Mathematics Notes

Note 60

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Evaluation of the Oblate Spheroidal Wave Functions

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Abstract

This paper describes a method of calculating the oblate spheroidal wave functions given in Interaction Note 352.

A. Evaluation of $\lambda_{mn}(c)$

given c, m, n $0 \le c$ $0 \le m \le n$

Define

$$\left.\begin{aligned} A_r &= \frac{(2m+r+2)(2m+r+1)c^2}{(2m+2r+3)\ (2m+2r+5)} \\ C_r &= \frac{r(r-1)c^2}{(2m+2r-1)(2m+2r-3)} \\ \bar{B}_r &= (m+r)\ (m+r+1) - \frac{[2r^2+2r(2m+1)+2m-1]c^2}{(2m+2r-1)(2m+2r+3)} \end{aligned}\right\} \quad r = \text{integer}$$

λ_{mn} is determined by iteration or power series from the equation

$$\cfrac{-A_{r-2}}{\lambda-\bar{B}_{r-2} - \cfrac{C_{r-2}A_{r-4}}{\lambda-\bar{B}_{r-4} - \cfrac{C_{r-4}A_{r-6}}{\lambda - \dots}}}$$

$$= \frac{-\lambda+\bar{B}_r}{C_r} - \cfrac{A_r/C_r}{\cfrac{-\lambda+\bar{B}_{r+2}}{C_{r+2}} - \cfrac{A_{r+2}/C_{r+2}}{\cfrac{-\lambda+\bar{B}_{r+4}}{C_{r+4}} - \cfrac{A_{r+4}/C_{r+4}}{\cfrac{-\lambda+\bar{B}_{r+6}}{C_{r+6}} - \dots}}}$$

(See Abramowitz[1] 21.15, 21.81, Stratton[2] pg. 13, equation 82.)

1. Abramowitz, M. and I. A. Stegun, Handbook of Mathematical Functions, Dover Pub. Inc., 1965, pp. 752-759.

2. Stratton, J. A., P. M. Morse, L. J. Chu, J. D. C. Little, F. J. Corbato, Spheroidal Wave Functions, The M.I.T. Press, 1956.

B. Evaluation of f Ratios

Define $B_r = -\bar{B}_r + \lambda_{mn}$

Since $A_{-2m-1} = A_{-2m-2} = 0$, then the ratio

$$\frac{f_{r-2}}{f_r} = -\frac{A_{r-2}}{B_{r-2} + \frac{C_{r-2}f_{r-4}}{f_{r-2}}}$$

can be calculated for $r = -2m, -2m+1, \ldots, 0, 1$

($f_r(c) = d_r(-ic)$ of Abramowitz).

Since $C_1 = C_o = 0$, these ratios can also be calculated from

$r = 2, 3, \ldots$ to infinity.

$$\frac{f_r}{f_o} = \frac{f_r}{f_{r+2}} \cdot \frac{f_{r+2}}{f_{r+4}} \cdots \frac{f_{-2}}{f_o} \qquad -2m \leq r < o, \quad r = \text{even}$$

$$\frac{f_r}{f_1} = \frac{f_r}{f_{r+2}} \frac{f_{r+2}}{f_{r+4}} \cdots \frac{f_{-1}}{f_1} \qquad -2m+1 \leq r < o, \quad r = \text{odd}$$

$$\frac{f_r}{f_o} = \frac{f_r}{f_{r-2}} \frac{f_{r-2}}{f_{r-4}} \cdots \frac{f_2}{f_o} \qquad \text{if } r = \text{even}, \quad o < r$$

$$\frac{f_r}{f_1} = \frac{f_r}{f_{r-2}} \frac{f_{r-2}}{f_{r-4}} \cdots \frac{f_3}{f_1} \qquad \text{if } r = \text{odd}, \quad o < r$$

$$f_r = 0 \qquad r < -2m$$

C. Normalization of f

$$f_o = \sum_{\substack{r=o \\ \text{even}}}^{\infty} f_r/f_o \; P_{m+r}^{m}(o) \Big/ P_n^m(o) \qquad \text{n-m=even}$$

