Mathematics Notes

Note 37

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SUBROUTINE CAUCHY: COMPLEX ROOTS OF A FUNCTION USING A CAUCHY INTEGRAL TECHNIQUE

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ABSTRACT

A description of the subroutine CAUCHY, the program for finding zeros of an analytic function \( f(z) \) within a contour in the \( z \)-plane, is given. Examples of types of functions for which this program is most advantageous as well as types of functions which show its weaknesses, are given.

ACKNOWLEDGEMENT


Approved for public release, distribution unlimited.
I. Introduction

The purpose of this report is to describe in detail the subroutine CAUCHY and its subprograms. This routine is used to find the zeros of an arbitrary function \( f(z) \) within a contour \( c \) in the complex \( z \)-plane.

The method used has been documented.\(^1\) It relies on calculating integrals of the form

\[
S_N = \frac{1}{2\pi i} \oint_C z^N \frac{f'(z)}{f(z)} \, dz = \sum_{i=1}^{M} z_i^N - \sum_{j=1}^{K} w_i^N
\]  

(1)

where \( z_i \) are the zeros and \( w_i \) are the poles of \( f \) within \( c \). Thus if there are no poles, \( S_0 \) gives the number of zeros, \( S_1 \) the sum of the zeros, \( S_2 \) the sum of their squares, etc.
This method becomes numerically practical when \( c \) contains a small number of zeros. Thus some knowledge of \( f \) is desirable, and in principle, if sufficiently small integration steps are taken so that the function is sufficiently smooth, the answer may be obtained to arbitrary accuracy. Since the method does not rely on iteration, it will work even when well-known iteration methods will not.

The problem is then to perform the integral with suitable accuracy. One straightforward method is found in the programs ISQNK1 and ISQNK2 from the Massachusetts Institute of Technology Information Processing Center. The method used by these programs is to integrate (1) by parts, and then use a Simpson's rule integration to perform the numerical integration around \( c \). The program is most useful when large numbers of points of \( c \) may be calculated and when a large number of roots may exist (\( M \leq 4 \) for this program). An automatic increasing or decreasing of step size along \( c \) is included in the program.

In many cases of numerical interest, it is costly to compute the function, and one would like to locate one or two zeros with fair accuracy with a minimum number of points. The presence of nearby poles or zeros in \( f \) introduces poles on the integrand of equation 1 and may make normal integration procedures impractical because of the large number of contour points required. It is these cases for which the subroutine CAUCHY is designed.

In the next section, we will discuss the method. In the third section, we describe the subroutines. The fourth section gives some sample results, and the fifth a listing of the subroutine and its subprograms.
II. METHOD

Suppose we are given three values of a function \( f_j \) at three points \( z_j \). We can form a best-fit binomial

\[
f_{\text{approx}} = a z^2 + b z + c
\]

by determining \( a, b, \) and \( c \). Now we may factor \( f_{\text{approx}} \)

\[
f_{\text{approx}} = (z - z_1)(z - z_2)
\]

where

\[
z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

Then

\[
f'_{\text{approx}} = (z - z_2) + (z - z_1)
\]

and

\[
\frac{f'_{\text{approx}}}{f_{\text{approx}}} = \frac{1}{z - z_1} + \frac{1}{z - z_2}
\]

So we may perform our integral over the segment of the contour \( c \) over which (2) is valid; the integrals are of the form

\[
\int_{z_a}^{z_b} \frac{z^n}{z - z_\alpha} \, dz = 0 \quad n = 0, 1, 2 \ldots
\]

\[
= \ln\left(\frac{z_b - z_\alpha}{z_a - z_\alpha}\right) \quad n = 0
\]

\[
= (z_b - z_a) - \ln\left(\frac{z_b - z_\alpha}{z_a - z_\alpha}\right) \quad n = 1
\]

\[
= \frac{1}{2} (z_b - z_a)(z_b + z_a - 2 z_\alpha) - 2(z_b - z_a) + \ln\left(\frac{z_b - z_\alpha}{z_a - z_\alpha}\right) \quad n = 2
\]

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However, (2) is not a good approximation to \( f \) if \( f \) has nearby
singularities. A much better approximation obtains if \( f_j \) is known at
five points \( z_j \):
\[
\frac{c_1 + c_2 z_j + c_3 z_j^2}{c_4 + c_5 z_j + c_6 z_j^2} \quad (6)
\]
We may factor both numerator and denominator
\[
\frac{(z-z_{n1})(z-z_{n2})}{(z-z_{d1})(z-z_{d2})} \quad (7)
\]
So
\[
\frac{f_f}{f_{\text{approx}}} = \frac{1}{z-z_{n1}} + \frac{1}{z-z_{n2}} - \frac{1}{z-z_{d1}} - \frac{1}{z-z_{d2}} \quad (8)
\]
Thus in calculating the integral (5), one adds the contributions from
the roots of the numerator and subtracts those of the denominator.

