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MEASUREMENT NOTES

NOTE 16

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A VOLUMETRIC METHOD FOR REDUCING EMP DATA PHOTOGRAPHS

ABSTRACT

A method for finding the trace within the EMP test data trace is proposed. Methods of calculating voltages using trace density readings and algorithms for double integrals are suggested. A plan for developing a code to calculate the voltages is offered.

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I. INTRODUCTION

When reducing EMP test photographs, readings are taken in the middle of the data trace. If more accurate data is needed, one could consider the data trace that is supposedly hidden within the data trace that appears on the photograph. In this measurement note a method for finding this hidden trace is proposed.

There are reasons for interest in this problem. More accurate data is desired, especially when it is needed for key parts of a test, better reliability, or greater confidence. The middle of the trace method has some bias toward the time axis for photographs where much blooming is present.

II. THE PROBLEM

The problem is finding the hidden data trace. To do this, one could use a computerized flying spot scanner, to make density readings d(x,y), for points (x,y) on the photograph. One might suspect that for a constant x, a cross-section of density readings would appear like figure 1 if no blooming was present and figure 2 if blooming was present. Because of the effects of blooming, the point a in figure 2 may be more indicative of the true data point than b. Experience with a flying spot scanner has shown that figure 3 is more typical of a data trace cross-section.

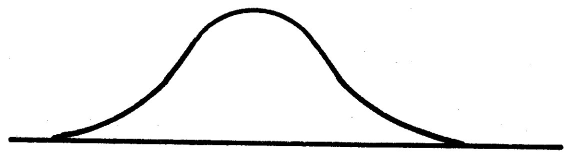

Fig. 1

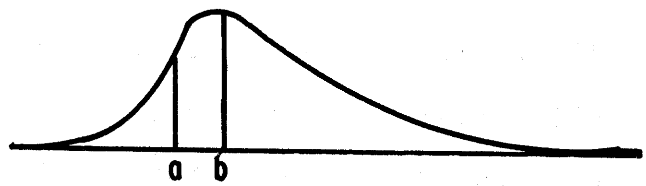

Fig. 2

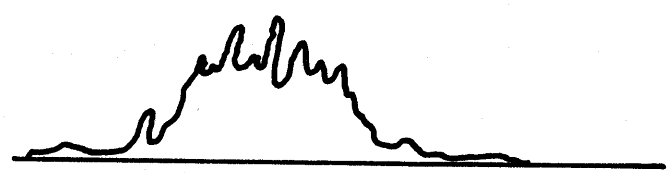

Fig. 3

The crux of the problem is finding a way to interpret density readings. Smoothing the curve given in figure 3 is difficult and subject to a variety of individual interpretations. For a given x, one could consider the cross-sections of density readings for time points within the interval (x - a, x + a) as being the data needed to determine the amplitude A(x) that represents the hidden trace at the time point x. Consider the volume V(s,y,a) of density under the function d(x,y) over the disc of radius a about the point (x,y). For constants x and a, and as y varies from the bottom of the trace to the top of the frace the function V(s,y,a) may trace a more interpretable curve than figure 3. In this case a sequence of discs like that pictured in figure 4 is used. One could define A(x) then as the maximum of V(x,y,a) as y varies or as d(x,y*) where V'(x,y*,a) = 0, if severe blooming is not present. If blooming is present another method must be used.

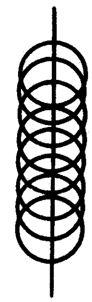

Fig. 4

Several theoretical and practical questions arise that can only be answered with experience. We have assumed that the density volume should be taken over a disc. One would guess that this base region should be fully symmetric, that is symmetric with respect to reflections about both axes and both 45-degree diagonals. The base region then could be a disc, a square, a regular diamond, or a regular octagon. Experience is needed to determine the optimal radius or the optimal parameters of the base region; the optimal distance s needed between consecutive base regions to get an adequate description of V(x,y,a) for a given x and a given base region; and the optimal spacing between time values, x, to achieve the best amplitude determinations, A(x). What is optimal may change as the trace width and other trace qualities change. A method must be found for calculating the desired volumes. Because we assume a flying spot scanner will be used, we are limited to density readings over an uniform grid. We assume that software can be created to handle data trace nonuniformities. We have assumed that the hidden trace does in fact exist in our test data and that it can be detected with a densitometer. We have assumed that for each time x that the curves V(x,y,a) can be interpreted into data more meaningful than that which the usual averaging technique yields. Several questions exist that can only be answered by experience. In the next section we propose numerical methods for calculating V(x,y,a).

III. CALCULATING V(x,y,r)

For simplicity of discussion, we assume that the base region is a disc of radius a. To calculate V(x,y,a), the 2 dimensional trapezoic rule could be used. This would involve no interpolation, but would involve a cumbersome number of points. The resulting program could be expensive in terms of time and core storage.

To minimize the number of density readings, quadrature formulas could be used. Such quadrature formulas would have to be for double integrals over fully symmetric regions. See the appendix. Most quadrature formulas of this type use the least number m of points $p(x_i,y_i)$ to estimate the integral of p(x,y) over our domain of integration, a disc. Using such minimum quadrature formulas would probably involve interpolation. Such formulas need values of $p(x_i,y_i)$ at certain preset, and usually irrational points. For instance $(0,1/\sqrt{3})$. The appendix, however, deals with near minimum quadrature formulas. Such formulas have one or more arbitrary parameters, which can be adjusted to the advantage of the user.

In the remainder of this section we will construct quadrature formulas for calculating V(s,y,a). We assume that an optimal a has been determined and that slight, but convenient, changes in the value of a are allowed. For convenience we assume (x,y) = (0,0). We assume that across a trace we can make 200 density readings in the direction of the y axis and 60 density readings in the direction of the x axis, all of these readings being equally spaced a distance h apart on a uniform grid. Investigatory use of near minimum quadrature formulas is not limited to the examples given below. Other formulas or variations of the given examples are possible.

The moments $I_{ij} = \int x^i y^i dxdy$ over a disc of radius a are:

$I_{00} = \pi a^2$, $\quad I_{20} = \pi a^4/4$, $\quad I_{40} = \pi a^6/8$

$I_{22} = \pi a^6/24$, $\quad I_{60} = 5\pi a^8/64$, $\quad I_{42} = \pi a^8/64$

$I_{40} + I_{22} = \pi a^6/6$, $\quad I_{40} - I_{22} = \pi a^6/12$

$I_{60} + 3I_{42} = \pi a^8/8$, $\quad I_{60} - I_{42} = \pi a^8/16$.

Formulas of third degree accuracy: By third degree accuracy, we mean that the estimate of the integral of any polynomial p(x,y) of degree 3 will be exact. THEOREM 5 of the appendix considers 4 points equally spaced on a circle of radius R where $R^2 = a^2/2$. Assume $a^2 = 2(17)^2h^2$. Then R = 17h and we could make our evaluations at the following points:

(17h,0) (-17h,0), (0,17h) and (0,-17h).

In this example, we used a minimum point formula, allowing a to vary as needed, so that a convenient R could be chosen.

Instead of starting with a, we could start with convenient choices of points of evaluation: (5h,15h), (-5h,-15h), (15h, -5h), (-15h,5h). These points could be used in THEOREM 6. Then, $R^2 = 250h^2$. Assume we want the weight of the center point A_2 to be twice the weight A_1. That is we want

$$I_{00} - 2I_{20}/R^2 = I_{20}/R^2 ,$$

which results in $a^2 = 4R^2/3 = 1{,}000h^2/3$. Then $A_1 = A_2/2$ and $A_2 = 1{,}000\pi h^2/9$. The formula then is

$$V(0,0,r) = \frac{1{,}000\pi h^2}{9} d(0,0)$$

$$+ \frac{1{,}000\pi h^2}{18}[d(5h,15h) + d(-5h,-15h) + d(15h,-5h) + d(-15h,5h)]$$

In THEOREM 6 we were free to select a and one of the weights.

<u>Formulas of fifth degree accuracy</u>: By fifth degree accuracy we mean that the estimate of the integral of any polynomial $p(x,y)$ of degree 5 will be exact. THEOREM 7 is a minimum point formula and may be too rigid to be applied to our problem. There is not enough freedom in choosing μ and ν. We must have that μ and ν be integral multiples of h yet have $\nu/\mu = \tan 30°$ which is an irrational number. Similar problems arise in applying THEOREM 8. We must have h and μ rational, yet have $\mu = h(2)^{1/2} \sin 15°$. THEOREM 9 requires that $\nu/\mu = [(\sqrt{2} - 1)/(\sqrt{2} + 1)^{1/2}]$; this number is irrational, and we need rational ν and μ.

If interpolation is not a problem THEOREMS 7, 8 and 9 can be applied. In particular the example of THEOREM 8 could be used if r is given a value of zero to make the domain of integration a disc. If the domain of integration is a square, the example of THEOREM 9 could be used. Use of THEOREM 10 allows us the freedom to choose R and a as we need. R can be chosen first and a can then be chosen in such a way to make r convenient. In this way r and R can be integral multiples of h. For this THEOREM, the weights are:

$$A_1 = \pi a^6/96R^4,$$

$$A_2 = 3\pi a^2[1 - a^2/6R^2]^2/8,$$

$$A_3 = I_{00} - \pi a^6/24R^4 - 3\pi a^2[1 - a^2/6R^2]^2/2.$$

We want $I_{22}/I_{20} < R^2 < a^2$. That is

$$a^2/6 < R^2 < a^2.$$

This last condition can be transformed to $R^2 < 5r^2/2$. These conditions put bounds on our choices for R and r. In terms of R and a,

$$r^2 = 2a^2R^2/(6R^2 - a^2).$$

Solving for a^2 yields,

$$a^2 = 6R^2r^2/(2R^2 + r^2).$$

With this formula and the previously given restrictions we may dictate any r and R we choose and obtain the proper a for the formula in THEOREM 10 to work. For instance if R was 20h and we considered the following values of r: 14h, 15h and 16h we would find that, to sliderule accuracy, a would be 21.7h, 23.0h and 24.1h, respectively. Exact computations to determine formula parameters should be done on a computer. The user of this formula should attempt to keep the weights, A_i, as close to equal as possible and attempt to keep both r and R less than a. It is possible that r could be greater than R.