$$f_1 = \sum_{\substack{r=1 \\ \text{odd}}}^{\infty} f_r/f_1 \; P_{m+r}^{m'}(o) \Big/ P_n^{m'}(o) \qquad \text{n-m=odd}$$

where

$$P_n^m(o) = \frac{(-1)^{\frac{n+m}{2}} (n+m)!}{2^n \left(\frac{n-m}{2}\right)! \left(\frac{n+m}{2}\right)!} \qquad \text{n-m=even}$$

$$P_n^{m'}(o) = \frac{(-1)^{\frac{n+m-1}{2}} (n+m+1)!}{2^n \left(\frac{n-m-1}{2}\right)! \left(\frac{n+m+1}{2}\right)!} \qquad \text{n-m=odd}$$

$$S_{mn}^{(1)}(-ic,o) = P_n^m(o) \qquad \text{if n-m=even}$$

$$S_{mn}^{(1)'}(-ic,o) = P_n^{m'}(o) \qquad \text{if n-m=odd}$$

$$\left.\begin{aligned} f_r &= \frac{f_r}{f_o} f_o \quad \text{if r=odd} \\ f_r &= \frac{f_r}{f_i} f_i \quad \text{if r=even} \end{aligned}\right\} \quad r \geq -2m$$

$$f_r = 0 \qquad \text{if } r<-2m$$

This determines all f_r.

D. <u>Evaluation of the g ratios</u>

$$D_m \equiv \frac{c^2}{(2m-1)(2m+1)} \qquad E_m \equiv \frac{c^2}{(2m-1)(2m-3)}$$

$$\frac{g_{-2m-2}}{f_{-2m}} = \cfrac{D_m}{B_{-2m-2} \cfrac{-C_{2m-2}\, A_{-2m-4}}{B_{-2m-4} \cfrac{-C_{-2m-4}\, A_{-2m-6}}{B_{-2m-6} - \cdots}}}$$

$$\frac{g_{-2m-1}}{f_{-2m+1}} = \cfrac{-E_m}{B_{-2m-1} \cfrac{-C_{-2m-1}\, A_{-2m-3}}{B_{-2m-3} \cfrac{-C_{-2m-3}\, -A_{-2m-5}}{B_{-2m-5} - \cdots}}}$$

This determines g_{-2m-1}, g_{-2m-2}

$$\frac{g_{-2m-4}}{g_{-2m-2}} = -\frac{B_{-2m-2}}{C_{-2m-2}} + \frac{D_m}{C_{-2m-2}\, g_{-2m-2}/f_{-2m}}$$

$$\frac{g_{-2m-3}}{g_{-2m-1}} = -\frac{B_{-2m-1}}{C_{-2m-1}} - \frac{E_m}{C_{-2m-1}\, g_{-2m-1}/f_{-2m+1}}$$

$$\frac{g_{r-2}}{g_r} = -\frac{B_r}{C_r} - \frac{A_r}{C_r g_r/g_{r+2}} \qquad r<-2m-2$$

$$\frac{g_{r-2}}{g_{-2m-1}} = \frac{g_{r-2}}{g_r}\,\frac{g_r}{g_{r+2}} \cdots \frac{g_{-2m-3}}{g_{-2m-1}} \qquad \text{if } r = \text{odd}$$

$$\frac{g_{r-2}}{g_{-2m-2}} = \frac{g_{r-2}}{g_r}\,\frac{g_r}{g_{r+2}} \cdots \frac{g_{-2m-4}}{g_{-2m-2}} \qquad \text{if } r = \text{even}$$

$$g_{r-2} = \frac{g_{r-2}}{g_{-2m-1}} \qquad g_{-2m-1} \qquad \text{if } r = \text{odd}$$

$$g_{r-2} = \frac{g_{r-2}}{g_{-2m-2}} \qquad g_{-2m-2} \qquad \text{if } r = \text{even}$$

