Mathematics Notes

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Representation and Product Integration of 2 x 2 Matrices

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Abstract

This paper considers the product integration of 2 x 2 matrices by using the quaternion decomposition of such matrices. This gives conditions for the solution in terms of sum integrals. This is applied to the solution of second-order linear differential equations, including the telegrapher equations.

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1. Introduction

A previous paper [6] has considered the decomposition of 2 x 2 matrices using the four quaternion units. This gives a convenient way to represent such matrices for product integration, as we shall see. Such matrices are also useful for casting a second-order linear differential equation as a first-order vector/matrix differential equation, the solution of which is representable (and calculable) by a product integral.

The properties of the product integral can then be explored to understand the possible solutions of second-order linear differential equations. Only in special cases can the solution of the product integral be expressed in terms of one or a few of the usual sum (Riemann) integrals. In effect, this gives conditions on the "sum integrability" of such equations. As a special case we consider the solution of the telegrapher equations for a single-conductor (plus reference) transmission line.

2. Quaternion Decomposition of 2 x 2 Matrices

Consider a 2 x 2 matrix (in general complex)

$$\left(f_{n,m}\right) \equiv \begin{pmatrix} f_{1,1} & f_{1,2} \\ f_{2,1} & f_{2,2} \end{pmatrix} \tag{2.1}$$

Let us write this in quaternion form [6].

For quarternions begin with

$$\begin{aligned} \hat{1} &\equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \hat{1}_0 \equiv \text{identity quaternion} \\ \hat{0} &\equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \equiv \text{zero quaternion} \end{aligned} \tag{2.2}$$

Then the four quaternion units are

$$\begin{aligned} &\hat{1}_0\,,\ \hat{1}_1\,,\ \hat{1}_2\,,\ \hat{1}_3 \\ &\hat{1}_1 = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix} = j\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - j\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = j\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &\hat{1}_2 = \begin{pmatrix} 0 & 1 \\ -1 & -0 \end{pmatrix} = \frac{j}{2}\begin{pmatrix} 1 \\ j \end{pmatrix}\begin{pmatrix} 1 \\ -j \end{pmatrix} - \frac{j}{2}\begin{pmatrix} 1 \\ -j \end{pmatrix}\begin{pmatrix} 1 \\ j \end{pmatrix} \\ &\hat{1}_3 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} = \frac{j}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{j}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = j\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &\mathrm{tr}(\hat{1}_1) = \mathrm{tr}(\hat{1}_2) = \mathrm{tr}(\hat{1}_3) = \\ &\det(\hat{1}_1) = \det(\hat{1}_2) = \det(\hat{1}_3) = \det(\hat{1}_0) = 1 \end{aligned} \tag{2.3}$$

From this we can find that the eigenvalues of each of $\hat{1}_1$, $\hat{1}_2$ and $\hat{1}_3$ are $+j$ and $-j$, while the eigenvalues of $\hat{1}_0$ are both +1.

These have various interrelationships as

$$
\begin{aligned}
-\hat{1}_0 &= \hat{1}_1^2 = \hat{1}_2^2 = \hat{1}_3^2 \\
\hat{1}_1 \bullet \hat{1}_2 &= \hat{1}_3 = -\hat{1}_2 \bullet \hat{1}_1 \\
\hat{1}_2 \bullet \hat{1}_3 &= \hat{1}_1 = -\hat{1}_3 \bullet \hat{1}_2 \\
\hat{1}_3 \bullet \hat{1}_1 &= \hat{1}_2 = -\hat{1}_1 \bullet \hat{1}_3
\end{aligned} \tag{2.4}
$$

Here we have used subscripts 1, 2, 3. This corresponds to *x, y, z* in [6]. In that case we were concerned with three spatial coordinates *x, y, z* (plus time *t*). Here we are concerned with one coordinate with 2 x 2 matrices as functions of that coordinate. In the previous case we also had

$$
\hat{1}_q = jq\hat{1} \quad , \quad q = \pm 1 \tag{2.5}
$$

as one of our units.