We must first solve for the \( c_1 \). We can make the numerical
computation more accurate by transforming to a coordinate system
where the center point of the five \( z_j, z_0 \), is zero; thus we know our
function at five points \( f_{-2}, f_{-1}, f_0, f_1, \) and \( f_2 \). We can then write
(6) as
\[
c_1 + c_2 z_j + c_3 z_j^2 = (c_4 + c_5 z_j + c_6 z_j^2) f_j \quad (9)
\]
We note that (9) is a set of five linear equations for the six \( c_i \).
We may write the \( c_1 \) as follows. We first note that at \( z_0 = 0 \),
\[
c_1 = c_4 f_0 \quad (10)
\]
Using this, we may write the other numerator coefficients \( c_2 \) and \( c_3 \)
in terms of the denominator coefficients:
\[ c_2 = \frac{z_1 z_{-1} (f_{-1} - f_1)}{z_1 - z_{-1}} c_6 + \frac{(z_{1} f_{-1} - z_{-1} f_1)}{z_1 - z_{-1}} c_5 + \] (11)

\[ + \frac{1}{z_1 - z_{-1}} \left[ \frac{z_1}{z_{-1}} (f_{-1} - f_0) - \frac{z_{-1}}{z_1} (f_1 - f_0) \right] c_4 \]

\[ c_3 = \frac{z_1 f_{-1} - z_{-1} f_1}{z_1 - z_{-1}} c_6 + \frac{f_{-1} - f_1}{z_1 - z_{-1}} c_5 + \frac{1}{z_1 - z_{-1}} \left[ \frac{f_{-1} f_0}{z_1} - \frac{f_1 f_0}{z_{-1}} \right] c_4 \] (12)

If we put \( c_1, c_2, \) and \( c_3 \) in (9) and divide by \( z_j \),

\[ A_j c_5 + B_j c_6 = C_j c_4 \quad (j = \pm 2) \] (13)

where

\[ A_j = \frac{z_j (f_{-1} - f_1)}{z_1 - z_{-1}} + \frac{(z_{1} f_{-1} - z_{-1} f_1)}{z_1 - z_{-1}} \quad f_j \] (14)

\[ B_j = \frac{z_j (z_{1} f_{-1} - z_{-1} f_1)}{z_1 - z_{-1}} + \frac{z_{-1} z_j f_{-1} f_1}{z_1 - z_{-1}} - z_j f_j \] (15)

\[ C_j = \frac{f_{-1} f_0}{z_j} - \frac{1}{z_1 - z_{-1}} \left[ \frac{z_1}{z_{-1}} (f_{-1} - f_0) - \frac{z_{-1}}{z_1} (f_1 - f_0) \right] 
- \frac{z_j}{z_1 - z_{-1}} \left[ \frac{f_{-1} f_0}{z_1} - \frac{f_1 f_0}{z_{-1}} \right] \] (16)

\[ A_j = B_j = C_j = 0 \text{ for } j = 0, \pm 1. \text{ We now solve (13) for } c_5 \text{ and } c_6: \]

\[ c_5 = \frac{c_4}{\text{DET}} \begin{vmatrix} c^2 & B^2 \\ A^2 & C^2 \end{vmatrix} \] (17)

\[ c_6 = \frac{c_4}{\text{DET}} \begin{vmatrix} A^2 & C^2 \\ A^2 & C^2 \end{vmatrix} \] (18)

where

\[ \text{DET} = A^2 B^{-2} - B^2 A^{-2} \] (19)

If we now set \( c_4 = \text{DET} \), we have a unique solution for the \( c_j \).

Problems do arise if \( \text{DET} = 0 \). This will happen when \( f \) is of the form
\[ f = \frac{\alpha_o + \alpha_1 z}{\beta_o + \beta_1 z} \]  

(20)

We examine three special cases:

1) \( \alpha_1 = \beta_1 = 0 \). In this case, \( A^j = B^j = C^j = 0 \). The simplest solution is \( c_4 = 1, c_5 = c_6 = 0 \).

2) \( \beta_1 = 0 \). In this case, \( A^j = C^j = 0 \neq B^j \). Again the simplest solution is to set \( c_4 = 1, c_5 = c_6 = 0 \).

3) \( \alpha_1 \neq 0 \neq \beta_1 \). In this case we are free to choose \( c_6 = 0 \); then from (13), \( A^j c_5 = C^j c_4 \), and since the determinant is zero \( (A^{-2}, C^{-2}) \) equals \( (A^2, C^2) \) within a constant factor so we can set \( c_5 + C^2 \) and \( c_4 = A^2 \). One could, of course, set \( c_5 \) and \( c_6 \) to any numbers not equal to 0; this would amount to multiplying numerator and denominator by the same factor, provided the overall polynomial remained of degree \( \leq 2 \).

Once the coefficients \( c_i \) have been obtained, the roots \( z_{n1}, z_{n2} \) are easily obtained. The integral is then performed from \( z_a \) to \( z_b \) where \( z_a = \frac{z_0 + z_1}{2} + \frac{z_0 - z_1}{2} \). If \( z_o \) is not a corner point, that is, if \( \Delta^2 z/\Delta z = 0 \), then the integral is performed in one step: \( z_a \) to \( z_b \). If \( \Delta^2 z/\Delta z \neq 0 \), it is performed in two steps, \( z_a \) to \( z_o \), and \( z_o \) to \( z_b \). The integrals are performed in the transformed coordinate system \( (z_3 = 0) \), the same one that the coefficients are found in.
III. DESCRIPTION OF THE SUBROUTINE

A. CAUCHY

The main subroutine begins by initializing IND (D = double precision), where N is defined in equation (1). The initialization is necessary, since the subroutine which performs the integral adds incrementally to IND.

Setting \( C(1) = 10^{15} \) signals COEF: that it is the first time that COEF has been called for a given contour (see description of COEF).

The array ZSC(I) is set equal to \( |\Delta z_i| \). If any ZSC(I) equals zero, an error message is printed out and a return initiated.

A check is made to see if the contour contains at least five points. If not, an error message is printed out and a return initiated.

The DO 160 loop is now started; this ultimately performs the incremental integral for each \( Z(I) \). Each set of five \( z_j = ZD(J) \) and \( f_j = FD(J) \), \( 1 \leq J \leq 5 \) are then set up, with special provisions at the "beginning" and "ending" of the contour. For these five \( ZD(J) \) and \( FD(J) \), the coefficients \( c_j = CD(L) \), \( 1 \leq L \leq 6 \), are calculated (see equation (6)) in subroutine COEF. The roots ZRN(1) and ZRN(2) of the numerator, and ZRD(1) and ZRD(2) of the denominator, are calculated in ROOT2.

