Mathematics Notes

Note 29

4 September 1973

Evaluation of the Surface Integral Occurring In the E-Field Integral Equations for Wire Antennas

F. M. Tesche
Science Applications, Inc.
P.O. Box 277
Berkeley, California 94701

Abstract

In this note, the method for performing an exact surface integration for the E-field integral equations is outlined, and a comparison with the often used thin-wire approximation is made. It is found that for pulse type basis functions, the thin-wire kernel is valid for cell sizes greater than about eight wire radii, with an overall error in the kernel less than one percent.

In the numerical determination of the current flowing on thin-wire antennas and scatterers, a form of the E-field integral equation is often used[1]. For a straight wire of length L and radius a, the Pocklington form of the equation is

$$\left(\frac{d^2}{dz^2} + k^2\right)\int_0^L\int_0^{2\pi} J(\theta',z')\frac{e^{-jkR}}{4\pi R}\, adz'd\theta' = -j\omega\varepsilon\, E^{inc}(z,\theta) \qquad (1)$$

and the Hallén form of the equation is

$$\int_0^L\int_0^{2\pi} J(\theta',z')\frac{e^{-jkR}}{4\pi R}\, adz'd\theta' = A\sin(kz) + B\cos(kz) - \frac{j\omega\varepsilon}{k}\int_0^z E^{inc}(\xi,\theta)\sin k(z-\xi)\, d\xi. \qquad (2)$$

where the distance R between source and observation points is given by $R = \sqrt{(z-z')^2 + 4a^2\sin^2\frac{\theta-\theta'}{2}}$ as shown in Figure 1. In these equations, $J(\theta',z')$ is the axial current density flowing on the wire, and A and B are constants of integration which are found by requiring that the total current at the wire ends be zero. Note that the effects of end-cap currents have been neglected in these equations.

In the usual manner, the current density is assumed to be independent of the θ' coordinate of the wire, and related to the total current at z' by

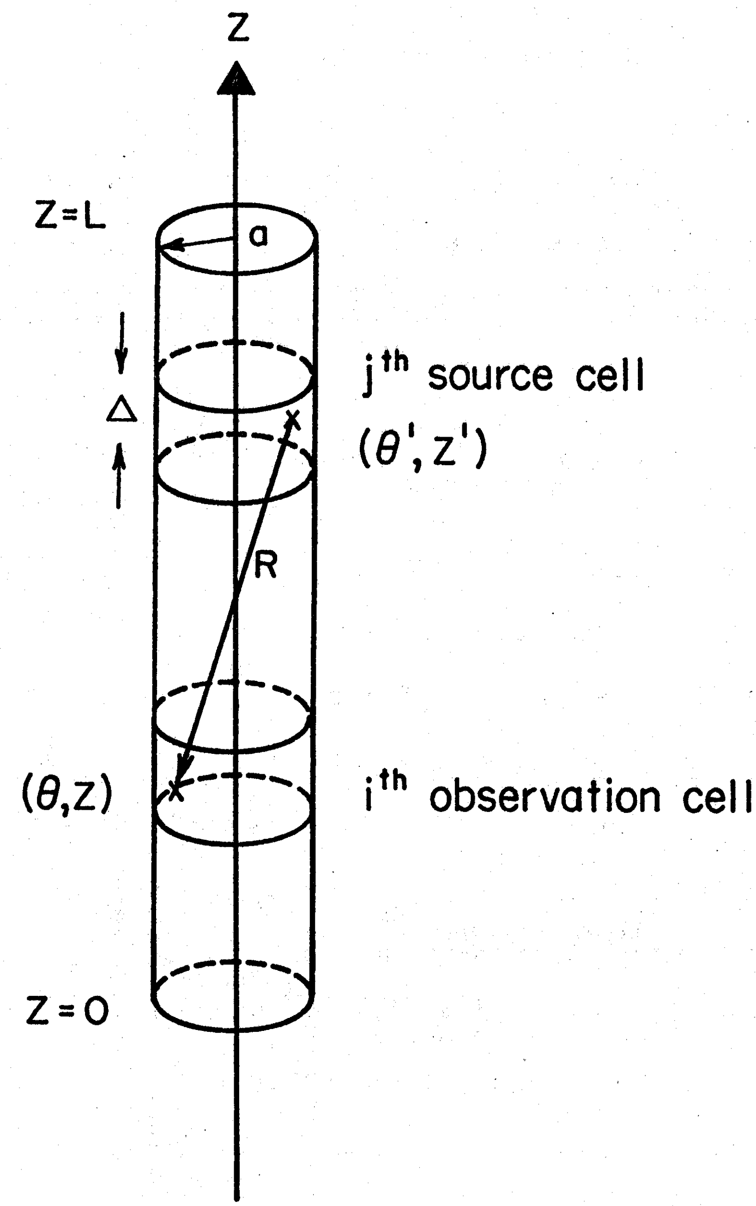

Figure 1. Geometry of the problem.

$$J(\theta',z') = I(z')/2\pi a \tag{3}$$

This is due to the assumption that the wire is thin compared to a wavelength. Both of these integral equations thus have an integral of the form

$$\frac{1}{2\pi a}\int_0^L\int_0^{2\pi} I(z')\,\frac{e^{-jkR}}{4\pi R}\,a\,dz'd\theta \tag{4}$$

to be evaluated.