Formulas of seventh degree accuracy: The formulas of seventh degree accuracy listed in the appendix do not have the necessary freedom to be applied to our problem without the need for interpolation. Both a and α can be chosen freely. But there is no readily apparent way to choose a and α such that r, R, and T are all integral multiples of h. Thus, a user desiring to use these more accurate formulas (THEOREMS 12 and 13) would probably have to use two dimensional interpolation to obtain results. Two formulas for the two by two square are given after THEOREM 12. Two other formulas for the disc of radius h are given after THEOREM 13.

We have attempted to suggest a way to use the formulas given in the appendix. Use of some of the formulas will require interpolation. Others are more convenient. The examples given for this latter case are incomplete. Perhaps a code can be created to create and test several such formulas.

IV. CODE DEVELOPMENT

Below we suggest a sequence of decisions, tests, and subroutines needed to develop the proposed code. In doing this it has been assumed that a flying spot scanner will be used.

1. Develop a code to detect top and bottom of trace for any time x.

2. Develop a code to gather d(x,y) readings for an organized given need.

3. Develop a code to calculate V(0,0,a) using a circle as the domain of integration.

   a. Test the trapezoid rule.

   b. Test minimum 3rd degree formula, then test minimum formulas of 5th and 7th degree.

   c. Determine an optimal a for the above quadrature formulas.

   d. Test near minimum, 3rd and 5th degree formulas. Determine optimal values for a, r, and R.

   e. Decide on a weight (A_i) policy. Should the larger weights be toward the center, so that values of d(x,y) in the center of the circle have a greater effect on the value of V(0,0,a), or should the weights be as nearly equal as possible so that errors in reading d(x,y) will have equal effects?

   f. Compare above methods.

   g. Decide if you want a to vary with the tracewidth or if you want to use only one a.

   h. Test other domains of integration: square, diamond.

4. Develop code to calculate V(0,y,a). Choose optimal distance s between consecutive values of y.

5. Determine effects of trace quality on codes.

6. Develop code to calculate V(x,y,a). Choose optimal distance between consecutive times x.

7. Test methods to determine A(x): simple maximum, derivative equals zero, polygonal smoothing, fitting data to a statistical distribution curve.

8. Determine reasonableness of (x,A(x)) data.

9. Identify traces for which this method does not work.

The above outline should not be considered perfect, complete or optimal. It is included as a suggested initial plan of attack.

V. ALTERNATE METHODS

Baum has suggested another way of determining A that makes use of the formulas given in the appendix. His method includes adjustment of the data for the effects of curvature and blooming, and a study of moments to determine A.

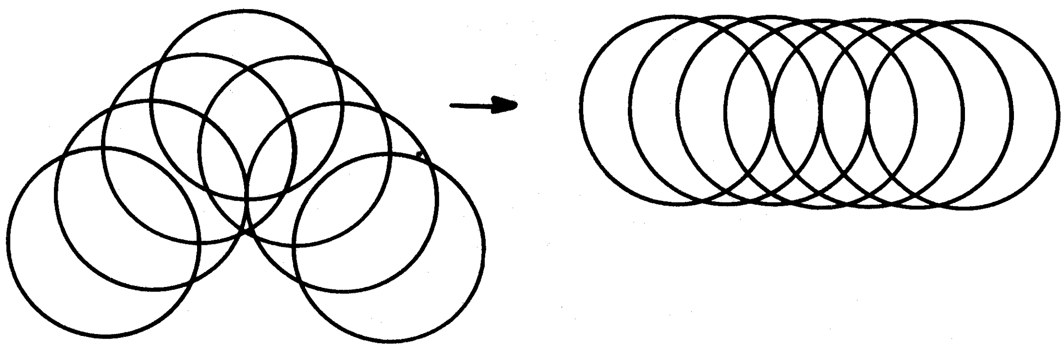

Fig. 5

A representation of the adjustment of the data for curvature and blooming is pictured in figure 5. The method given in the previous sections could be used to find the trace within the left and right legs of the data trace pictured in figure 5. Finding the data point at the top of the pictured trace is complicated by the greater accumulation of light at the bottom compared to the top. Defining the data point as the point of greatest density would lead to inaccurate data. Our point is better illustrated in figure 6. The true data trace has been approximated by connecting the centers of the circles that represent the trace. Point B is the data point for the maximum of this trace. Averaging the top, A and the D of the trace would yield C. Density considerations would lead one to choose a

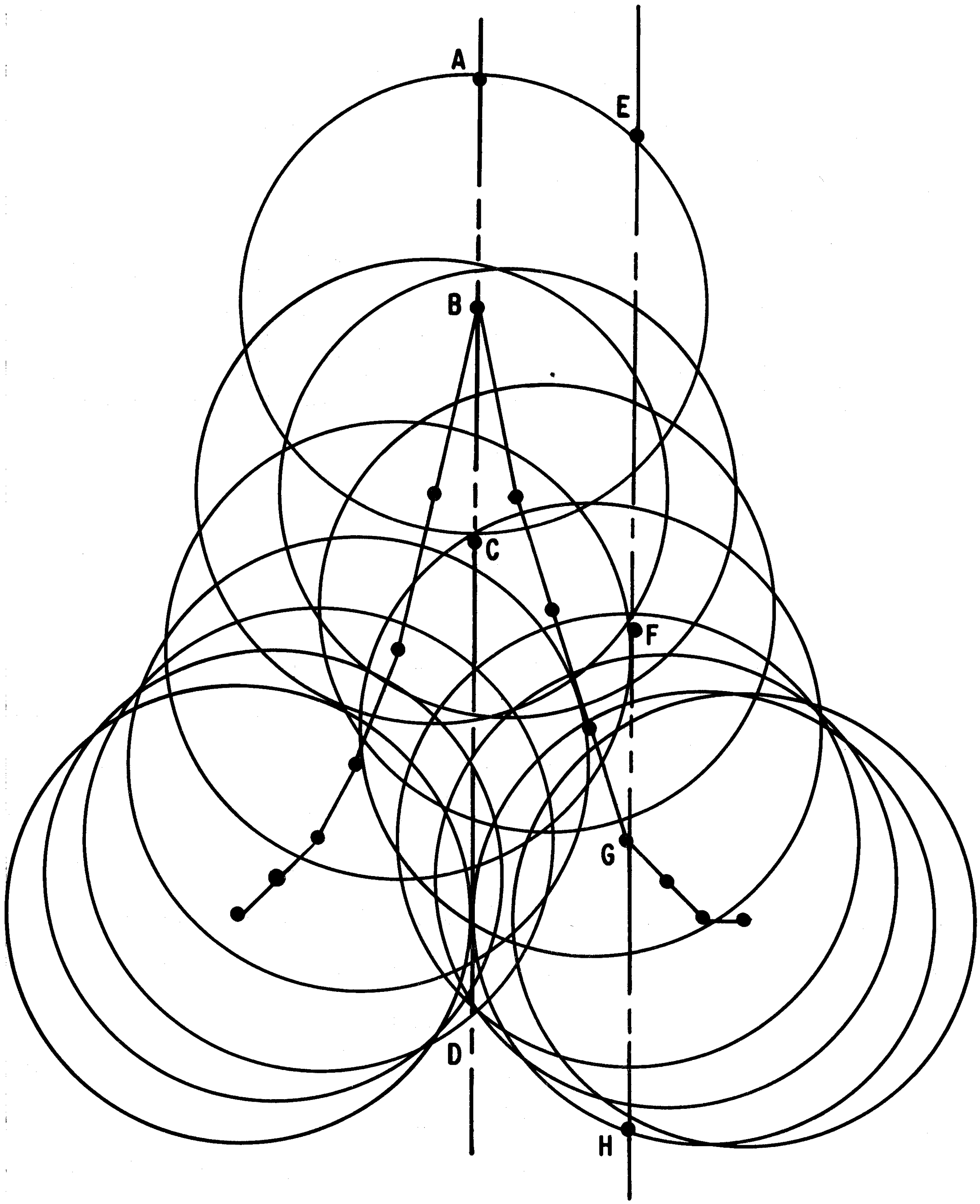

Fig. 6

point between B and D. Similar comments can be made about points E, F, G, and H, but in this case the true data point G appears to be in the middle of the area of maximum density, F to H.

The procedure for making this adjustment would involve iteration. The procedure would start with averaging the top and the bottom of the trace to gain a first approximation of center of the trace. Next, one would adjust the density data in the spirit of figure 5. The adjustment would be based on a model of electron beam image, its speed, its direction and the amount of blooming present. Several re-applications of this adjustment might be necessary until two successive applications resulted in little or no change.

Having corrected for curvature, a way to calculate A for a given x is now considered. To determine A, Baum recommends a study of the moments.

$$M_n = \int_{R(x_o')} c(x,y)d(x,y)|x' - x_o'|^n \, dxdy$$

where c(x,y) is the correction for curvature, $|x' - x_o'|$ is the distance from the center of the trace along a line perpendicular to the trace intersecting the center point we are interested in, and $R(x_o')$ is bounded by the top and bottom of the trace and two lines parallel to the x' axis and of equal distance from the x' axis.

If $R(x_o')$ is a rectangle, the parameters of $R(x_o')$ could be arranged so that it could be divided up into several squares. The formulas in the appendix could be applied to these squares. If $R(x_o')$ is a square, the previous method applies along with a second method. The second method involves dividing $R(x_o')$ up into several concentric square rings and one small center square. These regions are fully symmetric. The formulas given in the appendix apply.

Baum's suggestion raises several questions. Does the recording film possess inconsistencies that should be accounted for? How would one interpret the moments M_n to obtain A? The thoughts recorded here are based on figures 5 and 6. Is the representation assumed within these figures adequate? Are the circles always the same size? Does the oscilloscope always record with a sequence of dots? Further study is needed.

A topologist might suggest another method. He might consider figure 5 and suggest that the arrow be reversed. If our data trace is a straight line, points in the exact center of that trace represent the trace within the trace. A topological projection from this trace to the data trace could be created. This projection could account for blooming, narrow tracewidths, curves, etc. Such a projection could be a sum of several projections, each accounting for a trace quality. These qualities could be surmised from actual density readings, or smoothed density readings. Smoothed density readings could be calculated using the quadrature formulas given in the appendix. After the best projection had been calculated, it could be used to project the exact centerline of a straight data trace to a test data trace. This projection would then be the trace within the trace.

