Mathematics Notes

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Matrix Operators and Functions Thereof

Carl E. Baum
University of New Mexico
Department of Electrical and Computer Engineering
Albuquerque New Mexico 87131

Abstract

This paper summarizes some results of matrix operators and functions of these. Particular attention is given to commutators for which a new symbol, ©, is used.

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1. Introduction

Matrices, matrix operators, and functions of these are important areas of mathematical physics. Our interest has centered on the solution of the telegrapher equations for nonuniform multiconductor transmission lines, but the application is actually much broader [1-3].

In this paper we summarize some results (with some minor extensions). We also introduce some new notation for commutator operators.

2. Functions of Matrices

Consider some constant N x N matrix ($a_{n,m}$). If we have some scalar function *f(x)* which has a power (Taylor) series representation about x_0 as

$$f(x) = f(x_0) + \sum_{n=1}^{\infty} \left[\frac{d^n}{dx^n} f(x) \right] \Bigg|_{x=x_0} \frac{1}{n!} [x - x_0]^n \tag{2.1}$$

with some radius of convergence, which might be the entire complex *x*-plane. This generalizes to functions of N x N matrices ($a_{n,m}$) as

$$\begin{aligned} f\left(\left(a_{n,.m}\right)\right) &= f\left(\left(a_{n,m}\right)_0\right) + \sum_{\ell=1}^{\infty} f^{(\ell)}\left(\left(a_{n,m}\right)_0\right) \frac{1}{\ell!} \left[\left(a_{n,m}\right) - \left(a_{n,m}\right)_0\right]^{\ell} \\ f^{(\ell)}(x) &\equiv \frac{d^{\ell}}{dx^{\ell}} f(x) \end{aligned} \tag{2.2}$$

The radius of convergence about $x = x_0$ generalizes to the largest eigenvalue of $(a_{n,m}) - (a_{n,m})_0$.

Some common matrix functions include

$$\begin{aligned} e^{(a_{n,m})} &= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (a_{n,m})^{\ell} \\ \ell n\left(\left(1_{n,m}\right) + \left(a_{n,m}\right)\right) &= \sum_{\ell=0}^{\infty} (-1)^{\ell-1} \frac{(a_{n,m})^{\ell}}{\ell} \end{aligned} \tag{2.3}$$

and similarly for trigonometric and hyperbolic functions, etc.

3. Functions of Matrix Operators

An operator may be as simple as addition or multiplication by some *f(x)* in the scalar case. Going on to the calculus it may involve differentiation or integration. In general it can be represented as

$$F(x) = y$$

$$F(\cdots) \equiv \text{operator mapping the variable } x \text{ to the variable } y \tag{3.1}$$

Similarly we have for N x N matrices

$$F\left(\left(a_{n,m}\right)\right) = \left(b_{n,m}\right) \tag{3.2}$$

Now $(a_{n,m})$ and $(b_{n,m})$ may be functions of some one or more variables, such as *x* or *t*, or whatever.

We can have various examples of matrix operators such as

$$F(\cdots) = \left(c_{n,m}\right) \bullet (\cdots)$$

$$F(\cdots) = \frac{\partial}{\partial x}(\cdots) \text{ (derivative)}$$

$$F(\cdots) = \int_{x_0}^{x} (\cdots) dx' \text{ (integral)}$$

$$F(\cdots) = \prod_{x_0}^{x} e^{(\cdots)dx'} \text{ (product integral)} \tag{3.3}$$

$$F(\cdots) = D_t(\cdots) = \left[\frac{d}{dt}(\cdots)\right] \bullet (\cdots)^{-1} \text{ (product derivative)}$$

The basic idea is that $F(\cdots)$ provides a formula for mapping ($a_{n,m}$) (which can be in turn a function of various parameters) into ($b_{n,m}$).