A check is then made to see if \( \Delta^2 Z_3 / \Delta Z_3 = 0 \); if so, the point ZD(3) is not a "corner point", and the integral is performed from \( [ZD(2)+ZD(3)]/2 = Z1 \) to \( [ZD(3)+ZD(4)]/2 = Z2 \). If ZD(3) is a corner point, the integral is done in two steps: from Z1 to ZD(3), and from ZD(3) to Z2.

The integrals \( \text{IN} = \text{IND}, 0 \leq N \leq 2 \) are divided by \( 2\pi i \) and then a return is made to the calling program.
B. COEF

Given the function $F(I)$ at $Z(I)$, $1 \leq I \leq 5$, this subroutine calculates the $c_j = C(J)$, $1 \leq J \leq 6$ as defined in (6).

A check is first made to see if $C(1) = 10^{15}$. If so, it is the first call of COEF for a given contour, and all five $|f_1| = FM(I)$, $1 \leq I \leq 5$ are calculated; if not, the $FM(I)$ are set equal to the $FM(I+1)$, $1 \leq I \leq 4$, and $FM(5)$ is calculated from $F(5)$.

From the $FM(I)$, a scale factor is formed by finding the largest $FM(I)$. If Max ($FM(I)$) = 0, $F(I) = 0$, $1 \leq I \leq 5$, and the scale factor is not needed; otherwise the scale factor $SCF = 1/\max(FM(I))$.

All $FM(J)$ are now multiplied by $SCF$. Then the second largest $FM(I)$, TSCFM is found. If it is 0, then the largest $F(I) = F(J)$ is a singular point, and the points of the contour have to be redefined. Hence an error message and stop are provided. If two or more $FM(I)$ are equal to the maximum $FM(I)$, or if the TSCFM $\neq 0$, the program proceeds. Note that the presence of two singular points at two contour points (within five consecutive points) would cause the program to fail without any error message.

The program now proceeds to calculate the $A^j$, $B^j$, $C^j$ defined in equations (14), (15), and (16) ($A^{-2} = AM2$, etc.). From these, $C(4)$, $C(5)$, and $C(6)$ are calculated. We now test to see if $C(4) = \text{DET} = 0$. Since we have used the scaled $FP(I)$ to calculate $C(4)$, we define "0" as a small number to take care of roundoff errors in subtraction. If $|C(4)| = "0"$, then we have a function of the type shown in equation (20).
We then set C(5) = C2 and C(4) = A2 as discussed in case 3). In addition \(|C(5)| = 0\), we have either case 2) or case 1), and in either case, the denominator is a constant, set equal to 1. The numerator coefficients C(1), C(2), C(3) are then calculated, and the return is made to CAUCHY.

C. ROOT2

This program calculates the roots ZR(1) and ZR(2) of \(f = A*Z^2 + B*Z + C\). Roots at infinity are set to \(10^{27}\). The subroutine then calculates \(Z(1) = -B + \sqrt{B^2 - 4*A*C}\) and \(Z(2) = -B - \sqrt{B^2 - 4*A*C}\). If \(T_1 = |Z(1)| = 0\) and \(T_2 = |Z(2)| = 0\) special cases must be treated. In that case, if \(A = 0\), then \(f\) is constant, and the two roots are at "infinity"; a return is made. If \(C = 0\) also (which could not happen in CAUCHY, but might arise in some other use of ROOT2), an error stop and message are instituted. If \(A = 0\) when \(T_1 = T_2 = 0\), then both roots \(ZR(1) = ZR(2) = 0\).

If none of the special cases occurs, we determine the larger of \(T_1\) and \(T_2\). The larger \(Z(1,2)\) is then used to calculate \(ZR(1) = 2C/Z(1,2)\).

A check is made to see whether the second root is at "infinity", corresponding to \(A = 0\). If it is, a return occurs, if not the second root is calculated: \(ZR(2) + Z(1,2)/2A\).

D. INCIN

Incin increments the \(I_0\), \(I_1\) and \(I_2\) integrals by forming

\[
I_0 = I_0 + \text{FND} \cdot \sum_{j=1}^{\text{IR}} \Delta I_{0j},
\]

\[
I_1 = I_1 + \text{FND} \cdot \sum_{j=1}^{\text{IR}} \Delta I_{2j}.
\]
and
\[ I_2 = I_2 + FND \cdot \sum_{j=1}^{IR} \Delta I_{2j} \]  \hspace{1cm} (21.3)

where
\[ \Delta I_{0j} = \int_{Z_1}^{Z_2} \frac{dz}{z - Z_j} \]  \hspace{1cm} (22.1)
\[ \Delta I_{1j} = \int_{Z_1}^{Z_2} \frac{z \, dz}{z - Z_j} \]  \hspace{1cm} (22.2)

and
\[ \Delta I_{2j} = \int_{Z_1}^{Z_2} \frac{z^2 \, dz}{z - Z_j} \]  \hspace{1cm} (22.3)

\[ Z_1, Z_2, FND, I_0, I_1, I_2 \] and \[ IR \] are arguments of \[ \text{INCIN} \]. The \[ Z_j \]’s are obtained from
\[ Z_j = ZC + ZR(j) \]  \hspace{1cm} (23)

where \[ ZC \] and \[ ZR(j) \] are additional arguments of the subroutine.