APPENDIX

QUADRATURE FORMULAS OVER FULLY SYMMETRIC PLANAR REGIONS

CHARLES B. HUELSMAN, III

Abstract. The nonexistence of 10 point 7th degree quadrature formulas over fully symmetric regions is proven. Two, 13 point, 7th degree quadrature formulas over such regions are presented. The first can be a 12 point quadrature formula, if a condition is satisfied. The second can be also, if another condition is satisfied. Five quadrature formulas of 3rd and 5th degree are extended to fully symmetric regions.

1. Introduction. The purpose of this paper is to exhibit two dimensional quadrature formulas for fully symmetric regions. Polygonal accuracies of degrees 3, 5, and 7 are investigated. Integral inequalities needed to develop these formulas are given in section 2. Theorems about the minimal number of points for quadrature formulas over fully symmetric regions are in section 3. The next three sections contain formulas of 3rd, 5th and 7th degree accuracy. In some cases near minimum point formulas are given. These formulas contain at least one arbitrary parameter which can

be adjusted to the advantage of the user. All formulas are listed in section 7.

The following notation is used. By $(\pm x, \pm y)$ we mean the four points (x,y), $(-x,y)$, $(x,-y)$, $(-x,-y)$. By $(\pm x, 0)$ we mean $(x,0)$ and $(-x,0)$. $\Sigma p(\pm x, \pm y)$ and $\Sigma p(\pm x, 0)$ are sums over these points. By N we mean the positive non-zero integers. N_o is the positive integers including zero. D is fully symmetric if $(x,y) \varepsilon D$ implies $\{(\pm x, \pm y), (\pm y, \pm x)\} \subset D$. When we say D is fully symmetric, we assume D is measurable and that $0<m(D)<\infty$. For $i,j \varepsilon N_o$, $I_{ij} \equiv \int_D x^i y^j dxdy$.

If D is fully symmetric, I_{ij} has several convenient properties.

(i) $I_{ij} = I_{ji}$.

(ii) If i or j is odd, $I_{ij} = 0$.

(iii) For $i,j \varepsilon N$, $I_{2(i+j),0} > I_{2i,2j}$.

(iv) If $i,j,n \varepsilon N_o$ are even such that $i+j = n$, and $I_{n,0}<\infty$, then $0<I_{ij}<\infty$.

The proof of (iii) results from the inequality

$$(x^{2i} - y^{2i})(x^{2j} - y^{2j}) > 0$$

almost everywhere in D. From (iii) it is immediate that $I_{40}>I_{22}$, and $I_{60}>I_{42}$.

2. Integral inequalities. To justify the existence of certain quadrature formula parameters, and to prove contradictions necessary in section 3, a knowledge of relations between combinations of various moments, I_{ij}, is necessary. All of these relations are developed using Hölder's inequality.

THEOREM 1. *Let D be fully symmetric. If* $I_{40} < \infty$, *then*

$$I_{20}^2 < I_{00}I_{40} .$$

If $I_{60} < \infty$, *then*

$$I_{22}^2 < I_{20}I_{42} , \quad I_{40}^2 < I_{20}I_{60} , \quad I_{20}^3 < I_{00}^2 I_{60} ,$$

$$I_{22}^3 < I_{00}I_{42}^2 , \quad I_{22}^3 < I_{20}I_{40}I_{42} , \text{ and } I_{40}^3 < I_{00}I_{60}^2 .$$

Let n be even, and p and q be rational fractions. $I_{n0} < \infty$ implies for any p and q such that $p + q \leq n$, that

$$\int_D |x^p y^q| \, dxdy < \infty .$$

THEOREM 1 is proven by using this, Hölder's inequality and $2x^2y^2 < x^4 + y^4$ (almost everywhere in D). An example is given in the appendix by proving $I_{22}^2 < I_{20}I_{42}$.

THEOREM 2. *Let D be fully symmetric. If* $I_{40} < \infty$,

$$2I_{20}^2 < I_{00}(I_{40} + I_{22}) .$$

If $I_{60} < \infty$,

$$(I_{40} + I_{22})^2 < I_{20}(I_{60} + 3I_{42}) ,$$

$$2(I_{40} - I_{22})^3 < I_{00}(I_{60} - I_{42})^2 ,$$

$$\text{and } (I_{40} - I_{22})^2 < (I_{60} - I_{42})(I_{20} - I_{22}^2/I_{42}) .$$

The proof of the first two inequalities results from property (*i*) and Hölder's inequality. The proofs of the last two inequalities are given in the appendix.

3. Nonexistence of formulas. All three of our conclusions of nonexistence of quadrature formulas over fully symmetric regions are applications of the following theorem.

THEOREM 3. *Assume D is a measurable set in* R^2, *and* $m(D) > 0$. *Assume there exists a* $m = (n+1)(n+2)/2$ *point quadrature formula, exact for polynomials* $p(x,y) \in P_{2n+1}(x,y)$. *Then all* $n+1$ *degree polynomials that are orthogonal to* $p(x,y) \in P_n(x,y)$ *relative to* D *have* m *common zeros.*

Proof. We have assumed

$$\int_D p(x,y)\,dxdy = \sum_{i=1}^{m} w_i\, p(x_i,y_i) \qquad p(x,y) \in P_{2n+1}(x,y) .$$

Through m-1 points of $\{(x_i, y_i)\}_{i=1}^{m}$, excluding say (x_k, y_k), we put a curve $R_k(x,y)$ of degree n, such that

$$R_k(x_i, y_i) \begin{cases} \neq 0, & i = k \\ = 0, & i \neq k \end{cases}$$

thus defining m polynomials. Applying the assumed quadrature formula,

$$0 < \int_D R_k^2(x,y)\,dxdy = w_k R_k^2(x_k, y_k) \qquad k = 1, 2, \cdots, m .$$

As $R_k^2(x_k, y_k) > 0$ we have $w_k > 0$ and $R_k(x_k, y_k) \neq 0$. Let $K(x,y)$ be any polynomial of degree n+1, orthogonal to $p(x,y) \in P_n(x,y)$ relative to D. $K(x,y)R_k(x,y) \in P_{2n+1}(x,y)$. Thus,

$$0 = \int_D K(x,y)R_k(x,y)\,dxdy = w_k R_k(x_k, y_k) K(x_k, y_k)$$
$$k = 1, 2, \cdots, m .$$

Therefore $K(x_k, y_k) = 0$ for each k. All polynomials $K(x,y) \in P_{n+1}(x,y)$ such that

$$\int_D K(x,y)p(x,y)\,dxdy = 0 \qquad \text{for all } p(x,y) \in P_n(x,y)$$

then have m common zeros.

COROLLARY 1. *There does not exist a* 3 *point quadrature formula over fully symmetric regions, with polygonal accuracy of degree* 3.

Proof. Assume such a formula exists. By the previous theorem (n=1), the 3 following orthogonal polynomials have 3 common zeros:

$$K_1(x,y) = x^2 - I_{20}/I_{00},\; K_2(x,y) = xy,\; K_3(x,y) = y^2 - I_{20}/I_{00} .$$

Since $I_{20}/I_{00} \neq 0$, 3 common zeros do not exist.

More general results for quadrature formulas of accuracy degree 3 or less can be found in the writings of Georgiev [25, 26, 27] and Stroud [115]. Their proofs, however are more involved than the one just given.

COROLLARY 2. *There does not exist a* 6 *point quadrature formula over fully symmetric regions, with polygonal accuracy of degree* 5.

Proof. Again we apply THEOREM 3 (n=2). The 4 following third degree orthogonal polynomials have 6 common zeros:

$$K_1(x,y) = x(x^2 - I_{40}/I_{20}),\quad K_3(x,y) = x(y^2 - I_{22}/I_{20}) ,$$

$$K_2(x,y) = y(x^2 - I_{22}/I_{20}),\quad K_4(x,y) = y(y^2 - I_{40}/I_{20}) .$$

One of the zeros is (0,0). Let $a,b \neq 0$. Assuming $(a,0)$ is the second common zero, yields the contradiction $0 = I_{22}/I_{20}$. In a similar way $(0,b)$ is not a zero. Assuming (a,b) is a common zero yields the contradiction

$$I_{40}/I_{20} = a^2 = I_{22}/I_{20} < I_{40}/I_{20} .$$

Thus common zeros beyond (0,0) do not exist. We have the necessary contradiction.

More general results can be found in the writings of Radon [92], Mysovskikh [79], and Bykova and Mysovskikh [9]. Their proofs are more complex than the preceding.

COROLLARY 3. *There does not exist a* 10 *point quadrature formula over fully symmetric regions, with polygonal accuracy of degree* 7.

Proof. To apply THEOREM 3 ($n=3$) we consider the following 5 polynomials:

$$K_1(x,y) = Ax^4 + Bx^2 + Cy^2 + E \quad \text{where}$$

$$A = (I_{40}-I_{22})[I_{00}(I_{40}+I_{22})-2I_{20}^2] ,$$

$$B = -(I_{60}-I_{42})[I_{00}I_{40}-I_{20}^2] - (I_{40}-I_{22})[I_{00}I_{42}-I_{40}I_{20}] ,$$

$$C = (I_{60}-I_{42})[I_{00}I_{22}-I_{20}^2] - (I_{40}-I_{22})[I_{00}I_{42}-I_{40}I_{20}] ,$$

$$E = (I_{40}-I_{22})[I_{20}(I_{60}+I_{42})-I_{40}(I_{40}+I_{22})] ,$$

$$K_2(x,y) = xy(x^2 - I_{42}/I_{22}) ,$$

$$K_3(x,y) = Ax^2y^2 + F(x^2+y^2) + G \quad \text{where}$$

$$F = -(I_{40}-I_{22})[I_{00}I_{42}-I_{20}I_{22}] \ ,$$

$$G = (I_{40}-I_{22})[2I_{20}I_{42}-I_{22}(I_{40}+I_{22})] \ ,$$

$$K_4(x,y) = xy(y^2-I_{42}/I_{22}) \ ,$$

$$K_5(x,y) = Ay^4 + Cx^2 + By^2 + E \ .$$

Since $I_{40} > I_{22}$ by (*iii*) and $I_{00}(I_{40} + I_{22}) > 2I_{20}^2$ by Hölder's inequality, we have $A > 0$. Thus each of the above polynomials is in $P_4(x,y)$. Direct calculation shows that they are orthogonal.