We would like to write

$$
\left(f_{n,m}\right) = \sum_{p=0}^{3} F_p \hat{1}_p \tag{2.6}
$$

Thinking, for the moment, of the quaternions as 4-component vectors, the four elements of $(f_{n,m})$ can be readily written in terms of these four "units". First, we find

$$
\begin{aligned}
F_0 &= \frac{1}{2}\left[f_{1,1} + f_{2,2}\right] \\
\left(f_{n,m}\right) &= F_0 \hat{1} + \begin{pmatrix} f_0 & f_{1,2} \\ f_{2,1} & -f_0 \end{pmatrix} \\
f_0 &= \frac{1}{2}\left[f_{1,1} + f_{2,2}\right]
\end{aligned} \tag{2.7}
$$

Note that

$$
\operatorname{tr}\left(\begin{pmatrix} f_0 & f_{1,2} \\ f_{2,1} & -f_0 \end{pmatrix}\right) = 0 \tag{2.8}
$$

corresponding to the three traceless units. For completeness then we have the eigenvalues

$$
\begin{aligned}
&\lambda_1 = -\lambda_2 \text{ (zero trace)} \\
&\det\left(\begin{pmatrix} f_0 & f_{1,2} \\ f_{2,1} & -f_0 \end{pmatrix}\right) = \lambda_1\lambda_2 = -\lambda_\beta^2 , \ \beta = 1,2 \\
&\lambda_\beta = \pm\left[f_0^2 + f_{1,2}f_{2,1}\right]^{1/2}
\end{aligned}
\tag{2.9}
$$

These two eigenvalues could then be real, imaginary, or general complex.

Continuing the expansion we readily find

$$
\begin{aligned}
&\left(f_{n,m}\right) = F_0 \hat{1}_0 + F_1 \hat{1}_1 + F_2 \hat{1}_2 + F_3 \hat{1}_3 \\
&F_1 = -jf_0 = -\frac{j}{2}\left[f_{1,1} - f_{2,2}\right] \\
&F_2 = \frac{1}{2}\left[f_{1,2} - f_{2,1}\right] \\
&F_3 = -\frac{j}{2}\left[f_{1,2} - f_{2,1}\right]
\end{aligned}
\tag{2.10}
$$

While $(f_{n,m})$ can be complex, if it is real then F_1 and F_3 are imaginary.

3. Product Integral of 2 x 2 Matrices

Let us now consider a product integral of the form

$$\left(\Phi_{n,m}\left(z,z_0\right)\right) = \prod_{z_0}^{z} e^{\left(f_{n,m}(z')\right)dz'} \tag{3.1}$$

where our 2 x 2 matrix is a function of a real variable which we take as z (for example, the spatial coordinate along a transmission line). Let us consider the quaternion decomposition of $(f_{n,m}(z))$ as in Section 2 for the purpose of calculating this product integral.

First we find

$$\begin{aligned}
&\left(\Phi_{n,m}\left(z,z_0\right)\right) \\
&\quad = \left[\prod_{z_0}^{z} e^{F_0(z')\hat{1}_0\,dz'}\right] \bullet \left[\prod_{z_0}^{z} e^{\begin{pmatrix} f_0(z') & f_{1,2}(z') \\ f_{2,1}(z') & -f_0(z') \end{pmatrix} dz'}\right] \\
&\quad = e^{\int_{z_0}^{z} F_0(z')dz'}\,\hat{1}_0 \bullet \left(\Lambda_{n,m}\left(z,z_0\right)\right) \\
&\quad = e^{\int_{z_0}^{z} F_0(z')dz'}\left(\Lambda_{n,m}\left(z,z_0\right)\right) \\
&\left(\Lambda_{n,m}\left(z,z_0\right)\right) = \prod_{z_0}^{z} e^{\begin{pmatrix} f_0(z') & f_{1,2}(z') \\ f_{2,2}(z') & -f_0(z') \end{pmatrix} dz'}
\end{aligned} \tag{3.2}$$

since the identity commutes with all matrices. This reduces the problem to three functions.