The solution of Eqs. (1) or (2) is usually achieved by using the method of moments[2] to form a matrix equation which is then inverted numerically. One type of basis function for expanding the unknown current is the pulse function, which is non-zero only within a small zone or cell on the antenna. For adequate representation of the current, it is required that there be many such cells per wavelength. If delta functions are used as the testing functions, a typical integral that must be evaluated is

$$\frac{I(z_j)}{2\pi a}\int_{z_j-\frac{\Delta}{2}}^{z_j+\frac{\Delta}{2}}\int_0^{2\pi} \frac{e^{-jk\sqrt{(z_i-z')^2+4a^2\sin^2\frac{\theta'}{2}}}}{4\pi\sqrt{(z_i-z')^2+4a^2\sin^2\frac{\theta'}{2}}}\,a\,dz'd\theta' \tag{5}$$

where z_j is the source point and z_i is the observation point. The value $\theta=0$ has been chosen without loss of generality, and Δ is the size of the source cell. This double integral must then be evaluated for all z_i and z_j on the wire to form the matrix representation of Eq. (1) or (2).

For z_i outside the range of the z' integration of Eq. (5), the integrals are easily carried out numerically. For $z_i = z_j$, however, the integrand is singular at $\theta' = 0$ and cannot be evaluated numerically. Let the integral of this singular integrand be T. Then, with a slight change of variables, and omitting $I(z')$, T becomes

$$T = \frac{1}{2\pi a} \int_{-\frac{\Delta}{2}}^{\Delta/2} \int_{0}^{2\pi} \frac{e^{-jk\sqrt{z^2+4a^2\sin^2\frac{\theta}{2}}}}{4\pi\sqrt{z^2+4a^2\sin^2\frac{\theta}{2}}}\, a\,dz\,d\theta \qquad (6)$$

which is the singular integral to be evaluated.

One way of evaluating Eq. (6) is to use the thin-wire approximation which is

$$T_a = \int_{-\frac{\Delta}{2}}^{\Delta/2} \frac{e^{-jk\sqrt{z^2+a^2}}}{4\pi\sqrt{z^2+a^2}}\, dz \quad . \qquad (7)$$

This may be integrated numerically, or, as done by Harrington[(2)], approximated by

$$T_a \approx \frac{1}{2\pi} \ln\left(\frac{\Delta}{a}\right) - \frac{jk\Delta}{4\pi} \tag{8}$$

for $k\Delta << 1$ and $\Delta >> a$.

To accurately treat the integration in Eq. (6), one can integrate the singularity analytically. Interchanging the integral signs in Eq. (6) gives

$$T = \frac{1}{2\pi a} \int_0^{2\pi} a d\theta \int_{-\Delta/2}^{\Delta/2} \frac{e^{-jkR}}{4\pi R} dz \tag{9}$$

where R is now defined as $R = \sqrt{z^2+4a^2\sin^2\frac{\theta}{2}}$.

Expanding the term e^{-jkR} as $1-jkR+\cdots$ and retaining only the first two terms, yields for $k\Delta << 1$,

$$T \approx \frac{1}{2\pi a} \int_o^{2\pi} a d\theta \int_{-\frac{\Delta}{2}}^{\Delta/2} \frac{1-jkR}{4\pi R} dz \ . \tag{10}$$

The negative term of the integrand is easily integrated and by using 200.01 of Dwight[3], the integral over z may be performed on the $1/R$ term, yielding

$$T \approx -\frac{jk\Delta}{4\pi} + \frac{1}{4\pi^2} \int_0^{2\pi} \left(\ln\left(\frac{\Delta}{4a} + \sqrt{\left(\frac{\Delta}{4a}\right)^2 + \sin^2\frac{\theta}{2}}\right) - \ln \sin\left(\frac{\theta}{2}\right)\right) d\theta \ . \tag{11}$$

The second term of the integrand in the above relation is singular and must be integrated analytically. Using 865.41 of Dwight, Eq. (11) then becomes

$$T \approx -\frac{jk\Delta}{4\pi} + \frac{\ln 2}{2\pi} + \frac{1}{4\pi^2}\int_0^{2\pi} \ln\left(\frac{\Delta}{4a} + \sqrt{\left(\frac{\Delta}{4a}\right)^2 + \sin^2\frac{\theta}{2}}\right) d\theta . \quad (12)$$

This last integral is non-singular and easily determined by machine integration. Note that this result is valid for any size zone Δ compared with a, as long as $k\Delta >> 1$. As Δ/a becomes large, the value of (12) becomes close to the approximate value of Eq. (8).