An important aspect of such operators concerns their definition as power series. Let $f(x)$ be representable as

$$f(x) = x_0 + \sum_{n=1}^{\infty} \frac{d^n}{dx^n} f(x)\bigg|_{x=x_0} \frac{1}{n!}\left[x - x_0\right]^n \tag{3.4}$$

within some radius of convergence. Then the matrix operator can take various forms, such as

$$\begin{aligned} F(\cdots) &= \left(c_{n,m}\right) \bullet \quad \Rightarrow \quad F\left(\left(a_{n,m}\right)\right) = \left(c_{n,m}\right) \bullet \left(a_{n,m}\right) \\ F(\cdots) &= \frac{d^n}{dt^n} \text{ or } \left(1_{n,m}\right)\frac{d^n}{dt^n} \bullet \quad \Rightarrow \quad F\left(\left(a_{n,m}(t)\right)\right) = \frac{d^n}{dt^n}\left(a_{n,m}(t)\right) \end{aligned} \tag{3.5}$$

The operator can be defined by a series of operators as

$$F(\cdots) \equiv \sum_{n=0}^{\infty} F_n(\cdots) \tag{3.6}$$

Where F_n can assume various forms, such as

$$F_n(\cdots) = f_n \frac{d^n}{dx^n} \tag{3.7}$$

Note now that this is not a product of *n* terms, but the successive operation by *d/dt n* times. For example, we might have

$$e^{\frac{d}{dx}}(\cdots) = \left[\sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{dx^n}\right] \bullet (\cdots) \tag{3.8}$$

For both scalar and square matrix arguments.

Another important operator is the product integral or matrizant operator given by a series as

$$\begin{aligned} \prod_{z_0}^{z} e^{(\cdots)dz'} &= \sum_{\ell=0}^{\infty} \left(X_{n,m}(z, z_0)\right)_\ell \\ \left(X_{n,m}(z, z_0)\right)_0 &= \left(1_{n,m}\right) \\ \left(X_{n,m}(z, z_0)\right)_{\ell+1} &+ \int_{z_0}^{z} (\cdots) \bullet \left(X_{n,m}(z', z_0)\right)_\ell \, dz' \end{aligned} \tag{3.9}$$

Its inverse operator is the product derivative as in (3.3). This solves the differential equation

$$\begin{aligned}
\frac{d}{dz}\left(U_{n,m}(z,z_0)\right) &= \left(a_{n,m}(z)\right) \bullet \left(U_{n,m}(z,z_0)\right) \\
\left(U_{n,m}(z_0,z_0)\right) &= \left(1_{n,m}\right) \\
\left(U_{n,m}(z,z_0)\right) &= \prod_{z_0}^{z} e^{\left(a_{n,m}(z')\right)dz'}
\end{aligned} \tag{3.10}$$

4. Commutators

An important operation is the commutator of two N x N matrices, appearing often in problems of mathematical physics. This is often expressed by

$$\left[\left(a_{n,m}\right)\ ,\ \left(b_{n,m}\right)\right] \equiv \left(a_{n,m}\right)\bullet\left(b_{n,m}\right) - \left(b_{n,m}\right)\bullet\left(a_{n,m}\right) \tag{4.1}$$

If the two matrices commute, then the commutator is zero. For $n = 1$ (scalars) the commutator is always zero.

Thinking of this as an operator we might have

$$\left(a_{n,m}\right)© \ (\cdots) = \left[\left(a_{n,m}\right),\ (\cdots)\right] = \left(a_{n,m}\right)\bullet(\cdots) - (\cdots)\bullet\left(a_{n,m}\right) \tag{4.2}$$

In this case $(a_{n,m})©$ is the operator. Noting, however, that

$$\left(a_{n,m}\right)\ ©\ \left(b_{n,m}\right) = -\left(b_{n,m}\right)\ ©\ \left(a_{n,m}\right) \tag{4.3}$$

we can equally regard $(-b_{n,m})©$ as the operator. Here we use the symbol

$$© \ \equiv \quad \text{commutator} \tag{4.4}$$

The commutator has the important property [3]

$$tr\left(\left(a_{n,m}\right)\ ©\ (\cdots)\right) = \left(0_{n,m}\right) \tag{4.5}$$

With $(a_{n,m})©$ as the operator we can define recursively

$$\left[\left(a_{n,m}\right)©\ \right]^{n} \equiv \left(a_{n,m}\right)©\ \left[\left(a_{n,m}\right)©\ \right]^{n-1} \tag{4.6}$$

Again, this is not multiplication, but successive application. Note that this is defined for positive integer values of n. One starts with $n = 1$ for the first operation. This produces another matrix. Then operate by $(a_{n,m})©$ again for $n = 2$ to produce another matrix. Continue through the nth application. Note that (4.5) applies to this as well.