Having pulled out $F_0(z)$ (defined in (2.7)), we can similarly write

$$\left(\Lambda_{n,m}\left(z,z_0\right)\right) = \prod_{z_0}^{z} e^{\left[F_1(z')\hat{1}_1 + F_2(z')\hat{1}_2 + F_3(z')\hat{1}_3\right]dz'} \tag{3.3}$$

with the three functions defined in (2.10). First we note that zero trace for the exponent implies

$$
\begin{aligned}
&\det\left(\Lambda_{n,m}(z,z_0)\right) \\
&\quad = \Lambda_{1,1}(z,z_0)\Lambda_{2,2}(z,z_0) - \Lambda_{1,2}(z,z_0)\Lambda_{2,1}(z,z_0) \\
&\quad = 1
\end{aligned}
\tag{3.4}
$$

There are then only three "independent" matrix elements as we might expect.

We can next bring out one of the remaining quaternion units. Let us start with $\hat{1}_1$ for which we have

$$
\begin{aligned}
\left(g_{n,m}(z,z_0)\right)_1 &\equiv \prod_{z_0}^{z} e^{F_1(z')\hat{1}_1 dz'} = e^{G_1(z,z_0)\hat{1}_1} \\
G_1(z,z_0) &= \int_{z_0}^{z} F_1(z')dz' \quad , \quad \det\left(\left(g(z,z_0)\right)_1\right) = 1
\end{aligned}
\tag{3.5}
$$

Then applying the sum rule [2, 3] of the product integral gives

$$
\begin{aligned}
\left(\Lambda_{n,m}(z,z_0)\right) &= \left(g_{n,m}(z,z_0)\right) \bullet \left(\Xi_{n,m}(z,z_0)\right) \\
\left(\Xi_{n,m}(z,z_0)\right) &= \prod_{z_0}^{z} e^{\left(g_{n,m}(z',z_0)\right)_1^{-1}\left[F_2(z')\hat{1}_2 + F_3(z')\hat{1}_3\right] \bullet \left(g_{n,m}(z',z_0)\right)_1 dz'}
\end{aligned}
\tag{3.6}
$$

Now expand the exponential as

$$
\begin{aligned}
\left(g_{n,m}(z,z_0)\right) &= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} G_1^{\ell}(z,z_0)\hat{1}_1^{\ell} \\
&= \sum_{\ell=0}^{\infty,2} \frac{1}{\ell!} G_1^{\ell}(z,z_0)[-\hat{1}_0]^{\frac{\ell}{2}} + \sum_{\ell=1}^{\infty,2} \frac{1}{\ell!} G_1^{\ell}(z,z_0)\hat{1}_1[-\hat{1}_0]^{\frac{\ell-1}{2}} \\
&= \cos\left(G_1(z,z_0)\right)\hat{1}_0 + \sin\left(G_1(z,z_0)\right)\hat{1}_1 \\
\left(g_{n,m}(z,z_0)\right)^{-1} &= e^{-G_1(z,z_0)\hat{1}_1} \\
&= \cos\left(G_1(z,z_0)\right)\hat{1}_0 + \sin\left(G_1(z,z_0)\right)\hat{1}_1
\end{aligned}
\tag{3.7}
$$

Applying this to the exponent in the (3.6) product integral (the product integrand) gives

$$\left(g_{n,m}(z,z_0)\right)_1^{-1} \bullet \left[F_2(z)\hat{1}_2 + F_3(z)\hat{1}_3\right] \bullet \left(g_{n,m}(z,z_0)\right)_1$$

$$= F_2(z)\Big[\left[\cos^2\left(G_1(z,z_0)\right) - \sin^2\left(G_1(z,z_0)\right)\right]\hat{1}_2$$

$$- 2\cos\left(G_1(z,z_0)\right)\ \sin\left(z,z_0\right)\hat{1}_3\Big]$$

$$+ F_3(z)\Big[\left[\cos^2\left(G_1(z,z_0)\right) - \sin^2\left(G_1(z,z_0)\right)\right]\hat{1}_3$$

