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UNDERSTANDING QUADRATURE FORMULAS FOR PLANAR REGIONS

by

Charles B. Huelsman III
Capt USAF

AIR FORCE WEAPONS LABORATORY

ABSTRACT

Quadrature formulas for double integrals are described. Examples are used to give an idea of what should be proven about quadrature formulas. Proofs are simple and cover formulas of third- and fifth-degree polygonal accuracy. Several problems are provided and should be worked to gain some idea of why quadrature formulas work, and how they can be created or altered to suit ones' need.

FOREWORD

This report is based on a series of six lectures on numerical integration given at the Air Force Weapons Laboratory during the summer of 1971. Some of the research included in this report was performed at the Ohio State University under an AFIT doctoral program between June 1968 and June 1970. The remainder of the research was done at the Air Force Weapons Laboratory. Most of the calculations were done by Mr. Richard Stephens and Mr. Jeffery Merilatt.

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This technical report has been reviewed and is approved.

CHARLES B. HUELSMAN III
Captain, USAF
Project Officer

JOHN P. PORTASIK
Colonel, USAF
Chief, Electromagnetic Pulse Branch

CARL F. DAVIS
Colonel, USAF
Chief, Electronics Division

CONTENTS

TABLES

SECTION I

INTRODUCTION

There is a problem which involves a communication gap between mathematicians who create numerical integration techniques, and scientists and engineers who use them. This report attempts to bridge that gap. The report is restricted to the calculation of double integrals. In most cases the domain of integration is fully symmetric.

This introduction includes some history, a comparison of numerical integration methods, and a list of what is known about two-dimensional quadrature formulas. The basis of the accuracy of such quadrature formulas is discussed in section II. In section III, properties of integrals of monics, $x^p y^q$, are discussed. This is a foundation for the remaining sections. In section IV an attempt is made to give the reader some idea of what cannot be done with quadrature formulas. Quadrature formulas of polygonal accuracies 3 and 5 are given in sections V, and VI, respectively.

To use a numerical integration method on a computer, the method must be programable, and accurate. It cannot be time consuming. Types of methods include the trapezoid rule, Monte-Carlo methods, and quadrature formulas. These methods have the form

$$\int_D f(x,y)\,dxdy \approx \sum_{k=1}^{m} w_k f(x_k,y_k)$$

The three methods have different merits. In the Monte-Carlo method all of the weights w_k are positive. Sometimes all weights are equal. Unequal and sometimes negative weights are characteristic of quadrature formulas. With the quadrature formulas, errors in the functional evaluations, $f(x_k,y_k)$, are unequally weighted. Thus, any functional evaluation error plays a disproportionate role in the total error. For this reason, quadrature formulas with near equal and predominately positive weights are of interest.

Monte-Carlo methods use a substantial number of conveniently located points (x_k, y_k). Quadrature formulas employ a small number of inconveniently located points. Near minimum point quadrature formulas use a few more points but allow some freedom in choosing either the positions of some of the points or a value of one of the weights.

If f(x,y) is defined analytically, and only a few integrals need be approximated for a given region D, it is advantageous to use a Monte-Carlo method. The convenient placement of points simplifies programming. If, however, many integrals are to be approximated, it may be more economical to use a quadrature formula. For each integral significantly less computer time would be spent in calculating functional evaluations at a small number of inconveniently placed points rather than at thousands of simply placed points. Near minimum point quadrature formulas offer a variety of compromises between the above two extremes. While they have less potential for theoretical interest than the minimum point methods, computerwise they have potential for practicality. Minimizing the number of points for a given degree of polynomial accuracy is especially valuable if each functional evaluation must be obtained at a moderate expense. For instance, if such evaluations must be made empirically, the Monte-Carlo method becomes not only time consuming but expensive. If f(x,y) can be observed more economically and/or more accurately in some areas of D than others, then it is advantageous to have several different quadrature formulas for a given degree of accuracy. This is true even though some formulas may have more than the minimum number of points. A comparison of these methods is given in table I. Note that the qualities labled with an asterisk are rarely obtained simultaneously.

Quadrature formulas exist for a variety of planar regions. For discussion, the following definitions are helpful.

Definition: D is radially symmetric if and only if $(x,y) \in D$ implies $(-x,-y) \in D$.

Definition: By $(\pm x, \pm y)$ we mean the four points: (x,y), (x,-y), (-x,y), and (-x,-y).