Starting with equations (22), \[ \Delta I_{1j} \] can be expressed as
\[ \Delta I_{1j} = \int_{Z_1}^{Z_2} \frac{dz}{(z - Z_j)} \left[ (z - Z_j) + Z_j \right] \]
\[ = \int_{Z_1}^{Z_2} dz \left[ 1 + \frac{Z_j}{z - Z_j} \right] \]
\[ = (Z_2 - Z_1) + Z_j \Delta I_{10} \]
\[ = DZ + Z_j \Delta I_{10} \]  \hspace{1cm} (24.1)

and
\[ \Delta I_{2,j} = \sum_{z_1} \frac{z_2}{z_1} \left[ \frac{(z^2 - z_j^2) + z_j^2}{z - z_j} \right] \]

\[ = \sum_{z_1} dz \left[ z + z_j + \frac{z_j^2}{z - z_j} \right] \]

\[ = \frac{1}{2} (z_2^2 - z_1^2) + z_j (z_2 - z_1) + z_j^2 \Delta I_{0,j} \]

\[ = DZ \cdot Z_B + z_j \cdot DZ + z_j^2 \Delta I_{0,j} \]

\[ = DZ \cdot Z_B + z_j \cdot \Delta I_{1,j} \tag{24.2} \]

where

\[ DZ = z_2 - z_1 \tag{25.1} \]

and

\[ Z_B = \frac{1}{2} (z_1 + z_2). \tag{25.2} \]

So the only difficulty is the evaluation of \( \Delta I_0 \). If

\[ z = \frac{1}{2} DZ \cdot x + Z_B \]

is substituted into equation (2.1) it is seen that

\[ \Delta I_{0,j} = \int_{-1}^{1} \frac{dx}{x - \frac{2(z_j \cdot Z_B)}{DZ}} \tag{26} \]

or

\[ \Delta I_{0,j} = \int_{-1}^{1} \frac{dx}{x - a - i b} \tag{27} \]
where
\[ a + ib = \frac{2 (Z_j - Z_B)}{DZ} \]  

(28)

and
\[ a \text{ and } b \text{ are real.} \]

This can then be expressed as
\[
\Delta I_{0j} = \int_{-1}^{1} \frac{dx (x - a + ib)}{(x-a)^2 + b^2} 
\]

\[
\frac{1}{2} \ln \left[ \frac{(x-a)^2 + b^2}{(1-a)^2 + b^2} \right] + ib \int_{-1}^{1} \frac{dx}{(x-a)^2 + b^2} 
\]

(29)

\[
= \frac{1}{2} \ln \left[ \frac{(1-a)^2 + b^2}{(1+a)^2 + b^2} \right] + ib \int_{-1}^{1} \frac{dx}{(x-a)^2 + b^2} 
\]

and
\[
\text{Im } \Delta I_{0j} = b \int_{-1}^{1} \frac{dx}{(x-a)^2 + b^2} . 
\]

(30)

Now make the substitution
\[ x = a + |b| \tan \varphi \]  

(31)

with
\[ \tan \varphi_{-1} = -\frac{(1+a)}{|b|} , \]
\[ \tan \varphi_{1} = \frac{1-a}{|b|} , \]

and
\[ |\varphi| \leq \frac{\pi}{2} , \]

(32)

into equation (30).

It is now seen that
\[
\text{Im } \Delta I_{0j} = \frac{b}{|b|} \int_{\varphi_{-1}}^{\varphi_{1}} d\varphi 
\]

\[
= \frac{b}{|b|} (\varphi_{1} - \varphi_{-1}) . 
\]

(33)
Since \(-1 < l \) it follows from equation (31) and (32) that
\[ 0 \leq \varphi_1 - \varphi_{-1} \leq \pi \]
hence
\[ |\text{Im} \Delta I_{0j}| \leq \pi . \]  
(34)

Starting again with equation (26) it follows that
\[
\Delta I_{0j} = \int_{-1}^{1} \frac{dx}{x - \frac{2}{DZ} (Z_j - Z_B)} ,
\]
\[ = \ln \left( x - \frac{2}{DZ} (Z_j - Z_B) \right)_{-1}^{1} , \]  
(35)
\[ = \ln \left( \frac{1 - \frac{2}{DZ} (A_j - Z_B)}{-1 + \frac{2}{DZ} (Z_j - Z_B)} \right) , \]

with the branch specified by equation (34).

If
\[ \left| \frac{2(Z_j - Z_B)}{DZ} \right| \leq 1 \]
define
\[ R = \frac{2(Z_j - Z_B)}{DZ} \]  
(36)
and then
\[ \Delta I_{0j} = \ln \left( \frac{1-R}{1+R} \right) + \ln (-1) , \]

and
\[ \Delta I_{0j} = \ln \left( \frac{1-R}{1+R} \right) - i\pi . \]  
(37)

It makes no difference which branch of \( \ln (-1) \) is used here since
the imaginary part of \( \Delta I_{0j} \) is adjusted to satisfy equation (34) by
adding or subtracting \( 2\pi i \) to the final result.
If
\[ 1 \leq \left| \frac{2 \left( Z_j - Z_B \right)}{DZ} \right| \]
then \( R \) is defined as
\[ R = \frac{DZ}{2(Z_j - Z_B)} \]
and
\[ \Delta I_{0j} = \ln \left( \frac{1 - \frac{R}{1 + R}}{\frac{R}{1 + R}} \right) \]
\[ = \ln \left( \frac{1 - R}{1 + R} \right) \]
(38)

This is the same as equation (37) without the \(-im\) term.