This completely determines all g_r $\qquad r < -2m$

E. Evaluation of $P_n^m\ (i\xi)$ $\quad o\leq\xi<1$

$$P_n^m\ (i\xi) = \frac{(-1)^{n/2}(m+n+1)!\ \xi(\xi^2+1)^{-m/2}F(\frac{1-m-n}{2},\ 1+\frac{n-m}{2},\ \frac{3}{2},\ -\xi^2)}{2^n\ (\frac{n-m-1}{2})!\ (\frac{n+m+1}{2})!}$$

If $o\leq m<n$ $\quad n - m = \text{odd}$ $\quad o\leq\xi<1$

F is a polynomial

$$F\ (a,b,c,z) = 1 + \frac{ab}{c}\ z + \frac{a(a+1)b(b+1)}{c(c+1)}\ \frac{z^2}{2!} + \ldots$$

$$P_n^m\ (i\xi) = \frac{(-1)^{\frac{n}{2}}\ (m+n)!(\xi^2+1)^{-m/2}F(\frac{-m-n}{2},\ \frac{1+n-m}{2},\ \frac{1}{2},\ -\xi^2)}{2^n\ (\frac{n-m}{2})!\ (\frac{n+m}{2})!}$$

if $o\leq m\leq n$ $\quad n - m = \text{even}$ $\quad o\leq\xi<1$

$$P_n^m\ (\eta) = \frac{(m+n+1)!\ \eta(1-\eta^2)^{-m/2}(-1)^{\frac{m+n+1}{2}}}{2^n(\frac{m+n+1}{2})!(\frac{n-m-1}{2})!}\ F\ (\frac{1-m-n}{2},\ 1+\frac{n-m}{2},\frac{3}{2},\eta^2)$$

if $o\leq m<n$ $\quad n - m = \text{odd}$ $\quad o\leq\eta^2<1$

$$P_n^m\ (\eta) = \frac{(m+n)!(1-\eta^2)^{-m/2}(-1)^{\frac{m+n}{2}}}{(\frac{m+n}{2})!(\frac{n-m}{2})!\ 2^n}\ F\ (\frac{-m-n}{2},\ \frac{1+n-m}{2},\frac{1}{2},\ \eta^2)$$

if $o\leq m<n$ $\quad n - m = \text{even}$ $\quad o\leq\eta^2<1$

$P_n^m\ (1) = 0 \quad . \quad P_n^m\ (-1) = 0 \qquad \text{if } 1 \leq m \leq n$

if $1 \leq m \leq n$

$P_n\ (1) = 1 \qquad o \leq n \quad P_n\ (-1) = (-1)^n \quad o \leq n$

$$P_n\ (i\xi) = \frac{\pi^{\frac{1}{2}}(2n)!\,i^n \xi^{m+n}}{(n-m)!\,n!\,2^n}\ (1+\xi^2)^{-m/2} F\left(\frac{-m-n}{2}, \frac{1-m-n}{2}, \frac{1}{2} - n, \frac{1}{\xi^2}\right)$$

$1 < \xi$

$o \leq m \leq n$

$$P_n^m\ (i) = e^{\frac{-\pi im}{4}} \sum_{r=m}^{n} \frac{(n+k)!\ e^{\frac{3\pi ik}{4}}}{(n-k)!\,k!\,(k-m)!\,2^{k/2}}$$

F. <u>Evaluation of Γ</u>

$o \le m \quad o \le m$

$$\frac{\Gamma(\frac{1+m+n}{2})}{\Gamma(1+\frac{n-m}{2})} = 0 \text{ if } n<m, \quad n-m = \text{even}$$

$$= \frac{(m+n)!\sqrt{\pi}}{(\frac{n-m}{2})!\,(\frac{n+m}{2})!\,2^{m+n}} \qquad \text{if } n \ge m \text{ and } n-m = \text{even}$$