Problem: How would one define $(\pm t, 0)$, $p(\pm x, \pm y)$, and $\sum p(\pm x, \pm y)$?

Definition: D has axial symmetry if and only if $(x,y) \in D$ implies $(\pm x, \pm y) \in D$.

Definition: D is fully symmetric if and only if $(x,y) \in D$ implies $(\pm x, \pm y)$, $(\pm y, \pm x) \in D$.

<u>Problem</u>: Classify the following planar regions: disc, square, a pentagon, The Pentagon, hexagon, octagon, cross, ring, rectangle, trapezoid, a fat S, diamond, four-point star, six-point star, stove grill, stop sign, yield sign, shower drain, and modern clock face.

Table I

COMPARISON OF NUMERICAL INTEGRATION METHODS

Parameter	Methods			
	Trapozoid and Simpson's Rule	Monte-Carlo Methods	Minimum Point Quadrature Formulas	Near Minimum Point Quadrature Formulas
If integrand evaluations are expensive then method is	very costly	costly	economic	nearly economic
Placement of points is	convenient	convenient	inconvenient	some inconvenient and some convenient*
Are all weights equal?	yes	only locally	almost never	only some can be adjusted to be equal*
Are all weights positive?	yes	yes	usually	some can be negative. Depends on choice

* Rarely obtained simultaneously

In table II we have attempted to list what types of quadrature formulas are known. With the great amount of research going on in quadrature formulas, this table will probably be out of date by the time this report is published. If a number p is listed under EX. it means the existence of such a formula has been proven and the minimum number of points is p. A blank means the minimum number of points is still under discussion. If a number q is listed under m it means that a formula of use to scientists and engineers has been published and that it has q points.

Table II

TYPES OF KNOWN QUADRATURE FORMULAS

Degree of Accuracy	Fully Sym.		Axial Sym.		Radial Sym.		Arbitrary Region	
	EX.	m	EX.	m	EX.	m	EX.	m
1	1	1	1	1	1	1	1	1
2	3	3	3	3	3	3	3	3
3	4	4,5	4	4	4	4	---	---
4	6	---	6	---	6	---	---	---
5	7	7,9	7	7	7	7	---	---
6	10	---	10	---	10	---	---	---
7	12*	12*,13,17**	12*	12	---	---	---	---
8	15	---	15	---	15	---	---	---
9	---	***	---	***	---	---	---	---

* Existence of formulas known for all regions, but in some cases not all all of the points are real. See reference 7.

** Unpublished, but known to the author.

*** Formulas are known for special regions such as circles.

SECTION II

WHAT IS A QUADRATURE FORMULA?

This section will discuss what is a quadrature formula for double integrals and what is meant by the term polygonal accuracy. Throughout this section and the remainder of the report, only quadrature formulas with real parameters are considered.

A quadrature formula is an approximation of an integral by a summation of weighted values of the integrand.

$$\int_D f(x,y)dxdy \approx \sum_{k=1}^{m} w_k f(x_k,y_k)$$

Quadrature formulas are usually judged by the number of integrand evaluations they require for being accurate for all polynomials $p(x,y) \in P_n(x,y)$ for some n.

<u>Example</u>: Let us attempt to construct a quadrature formula that would use the points (a,a) and (-a,-a) and be accurate for any polynomial of the type $dx^2 + by + c = p(x,y)$ for integrals taken over the square with corners at $(\pm 1, \pm 1)$. Note that d, b, and c are constants. We want

$$\int_{-1}^{1}\int_{-1}^{1} p(x,y)dxdy = w_1 p(a,a) + w_2 p(-a,-a)$$

$$= w_1(da^2 + ba + c) + w_2(da^2 - ba + c)$$

Matching powers on both sides of the above equation yields the following equations:

$$4c = \int_{-1}^{1}\int_{-1}^{1} c\,dxdy = w_1 c + w_2 c = c(w_1 + w_2)$$

$$0 = \int_{-1}^{1}\int_{-1}^{1} by\,dx\,dy = w_1 ba + w_2 b(-a) = ab(w_1 - w_2)$$

$$4d/3 = \int_{-1}^{1}\int_{-1}^{1} dx^2\,dx\,dy = w_1 da^2 + w_2 d(-a)^2 = a^2 d(w_1 + w_2)$$

which is equivalent to

$$w_1 + w_2 = 4$$

$$w_1 - w_2 = 0$$

$$a^2(w_1 + w_2) = 4/3$$

Inspection yields $w_1 = w_2 = 2 \equiv w$, and $a^2 = 1/3$. Thus, $a = 1/\sqrt{3}$