In comparing Eqs. (8) and (12), it is seen that the imaginary values of the integrals are identical due to the consequence of assuming $k\Delta$ small. The real parts are different and the percent difference of the thin-wire value relative to the exact value is shown in Figure 2 as a function of Δ/a.

As seen from this figure, the deviation between the two methods is under 1 percent for $\Delta \geq 8a$. As Δ becomes smaller, there may be appreciable error in the kernel and at high frequencies where Δ is necessarily small to adequately sample the current, the numerical results of the entire problem may be in question[4].

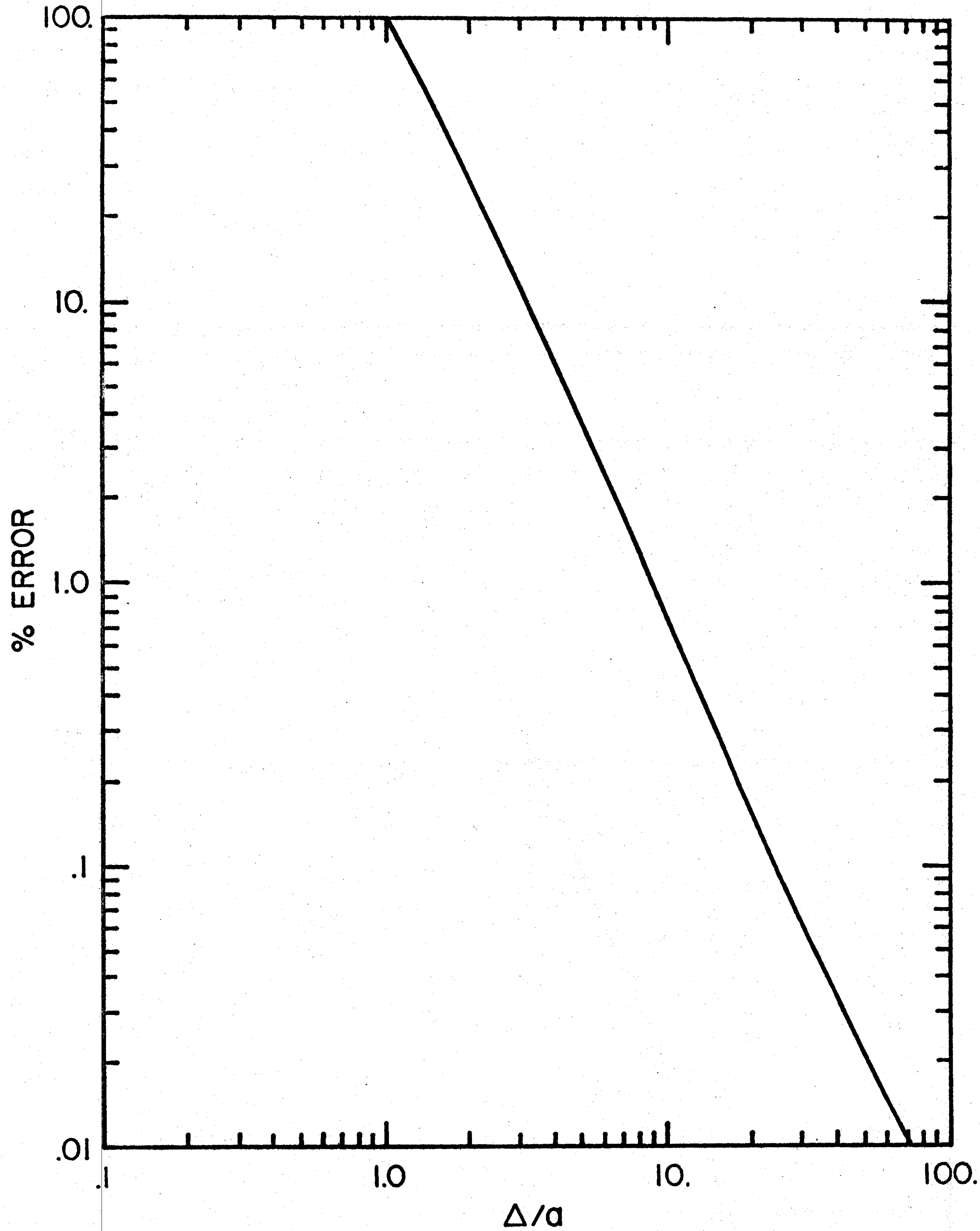

Figure 2. Percent error of thin-wire integration relative to the exact value, as a function of the cell size Δ/a.

REFERENCES

1. K.K. Mei, "On the Integral Equations of Thin-Wire Antennas", IEEE Trans. A.P., Vol. AP-13, May 1965, pp. 374-378.

2. R. F. Harrington, Field Computation by Moment Methods, MacMillan, 1962.

3. H. B. Dwight, Tables of Integrals and Other Mathematical Data, MacMillan, 1965.

4. F. M. Tesche, "The Effect of the Thin-Wire Approximation and the Source Gap Model on the High Frequency Integral Equation Solution of Radiating Antennas", IEEE Trans. A.P., Vol. AP-20, March 1972, pp. 210-211.