The program sets \( R \) according to the value of \( \left| \frac{Z_j - Z_B}{DZ} \right| \) to either the R.H.S. of equation (36) or (38) and \( \Delta I_{0j} \) to \(-im\) or 0. Then in either case the fortran statement
\[ \Delta I_{0j} = \Delta I_{0j} + \text{CLOG} \left( \frac{1 - R}{1 + R} \right) \]
produces \( \Delta I_{0j} \). If \( R \) is less than .1 an expansion of the \( \ln \left( \frac{1 - R}{1 + R} \right) \) is used otherwise a fortran supplied subroutine computes the value. The imaginary part of \( \Delta I_{0j} \) is adjusted to satisfy equation (34) by adding or subtracting 2 \( mi \). Then \( \Delta I_{1j} \) and \( \Delta I_{2j} \) are computed from equations (24) and the integrals are incremented.
IV. SAMPLE RESULTS

In Table 1, we show results for various functions when integrated around a square in the complex Z-plane

$$-2 \leq \text{Re}, \text{Im}(Z) \leq 2$$

While the function does not provide an exhaustive examination, it does illustrate several important points

1) There is a dramatic change in the winding number (N) as a zero or pole crosses a contour. In all examples, this zero or pole was chosen to lie halfway between two grid points by choosing an appropriate y.

2) The error in any two runs should decrease as \( \left( \frac{\Delta Z}{|\Delta Z|} \frac{\text{Run B}}{\text{Run A}} \right) \); for the larger errors, this is borne out in practice.

3) Even a crude number of points, with zeros (or poles) right on the contour yield an interpretable N.

In Table 1, the first column shows the function calculated. Results are listed for three different \( \Delta Z \). Below the calculated result is the correct result for comparison.
Table 1A