Thus

$$\int_{-1}^{1}\int_{-1}^{1} p(x,y)\,dx\,dy = 2\left[p(1/\sqrt{3},\ 1/\sqrt{3}) + p(-1/\sqrt{3},\ -1/\sqrt{3})\right]$$

For example, if $d = 2$, $b = -1$, and $c = -3$, then for a check answer we would have

$$\int_{-1}^{1}\int_{-1}^{1} (2x^2 - y - 3)\,dx\,dy = \frac{2x^3y}{3}\Bigg|_{-1}^{1}\Bigg|_{-1}^{1} - \frac{y^2x}{2}\Bigg|_{-1}^{1}\Bigg|_{-1}^{1} - \frac{3xy}{2}\Bigg|_{-1}^{1}\Bigg|_{-1}^{1}$$

which is -28/3. By the formula we would have

$$\int_{-1}^{1}\int_{-1}^{1} (2x^2 - y - 3)\,dx\,dy = 2\left[2(1/\sqrt{3})^2 - 1/\sqrt{3} - 3 + 2(-1\sqrt{3})^2 + 1/\sqrt{3} - 3\right]$$

which is also -28/3.

From this example four natural questions arise.

(1) Can this method be generalized for any $p(x,y) \epsilon P_n(x,y)$, and furthermore for any n?

(2) Can this method be further generalized for approximations of integrals over various regions?

(3) Is there a relationship between n, m, and the dimension s of the integral?

(4) Are there methods that offer a more convenient placement of points at which integrand evaluations are made?

Questions 1 and 2 are answered with the theorem that follows. Question 3 has been answered by a few authors including Mysovskikh (ref. 6). An attempt to answer question 4 was made in AFWL-TR-71-162 (ref. 4).

Definition:

$$I_{ij} = \int_D x^i y^j dxdy$$

Theorem:

Let D be any planar region such that for every $0 \le i + j \le n$; $i, j \epsilon N \equiv \{0, 1, 2, \ldots\}$, we have $0 \le I_{ij} < \infty$. Let $p(x,y) \epsilon P_n(x,y)$. If there exists a set of points $\left\{(x_k, y_k)\right\}_{k=1}^{m}$ and m associated weights w_k such that

$$I_{ij} = \sum_{k=1}^{m} w_k x_k^i y_k^j \qquad 0 \le i + j \le n \tag{1}$$

Then

$$\int_D p(x,y)dxdy = \sum_{k=1}^{m} w_k p(x_k, y_k) \tag{2}$$

Proof: By definition

$$p(x,y) = \sum_{i+j\le n} c_{ij} x^i y^j$$

where some $c_{i,n-i} \neq 0$. Multiplying each I_{ij} by c_{ij} and summing we have by equation (1) that

$$\sum_{i+j\leq n} c_{ij} I_{ij} = \sum_{i+j\leq n} c_{ij} \sum_{k=1}^{m} w_k \, x_k^i \, y_k^j$$

On the left $I_{ij} \equiv \int_D x^i y^j dxdy$, and on the right the double sum is finite.

Thus

$$\int_D \sum_{i+j\leq n} c_{ij} x^i y^j dxdy = \sum_{k=1}^{m} w_k \sum_{i+j\leq n} c_{ij} \, x_k^i \, y_k^j$$

but

$$p(x,y) \equiv \sum_{i+j\leq n} c_{ij} \, x^i y^j$$

implying

$$\int_D p(x,y)dxdy = \sum_{k=1}^{m} w_k p(x_k, y_k)$$

as needed. Thus, our search for quadrature formulas over some region D for some level of polynomial accuracy n becomes a search for points and weights satisfying equation (1).