$$+ 2\cos\left(G_1(z,z_0)\right)\ \sin\left(z,z_0\right)\hat{1}_2\Big]$$

$$= F_2(z)\left[\cos\left(2G_1(z,z_0)\right)\hat{1}_2 - \sin\left(2G_1(z,z_0)\right)\hat{1}_3\right]$$

$$+ F_3(z)\left[\cos\left(2G_1(z,z_0)\right)\hat{1}_3 + \sin\left(2G_1(z,z_0)\right)\hat{1}_2\right]$$

$$= \left[F_2(z)\cos\left(2G_1(z,z_0)\right) + F_3(z)\sin\left(2G_1(z,z_0)\right)\right]\hat{1}_2$$

$$+ \left[F_3(z)\cos\left(2G_1(z,z_0)\right) - F_2\sin\left(2G_1(z,z_0)\right)\right]\hat{1}_3 \qquad (3.8)$$

If we attempt to further reduce the product integral, say by pulling out $\hat{1}_2$, we will have a remaining product integrand which will reintroduce $\hat{1}_1$ in general. This leaves, of course, special cases in which the coefficient of $\hat{1}_2$ or $\hat{1}_3$ in (3.8) is zero, implying a constraint between $F_2(z)$ and $F_3(z)$ (and $F_1(z)$ through $G_1(z,z_0)$) of the form

$$\frac{F_2(z)}{F_3(z)} = \pm\tan\left(2G_1(z,z_0)\right) \qquad (3.9)$$

While we have chosen to first pull out $F_1(z)$, the cyclic symmetry of the set of the $F_n(z)$ (n = 1, 2, 3) says that one could first pull out F_2 or F_3 and get similar results. So, except for special cases, we conclude that the product integral does not reduce to the sum integral.

4. Relation to Second Order Linear Differential Equations

A previous paper [5] shows how product integrals of 2 x 2 matrices can be constructed in special cases. Basically one can take known solutions of second-order linear differential equations and manipulate them into product-integral form, obtaining functions such as Bessel functions in the solutions. Such cases are found [1, 4] in the solutions for nonuniform transmission lines (impedance and admittance per-unit-length matrices varying along the line). One should not expect that, in general, such known solutions should be found in the form of Riemann (sum) integrals over the transmission-line parameters.

Consider a linear second-order homogeneous differential equation of the canonical form [7]

$$\frac{d^2u(z)}{dz^2} + b_1(z)\frac{du(z)}{dz} + b_2(z)u(z) = 0 \tag{4.1}$$

This is cast as a first-order vector equation as

$$\frac{d}{dz}\begin{pmatrix} u(z) \\ \frac{du(z)}{dz} \end{pmatrix} = \left(\phi_{n,m}(z)\right) \bullet \begin{pmatrix} u(z) \\ \frac{du(z)}{dz} \end{pmatrix}, \quad \left(\phi_{n,m}(z)\right) \equiv \begin{pmatrix} 0 & 1 \\ -b_2(z) & -b_1(z) \end{pmatrix} \tag{4.2}$$

This formally has a solution via the product integral as

$$\begin{pmatrix} u(z) \\ \frac{du(z)}{dz} \end{pmatrix} = \left(\Phi_{n,m}(z,z_0)\right) \bullet \begin{pmatrix} u(z_0) \\ \left.\frac{du(z)}{dz}\right|_{z=z_0} \end{pmatrix}$$

$$\left(\Phi_{n,m}(z,z_0)\right) = \prod_{z_0}^{z} e^{\left(\phi_{n,m}(z')\right)dz'} \quad \text{(product integral or matrizant)} \tag{4.3}$$

$$\left(\Phi_{n,m}(z_0,z_0)\right) = \left(1_{n,m}\right) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{(boundary condition)}$$

In this form we can see what it takes for a "closed-form" solution, where by this term we mean a solution expressed in terms of the usual sum (Riemann) integral over terms including the coefficient functions of z. Referring back to the previous section we need the product integral to be solved in terms of the sum integral (with a finite number of terms). This reduces the problem to the separability of $(\phi_{n,m}(z))$ into the quaternion units as in Section 3. After removing the identity as

$$\left(\phi_{n,m}(z)\right) = -\frac{1}{2}b_1(z)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}\frac{b_1(z)}{2} & 1\\ -b_2(z) & -\frac{b_1(z)}{2}\end{pmatrix} \tag{4.4}$$

leaving a matrix with zero trace, we are left with the decomposition in terms of $\hat{1}_1$, $\hat{1}_2$ and $\hat{1}_3$. This restricts the forms $b_1(z)$ and $b_2(z)$ can take so as to leave this matrix expressible in terms of only one of these units, thereby permitting a sum-integral solution of the product integral. Of course, there are special cases, such as a constant matrix which also gives a sum-integral result.