Let us denote the 10 zeros by (x_i,y_i) for $i = 1, 2, \cdots, 10$. When $K_2(x_i,y_i) = 0$ and $K_4(x_i,y_i) = 0$, we must have one of the coordinates x_i or y_i equal to zero, or have the absolute values of both coordinates assume the value $(I_{42}/I_{22})^{1/2}$. Four cases result.

Case 1 $\quad x_1 = 0$ and $y_1 = 0$,

Case 2 $\quad x_i = 0$ and $y_i \neq 0$,

Case 3 $\quad y_i = 0$ and $x_i \neq 0$,

Case 4 $\quad |x_i|, |y_i| = (I_{42}/I_{22})^{1/2}$.

Case 1: If $(0,0)$ is a zero of K_j, $j = 1, 3$, then

$$0 = K_1(0,0) + K_3(0,0)$$

$$= (I_{40} - I_{22})[I_{20}(I_{60} + 3I_{42}) - (I_{40} + I_{22})^2] > 0$$

where $I_{20}(I_{60} + 3I_{42}) > (I_{40} + I_{22})^2$ results from Hölder's inequality. Thus contradiction and $(0,0)$ is not a zero of $K_j(x,y)$, $j = 1, 2, \cdots, 5$.

Case 2: If $x_i = 0$ and $y_i \neq 0$, we have:

$$Cy_i^2 = -E, \quad Fy_i^2 = -G, \quad Ay_i^4 + By_i^2 = -E .$$

Three subcases result:

Subcase 2.1: If $E = 0$, then

(3.1) $$I_{20}(I_{60} + I_{42}) = I_{40}(I_{40} + I_{22}) ,$$

and

(3.2) $$(I_{60}-I_{42})(I_{00}I_{22}-I_{20}^2) = (I_{40}-I_{22})(I_{00}I_{42}-I_{40}I_{20}) .$$

By *(iii)* $I_{60} > I_{42}$. The right most factors of both sides of (3.2) must be zero or have the same sign. If both are zero then

(3.3) $$I_{20}I_{42} = I_{40}I_{22} .$$

Deducting this last equality from (3.1) leaves $I_{20}I_{60} = I_{40}^2$ in contradiction to a result of Hölder's inequality, $I_{20}I_{60} > I_{40}^2$, listed in THEOREM 1. If these two factors are non-zero, we reach (3.3) by adjusting each side of (3.1) by $-2I_{20}I_{42}$, substituting the result into (3.2), and cancelling.

Subcase 2.2: If $G = 0$, then $F = 0$, yielding $2I_{20}I_{42} = I_{22}(I_{40} + I_{22}) > 0$ and $I_{20}I_{22} = I_{00}I_{42} > 0$. Multiplication yields $2I_{20}^2 = I_{00}(I_{40} + I_{22})$ which contradicts $2I_{20}^2 < I_{00}(I_{40} + I_{22})$ of THEOREM 2.

Subcase 2.3: Assume $E, G \neq 0$. Thus y_i^2 must simultaneously satisfy the 3 given conditions. This occurs only when the region D is such that

$$-E/C = -G/F = (C - B)/A > 0 \ . \qquad (3.4)$$

Let $a^2 = -E/C$. In terms of the moments, none of the above expressions is algebraically identical to any of the remaining ones. When (3.4) is satisfied, there are 2 common zeros $(0, \pm a)$. Otherwise, none. Thus *case* 2 yields at most 2 zeros.

Case 3: If $y_i = 0$ and $x_i \neq 0$, we have:

$$Ax_i^4 + Bx_i^2 = -E, \quad Fx_i^2 = -G, \quad Cx_i^2 = -E \ .$$

By substituting y_i for x_i we have the equations for *case* 2. The same conclusions apply. If condition (3.4) is not satisfied, we have no zeros. If it is satisfied, there are 2 zeros. Thus at most 2 zeros result from *case* 3.

Case 4: If $x_i, y_i = (I_{42}/I_{22})^{1/2}$ we have the condition on the region D that

$$(3.5) \qquad (B + C - 2F) I_{42}/I_{22} = G - E \; .$$

In terms of the moments, the two expressions above are algebraically unequal. If condition (3.5) is satisfied, *case* 4 yields only 4 zeros:

$$(\pm (I_{42}/I_{22})^{1/2}, \; \pm (I_{42}/I_{22})^{1/2}) \; .$$

If condition (3.5) is not satisfied, no zeros result.

Even in the questionable case when conditions (3.4) and (3.5) are both satisfied, we still only have a total of 8 zeros. The desired contradiction is attained.

4. Formulas of third degree accuracy. In the remaining sections we assume the following theorem is obvious.

THEOREM 4: *Let p be a polynomial of odd degree n. Let D be fully symmetric, and $I_{n-1,0} < \infty$. If there exists a set of points $\{(x_k, y_k)\}_{k=1}^{m}$ and m associated weights A_k, such that*

$$(4.1) \qquad I_{ij} = \sum_{k=1}^{m} A_k x_k^i y_k^j \qquad 0 \le i + j \le n$$

then,

$$\int_D p(x,y)\,dxdy = \sum_{k=1}^{m} A_k p(x_k,y_k) .$$

It is well known that in the 3rd degree case the minimum number of points is 4 [27]. The formulas given in this section are simple extensions of ones well known for simple regions [3].

THEOREM 5, 3RD DEGREE, 4 POINT FORMULA. *Let* $p(x,y) \in P_3(x,y)$. *If D is fully symmetric, and* $I_{20} < \infty$, *then*

$$\int_D p(x,y)\,dxdy = \frac{I_{00}}{4}\, p[(\mu,\nu) + p(-\mu,-\nu) + p(\nu,-\mu) + p(-\nu,\mu)]$$

where μ *and* ν *can be any pair of real positive numbers satisfying the relation* $\mu^2 + \nu^2 = 2I_{20}/I_{00}$.

Proof. Consider the following points and weights:

weight	A	A	A	A
x-coord.	μ	$-\mu$	ν	$-\nu$
y-coord.	ν	$-\nu$	$-\mu$	μ

From equations (4.1) for $n = 3$ the following equations need to be satisfied: $4A = I_{00}$, $2A(\mu^2 + \nu^2) = I_{20}$. These equations are obviously satisfied by $A = I_{00}/4$, and $\mu^2 + \nu^2 = 2I_{20}/I_{00}$.

With this method one is free to select one point on the radius of a predetermined circle. If $\mu = \nu$ the method is the well known Gaussian 4 point cross-product formula. The case where $\nu = 0$ is also well known. The following formula is an extension of a formula published by McNamee and Stenger [69].

THEOREM 6, 3RD DEGREE, 5 POINT FORMULA. *Let* $p(x,y) \in P_3(x,y)$. *If* D *is fully symmetric, and* $I_{20} < \infty$ *then,*

$$\int_D p(x,y)\,dxdy = (I_{00} - 2I_{20}/R^2)\,p(0,0)$$

$$+ (I_{20}/2R^2)\,[p(\mu,\nu)+p(-\mu,-\nu)+p(\nu,-\mu)+p(-\nu,\mu)]$$

where $R > 0$ *and* $\mu, \nu \geq 0$ *such that* $\mu^2 + \nu^2 = R^2$.

Proof. Consider the points and weights:

weight	A_2	A_1	A_1	A_1	A_1
x-coord.	0	μ	$-\mu$	ν	$-\nu$
y-coord.	0	ν	$-\nu$	$-\mu$	μ

From equations (4.1) for $n = 3$ the following equations need to be satisfied: $4A_1 + A_2 = I_{00}$, $2A_1(\mu^2 + \nu^2) = I_{20}$. Substituting the claimed solution into these equations yields the necessary verification.

When $R^2 > 2I_{20}/I_{00}$, $A_2 > 0$. With this method one is free to select any one point as an evaluation point, or the value of any one weight. If equal weights are desired one should use the 4 point method or $R = (5I_{20}/2I_{00})^{1/2}$. If positive weights are desired one should use $R > (2I_{20}/I_{00})^{1/2}$.

5. Formulas of 5th degree accuracy. For formulas of 5th degree accuracy, the nonexistence of a 6 point formula has only been proven for regions of radial symmetry [9]. A 7 point formula exists for such regions. See Radon [92]. For completeness we include Radon's formula restricted to fully symmetric regions. Equations (4.1) are solved yielding a solution that agrees with the results of Radon.

THEOREM 7, 7 POINT, 5TH DEGREE FORMULA. *Let* $p(x,y) \varepsilon P_5(x,y)$. *If* D *is fully symmetric and* $I_{40} < \infty$, *then*

$$\int_D p(x,y)\,dxdy = A_1\sum p(\pm\lambda,0) + A_2\sum p(\pm\mu,\pm\nu) + A_3 p(0,0) ,$$

where: $$\mu = \left(\frac{I_{22}}{I_{20}}\right)^{1/2}, \quad \nu = \left(\frac{I_{40}}{I_{20}}\right)^{1/2}, \quad \lambda = \left(\frac{I_{40}+I_{22}}{I_{20}}\right)^{1/2},$$

$$A_1 = \frac{1}{2}\frac{I_{20}^2}{I_{40}}\frac{I_{40}-I_{22}}{I_{40}+I_{22}}, \quad A_2 = \frac{I_{20}^2}{4I_{40}}, \quad A_3 = I_{00} - \frac{2I_{20}^2}{I_{40}+I_{22}},$$

and all $A_i > 0$.