Let us illustrate this proof by example. Consider a 4-point, third-degree formula. We desire a quadrature formula that is accurate for any polynomial of the type

$$c_{30}x^3 + c_{21}x^2y + c_{12}xy^2 + c_{03}xy^3 + c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{10}x + c_{01}y + c_{00}$$

or

$$\sum_{i+j\leq 3} c_{ij} \, x^i \, y^j$$

over the square with corners at ($\pm 1, \pm 1$). The natural question is what do we need to get

$$\int_{-1}^{1}\int_{-1}^{1}\sum_{i+j\le 3} c_{ij}\, x^i y^j = \sum_{k=1}^{m} w_k \sum_{i+j\le 3} c_{ij}\, x_k^i\, y_k^j$$

This last line can be restated. Using

$$I_{ij} \equiv \int_{-1}^{1}\int_{-1}^{1} x^i y^j dxdy$$

we have

$$\sum_{i+j\le 3} c_{ij}\, I_{ij} = \sum_{i+j\le 3} c_{ij} \sum_{k=1}^{m} w_k\, x_k^i\, y_k^j$$

That is with natural pairings

$$c_{30}\int_{-1}^{1}\int_{-1}^{1} x^3 dxdy = c_{30}\sum_{k=1}^{m} w_k\, x_k^3$$

$$c_{21}\int_{-1}^{1}\int_{-1}^{1} x^2 y dxdy = c_{21}\sum_{k=1}^{m} w_k\, x_k^2\, y_k$$

$$\vdots \qquad\qquad \vdots$$

$$c_{01}\int_{-1}^{1}\int_{-1}^{1} y dxdy = c_{01}\sum_{k=1}^{m} w_k\, y_k$$

$$c_{00}\int_{-1}^{1}\int_{-1}^{1} dxdy = c_{00}\sum_{k=1}^{m} w_k$$

The c_{ij}'s cancel leaving us with the condition we need to satisfy equation (1) in order to get what we want which is equation (2). Before listing these conditions, matters are simplified if we evaluate the integrals of the monics $x^i y^j$. These are commonly called moments.

$$\int_{-1}^{1}\int_{-1}^{1} x^i y^j dxdy = \frac{\left(1 - (-1)^{i+1}\right)\left(1 - (-1)^{j+1}\right)}{(i + 1)\quad\quad (j + 1)}$$

Thus, if i or j is odd, we have $I_{ij} = 0$. If i and j are even, we have $I_{ij} > 0$. And in any case $I_{ij} = I_{ji}$. So we have $I_{30}, I_{21}, I_{12}, I_{03}, I_{11}, I_{10}, I_{01} = 0$; $I_{20}, I_{02} = 4/3$; and $I_{00} = 4$.

We are looking for points (x_k, y_k) and associated weights w_k satisfying the following equations.

$$0 = \sum_{k=1}^{m} w_k y_k^3 \qquad\qquad 0 = \sum_{k=1}^{m} w_k x_k^2 y_k$$

$$0 = \sum_{k=1}^{m} w_k x_k y_k^2 \qquad\qquad 0 = \sum_{k=1}^{m} w_k x_k^3$$

$$4/3 = \sum_{k=1}^{m} w_k y_k^2 \qquad 0 = \sum_{k=1}^{m} w_k x_k y_k \qquad 4/3 = \sum_{k=1}^{m} w_k x_k^2$$

Do the following points and weights satisfy equations 3? Obviously $a \neq 0$.

weight	w	w	w	w
x-coord	a	a	-a	-a
y-coord	a	-a	a	-a

Listing the equations we have

$$I_{30}, I_{12} = a^3 (w_1 + w_2 - w_3 - w_4) = 0$$

$$I_{03}, I_{21} = a^3 (w_1 - w_2 + w_3 - w_4) = 0$$

$$I_{20}, I_{02} = a^2 (w_1 + w_2 + w_3 + w_4) = 4/3$$

$$I_{11} = a^2 (w_1 - w_2 - w_3 + w_4) = 0$$

$$I_{10} = a (w_1 + w_2 - w_3 - w_4) = 0$$

$$I_{01} = a (w_1 - w_2 + w_3 - w_4) = 0$$

$$I_{00} = w_1 + w_2 + w_3 + w_4 = 4$$

From $0 = I_{10}/a \pm I_{01}/a$, we have that $w_1 = w_4$ and $w_2 = w_3$. From $0 = I_{11}/a + I_{10}/a$, we have $w_1 = w_3$. Therefore, all four weights are equal. Let us denote the one common weight by w. Equations I_{30}, I_{21}, I_{12}, I_{03}, I_{11}, I_{10}, and I_{01} are satisfied. We are left with

$$4wa^2 = 4/3$$

$$4w = 4$$

Obviously $w = 1$ and $a = 1/\sqrt{3}$.