| $|\Delta z| = 1$ |

<table>
<thead>
<tr>
<th>$f(z)$</th>
<th>I0</th>
<th>I1</th>
<th>I2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 - z$</td>
<td>1.0000+0.0000i</td>
<td>1.0000+0.0000i</td>
<td>1.0000+0.0000i</td>
</tr>
<tr>
<td></td>
<td>1+0i</td>
<td>1+0i</td>
<td>1+0i</td>
</tr>
<tr>
<td>$\sin (z/4)$</td>
<td>1.0001+0.0000i</td>
<td>-0.0000+0.0000i</td>
<td>-0.0000+0.0000i</td>
</tr>
<tr>
<td></td>
<td>1+0i</td>
<td>0+0i</td>
<td>0+0i</td>
</tr>
<tr>
<td>$\sin (z/4) / (z - 1)$</td>
<td>-0.0002-0.0000i</td>
<td>-1.0003+0.0000i</td>
<td>-1.005+0.0000i</td>
</tr>
<tr>
<td></td>
<td>0+0i</td>
<td>-1+0i</td>
<td>-1+0i</td>
</tr>
<tr>
<td>$\sin \left[\frac{z-1.99-iy}{4}\right] / (z - 1)$</td>
<td>-0.0571+0.0048i</td>
<td>0.8695+0.4814i</td>
<td>2.468+1.892i</td>
</tr>
<tr>
<td></td>
<td>0+0i</td>
<td>0.99+.51</td>
<td>2.710+1.991</td>
</tr>
<tr>
<td>$\sin \left[\frac{z-2.01-iy}{4}\right] / (z - 1)$</td>
<td>-0.8860-0.0608i</td>
<td>-0.7420-0.0659i</td>
<td>-0.457-0.0051</td>
</tr>
<tr>
<td></td>
<td>-1+0i</td>
<td>-1+0i</td>
<td>-1+0i</td>
</tr>
<tr>
<td>$\sin \left[\frac{z - 3}{4}\right] / (z - 1)$</td>
<td>-1.0020+0.0000i</td>
<td>-1.0026-0.0000i</td>
<td>-1.004+0.0000i</td>
</tr>
<tr>
<td></td>
<td>-1+0i</td>
<td>-1+0i</td>
<td>-1+0i</td>
</tr>
<tr>
<td>$\sin (z/4) / (z - 1.99 - iy)$</td>
<td>-0.0329+0.0041i</td>
<td>-2.0586-0.5081i</td>
<td>-3.850-2.041i</td>
</tr>
<tr>
<td></td>
<td>0+0i</td>
<td>-1.99-.51</td>
<td>-3.71-1.991</td>
</tr>
<tr>
<td>$\sin (z/4) / (z - 2.01 - iy)$</td>
<td>1.0332-0.0037i</td>
<td>0.0675+0.0091i</td>
<td>0.124+0.0521</td>
</tr>
<tr>
<td></td>
<td>1+0i</td>
<td>0+0i</td>
<td>0+0i</td>
</tr>
<tr>
<td>$\sin (z/4) / (z - 1)(z + 1)$</td>
<td>-1.0009+0.0000i</td>
<td>0.0000-0.0000i</td>
<td>-1.985+0.0000i</td>
</tr>
<tr>
<td></td>
<td>-1+0i</td>
<td>0+0i</td>
<td>-2+0i</td>
</tr>
<tr>
<td>$\sin (z/4) / (z - 1)^2$</td>
<td>-1.0002+0.0000i</td>
<td>-1.9992+0.0000i</td>
<td>-1.979-0.0000i</td>
</tr>
<tr>
<td></td>
<td>-1+0i</td>
<td>-2+0i</td>
<td>-2+0i</td>
</tr>
<tr>
<td>$f(z)$</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>-----------------</td>
<td>------</td>
<td>------</td>
<td>------</td>
</tr>
<tr>
<td>1 - z</td>
<td>1.000+0.000i</td>
<td>1.000+0.000i</td>
<td>1.000+0.000i</td>
</tr>
<tr>
<td></td>
<td>1 + 0i</td>
<td>1 + 0i</td>
<td>1 + 0i</td>
</tr>
<tr>
<td>sin(z/4)</td>
<td>1.0000+0.0000i</td>
<td>-0.0000+0.0000i</td>
<td>-0.0000+0.0000i</td>
</tr>
<tr>
<td></td>
<td>1 + 0i</td>
<td>0 + 0i</td>
<td>0 + 0i</td>
</tr>
<tr>
<td>sin(z/4)</td>
<td>-0.0000+0.0000i</td>
<td>-1.0000+0.0000i</td>
<td>-1.0000+0.0000i</td>
</tr>
<tr>
<td></td>
<td>0 + 0i</td>
<td>-1 + 0i</td>
<td>-1 + 0i</td>
</tr>
<tr>
<td>$\frac{\sin(z-1.99-i\pi)}{z-1}$</td>
<td>-0.0052+0.0010i</td>
<td>0.9793+0.2508i</td>
<td>2.875+0.994i</td>
</tr>
<tr>
<td></td>
<td>0 + 0i</td>
<td>.99 + .25i</td>
<td>2.898 + 0.995i</td>
</tr>
<tr>
<td>$\frac{\sin(z-2.01-i\pi)}{z-1}$</td>
<td>-0.9941-0.0012i</td>
<td>-0.9878-0.0011i</td>
<td>-0.976+0.001i</td>
</tr>
<tr>
<td></td>
<td>-1 + 0i</td>
<td>-1 + 0i</td>
<td>-1 + 0i</td>
</tr>
<tr>
<td>$\frac{\sin(z-3)}{z-1}$</td>
<td>-1.001+0.0000i</td>
<td>-1.0002-0.0000i</td>
<td>-1.000-0.000</td>
</tr>
<tr>
<td></td>
<td>-1 + 0i</td>
<td>-1 + 0i</td>
<td>-1 + 0i</td>
</tr>
<tr>
<td>$\frac{\sin(z)}{z-1.99-i\pi}$</td>
<td>-0.0011+0.003i</td>
<td>-1.9923-0.2498i</td>
<td>-3.903-0995i</td>
</tr>
<tr>
<td></td>
<td>0 + 0i</td>
<td>-1.99 - 0.25i</td>
<td>-3.898-0.995i</td>
</tr>
<tr>
<td>$\frac{\sin(z)}{z-2.01-i\pi}$</td>
<td>1.0012-0.0003i</td>
<td>0.0025-0.0003i</td>
<td>0.005+0.000i</td>
</tr>
<tr>
<td></td>
<td>1 + 0i</td>
<td>0 + 0i</td>
<td>0 + 0i</td>
</tr>
<tr>
<td>$\frac{\sin(z)}{(z-1)(z+1)}$</td>
<td>-1.0002-0.0000i</td>
<td>0.0000-0.0000i</td>
<td>-1.999-0.0001</td>
</tr>
<tr>
<td></td>
<td>- 1 + 0i</td>
<td>0 + 0i</td>
<td>-2 + 0</td>
</tr>
<tr>
<td>$\frac{\sin(z)}{(z-1)^2}$</td>
<td>-1.0001+0.0000i</td>
<td>-2.0001-0.0000i</td>
<td>-1.998-0.0001</td>
</tr>
<tr>
<td></td>
<td>- 1 + 0i</td>
<td>-2 + 0i</td>
<td>-2 + 0i</td>
</tr>
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<td>Table 1C</td>
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<td></td>
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<td>-----------</td>
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</tr>
<tr>
<td></td>
<td>[Δz] .25</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>IO</td>
<td>II</td>
<td>I2</td>
</tr>
<tr>
<td>1 - z</td>
<td>1 + 0i</td>
<td>1 + 0i</td>
<td>1 + 0i</td>
</tr>
<tr>
<td>sin(z/4)</td>
<td>1 + 0i</td>
<td>0 + 0i</td>
<td>0 + 0i</td>
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<td>sin(z/4)</td>
<td>1 + 0i</td>
<td>0 + 0i</td>
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<tr>
<td>sin(z/4)</td>
<td>-0.0003+0.0001i</td>
<td>0.9894+0.1251i</td>
<td>2.943+0.498i</td>
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<td>0 + 0i</td>
<td>.99 + .125i</td>
<td>2.944+0.498i</td>
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<td>sin(z/4)</td>
<td>0.9996-0.0001i</td>
<td>-0.9993-0.0001i</td>
<td>-0.999-0.0001i</td>
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<tr>
<td>sin(z/4)</td>
<td>-1 + 0i</td>
<td>-1 + 0i</td>
<td>-1 + 0i</td>
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<td>-1 + 0i</td>
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<td>-1 + 0i</td>
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<tr>
<td>sin(z/4)</td>
<td>-1 + 0i</td>
<td>-1 + 0i</td>
<td>-1 + 0i</td>
</tr>
<tr>
<td>sin(z/4)</td>
<td>0.0000+0.0000i</td>
<td>-1.9991-0.1250i</td>
<td>-3.945-0.4981</td>
</tr>
<tr>
<td>sin(z/4)</td>
<td>0 + 0i</td>
<td>1.99-0.25i</td>
<td>-3.944-0.4981</td>
</tr>
<tr>
<td>sin(z/4)</td>
<td>1 + 0i</td>
<td>0 + 0i</td>
<td>0 + 0i</td>
</tr>
<tr>
<td>sin(z/4)</td>
<td>1 + 0i</td>
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<td>1 + 0i</td>
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</tr>
<tr>
<td>sin(z/4)</td>
<td>-1 + 0i</td>
<td>-2 + 0i</td>
<td>-2 + 0i</td>
</tr>
<tr>
<td>sin(z/4)</td>
<td>-1 + 0i</td>
<td>-2 + 0i</td>
<td>-2 + 0i</td>
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<tr>
<td>sin(z/4)</td>
<td>-1 + 0i</td>
<td>-2 + 0i</td>
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<tr>
<td>sin(z/4)</td>
<td>-1 + 0i</td>
<td>-2 + 0i</td>
<td>-2 + 0i</td>
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</table>
REFERENCES