$$= \frac{(\frac{m+n-1}{2})!\ (\frac{n-m+1}{2})!\,2^{n-m+1}}{(n-m+1)!\ \sqrt{\pi}} \qquad \text{if } n>m \text{ and } n-m = \text{odd}$$

$$= \frac{(-1)^{\frac{m-n+1}{2}}(\frac{m+n-1}{2})!\,(m-n+1)!}{\pi^{\frac{1}{2}}(\frac{m-n+1}{2})!\ 2^{m-n+1}} \qquad \text{if } n<m \text{ and } n-m = \text{odd}$$

$$\frac{\Gamma\ (1+\frac{m+n}{2})}{\Gamma\ (\frac{1+n-m}{2})} = 0 \quad \text{if } n<m \text{ and } n-m = \text{odd}$$

$$= \frac{\sqrt{\pi}\ (m+n+1)!}{(\frac{m+n+1}{2})!\,2^{m+n+1}\,(\frac{n-m-1}{2})!} \qquad \text{if } n>m \text{ and } n-m = \text{even}$$

$$= \frac{(\frac{m+n}{2})!\,(\frac{n-m}{2})!\ 2^{n-m}}{(n-m)!\ \pi^{\frac{1}{2}}} \qquad \text{if } n>m \text{ and } n-m = \text{even}$$

$$= \frac{(-1)^{\frac{n-m}{2}}(\frac{m+n}{2})!\,(m-n)!}{\pi^{\frac{1}{2}}\ (\frac{m-n}{2})!\,2^{m-n}} \qquad m \ge n \text{ and } n-m = \text{even}$$

G. Evaluation of $Q_n^m\ (i\xi)$ $\quad o\leq\xi<1$

$o\leq m \quad o\leq n$

$$Q_n^m\ (i\xi) = \pi^{\frac{1}{2}} 2^m (1+\xi^2)^{-m/2} i^m$$

$$\left[\frac{\Gamma(\frac{1+m+n}{2})}{2\Gamma(1+\frac{n-m}{2})}\ i^{m-n-1}\ F\left(-(\frac{m+n}{2}),\ \frac{1+n-m}{2},\ \frac{1}{2},\ -\xi^2 \right) \right.$$

$$\left. + \frac{\xi\Gamma(1+\frac{m+n}{2})}{\Gamma(\frac{1+n-m}{2})}\ i^{m-n+1}\ F\left(\frac{1-n-m}{2},\ 1+\frac{n-m}{2},\ \frac{3}{2},\ -\xi^2\right) \right]$$

$$Q_n^m\ (i\xi) = \frac{2^{n+1}(m+n)!\,(1+\xi^2)^{m/2}}{(2n+2)!\,i^{n+1}\xi^{m+n+1}}\ F\left(1+\frac{m+n}{2},\ \frac{1+m+n}{2},\ n+\frac{3}{2},\ -\frac{1}{\xi^2}\right)$$

$1\leq\xi$

$o\leq m+n$

H. Evaluation of $S_{mn}^{(2)}(-ic, i\xi)$

$$S_{mn}^{(2)}(-ic, i\xi) = \sum_{r=-\infty}^{-2m-1}{}' g_r P_m^m{}_{-r-1}(i\xi)$$

$$+ \sum_{r=0}^{m-1}{}' [f_{r-m} + f_{-r-m-1}] Q_r^m (i\xi)$$

$$+ \sum_{r=0}^{\infty}{}' f_r Q_{m+r}^m (i\xi)$$

The second term is absent if m=0. Prime indicates summation over even or odd, depending on whether n-m = even or odd.