Thus, we have the following four-point, third-degree formula over the square with corners at $(\pm 1, \pm 1)$

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)dxdy \approx \sum f\,(\pm\, 1/\sqrt{3}, \pm\, 1/\sqrt{3})$$

Some applications of this formula are listed below.

Application:	Error:
$\int_{-1}^{1}\int_{-1}^{1}\left[\lvert x\rvert + \lvert y\rvert\right] dxdy \approx 4.218\ 802\ 152$	0.218 802 152
$\int_{-1}^{1}\int_{-1}^{1} \exp(x+y)\ dxdy \approx 5.488\ 224\ 960$	0.036 466 422

The objective of this section was to give the reader some idea of what is a quadrature formula for double integrals. The most important part of this section to remember is the realization that equation (1) must be solved to obtain the parameters for a quadrature formula (equation 2).

SECTION III

PROPERTIES OF MOMENTS

You have already seen the importance of solving equations involving moments I_{ij}. It is, thus, necessary to derive several properties of moments. Within the discussion below, we will use the term measurable. For the mathematician, we mean measurable in the sense of Lebesque. It would suffice, however, to think of the measure m(D) of a set D to be the area of D and to think that D is measurable if a way exists of finding the area of D. Henceforth, when we say D is fully symmetric, we assume D is two-dimensional and measureable such that $0 < m(D) < \infty$.

Definition:

$$N \equiv \{1, 2, 3, \ldots\}, \; N_0 \equiv \{0, 1, 2, \ldots\}$$

Definition: If D is a fully symmetric set, define

$$S = \{(x,y) \varepsilon D : |x| > |y|\}$$

$$U = \{(x,y) \varepsilon D : |x| = |y|\}$$

$$T = \{(x,y) \varepsilon D : |x| < |y|\}$$

Since D is measurable, S, U, and T are measurable. Obviously m(U) = 0. Thus,

$$\int_U x^i y^j \, dxdy = 0 \text{ for } i,j \; \varepsilon \; N_0$$

By reversing the order of the coordinates of the points is S, we have T. Thus,

$$\int_S f(x,y)dxdy = \int_T f(y,x)dxdy$$

or for monomials

$$\int_S x^i y^j \, dxdy = \int_T x^j y^i \, dxdy \qquad \text{for } i,j \in N_o$$

<u>Proposition</u>: If D is fully symmetric then $I_{ij} = I_{ji}$.

Using the above equality

$$I_{ij} \equiv \int_D x^i y^j \, dxdy = \int_S x^i y^j \, dxdy + \int_T x^i y^j \, dxdy$$

$$= \int_T x^j y^i \, dxdy + \int_S x^j y^i \, dxdy$$

$$= \int_D x^j y^i \, dxdy \equiv I_{ji}$$

<u>Proposition</u>: If D is fully symmetric and one of the numbers i or j is odd, then $I_{ij} = 0$.

If i is odd and j is even, then define

$$D_+ \equiv \{(x,y) \in D : x > 0\}; \; D_- \equiv \{(x,y) \in D : x < 0\}$$

$$D_o \equiv \{(x,y) \in D : x = 0\}$$

Obviously $m(D_o) = 0$ and $\int_{D_o} x^i y^j \, dxdy = 0$. Both D_+ and D_- are measurable.

$$I_{ij} = \int_D x\left(x^{2p} y^{2q}\right) dxdy$$

$$= \int_{D_+} x\left(x^{2p} y^{2q}\right) dxdy + \int_{D_-} x\left(x^{2p} y^{2q}\right) dxdy$$

where $i = 2p + 1$ and $j = 2q$; $p, q \in N_o$.

Since $(x,y) \varepsilon D_+$ implies $(-x,y) \varepsilon D_-$ we have

$$I_{ij} = \int_{D_+} x\left(x^{2p}y^{2q}\right)dxdy + \int_{D_+} (-x)\left(x^{2p}y^{2q}\right)dxdy = 0$$

With other partitions of D, $I_{ij} = 0$ can be proven for the cases i even and j odd, and both i and j odd.

Proposition: If D is fully symmetric, and m, k ε N, then

$$I_{2(m+k),0} > I_{2m,\ 2k}$$

For $(x,y) \varepsilon D$, it is obvious that

$$\left(x^{2m}-y^{2m}\right)\left(x^{2k}-y^{2k}\right) > 0$$

almost everywhere in D.