Discussion. Consider the points and weights:

weight	A_1	A_1	A_2	A_2	A_2	A_2	A_3
x-coord.	λ	$-\lambda$	μ	μ	$-\mu$	$-\mu$	0
y-coord.	0	0	ν	$-\nu$	ν	$-\nu$	0

From equations (4.1) applied to these points and weights, the following equations remain to be solved:

$$2A_1 + 4A_2 + A_3 = I_{00} ,$$

$$2A_1\lambda^2 + 4A_2\mu^2 = I_{20} ,$$

$$4A_2\nu^2 = I_{20} ,$$

$$2A_1\lambda^4 + 4A_2\nu^4 = I_{40} ,$$

$$4A_2\nu^4 = I_{40} ,$$

$$4A_2\mu^2\nu^2 = I_{22} .$$

The claimed values of μ, ν and A_2 are immediate. By subtraction

$$2A_1\lambda^2 = 4A_2(\nu^2 - \mu^2) ,$$

$$2A_1\lambda^4 = 4A_2(\nu^2 - \mu^2)(\nu^2 + \mu^2) ,$$

and thus

$$\lambda^2 = \nu^2 + \mu^2 = (I_{40} + I_{22})/I_{20} .$$

Simple computations yield the claimed values for A_1 and A_3, which are positive by the previously used inequalities.

It is possible to rotate 6 of Radon's points 45° and still obtain a quadrature formula. Albrecht and Collatz [3] noted this for the square. The theorem below generalizes their quadrature formula to fully symmetric domains of integration.

THEOREM 8, 5TH DEGREE, 7 POINT FORMULA. *Let* $p(x,y) \in P_5(x,y)$. *If* D *is fully symmetric, and* $I_{40} < \infty$, *then*

$$\int_D p(x,y)\,dxdy = A_1[p(\lambda,\lambda)+p(-\lambda,-\lambda)]$$

$$+ A_2[p(\mu,-\nu)+p(-\mu,\nu)+p(\nu,-\mu)+p(-\nu,\mu)]$$

$$+ A_3 p(0,0)$$

where: *all* $A_i > 0$, $\lambda = \left(\frac{I_{40} + I_{22}}{2I_{20}}\right)^{1/2}$,

$$\mu,\nu = \left(\frac{I_{40}+I_{22}}{2I_{20}} \pm \frac{1}{2I_{20}} \left((I_{40}+3I_{22})(I_{40}-I_{22})\right)^{1/2}\right)^{1/2} ,$$

$$A_1 = \frac{2I_{20}^2 I_{22}}{(I_{40}+3I_{22})(I_{40}+I_{22})},\quad A_2 = \frac{\frac{1}{2}\, I_{20}^2}{I_{40}+3I_{22}},\quad A_3 = I_{00} - \frac{2I_{20}^2}{I_{40}+I_{22}} .$$

Proof. Since $0 < I_{ij} < \infty$ for $i + j < 5$, i and j even, we have that λ, A_1, and A_2 are positive. A_3 is positive by Hölder's inequality. From $I_{22}^2 > -3I_{22}^2$ we can obtain

$$(I_{40}+I_{22})^2 > (I_{40}+3I_{22})(I_{40}-I_{22}) \; .$$

Thus μ and ν are well defined. When the points and weights

weights	A_1	A_1	A_2	A_2	A_2	A_2	A_3
x-coord.	λ	$-\lambda$	μ	$-\mu$	ν	$-\nu$	0
y-coord.	λ	$-\lambda$	$-\nu$	ν	$-\mu$	μ	0

are considered, the following equations of (4.1) remain to be verified:

$$2A_1 + 4A_2 + A_3 = I_{00} \; ,$$

$$2A_1\lambda^2 + 2A_2(\mu^2+\nu^2) = I_{20} \; ,$$

$$2A_1\lambda^2 - 4A_2\mu\nu = 0 \; ,$$

$$2A_1\lambda^4 + 2A_2(\mu^4+\nu^4) = I_{40} \; ,$$

$$2A_1\lambda^4 - 2A_2\mu\nu(\mu^2+\nu^2) = 0 \; ,$$

$$2A_1\lambda^4 + 4A_2\mu^2\nu^2 = I_{22} \; .$$

The first equation is verified by routine computations. Verification of the second is clear after noting $2\lambda^2 = \mu^2 + \nu^2$. Verification of the fourth equation is a matter of computation after calculating

$$\mu^4 + \nu^4 = \frac{(I_{40}+I_{22})^2 + (I_{40}+3I_{22})(I_{40}-I_{22})}{2I_{20}^2}$$

To solve these six equations, we observe that $\lambda^2/2 = (\mu^2 + \nu^2) = (I_{40} + I_{22})/I_{20}$ and $A_3 = I_{00} - I_{20}/\lambda^2$ are immediate. Treating $\mu\nu$ as one variable we then can solve the following 3 resulting equations:

$$4A_2 = I_{20}/\lambda^2 - 2A_1 ,$$

$$16A_2^2\mu^2\nu^2 = 4A_1^2\lambda^4 ,$$

$$4A_2\mu^2\nu^2 = I_{22} - 2A_1\lambda^4 .$$

The claimed values for A_1 and A_2 are attained, and $\mu\nu = I_{22}/I_{20}$. The system $\mu\nu = I_{22}/I_{20}$, $\mu^2 + \nu^2 = (I_{40} + I_{22})/I_{20}$ is easily solved to acquire the claimed values of μ, ν from a quadratic in μ^2.

If we let the domain of integration be the area of a torus between 2 circles of radii $r < R$, and define

$$\eta \equiv \frac{R^4+R^2r^2+r^4}{R^2+r^2} ,$$

then the parameters of THEOREM 8 become:

$$A_1 = A_2 = \pi(R^4-r^4)/8\eta \ , \quad A_3 = \pi(R^2-r^2)(1-3(R^2+r^2)/4\eta) \ ,$$

$$\lambda = (\eta/3)^{1/2}, \quad \mu = ((2+\sqrt{3})\eta/6)^{1/2}, \quad \nu = ((2-\sqrt{3})\eta/6)^{1/2}.$$

THEOREM 9, 5TH DEGREE, 9 POINT, FORMULA. *Let* $p(x,y) \in P_5(x,y)$. *If* D *is fully symmetric, and* $I_{40} < \infty$ *then*

$$\int_D p(x,y)\,dxdy = A_1[\sum p(\pm\mu,\pm\nu)+\sum p(\pm\nu,\pm\mu)] + A_2 p(0,0)$$

where: all $A_i > 0$, $A_1 = \dfrac{\frac{1}{4}I_{20}^2}{I_{40}+I_{22}}$, $A_2 = I_{00} - \dfrac{2I_{20}^2}{I_{40}+I_{22}}$,

$$\mu,\nu = \left(\frac{I_{40}+I_{22}}{2I_{20}} \pm \frac{1}{2I_{20}}((I_{40}+I_{22})(I_{40}-I_{22}))^{1/2}\right)^{1/2} .$$

Proof. By previous inequalities $A_1, A_2 > 0$. Parameters μ and ν are well defined as $(I_{40}+I_{22}) > (I_{40}-I_{22})$ implies the necessary inequality. We consider the points:

weights	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_2
x-coord.	μ	μ	$-\mu$	$-\mu$	ν	ν	$-\nu$	$-\nu$	0
y-coord.	ν	$-\nu$	ν	$-\nu$	μ	$-\mu$	μ	$-\mu$	0

The following equations of (4.1) remain to be verified:

$$8A_1 + A_2 = I_{00} ,$$

$$4A_1(\mu^2+\nu^2) = I_{20} ,$$

$$4A_1(\mu^4+\nu^4) = I_{40} ,$$

$$8A_1\mu^2\nu^2 = I_{22} .$$

Elementary calculations show the claimed solution solves these equations. Defining $R^2 = \mu^2 + \nu^2$, reduces the last 3 equations to $4A_1R^2 = I_{20}$ and $4A_1R^4 = I_{40} + I_{22}$. Thus $R^2 = (I_{40} + I_{22})/I_{20}$ and A_1 and A_2 are the values we have claimed. The remaining equations, $\mu^2 + \nu^2 = R^2$ and $\mu^2\nu^2 = I_{22}/8A_1$ yield a quadratic in μ^2. The solution to this quadratic yields the claimed values of μ and ν.

If we allow D to be a $2h$x$2h$ square about (0,0) we have:

$$\int_{-h}^{h}\int_{-h}^{h} p(x,y)\,dxdy = \frac{8h^2}{7}\,p(0,0)$$

$$+ \frac{5h^2}{14}\left[\sum p\left(\pm h\left(\frac{7+\sqrt{14}}{15}\right)^{1/2},\ \pm h\left(\frac{7-\sqrt{14}}{15}\right)^{1/2}\right)\right.$$

$$\left. + \sum p\left(\pm h\left(\frac{7-\sqrt{14}}{15}\right)^{1/2},\ \pm h\left(\frac{7+\sqrt{14}}{15}\right)^{1/2}\right)\right] .$$

The following formula is a generalization of one published by McNamee and Stenger [69]. It is included in this section to increase the variety of formulas.

THEOREM 10, 9 POINT, 5TH DEGREE, FORMULA. *Let* $p(x,y) \in P_5(x,y)$. *If* D *is fully symmetric, and* $I_{40} < \infty$, *then for any* $R > (I_{22}/I_{20})^{1/2}$,

$$\int_D p(x,y)\,dxdy = A_1 \sum p(\pm R, \pm R) + A_2 \sum [p(\pm r, 0) + p(0, \pm r)] + A_3 p(0,0)$$

where: $A_1, A_2 > 0$,

$$A_1 = I_{22}/4R^4 \;, \quad A_2 = [I_{20} - I_{22}/R^2]^2 / 2(I_{40} - I_{22}) \;,$$

$$A_3 = I_{00} - \frac{2I_{22}}{I_{40}-I_{22}} \left[\frac{I_{20}^2}{I_{22}} - \frac{2I_{20}}{R^2} + \frac{I_{40}+I_{22}}{2R^4} \right], \quad r = \left(\frac{I_{40}-I_{22}}{I_{20}-I_{22}/R^2} \right)^{1/2}.$$

Proof. Clearly $A_1 > 0$. As $R^2 > I_{22}/I_{20}$ and $I_{40} > I_{22}$, one has $A_2 > 0$ and that r is well defined. We consider the points:

weight	A_1	A_1	A_1	A_1	A_2	A_2	A_2	A_2	A_3
x-coord.	R	R	$-R$	$-R$	r	$-r$	0	0	0
y-coord.	R	$-R$	R	$-R$	0	0	r	$-r$	0

Equations (4.1) yield:

$$4A_1 + 4A_2 + A_3 = I_{00} \;,$$

$$4A_1 R^2 + 2A_2 r^2 = I_{20} \;,$$

$$4A_1 R^4 + 2A_2 r^4 = I_{40} \;,$$

$$4A_1R^4 \qquad = I_{22} .$$

Elementary computations verify the validity of the claimed solution. We have four equations in five variables. Selecting R as a variable, we immediately have the claimed value of A_1. The equations $2A_2r^2 = I_{20} - I_{22}/R^2$ and $2A_2r^4 = I_{40} - I_{22}$ are readily solved to obtain the claimed values. The first equation then yields a value for A_3.