SUBROUTINE CAUCHY(IMAX,Z,F,IO,II,IZ)

CAUCHY CALCULATES THE INTEGRALS IO=INT(D(LN(F))/DZ)/2*PI*I,
II=INT(Z*D(LN(F))/DZ)/2*PI*I, AND IZ=INT(Z*Z*D(LN(F))/DZ)/2*PI*I
WHERE F IS DEFINED AT IMAX POINTS ON THE CONTOUR Z. IN COEF
THE CURVE IS FIRST FIT TO A FORM (A*X**2+B*X+C)/(D*X**2+E*X+F),
THE ROOTS OF THE NUMERATOR AND DENOMINATOR ARE OBTAINED (ROOT2),
AND THEN THE INTEGRAL IS PERFORMED (INCIN)

COMPLEX*16 ZD(5),FD(5),CD(6),ZRZ(2),ZRD(2),Z1,Z2,IOD,IOD,IOD
COMPLEX*8 Z(101),F(101),IO,II,IZ
REAL*4 ZBC(101)

100 IOD=(0,0,0,0)
110 IOD=(0,0,0,0)
120 IOD=(0,0,0,0)
CD(1)=(1.015,0,0)
DO 110 I=2,IMAX
ZBC(I)=CABS(Z(I)-Z(I-1))
IF(ZBC(I),NE,0.) GO TO 110
WRITE (1,501)
501 FORMAT (19M ERROR IN Z-CONTOUR)
RETURN
110 CONTINUE
IF(IMAX,GE,5) GO TO 120
WRITE (1,502)
502 FORMAT (10M IMAX,LT,5)
RETURN
120 IMAXM1=IMAX-1
DO 130 J=1,IMAXM1
JJ=J+1
IF(JJ,EQ,1) JJ=IMAX-2
IF(JJ,EQ,0) JJ=IMAX-1
IF(JJ,GT,IMAX) JJ=2
ZD(J)=Z(JJ)
130 FD(J)=F(JJ)
CALL COEF(ZD,FD,CD)
CALL ROOT2(CD(1),CD(2),CD(3),ZRZ)
CALL ROOT2(CD(4),CD(5),CD(6),ZRD)
IP=I+1
IM=I+1
IF(IM,EQ,0) IM=IMAX-1
IF(CABS(Z(IP)+Z(IM)=2.*Z(I)) .LT. ZBC(IP)*1.E-6) GO TO 140
Z1=(Z(I)+Z(IM))*0.5
Z2=Z(I)
CALL INCIN(Z1,Z2,ZD(3),ZRZ,2,IOD,IOD,IOD,1.0D)
CALL INCIN(Z1,Z2,ZD(3),ZRD,2,IOD,IOD,IOD,-1.0D)
Z1=Z(I)
Z2=(Z(IP)+Z(I))*0.5
GO TO 150
140 Z1=(Z(I)+Z(IM))*0.5
Z2=(Z(IP)+Z(I))*0.5
150 CALL INCIN(Z1,Z2,ZD(3),ZRZ,2,IOD,IOD,IOD,1.0D)
CALL INCIN(Z1,Z2,ZD(3),ZRD,2,IOD,IOD,IOD,-1.0D)
160 CONTINUE
IO=IOD/(0,0,0,6.2831853D0)
II=IOD/(0,0,0,6.2831853D0)
IZ=IOD/(0,0,0,6.2831853D0)
RETURN
END
SUBROUTINE INCIN(Z1,Z2,ZC,ZR,IR,I0,I1,I2,FND)
IMPLICIT COMPLEX*16 (A-H,O-Z)
INCIN ADDS THE INCREMENTAL INTEGRALS FROM Z1 TO Z2 TO THE
INTEGRALS I0, I1, AND I2.
I=SUB-N * I=SUB-N + FND * SUM OVER J, J=1,IR OF INT(J), WHERE
INT(J) = INTEGRAL FROM Z1 TO Z2 OF (dz*(z-zr(j))
REAL*8 AR,PI,TWOP1,FND,TEST
COMPLEX*16 I0,I1,I2,ZR(I)
DATA PI/3.141592653589793/.,TWOP1/6.2831853071795860/
DO=Z2-Z1
ZB=.900*(Z2+Z1)
DO 50 J=1,IR
ZJ=ZR(J)+ZC
ZRB=ZD0=(ZJ-ZB)
R=ZR&B/DZ
AR=CDABS(R)
IF(AR.GT.1.D0) GO TO 10
DI=DCMPLX(0.D0,-PI)
GO TO 20
10 R=DZ/ZRB
AR=1.D0/AR
DI=(0.D0,0.D0)
20 IF(AR.GT.0.1D0) GO TO 30
IF(AR.LT.1.D-25) GO TO 50
RS=R*R
DI=DI=R/(2.D0+((.6666666666666666D0+((.4D0+((.2857142857142857D0
1 +(.22222222222222D0+.18181818181818D0*R8)*R8)*R8)*R8)
TEST=DIAG(DI)
IF(TEST.LT.-PI) DI=DI+DCMPLX(0.D0,TWOP1)
GO TO 40
30 DI=DI+COLOG((1.D0-R)/(1.D0+R))
TEST=DIAG(DI)
IF(TEST.GT.PI) DI=DI+DCMPLX(0.D0,TWOP1)
IF(TEST.LT.-PI) DI=DI+DCMPLX(0.D0,TWOP1)
40 I0=I0+FND+DI
DO=ZD-ZJ+DI
II=I1+FND+DI
I2=I2+FND+(DZ*ZB-ZJ+DI)
50 CONTINUE
RETURN
END
SUBROUTINE ROOT2(C, B, A, ZR)
FINDS ROOTS OF A*Z**2+B*Z+C