Hence,

$$I_{2(m+k),0} + I_{0,2(m+k)} > I_{2m,2k} + I_{2k,2m}$$

Applying the first propositon

$$I_{2(m+k),0} > I_{2m,2k}$$

Corollary: $I_{40} > I_{22}$ and $I_{60} > I_{42}$.

Proposition: If D is fully symmetric such that $I_{n,0} < \infty$; and i, j, and n are even such that $i + j \leq n$, then $0 < I_{ij} < \infty$.

Define

$$B \equiv \{(x,y) \varepsilon R^2 : 0 \leq |x|,\ |y| \leq 1\}$$

$$D_1 \equiv D \cap B$$

$$D_2 \equiv D \cap cB$$

Clearly D_1 and D_2 are fully symmetric and measurable. As $m(D_1) \leq 4$, and $x^i y^j$ is bounded on D_1, for $i + j \leq n$

$$\int_{D_1} x^i y^j \, dxdy < \infty$$

To show $I_{ij} < \infty$, we need only show

$$\int_{D_2} x^i y^j \, dxdy < \infty \quad \text{for} \quad i + j \leq n$$

If $m(D_2) = 0$, we are done. If $m(D_2) > 0$, we have for every $(x,y) \in D_2$, that $|x| > 1$, and thus

$$\int_{D_2} dxdy < \int_{D_2} x^2 dxdy < \ldots < \int_{D_2} x^n dxdy \leq I_{n,0} < \infty$$

Thus, I_{00}, I_{20}, $I_{40}, \ldots, I_{n0} < \infty$, and by the first proposition, I_{02}, $I_{04}, \ldots,$ $I_{0n} < \infty$.

We are left with the cases where $i, j \neq 0$. D_2 is fully symmetric. The previous proposition applies as $i, j \neq 0$. For $i + j \leq n$

$$\int_{D_2} x^i y^j \, dxdy < \int_{D_2} x^{i+j} < \infty$$

As $m(D) > 0$, and for i and j even such that $i + j \leq n$, we have $x^i y^j > 0$ almost everywhere. We conclude that

$$\int_D x^i y^j > 0$$

To justify the existence of some quadrature formula parameters in future sections, a knowledge of several integral inequalities is necessary. Rather than prove some or all of the following inequalities, we list the inequalities below and refer the reader to section II and the appendix of AFWL-TR-71-162 (ref. 4). The inequality $2I_{20}^2 < I_{00}(I_{40} + I_{22})$ is proven on page 30.

If D is fully symmetric and $I_{40} < \infty$, then

$$I_{20}^2 < I_{00} I_{40}$$

and

$$2I_{20}^2 < I_{00}(I_{40} + I_{22})$$

If D is fully symmetric and $I_{60} < \infty$, then

$$I_{22}^2 < I_{20} I_{42}$$

$$I_{40}^2 < I_{20} I_{60}$$

$$I_{20}^3 < I_{00}^2 I_{60}, \quad I_{22}^3 < I_{00} I_{42}^2$$

$$I_{22}^3 < I_{20} I_{40} I_{42}$$

$$I_{40}^3 < I_{00} I_{60}^2$$

$$(I_{40} + I_{40})^2 < I_{20}(I_{60} + 3I_{42})$$

$$2(I_{40} - I_{22})^3 < I_{00}(I_{60} - I_{42})^2$$

and

$$(I_{40} - I_{22})^2 < (I_{60} - I_{42})(I_{20} - I_{22}^2/I_{42})$$

Problem: Create a formula for the moments over a square with corners at $(\pm d, \pm d)$. Hint: see formula for d=1 in last part of section II.

Problem: Create a formula for the moments over a disc of radius h.

Problem: Calculate I_{22} over the four-point star made by connecting the points $(0, \pm 1)$, $(\pm 1, 0)$, $(\pm t, \pm t)$, $t < 1$.

Problem: Prove $I_{20}^2 < I_{00} I_{40}$. Hint: use Hölder's inequality.

SECTION IV

THE NUMBER OF POINTS

In this section we attempt to give the reader some idea of what forms quadrature formulas cannot take. The minimum number of points a formula can use is discussed, as well as where these points can be placed. Problems involving the nonexistence of certain quadrature formulas are given at the end of this section.