Depending on the set D and the choice of R, A_3 may be positive or negative. For example, if $R = ((I_{40} + I_{22})/2I_{20})^{1/2}$, then $A_3 = I_{00} - 2I_{20}^2/(I_{40} + I_{22}) > 0$. In this special case we have $2R^2 = r^2$. Thus the outer 8 points lie on a circle of radius R.

6. Formulas of 7th degree accuracy. Tyler has published a 12 point method of 7th degree accuracy for rectangles [133]. This formula can be applied to a disc. By rotating these points 45°, a different array of 12 points results. This array was used by Hammer for a formula over the disc [41]. Mysovskikh applied Hammer's 12 point array to the square [76]. We gain generalizations of these two formulas by adding a point to the center. If any one of two conditions on D is satisfied, a 12 point formula results.

THEOREM 11. *Let D be fully symmetric, and $I_{60} < \infty$. If the moments over D satisfy any one of the following inequalities:*

(6.1) $$I_{22}[I_{00}(I_{60}-I_{42})^2-2(I_{40}-I_{22})^3]$$

$$> [I_{20}(I_{60}-I_{42})-(I_{40}-I_{22})^2]^2 ,$$

(6.2) $$(I_{40}-I_{22})[I_{00}I_{42}^2-I_{22}^3] > 2[I_{20}I_{42}-I_{22}^2]^2 ,$$

then there exists a 12 *point quadrature formula of* 7*th degree accuracy over D. In any case there exist several* 13 *point quadrature formulas of* 7*th degree accuracy over D.*

The proof of this theorem is contained within the proofs of the two following theorems. Much of the algebra in these two proofs is identical. Thus the proof of the second is abbreviated.

THEOREM 12, 7TH DEGREE, 13 POINT, FORMULA. *Let* $p(x,y) \varepsilon P_7(x,y)$. *If D is fully symmetric, and* $I_{60} < \infty$, *then for every* $\alpha > \beta^2/\gamma$ *there is a quadrature formula*

$$\int_D p(x,y)\,dxdy = A_0 p(0,0) + A_1 \sum p(\underline{+}R,\underline{+}R)$$

$$+ A_2 \sum (p(\underline{+}T,0)+p(0,\underline{+}T)) + A_3 \sum p(\underline{+}r,\underline{+}r)$$

where $A_1, A_2, A_3 > 0$. *If we define:*

$$\beta \equiv I_{20} - \frac{(I_{40}-I_{22})^2}{I_{60}-I_{42}} , \qquad \gamma \equiv I_{22} , \qquad \delta \equiv I_{42} ,$$

the parameters are:

$$A_0 = I_{00} - 2\,\frac{(I_{40}-I_{22})^3}{(I_{60}-I_{42})^2} - \alpha\ , \qquad A_1 = \frac{1}{4}\,\frac{\beta-\alpha r^2}{R^2-r^2}\ ,$$

$$A_2 = \frac{1}{2}\,\frac{(I_{40}-I_{22})^3}{(I_{60}-I_{42})^2}\ , \qquad A_3 = \frac{1}{4}\,\frac{\alpha R^2-\beta}{R^2-r^2}\ , \qquad T = \left(\frac{I_{60}-I_{42}}{I_{40}-I_{22}}\right)^{1/2}\ ,$$

where:

$$R,r = \left[\frac{\alpha\delta-\beta\gamma}{2(\alpha\gamma-\beta^2)} \pm \frac{1}{2(\alpha\gamma-\beta^2)}\left((\alpha\delta-\beta\gamma)^2-4(\alpha\gamma-\beta^2)(\beta\delta-\gamma^2)\right)^{1/2}\right]^{1/2} .$$

Proof. First we show that r, R and T are well defined and that A_1, A_2 and A_3 are positive. Initially, we want to show $\gamma^2 < \beta\delta$. It has been claimed in THEOREM 2 that:

$$(I_{40}-I_{22})^2 < (I_{20}-I_{22}^2/I_{42})(I_{60}-I_{42}) .$$

By solving this inequality for I_{22}^2/I_{42} and multiplying by I_{42} we have $\gamma^2 < \beta\delta$. The above inequality also shows that $\beta > 0$. To prove that r and R are well defined we need to show that

$$(\alpha\delta-\beta\gamma)^2 - 4(\alpha\gamma-\beta^2)(\beta\delta-\gamma^2) > 0$$

for any α. The above expression is a quadratic in α:

$$\delta^2\alpha^2 - 2\gamma(3\beta\delta-2\gamma^2)\alpha + \beta^2(4\beta\delta-3\gamma^2) .$$

Using calculus we obtain that the minimum value of this quadratic is $4(\beta\delta-\gamma^2)^3 > 0$. The product of $\alpha\gamma > \beta^2$ and $\beta\delta > \gamma^2$ yields $\alpha\delta > \beta\gamma$. Thus r and R are well defined. We have observed that $I_{40} > I_{22}$ and $I_{60} > I_{42}$. Thus $A_2 > 0$ and T is real and positive. To show $A_1, A_3 > 0$ we need only show $R^2 > \beta/\alpha > r^2$. This inequality can be obtained from

$$[(3\alpha\beta\gamma-2\beta^3-\alpha^2\delta)/\alpha]^2 < (\alpha\delta-\beta\delta)^2 - 4(\alpha\gamma-\beta^2)(\beta\delta-\gamma^2) \ .$$

By cancellation we find that proving the above inequality is equivalent to proving that $4(\gamma\alpha-\beta^2)^3 > 0$. This is obviously true for $\alpha > \beta^2/\gamma$.

We have the following points and weights:

weight	A_0	A_1	A_1	A_1	A_1	A_2	A_2	A_2	A_2	A_3	A_3	A_3	A_3
x-coord.	0	R	R	$-R$	$-R$	T	$-T$	0	0	r	r	$-r$	$-r$
y-coord.	0	R	$-R$	R	$-R$	0	0	T	$-T$	r	$-r$	r	$-r$

From equations (4.1) the following equations need to be verified:

$$A_0 + 4A_1 + 4A_2 + 4A_3 = I_{00} \ ,$$

$$4A_1R^2 + 2A_2T^2 + 4A_3r^2 = I_{20} \ ,$$

(6.3) $$4A_1R^4 + 2A_2T^4 + 4A_3r^4 = I_{40} \ ,$$

$$4A_1R^4 \qquad\qquad + 4A_3r^4 = I_{22} ,$$

$$4A_1R^6 + 2A_2T^6 + 4A_3r^6 = I_{60} ,$$

$$4A_1R^6 \qquad\qquad + 4A_3r^6 = I_{42} .$$

After noting that $4A_1 + 4A_3 = \alpha$, verification of the first equation is a matter of computation. Noting that $2A_2T^2 = (I_{40}-I_{22})^2/(I_{60}-I_{42})$ leaves $4A_1R^2 + 4A_3r^2 = \beta$ to be verified in the second equation. Again this is a matter of computations. The fourth equation can be simplified to $\beta(R^2+r^2) - \alpha R^2r^2 = \gamma$. After noting that

$$R^2r^2 = (\beta\delta-\gamma^2)/(\alpha\gamma-\beta^2) \text{ , and}$$

(6.4)

$$R^2 + r^2 = (\alpha\delta-\beta\gamma)/(\alpha\gamma-\beta^2) ,$$

confirmation of the fourth equation becomes routine. Having proven $4A_1R^4 + 4A_3r^4 = \gamma$, to satisfy the third equation, we need only show that $2A_2T^4 = I_{40} - I_{22}$. But this is obvious. The sixth equation can be simplified to:

$$\beta(R^4+r^2R^2+r^4) - \alpha r^2R^2(R^2+r^2) = \delta$$

Recalling (6.4) and observing that

$$R^4 + r^2R^2 + r^4 = (R^2+r^2)^2 - r^2R^2 ,$$

simplifies verification into routine calculations. To verify the fifth equation we need only prove $2A_2T^6 = I_{60} - I_{42}$, which is obvious. The sign of A_0 is a complex matter involving several cases.

To solve equations (6.3) we first remark that the claimed values of A_2 and T are easily obtained. This leaves the following 4 equations in 5 variables:

$$
\begin{aligned}
4A_1 + 4A_3 &= I_{00} - 4A_2 - A_0 , \\
4A_1R^2 + 4A_3r^2 &= \beta , \\
4A_1R^4 + 4A_3r^4 &= \gamma , \\
4A_1R^6 + 4A_3r^6 &= \delta .
\end{aligned}
\tag{6.5}
$$

Defining $\alpha \equiv I_{00} - 4A_2 - A_0$ we allow α to be a variable. Solving for α and r^2 we obtain:

$$(\alpha\gamma-\beta^2)r^4 - (\alpha\delta-\beta\gamma)r^2 + (\beta\delta-\gamma^2) = 0 .$$

The solution to this quadratic in r^2 is

$$\frac{\alpha\delta-\beta\gamma}{2(\alpha\gamma-\beta^2)} \pm \frac{1}{2(\alpha\gamma-\beta^2)}[(\alpha\delta-\beta\gamma)^2-4(\alpha\gamma-\beta^2)(\beta\delta-\gamma^2)]^{1/2} .$$

At the beginning of this proof we showed that the two above values were real and positive. By symmetry of equations

(6.5) these two solutions are R^2 and r^2. With further algebraic manipulations we acquire the claimed values of A_1 and A_3. The value of A_0 depends on the value we select for α. Within this proof the only restriction on α has been $\alpha > \beta^2/\gamma$. Thus for each such α we have a quadrature formula of 7th degree accuracy.

