TPI
300  I4=1
305  IOS=INS
310  PR=P/BE
315  ORELER=ABS(OR-PR)/(PR+OR)
320  IF (ORELEK=RELER) 205,210,210
325  IOS=-IOS
330  GO TO 160
335  P=BE+P
340  OR=PR
345  ORELER=RELER
350  TF=W*(PLUS+SIGNEM*(P/(PZ*AA(1ISTOP))))
355  CALL TEK (0*RH*EK*EE)
360  SUMEM=0.
365  GO 220 IK=2,ISTOP
370  SUMEM=SUMEM+CC(1K)*SIN(PSAV(1K))
375  TF=W*(PLUS1+SIGNEM*(EE/EK*TF+SUMEM))
380  RETURN
385  END
Appendix B:

Listing of subroutine TEK
SUBROUTINE TEL (ID, RM, EK, E)  
DIMENSION RKP(60)  
IF (ID) 60, 5, 60  
5 IF (RM=1) 30, 20, 10  
10 PRINT 15, RM  
15 FORMAT (5X, 9H***********, 3X, 13H LOOK OUT M =, F8.3, 3X, 9H***********)  
RETURN  
20 EK=1.E75  
E=1.E0  
25 RETURN  
30 EK=1.57079632679489  
E=EK  
IF (RM) 10, 25, 35  
35 IF (RM=999) 40, 40, 65  
40 RKN=SQRT(RM)  
DO 45 I=1, 60  
45 RKP(I)=SQRT(1.-RKN*RKN)  
RKN=(1.-RKP(I))/(1.+RKP(I))  
IF (I.GE.2.AND.RKN.LT.1.E-20) GO TO 50  
45 CONTINUE  
I=60  
50 N=I-1  
DO 55 J=1, N  
51 T1=1.*RKP(I-J)  
EK=2.*EK/T1  
55 E=T1*E-EK*RKP(I-J)  
RETURN  
50 RPK=SQRT(RM)  
GO TO 70  
65 RPK=SQRT(RM)  
70 PK2=RPK*RPK  
PKP=PK2  
GOL=ALOG(4.*/RPK)  
GK=GOL-1.E0  
FK=2.5  
FE=2.5  
EK=GOL+FK*GK*PKP  
E=1.+5.*(GOL-5.)*PKP  
GE=GK  
DO 85 I=2, 2000  
R=FLOAT(I+1)  
D=R-1.E0  
PKP=PKP*PK2  
C=D/R  
FK=FK*D/(R*R)  
FE=FE*C  
H=1./(D*R)  
GK=GK-1./(D*FLOAT(I))  
GE=GE-H  
T1=FK*GK*PKP  
EK=T1+EK  
T2=FE*GE*PKP  
E=T2+E  
IF (T1=1.E-15) 75, 75, 80  
75 IF (T2=1.E-15) 90, 90, 80  
-22-
80  FE*FE*C
85  GE*GE=H
90  RETURN
END