COMPLEX*16 A, B, C, ZR(2), Z(2), RAD
REAL*4 T1, T2, TOVFL

INFINITY = 1.0D7

ZR(1) = (1.0D7, 0.00)
ZR(2) = (1.0D7, 0.00)
RAD = CSQRT(B**2-4*A*C)
Z(1) = -B + RAD
Z(2) = -B - RAD
T1 = CDABS(Z(1))
T2 = CDABS(Z(2))
IF(T1*T2 .NE. 0.0) GO TO 30
IF(CDABS(A) .NE. 0.0) GO TO 20
IF(CDABS(C) .NE. 0.0) GO TO 10
WRITE (51, 501)

501 FORMAT (* ERROR STOP - A = B = C = 0*)
STOP

10 RETURN

20 ZR(1) = (0.00, 0.00)
ZR(2) = (0.00, 0.00)
RETURN

30 I = 1
IF(T2 .GT. T1) I = 2
ZR(1) = (C+C)/Z(I)
TOVFL = CDABS(1.0D7*Z(I)) - CDABS(A*A)
IF(TOVFL .GT. 0.0) RETURN
ZR(2) = Z(I)/(A+A)
RETURN
END

/*
SUBROUTINE COEF(Z,F,C)

PROGRAM TO FIND COEFFICIENTS FOR THE FUNCTION
\[ F(z) = \frac{C(1) + C(2)z + C(3)z^2}{C(4) + C(5)z + C(6)z^2} \]
GIVEN THE FUNCTION \( F \) AT FIVE POINTS \( z \).

IMPLICIT COMPLEX=16 (A-H,O-Z)
DIMENSION Z(5),F(5),C(6),FF(5)
REAL*8 ZERO, ONE, 1D-10, 3CF, FM(5), TSCF(5), TSCFM

DETERMINE SCALE FACTOR (SCF) FOR \( F \) SO \( F \) IS OF ORDER ONE

IF(CDABS(C(1)).NE.CDABS((1.D0,15,0.D0))) GO TO 20
DO 10 I=1,5
10 FM(I)=CDABS(F(I))
GO TO 40
20 DO 30 I=1,4
30 FM(I)=FM(I+1)
FM(5)=CDABS(F(5))
40 SCF=DFMAX1(FM(1),FM(2),FM(3),FM(4),FM(5))
IF(SCF.NE.ZERO) SCF=1.0D0/SCF

DETERMINE IF FUNCTION IS SINGULAR (TSCFM=0). IF ITSCF.GT.1, FUNCTION IS CONSTANT AT TWO OR MORE POINTS.

ITSCF=0
DO 50 I=1,5
TSCF(I)=SCF*FM(I)
IF(CDABS(TSCF(I)-1.0D0).GT.ZERO) GO TO 50
ITSCF=ITSCF+1
J=1
50 CONTINUE
IF(ITSCF.GE.2) GO TO 70
TSCFM=0.0D0
DO 60 I=1,5
IF(J.EQ.I) GO TO 60
IF(TSCF(I).GT.TSCFM) TSCFM=TSCF(I)
60 CONTINUE
IF(TSCFM.GT.ZERO) GO TO 70
WRITE (61,501) (F(I),I=1,5)
501 FORMAT (' ERROR STOP - SINGULAR FUNCTION:',/1PE12.4)
STOP
70 DO 80 I=1,5
80 FF(I)=SCF*F(I)

Z2=Z(1)-Z(3)
ZN1=Z(2)-Z(3)
Z1=Z(4)-Z(3)
Z2=Z(5)-Z(3)

GN2=(FF(1)-FF(3))/Z2
GN1=(FF(2)-FF(3))/ZN1
G1=(FF(4)-FF(3))/Z1
G2=(FF(5)-FF(3))/Z2
C

ZDENOM = 1.00 / (Z1 - ZN1)
F1M1 * FF(4) = FF(2)
ZF1 * Z1 * FF(4)
ZF1M1 * ZF1 = ZN1 * FF(2)
G1M1 * G1 = GN1
RAT = (ZN2 - Z1) * ZDENOM
AN2 = RAT * F1M1 * FF(1) + FF(4)
BN2 = RAT * ZF1M1 = ZN2 * FF(1) + ZF1
CN2 = GN2 = G1 = RAT * G1M1
RAT = (Z2 - Z1) * ZDENOM
A2 = RAT * F1M1 = FF(5) + FF(4)
B2 = RAT * ZF1M1 = Z2 * FF(5) + ZF1
C2 = G2 = G1 = RAT * G1M1

C

C(4) = A2 * BN2 = AN2 * B2
C(5) = C2 * BN2 = CN2 * B2
C(6) = A2 * CN2 = AN2 * C2

TEST FOR CASES WHERE DENOMINATOR COEFFICIENTS VANISH.

IF (CDABS(C(4)), GT, ZERO) GO TO 90
C(4) = A2
C(5) = C2
C(6) = (0.00, 0.00)
IF (CDABS(C(5)), LE, ZERO) C(4) = (1.00, 0.00)

90 C(5) = ZDENOM * (G1M1 * C(4) + F1M1 * C(5) + ZF1M1 * C(6))
C(2) = Z1 * C(3) + C1 * C(4) + FF(4) * (C(5) + Z1 * C(6))
C(1) = FF(3) * C(4)
RETURN
END