To prove the first part of THEOREM 11, we remark that if $I_{00} - 4A_2$ is an acceptable value of α, A_0 becomes zero and THEOREM 12 yields a 12 point quadrature formula. The needed condition is:

$$I_{00} - 2\,\frac{(I_{40}-I_{22})^3}{(I_{60}-I_{42})^2} > \frac{1}{I_{22}}\left[I_{20}-\frac{(I_{40}-I_{22})^2}{I_{60}-I_{42}}\right]^2 .$$

This is (6.1).

Users of this quadrature formula should note that for relatively large values of α, the sign of A_0 will be negative and the magnitude of A_0 may be unacceptably large.

For an example of this formula select D to be the square with corners at $(\pm 1, \pm 1)$. When $\alpha = \beta^3\delta/\gamma^3$, we have the following formula.

$$\int_{-1}^{1}\int_{-1}^{1} p(x,y)\,dx\,dy = \frac{34,336}{50,625}\,p(0,0)$$

$$+ \frac{98}{405}\left[\sum\left(p\left(\pm(6/7)^{1/2},0\right)+p\left(0,\pm(6/7)^{1/2}\right)\right)\right]$$

$$+ A_1 \sum p(\pm R, \pm R) + A_3 \sum p(\pm r, \pm r)$$

where:

$$A_3, A_1 = \frac{961(1643 \pm 81\sqrt{159})}{5,366,250} \approx 0.477141516,\ 0.111322681$$

$$R, r = \left(\frac{84 \pm 3\sqrt{159}}{155}\right)^{1/2} \approx 0.886561173,\ 0.545784072\ .$$

Tyler's 12 point formula [133] has the form

$$\int_{-1}^{1} \int_{-1}^{1} p(x,y)\,dxdy = \frac{98}{405}\left[\sum p\left(\pm(6/7)^{1/2}, 0\right) + p\left(0, \pm(6/7)^{1/2}\right)\right]$$

$$+ A_1' \sum p(\pm R', \pm R') + A_3' \sum p(\pm r', \pm r')$$

where:

$$A_3', A_1' = \frac{178,981 \pm 2769\sqrt{583}}{472,230} \approx 0.520592916,\ 0.237431774$$

$$R', r' = \left(\frac{114 \pm 3\sqrt{583}}{287}\right)^{1/2} \approx 0.805979782,\ 0.380554433\ .$$

The two formulas are similar. Both sets of parameters are solutions to equations (6.3) for the considered square.

THEOREM 13, 7TH DEGREE, 13 POINT FORMULA. *Let* $p(x,y) \in P_7(x,y)$. *If* D *is fully symmetric, and* $I_{60} < \infty$, *then for every* $\alpha > \beta^2/\gamma$ *there is a quadrature formula*

$$\int_D p(x,y)\,dx\,dy = A_0 p(0,0) + A_1 \sum (p(\pm R,0) + (0,\pm R))$$

$$+ A_2 \sum p(\pm T,\pm T) + A_3 \sum (p(\pm r,0) + p(0,\pm r)) ,$$

where $A_1, A_2, A_3 > 0$. *If we define:*

$$\beta \equiv I_{20} - I_{22}^2/I_{42} , \quad \gamma \equiv I_{40} - I_{22} , \quad \delta \equiv I_{60} - I_{42} ,$$

the parameters are:

$$A_0 = I_{00} - I_{22}^3/I_{42}^2 - 2\alpha , \quad A_1 = \frac{1}{2}\frac{\beta - \alpha r^2}{R^2 - r^2} ,$$

$$A_2 = I_{22}^3/4I_{42}^2 , \quad A_3 = \frac{1}{2}\frac{\alpha R^2 - \beta}{R^2 - r^2} , \quad T = (I_{42}/I_{22})^{1/2} ,$$

where:

$$R,r = \left[\frac{\alpha\delta - \beta\gamma}{2(\alpha\gamma - \beta^2)} + \frac{\pm 1}{2(\alpha\gamma - \beta^2)}((\alpha\delta - \beta\gamma)^2 - 4(\alpha\gamma - \beta^2)(\beta\delta - \gamma^2))^{1/2}\right]^{1/2} .$$

Proof. We have already observed that property (*iii*) implies that γ and δ are positive. Hölder's inequality and $4x^2y^2 < (x^2+y^2)^2$ can be used to prove that β is positive. The last integral inequality of THEOREM 2 is $\gamma^2 < \beta\delta$. The claimed values of R and r are algebraically the same here as in THEOREM 12. As we have $\gamma^2 < \beta\delta$, the proof that R and r are well defined is identical to the proof given in THEOREM 12. This proof is thus omitted. To prove $A_1, A_3 > 0$ we need only prove $R^2 > \beta/\alpha > r^2$. The proof of this in THEOREM 12 was dependent on $\gamma^2 < \beta\delta$ and ended requiring that $\alpha > \beta^2/\gamma$. As the algebra is the same, this part of the proof can also be omitted.

We have the following points and weights:

weight	A_0	A_1	A_1	A_1	A_1	A_2	A_2	A_2	A_2	A_3	A_3	A_3	A_3
x-coord.	0	R	$-R$	0	0	T	T	$-T$	$-T$	r	$-r$	0	0
y-coord.	0	0	0	R	$-R$	T	$-T$	T	$-T$	0	0	r	$-r$

Then equations (4.1) assume the form:

$$A_0 + 4A_1 + 4A_2 + 4A_3 = I_{00} ,$$

$$2A_1R^2 + 4A_2T^2 + 2A_3r^2 = I_{20} ,$$

$$2A_1R^4 + 4A_2T^4 + 2A_3r^4 = I_{40} ,$$

(6.6)

$$4A_2T^4 = I_{22} ,$$

$$2A_1R^6 + 4A_2T^6 + 2A_3r^6 = I_{60} ,$$

$$4A_2T^6 = I_{42} .$$

Verification of the first equation becomes routine after observing that $4A_1 + 4A_3 = 2\alpha$. It is obvious that

$$4A_2T^2 = \frac{I_{22}^2}{I_{42}}, \quad 4A_2T^4 = I_{22}, \quad 4A_2T^6 = I_{42}.$$

The last two equations verify the fourth and sixth equations of (6.6). In view of the above results to verify the second, third and fifth equations we only need show:

$$2A_1R^2 + 2A_3r^2 = \beta,$$

$$2A_1R^4 + 2A_3r^4 = \gamma,$$

$$2A_1R^6 + 2A_3r^6 = \delta.$$

After the claimed values for A_1, A_3, R and r are substituted in the above equations, the resulting expressions are identical to those appearing in the proof of THEOREM 12. The algebra of THEOREM 12 applies and yields the desired results. Complex rules can be developed to determine the sign of A_0.

Next we demonstrate how equations (6.6) are solved. Values for A_2 and T are immediate. Simplification by subtraction produces:

$$2A_1 \quad + 2A_3 \quad = (I_{00}-4A_2-A_0)/2$$

$$2A_1R^2 + 2A_3r^2 = \beta$$

$$2A_1R^4 + 2A_3r^4 = \gamma$$

$$2A_1R^6 + 2A_3r^6 = \delta$$

Defining $\alpha \equiv (I_{00}-4A_2-A_0)/2$, we have equations (6.5) of THEOREM 12 when each coefficient, 2, is replaced by a 4. Again α is a variable. The algebra of THEOREM 12 applies yielding the claimed solutions for R, r, A_1 and A_3. The value of A_0 depends on the value we select for α. Thus for every $\alpha > \beta^2/\gamma$ we have a quadrature formula of 7th degree accuracy.

To prove the last part of THEOREM 11, we note that if $(I_{00}-4A_2)/2$ is an acceptable value of α, A_0 becomes zero and THEOREM 13 yields a 12 point quadrature formula. The condition is:

$$(I_{00}-I_{22}^3/I_{42}^2)/2 > [I_{20}-I_{22}^2/I_{42}]^2/(I_{40}-I_{22}) .$$

This is (*6.2*).

Again users should beware of what large values of α do to A_0. Various selections of α within both of the previous theorems offer the user a variety of formulas. For an example allow $\alpha = \beta^3\delta/\gamma^3$. If D is the disc $S(0,h)$ of radius h about (0,0), then we have

$$\int_{S(0,h)} p(x,y)\,dx\,dy = \pi h^2\Big[\frac{1}{8}\,p(0,0) + \frac{2}{27}\sum p(\pm h\sqrt{3}/\sqrt{8}, \pm h\sqrt{3}/\sqrt{8})$$

$$+ A_1\sum(p(\pm R,0)+p(0,\pm R))$$

$$+ A_3\sum(p(\pm r,0)+p(0,\pm r))\Big],$$

where:

$$A_3, A_1 = \frac{25(5\pm 13/\sqrt{17})}{1728} \approx 0.117953749,\ 0.026722177\ ,$$

$$R, r = \frac{h}{2}\left(\frac{27\pm 3\sqrt{17}}{10}\right)^{1/2} \approx 0.992085138h,\ 0.604786804h\ .$$

Hammer's 12 point formula for this disc takes the form

$$\int_{S(0,h)} p(x,y)\,dx\,dy = \pi h^2\Big[\frac{2}{27}\sum p(\pm h\sqrt{3}/\sqrt{8}, \pm h\sqrt{3}/\sqrt{8})$$

$$+ A_1'\sum(p(\pm R',0)+p(0,\pm R'))$$

$$+ A_3'\sum(p(\pm r',0)+p(0,\pm r'))\Big]$$

where:

$$A_3', A_1' = \frac{97{,}983\pm 1{,}107\sqrt{573}}{1{,}113{,}912} \approx 0.111751846,\ 0.064174080\ ,$$

$$R', r' = \left(\frac{27\pm\sqrt{573}}{52}\right)^{1/2} h \approx 0.989730134h,\ 0.242684568h\ .$$

Again the two formulas are similar. Hammer's parameters also solve equations (6.6) for the disc, where A_0 is equal to zero.