We have already seen the minimum number of points for various degrees of polygonal accuracy in table II. For fully symmetric regions, 1-, 3-, 4-, and 7-points are needed for polygonal accuracies 1, 2, 3, and 5, respectively. If complex parameters are allowed, 12 points are needed for a seventh degree formula over any fully symmetric region (ref. 7). But if real formula parameters are demanded, the minimum number may vary with the type of region. For some regions, such as the square or the disc, 12-point formulas are known. Discussions between Franke of the Naval Postgraduate School and this author have lead to the conjecture that no 12-point formula with strictly real parameters exists for some 4-point star region of the type described in the next to the last problem of the previous section. Formulas containing 12 and 13 points are discussed in AFWL-TR-71-162 (ref. 4) and a paper by Dr. Franke (ref. 7). The minimum number of points for quadrature formulas over fully symmetric regions is covered in more detail in section III of AFWL-TR-71-162.

Formula point placement is the next topic. Proofs of the propositions to follow are variations of the same method. Therefore, only a few proofs are given. Below we use the term connected. By D connected we mean that for any two points in D we can draw some line from one point to the other point, and the line will still be in D. The more formal definition can be used, but for simplicity we avoid it.

<u>Proposition</u>: Let D be an arbitrary, connected region of positive measure over which there exists a $s + 2$-point quadrature formula of third-degree polygonal accuracy, $s = 2, 3, \ldots$. Then s of the $s + 2$ points cannot lie outside of the interior of D in such a way that a line $L(x,y) = 0$, $L(x,y) \in P$, (x,y) passing through these s points does not intersect the interior of D.

Proposition: Let D be as before. If there exists a $s + 5$-point quadrature formula of fifth-degree polynomial accuracy over D, $s = 2, 3, \ldots$, then s of the $s + 5$ points cannot lie outside the interior of D in such a way that a line $L(x,y) = 0$, $L(x,y) \in P_1(x,y)$, passing through the s points does not intersect the interior of D.

Problem: State a similar proposition for quadrature formulas of seventh-degree polynomial accuracy. Hint: proof of this proposition is given below.

Proof by contradiction: Assume s of the points $\{(x_i,y_i)\}_{i=10}^{9+s}$ do lie on a line $L(x,y)$ outside of the interior of D. Since D is connected, $L(x,y)$ has the same sign almost everywhere on D; that is everywhere on D except over a subset of B of D whose measure, or area, is zero. Without loss of generality $L(x,y) > 0$ almost everywhere on D. Through the remaining 9 points $\{(x_i, y_i)\}_{i=1}^{9}$, we can construct a cubic $r(x,y) \in P_3(x,y)$ such that $r(x_i,y_i) = 0$, $i = 1, 2, \ldots, 9$. Then $r^2(x,y)$, $L(x,y) > 0$ almost everywhere on D. As $r^2(x,y)L(x,y) \in P_7(x,y)$, and $m(D) > 0$,

$$0 < \int_D r^2(x,y)\, L(x,y)\, dxdy = \sum_{i=1}^{9+s} w_i r^2(x_i,y_i)\, L(x_i y_i) = 0$$

which is the needed contradiction.

Proposition: Let D be as before and $n \in N$. If there exists a quadrature formula of polygonal accuracy $4n$ over D, then the points of the formula cannot lie on only n ellipses.

Proposition: Let D be as before and $n \in N$. Assume there exists a quadrature formula of polygonal accuracy $4n + 2$ over D. If one of the quadrature formula points is $(0,0)$, then the remaining points cannot lie on only n ellipses.

Proof by contradiction: Assume the remaining points lie on the ellipses

$$x^2/a_i^2 + y^2/b_i^2 = 1 \quad \text{for} \quad i = 1, \ldots, n$$

Then

$$q(x,y) = \prod_{i=1}^{n} \left(\frac{x^2}{a_i^2} + \frac{y^2}{b_i^2} - 1\right)^2 \in P_{4n}(x,y)$$

and $q(x,y)$ is positive almost everywhere on D and $q(x,y)$ is zero at every point of each of the n ellipses. Thus,

$$0 < \int_D x^2 q(x,y)\,dx\,dy = \sum_{i=1}^{m} w_i x_i^2 q\,(x_i,y_i) = 0$$

and we have the necessary contradiction.