The Jacobi identity is

$$\left(a_{n,m}\right)© \left[\left(b_{n,m}\right)© \left(c_{n,m}\right)\right] + \left(b_{n,m}\right)© \left[\left(c_{n,m}\right)© \left(a_{n,m}\right)\right] + \left(c_{n,m}\right)© \left[\left(a_{n,m}\right)© \left(b_{n,m}\right)\right] = \left(0_{n,m}\right) \tag{4.7}$$

Commutators can also be applied to supermatrices. (See [1 (Appendix A)] for appropriate definitions.) For the direct sum we have

$$\begin{aligned}
&\left[\left(a_{n,m}\right)\oplus\left(c_{n,m}\right)\right] © \left[\left(b_{n,m}\right)\oplus\left(d_{n,m}\right)\right] \\
&\quad = \left[\left(a_{n,m}\right)\bullet\left(b_{n,m}\right)\right] \oplus \left[\left(c_{n,m}\right)\bullet\left(d_{n,m}\right)\right] \\
&\qquad - \left[\left(b_{n,m}\right)\bullet\left(a_{n,m}\right)\right] \oplus \left[\left(d_{n,m}\right)\bullet\left(c_{n,m}\right)\right] \\
&\quad \left(a_{n,m}\right) \text{ and } \left(b_{n,m}\right) \text{ are } M_1 \times M_1 \\
&\quad \left(c_{n,m}\right) \text{ and } \left(d_{n,m}\right) \text{ are } M_2 \times M_2 \\
&\oplus \equiv \text{ direct sum producing block-diagonal matrix } (M_1) \text{ x } (M_2) \\
&\odot \equiv \text{ generalized dot product}
\end{aligned} \tag{4.8}$$

For the direct product we have

$$\begin{aligned}
&\left[\left(a_{n,m}\right)\otimes\left(c_{n,m}\right)\right] © \left[\left(b_{n,m}\right)\otimes\left(d_{n,m}\right)\right] \\
&\quad = \left[\left(a_{n,m}\right)\otimes\left(c_{n,m}\right)\right] \odot \left[\left(b_{n,m}\right)\otimes\left(d_{n,m}\right)\right] \\
&\quad - \left[\left(b_{n,m}\right)\otimes\left(d_{n,m}\right)\right] \odot \left[\left(a_{n,m}\right)\otimes\left(c_{n,m}\right)\right] \\
&\quad = \left[\left(a_{n,m}\right)\bullet\left(b_{n,m}\right)\right] \otimes \left[\left(c_{n,m}\right)\bullet\left(d_{n,m}\right)\right] \\
&\quad - \left[\left(b_{n,m}\right)\bullet\left(a_{n,m}\right)\right] \otimes \left[\left(d_{n,m}\right)\bullet\left(c_{n,m}\right)\right] \\
&\quad = \left[\left(a_{n,m}\right)\bullet\left(b_{n,m}\right)\right] \otimes \left[\left(c_{n,m}\right)\bullet\left(d_{n,m}\right) - \left(d_{n,m}\right)\bullet\left(c_{n,m}\right)\right] \\
&\qquad\qquad \text{if } (a_{n,m}) \text{ and } (b_{n,m}) \text{ commute} \\
&= \left(0_{n,m}\right) \otimes \left(0_{n,m}\right) \\
&\qquad\qquad \text{if } (c_{n,m}) \text{ and } (d_{n,m}) \text{ also commute} \\
&\otimes \equiv \text{ direct product producing } [M_1 \times M_2] \text{ by } [M_1 \times M_2] \text{ matrices} \\
&\left(a_{n,m}\right) \text{ and } \left(b_{n,m}\right) \text{ are } M_1 \text{ by } M_1 \\
&\left(c_{n,m}\right) \text{ and } \left(d_{n,m}\right) \text{ are } M_2 \text{ by } M_2
\end{aligned} \tag{4.9}$$