7. Summary. For the convenience of the user, we list in this section all the formulas within this paper. We have assumed that D is fully symmetric, that all moments $I_{ij} \equiv \int_D x^i y^j dxdy$ are finite, and that $p(x,y)$ is a polynomial of the indicated degree.

3rd Degree Formula, 4 or 5 Points.

$$\int_D p(x,y)\,dxdy = (I_{00}-2I_{20}/R^2)p(0,0)$$

$$+ (I_{20}/2R^2)[p(\mu,\nu)+p(-\mu,-\nu)+p(\nu,-\mu)+p(-\nu,\mu)]$$

for any μ, ν and $R > 0$ such that $\mu^2 + \nu^2 = R^2$.

5th Degree Formula I, 7 Points, Radon.

$$\int_D p(x,y)\,dxdy = A_1\sum p(\pm\lambda,0) + A_2\sum p(\pm\mu,\pm\nu) + A_3 p(0,0)$$

where: $\mu = \left(\frac{I_{22}}{I_{20}}\right)^{1/2}$, $\nu = \left(\frac{I_{40}}{I_{20}}\right)^{1/2}$, $\lambda = \left(\frac{I_{40}+I_{22}}{I_{20}}\right)^{1/2}$,

$$A_1 = \frac{1}{2}\,\frac{I_{20}^2}{I_{40}}\,\frac{I_{40}-I_{22}}{I_{40}+I_{22}}, \quad A_2 = \frac{I_{20}^2}{4I_{40}}, \quad A_3 = I_{00} - \frac{2I_{20}^2}{I_{40}+I_{22}}.$$

5th Degree Formula I, 7 Points.

$$\int_D p(x,y)\,dxdy = A_1[p(\lambda,\lambda)+p(-\lambda,-\lambda)]$$

$$+ A_2[p(\mu,-\nu)+p(-\mu,\nu)+p(\nu,-\mu)+p(-\nu,\mu)] + A_3p(0,0)$$

where:

$$\lambda = \left(\frac{I_{40}+I_{22}}{2I_{20}}\right)^{1/2},$$

$$\mu,\nu = \left(\frac{I_{40}+I_{22}}{2I_{20}} \pm \frac{1}{2I_{20}}((I_{40}+3I_{22})(I_{40}-I_{22}))^{1/2}\right)^{1/2},$$

$$A_1 = \frac{2I_{20}^2 I_{22}}{(I_{40}+3I_{22})(I_{40}+I_{22})}, \quad A_2 = \frac{\frac{1}{2}\,I_{20}^2}{I_{40}+3I_{22}},$$

$$A_3 = I_{00} - \frac{2I_{20}^2}{I_{40}+I_{22}}.$$

5th Degree Formula II, 9 Points.

$$\int_D p(x,y)\,dxdy = A_1[\sum p(\pm\mu,\pm\nu)+\sum p(\pm\nu,\pm\mu)] + A_2p(0,0)$$

where:

$$\mu,\nu = \left(\frac{I_{40}+I_{22}}{2I_{20}} \pm \frac{1}{2I_{20}}((I_{40}+I_{22})(I_{40}-I_{22}))^{1/2}\right)^{1/2},$$

$$A_1 = \frac{\frac{1}{4} I_{20}^2}{I_{40}+I_{22}} , \quad A_2 = I_{00} - \frac{2I_{20}^2}{I_{40}+I_{22}} .$$

5th Degree Formula III, 9 Points.

$$\int_D p(x,y)\,dxdy = A_1 \sum p(\pm R, \pm R)$$

$$+ A_2 \sum [p(\pm r,0)+p(0,\pm r)] + A_3 p(0,0)$$

for any $R > (I_{22}/I_{20})^{1/2}$ where:

$$r = \left(\frac{I_{40}-I_{22}}{I_{20}-I_{22}/R^2}\right)^{1/2} , \quad A_1 = I_{22}/4R^4 ,$$

$$A_2 = [I_{20}-I_{22}/R^2]^2/2(I_{40}-I_{22}) ,$$

$$A_3 = I_{00} - \frac{2I_{22}}{I_{40}-I_{22}}\left[\frac{I_{20}^2}{I_{22}} - \frac{2I_{20}}{R^2} + \frac{I_{40}+I_{22}}{2R^4}\right] .$$

7th Degree Formula I, 12 or 13 Points.

Define $\beta \equiv I_{20} - \frac{(I_{40}-I_{22})^2}{I_{60}-I_{42}}$, $\gamma \equiv I_{22}$, $\delta \equiv I_{42}$.

$$\int_D p(x,y)\,dxdy = A_0 p(0,0) + A_1 \sum p(\pm R, \pm R)$$

$$+ A_2 \sum (p(\pm T,0)+p(0,\pm T)) + A_3 \sum p(\pm r, \pm r)$$

for every $\alpha > \beta^2/\gamma$ where:

$$R,r = \left[\frac{\alpha\delta-\beta\gamma}{2(\alpha\gamma-\beta^2)} \pm \frac{1}{2(\alpha\gamma-\beta^2)}((\alpha\delta-\beta\gamma)^2-4(\alpha\gamma-\beta^2)(\beta\delta-\gamma^2))^{1/2}\right]^{1/2},$$

$$T = \left(\frac{I_{60}-I_{42}}{I_{40}-I_{22}}\right)^{1/2}, \quad A_0 = I_{00} - 2\frac{(I_{40}-I_{22})^3}{(I_{60}-I_{42})^2} - \alpha$$

$$A_1 = \frac{1}{4}\frac{\beta-\alpha r^2}{R^2-r^2}, \quad A_2 = \frac{1}{2}\frac{(I_{40}-I_{22})^3}{(I_{60}-I_{42})^2}, \quad A_3 = \frac{1}{4}\frac{\alpha R^2-\beta}{R^2-r^2}.$$

7th Degree Formula 11, 12 or 13 Points.

Define $\beta \equiv I_{20}-I_{22}^2/I_{42}$, $\gamma \equiv I_{40}-I_{22}$, $\delta \equiv I_{60}-I_{42}$.

$$\int_D p(x,y)\,dxdy = A_0 p(0,0) + A_1\sum(p(\pm R,0)+p(0,\pm R))$$

$$+ A_2\sum p(\pm T,\pm T) + A_3\sum(p(\pm r,0)+p(0,\pm r))$$

for every $\alpha > \beta^2/\gamma$ where the formula given previously for R and r applies here and:

$$T = (I_{42}/I_{22})^{1/2}, \quad A_0 = I_{00} - I_{22}^3/I_{42}^2 - 2\alpha$$

$$A_1 = \frac{1}{2}\frac{\beta-\alpha r^2}{R^2-r^2}, \quad A_2 = I_{22}^3/4I_{42}^2, \quad A_3 = \frac{1}{2}\frac{\alpha R^2-\beta}{R^2-r^2}.$$

8. Appendix. For an example of the methods used to prove THEOREM 1, we prove $I_{22}^2 < I_{20}I_{42}$. Since $x^2y^2(x^2+y^2)$ and $x^2y^2/(x^2+y^2)$ are unequal almost everywhere, applying Hölder's inequality to these functions yields a strict inequality.

$$I_{22}^2 < \int_D x^2y^2(x^2+y^2)\,dxdy \int_D (x^2y^2/(x^2+y^2))\,dxdy$$

$$< (I_{42}+I_{24})\frac{1}{4}\int_D (x^2+y^2)\,dxdy = I_{20}I_{42} .$$

Above we have used the inequality $4x^2y^2 < (x^2+y^2)^2$ which holds almost everywhere in D.

DEFINITION. If D is fully symmetric,

$$S \equiv \{(x,y) \in D : |x| > |y|\} .$$

Proof of THEOREM 2. To prove the third inequality we consider $(x,y) \in S$. Clearly

$$(x^2-y^2)^2 < |x^2+y^2|^{2/3} \; |x^2-y^2|^{4/3}$$

almost everywhere in S and thus

$$(I_{40}-I_{22}) = \int_D x^2(x^2-y^2) = \int_S (x^2-y^2)^2$$

$$< \int_S \left| (x^2+y^2)(x^2-y^2)^2 \right|^{2/3} dxdy .$$

Preparing to use Hölder's inequality we observe that 1 and

$$\left|(x^2+y^2)(x^2-y^2)^2\right|^{2/3}$$

are unequal almost everywhere on S, $I_{00} < \infty$ and that

$$\int_S (x^2+y^2)(x^2-y^2)^2\,dxdy = I_{60}-I_{42} < \infty .$$

Thus we have

$$2(I_{40}-I_{22})^3 < 2\left[\int\!\!\int_S \left|(x^2+y^2)(x^2-y^2)^2\right|^{2/3} dxdy\right]^3$$

$$< 2\int_S dxdy\left[\int\!\!\int_S \left[\left|(x^2+y^2)(x^2-y^2)^2\right|^{2/3}\right]^{3/2} dxdy\right]^2$$

$$= I_{00}(I_{60}-I_{42})^2 .$$

Consider the last inequality. The two functions $(x^2+y^2)(x^2-y^2)^2$ and $(x^2-y^2)^2/(x^2+y^2)$ are unequal almost everywhere.

$$\int_S \frac{(x^2-y^2)^2}{x^2+y^2}\,dxdy < \int_S \frac{(x^2+y^2)^2}{x^2+y^2}\,dxdy = I_{20} < \infty .$$

proves boundedness. Applying Hölder's inequality we have

$$(I_{40}-I_{22})^2 = \left[\int_S (x^2-y^2)^2 dxdy\right]^2$$

$$< \int_S (x^2+y^2)(x^2-y^2)^2 dxdy \int_S \frac{(x^2-y^2)^2+4x^2y^2-4x^2y^2}{x^2+y^2} dxdy$$

$$< (I_{60}-I_{42})\left[\int_S \frac{(x^2+y^2)^2}{x^2+y^2} dxdy - \int_S \frac{4x^2y^2}{x^2+y^2} dxdy\right]$$

$$< (I_{60}-I_{42})[I_{20}-I_{22}^2/I_{42}] .$$

Between the last two lines of the above we used

$$\frac{I_{22}^2}{I_{42}} < \int_S \frac{2x^2y^2}{x^2+y^2} dxdy .$$

This was proven at the beginning of this appendix.

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