I. Evaluation of $R_{mn}^{(2)}(-ic,i\xi)$

$$k_{mn}^{(2)}(-ic) = \frac{2^{n-m}(2m)!\,(\frac{n-m}{2})!\,(\frac{n+m}{2})!\ f_{-2m}^{mn}}{(2m-1)m!\,(n+m)!\,(-ic)^{m-1}}$$

$$\sum_{\substack{r=0\\ \text{even}}}^{\infty} \frac{(2m+r)!}{r!}\, f_r$$

if n-m = even

$$k_{mn}^{(2)}(-ic) = -\frac{2^{n-m}(2m)!\,(\frac{n-m-1}{2})!\,(\frac{n+m+1}{2})!\ f_{-2m+1}^{mn}(c)}{(2m-3)(2m-1)m!\,(m+n+1)!\,(ic)^{m-2}}$$

$$\sum_{\substack{r=1\\ \text{odd}}}^{\infty} \frac{(2m+r)!}{r!}\, f_r$$

if n-m = odd

$$R_{mn}^{(2)}(-ic,i\xi) = S_{mn}^{(2)}(-ic,i\xi)/k_{mn}^{(2)}(-ic)$$

J. Evaluation of $R_{mn}^{(1)}(-ic, i\xi)$

$$R_{mn}^{(1)}(-ic,i\xi) = \frac{1}{N}\left(\frac{\xi^2+1}{\xi^2}\right)^{m/2} \sum_{r=0,1}^{\infty}{}' \frac{(2m+r)!}{r!}\, i^{n-m-r} f_r^{mn}\, j_{m+r}(c\xi)$$

where

$$N = \sum_{r=0,1}^{\infty}{}' \frac{(2n+r)!}{r!} f_r^{mn}$$

Abram 756/21.9.1

and where

$$j_n(z) = \frac{z^n}{(2n+1)!!}\left\{1 - \frac{z^2/2}{1!(2n+3)} + \frac{(z^2/2)^2}{2!(2n+3)(2n+5)} - \dots\right\}$$

if $o<\xi$

$$R_{mn}^{(1)}(-ic,io) = 0 \quad \text{if } n-m = \text{odd}$$

$$= \frac{(m+1)!\,2^{m+1} i^{n-m} f_o^{mn} c^m}{(2m+2)(2m+1)N} \quad \text{if } n-m = \text{even}$$

$$R_{mn}^{(1)\prime}(-ic,,io) = 0 \text{ if } n-m = \text{even}$$

$$R_{mn}^{(1)\prime}(-ic,io) = \frac{1}{N}\,\frac{i^{m-n}(2m+1)!\,f_1^{mn} c^{m+1}}{(2m+3)!!}$$

$$= \frac{1}{N}\,\frac{i^{m-n}(m+2)!\,2^{m+2}(2m+1)!\,f_1^{mn} c^{m+1}}{(2m+4)!}$$

if $n-m$ = odd.

K. Evaluation of $S_{mn}^{(1)}(-ic,\eta)$

$$S_{mn}^{(1)} (-ic,\eta) = \sum_{r=0,1}^{\infty}{}' f_r^{mn} P_{m+r}^{m} (\eta)$$

L. Evaluation of $R^{(3)\prime}(-ic,io)$ and $R^{(3)}(-ic,io)$

$$R_{mn}^{(3)\prime} (-ic,io) = + i /c R_{mn}^{(1)} (-ic,io) \qquad n-m = \text{even}$$

$$R_{mn}^{(3)} (-ic,io) = -i/c R_{mn}^{(1)\prime} (-ic,io) \qquad n-m = \text{odd}$$

M. Evaluation of $R_{mn}^{(1)}(-ic,i\xi)$

$$R_{mn}^{(3)} (-ic,i\xi) = R_{mn}^{(1)} (-ic,i\xi) + iR_{mn}^{(2)} (-ic,i\xi)$$

N. Evaluation of $I_{n\bar{n}}$, $L_{n\bar{n}}$

$$I_{n\bar{n}} = \sum_{\substack{r=1\\ \text{odd}}}^{\infty} \frac{f_r^{on} (c_1) \, f_r^{o\bar{n}} (c_2)}{2r+1}$$

$n,\bar{n}$ = odd

$$L_{n\bar{n}} = \sum_{\substack{r=0\\ \text{even}}}^{\infty} \frac{f^{ln} (c_1) \, f^{l\bar{n}} (c_2)}{2r+1}$$

n,n = odd