5. Functions of Commutators

Having defined powers of commutators in (4.4), we can now consider functions of commutators if they are expressible by power series as in Section 3. A commonly used one is the exponential as [2]

$$e^{\left(a_{n,m}\right)©}\left(\cdots\right) = \left(1_{n,m}\right) + \sum_{\ell=1}^{\infty}\frac{1}{\ell!}\left[\left(a_{n,m}\right)©\right]^{\ell}\left(\cdots\right) \tag{5.1}$$

From this one can define hyperbolic commutators as

$$\begin{aligned}
\cosh\left(\left(a_{n,m}\right)©\right)\ \left(\cdots\right) &\equiv \frac{1}{2}\left[e^{\left(a_{n,m}\right)©}\left(\cdots\right) + e^{-\left(a_{n,m}\right)©}\left(\cdots\right)\right] \\
&= \left(1_{n,m}\right) + \sum_{\ell=2}^{\infty,2}\frac{1}{\ell!}\left[\left(a_{n,m}\right)©\right]^{\ell}\left(\cdots\right) \\
\sinh\left(\left(\tilde{a}_{n,m}\right)©\right)\ \left(\cdots\right) &\equiv \frac{1}{2}\left[e^{\left(a_{n,m}\right)©}\left(\cdots\right) - e^{-\left(a_{n,m}\right)©}\left(\cdots\right)\right] \\
&= \sum_{\ell=2}^{\infty,2}\frac{1}{\ell!}\left[\left(a_{n,m}\right)©\right]^{\ell}\left(\cdots\right)
\end{aligned} \tag{5.2}$$

This is also readily extended to the corresponding trigonometric functions. We then have the result from [3]

$$e^{\left(a_{n,m}\right)} \bullet \left(\cdots\right) \bullet e^{-\left(a_{n,m}\right)} = e^{\left(a_{n,m}\right)©}\left(\cdots\right) \tag{5.3}$$

which can be applied to the cosh and sinh as

$$\begin{aligned}
\cosh\left(\left(a_{n,m}\right)©\right)\left(\cdots\right) &= \frac{1}{2}\left[e^{\left(a_{n,m}\right)}\bullet\left(\cdots\right)\bullet e^{-\left(a_{n,m}\right)} + e^{-\left(a_{n,m}\right)}\bullet\left(\cdots\right)\bullet e^{\left(a_{n,m}\right)}\right] \\
\sinh\left(\left(a_{n,m}\right)©\right)\left(\cdots\right) &= \frac{1}{2}\left[e^{\left(a_{n,m}\right)}\bullet\left(\cdots\right)\bullet e^{-\left(a_{n,m}\right)} - e^{-\left(a_{n,m}\right)}\bullet\left(\cdots\right)\bullet e^{\left(a_{n,m}\right)}\right]
\end{aligned} \tag{5.4}$$

Similar formulas apply to sine and cosine.

The logarithm operator has

$$\ell n\left(\left[\left(1_{n,m}\right)+\left(a_{n,m}\right)\right]©\right) = \sum_{\ell=1}^{\infty}\left(-1\right)^{\ell-1}\ \frac{\left[\left(a_{n,m}\right)©\right]^{\ell}}{\ell} = \ell n\left(\left(1_{n,m}\right)+\left(a_{n,m}\right)©\right) \tag{5.5}$$

noting that the identity commutes with every matrix.

6. Concluding Remarks

Here we have a summary of some important properties of matrix operators, with some small extensions. Particular attention is given to the commutator operator.

References

1. C. E. Baum, “Symmetric Renormalization of the Nonuniform Multiconductor-Tranmission-Line Equations with a Single Modal Speed for Analytically Solvable Sections”, Interaction Note 537, January 1998.

2. S. Steinberg and C. E. Baum, “Solutions of the Transmission Line Equations Using Product Integrals of Variable Matrices“, Interaction Note 557, February 2000.

3. S. Steinberg and C. E. Baum, “Lie-Algebraic Representations of Product Integrals of Variable Matrices”, Mathematics Note 92, September 1998.