Having removed the identity part, we are left with

$$\left(\Phi'_{n,m}(z)\right) = \begin{pmatrix}\frac{b_1(z)}{2} & 1\\ -b_2(z) & -\frac{b_1(z)}{2}\end{pmatrix} \tag{4.5}$$

This has

$$\operatorname{tr}\left(\left(\phi'_{n,m}(z)\right)\right) = 0 \tag{4.6}$$

giving

$$\det\left(\Phi'_{n,m}(z,z_0)\right) = 1 = \Phi'_{1,1}(z,z_0) - \Phi'_{2,2}(z,z_0) - \Phi'_{1,2}(z,z_0)\Phi'_{2,1}(z,z_0)$$

$$\left(\Phi'_{n,m}(z,z_0)\right) = \prod_{z_0}^{z} e^{\left(\Phi'_{n,m}(z')\right)dz'} \quad \text{(reduced product integral or matrizant)} \tag{4.7}$$

This determinant is the Wronskian of the two solutions of the associated differential equation.

Referring back to the original form in (4.2) we have [7]

$$\begin{aligned}\operatorname{tr}\left(\left(\phi_{n,m}(z)\right)\right) &= -b_1(z)\\ \det\left(\left(\Phi_{n,m}(z,z_0)\right)\right) &= e^{\int_{z_0}^{z} tr\left(\left(\phi_{n,m}(z')\right)\right)dz'}\\ &= e^{-\int_{z_0}^{z} b_1(z')dz'}\end{aligned} \tag{4.8}$$

The Wronskian is this determinant

$$
\begin{aligned}
W\left(u_1(z)\ ,\ \ u_2(z)\right) &= u_1(z)\left[\frac{d}{dz}u_2(z)\right] - \left[\frac{d}{dz}u_1(z)\right]u_2(z) \\
&= \det\left(\left(\Phi_{n,m}(z,z_0)\right)\right) \\
&= \Phi_{1,1}(z,z_0)\Phi_{2,2}(z,z_0) - \Phi_{1,2}(z,z_0)\Phi_{2,1}(z,z_0) \\
&= \det\left(\begin{pmatrix} u_1(z) & u_2(z) \\ \frac{d}{dz}u_1(z) & \frac{d}{dz}u_1(z) \end{pmatrix}\right)
\end{aligned}
\tag{4.9}
$$

So we have a closed-form relation between the two independent solutions of (4.1). If we know one, then the Wronskian will give us the second. It is this relationship that we know in terms of a sum integral, not the two solutions separately.

Now we can write the product integral as

$$
\begin{aligned}
\left(\Phi_{n,m}(z,z_0)\right) &= \begin{pmatrix} u_1(z) & u_2(z) \\ \frac{d}{dz}u_1(z) & \frac{d}{dz}u_2(z) \end{pmatrix} \bullet \begin{pmatrix} u_1(z_0) & u_2(z_0) \\ \left.\frac{d}{dz}u_1(z)\right|_{z=z_0} & \left.\frac{d}{dz}u_2(z)\right|_{z=z_0} \end{pmatrix}^{-1} \\
&= \prod_{z_0}^{z} e^{\left(\phi_{n,m}(z')\right)dz'} \\
\begin{pmatrix} u(z) \\ \frac{d}{dz}u(z) \end{pmatrix} &= \left(\Phi_{n,m}(z,z_0)\right) \bullet \begin{pmatrix} u(z_0) \\ \left.\frac{d}{dz}u(z)\right|_{z=z_0} \end{pmatrix} \\
\left(\Phi_{n,m}(z_0,z_0)\right) &= \left(1_{n,m}\right)
\end{aligned}
\tag{4.10}
$$

Note that

$$
\begin{pmatrix} u_1(z_0) & u_2(z_0) \\ \left.\frac{d}{dz}u_1(z)\right|_{z=z_0} & \left.\frac{d}{dz}u_2(z)\right|_{z=z_0} \end{pmatrix}^{-1} = \frac{1}{W\left(u_1(z_0),\ u_2(z_0)\right)} \begin{pmatrix} \left.\frac{d}{dz}u_2(z)\right|_{z=z_0} & -u_2(z_0) \\ -\left.\frac{d}{dz}u_1(z)\right|_{z=z_0} & u_1(z_0) \end{pmatrix}
\tag{4.11}
$$

which can be used in (4.10) to verify the formula by choosing $u_1(z_0)$ and $u_2(z_0)$ as two separate excitations to produce $u_1(z)$ and $u_2(z)$.

5. Relation to the Theory of a Nonuniform Transmission Lines

Consider the usual telegrapher equations for a single-conductor (plus reference conductor) transmission line as

$$\frac{d}{dz}\tilde{V}(s,z) = -s\mathrm{L}'(z)\tilde{I}(s,z)$$
$$\frac{d}{dz}\tilde{I}(s,z) = -s\mathrm{C}'(z)\tilde{V}(s,z)$$

$\sim\ \equiv$ Laplace transform (two-sided) over time t

$s = \Omega + j\omega \equiv$ Laplace-transform variable or complex frequency

$z =$ spatial coordinate

$\tilde{V}(s,z) \equiv$ voltage

$\tilde{I}(s,z) \equiv$ current

$\mathrm{L}'(z) \equiv$ inductance per unit length (5.1)

$\mathrm{C}'(z) \equiv$ capacitance per unit length

The telegrapher equations can be cast as a first-order vector/matrix differential equation in various ways. One such way is

$$\frac{d}{dz}\begin{pmatrix}\tilde{V}(s,z)\\ \tilde{I}(s,z)\end{pmatrix} = \begin{pmatrix}0 & -s\mathrm{L}'(z)\\ -s\mathrm{C}'(z) & 0\end{pmatrix}\bullet\begin{pmatrix}\tilde{V}(z,s)\\ \tilde{I}(z,s)\end{pmatrix}$$
$$\begin{pmatrix}\tilde{V}(s,z)\\ \tilde{I}(s,z)\end{pmatrix} = \left(\mathrm{X}_{n,m}(z,z')\right)\bullet\begin{pmatrix}\tilde{V}(s,z_0)\\ \tilde{I}(s,z_0)\end{pmatrix}$$
$$\left(\mathrm{X}_{n,m}(z,z')\right) = \prod_{z_0}^{z} e^{\begin{pmatrix}0 & -s\mathrm{L}'(z')\\ -s\mathrm{C}'(z') & 0\end{pmatrix}dz'} \tag{5.2}$$
$$\mathrm{tr}\left(\begin{pmatrix}0 & -s\mathrm{L}'(z)\\ -s\mathrm{C}'(z) & 0\end{pmatrix}\right) = 0$$
$$\det\left(\mathrm{X}_{n,m}(z,z_0)\right) = 1$$

(An alternate form could use $\tilde{V}$ and $d\tilde{V}/dz$, for example.) In the above form, one can decompose the product integrand into $\hat{1}_2$ and $\hat{1}_3$ terms, but, as we have seen, we cannot pull one of these terms out by the sum rule without introducing $\hat{1}_1$, except in special cases. As we have also seen, while we cannot, in general, reduce such a problem to a sum integral, at least the two characteristic vector solutions are related by a Wronskian relation which is 1 in this particular case.

6. Concluding Remarks

The product integral, together with quaternion decomposition, of 2 x 2 matrices gives us some insight into the solution of second-order linear differential equations. In particular this shows when solutions can be expressed in terms of one or a few sum integrals. This in turn applies to the solution of the telegrapher equations for transmission lines.

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