For those readers who are serious about learning two-dimensional quadrature formulas, it is time to stop reading and start working simple problems. Applying the words of the noted educational psychologist, John Dewey, "We learn what we do" (ref.8), the way to learn to apply quadrature formulas is to apply them. Several problems are provided to assist the reader in gaining experience with quadrature formulas and their application.

Problem: Create a one-point quadrature formula having polygonal accuracy one for any planar region D where $m(D) > 0$.

Answer: $w = I_{00} = m(D)$, $x_0 = I_{10}/w$, and $y_0 = I_{01}/w$

Problem: Can a quadrature formula of polygonal accuracy 4 over a fully symmetric region have all of its points on a circle? Hint: use the fourth proposition given in this section. The answer is no.

Problem: Prove there does not exist a one-point quadrature formula of polygonal accuracy 2 for fully symmetric regions.

Solution: Let (a,b) be the one point with weight w. Then equation (1) takes the form:

$$w = I_{00} = m(D) > 0$$

$$wa = I_{10} = 0$$

$$wb = I_{01} = 0$$

$$wa^2 = I_{20} > 0$$

$$wb^2 = I_{02} > 0$$

$$wab = I_{11} = 0$$

Finding a contradiction is left to the reader.

Problem: Prove there does not exist a two-point quadrature formula of polygonal accuracy 2 for fully symmetric regions. Hint: expand previously given solution, $w_1 + w_2 = I_{00}$, etc., and reduce the equations to $w_2cd\ (w_2/w_1 + 1) = 0$.

Problem: Construct a quadrature formula of polygonal accuracy 2 for the square with corners at $(\pm 1,\pm 1)$ using the points (a,-b), (b,-a), and (-c,-c) where all of the weights are equal.

Answer: $a,\ b = (1/3 \pm 1/2\ \sqrt{3})^{1/2}$, $c = 1/\sqrt{3}$, $w = 4/3$

Problem: If the three points in the above problem are connected with straight lines, is the resulting triangle equilateral? The answer is no.

Problem: Find a 3-point quadrature formula of polygonal accuracy 2 for fully symmetric regions. Can the 3 points be colinear? The answer is no. Hint: consider $\int_D L^2\ (x,y)dxdy$.

Problem: Can one construct a four-point quadrature formula of polygonal accuracy 3 for fully symmetric regions using the points $(\pm a,\ \pm b)$ where $a \neq b$. The answer is no.

Problem: Apply any quadrature formula that you have created thus far to the calculation of an integral of a first-degree polynomial, a fourth-degree polynomial, and a more complex expression of your own choosing.

Problem: Construct a 12-point, fifth-degree quadrature formula with equal weights for the square with corners at $(\pm 1,\ \pm 1)$ that uses the following points: $(\pm a,\pm b)$, $(\pm b,\pm a)$, $(\pm r,\ 0)$, and $(0,\pm r)$. Are the first eight points within the square? The answer is no. $a \approx 0.22$ and $b \approx 1.62$.

Review: The following points should have been learned:

(1) Creating quadrature formulas of polygonal accuracy n is a matter of pairing components of both sides of

$$\int_D p(x,y)\ dxdy = \sum_{k=1}^{m} w_k p(x_k,y_k), \qquad p \in P_n$$

to obtain

$$I_{ij} = \sum_{k=1}^{m} w_k x_k^i y_k^j$$

for $0 \leq i + j \leq n$, which are then solved for the parameters of w_k, x_k, and y_k.

(2) For real formula parameters, points should be symmetrically placed so that when i or j are odd, we have

$$I_{ij} = 0 \text{ and } 0 = \sum_{k=1}^{m} w_k x_k^i y_k^j$$

(3) Weights for symmetrically placed points should be equal.

SECTION V

FORMULAS OF THIRD-DEGREE ACCURACY

Formulas of third-degree accuracy are considered in this section. To obtain a quadrature formula of third-degree polygonal accuracy, we search for solutions to equation (1). A solution resulting in a near minimum quadrature formula is described. Advantages of this near minimum formula are discussed. The effects of negative weights are considered. The formula is applied.

Theorem: Third-degree, 5-point formula. Let $p(x,y) \in P_3(x,y)$. If D is fully symmetric and if for every i and j such that $0 \leq i + j \leq 3$ we have $0 \leq I_{ij} < \infty$, then

$$\int_D p(x,y)\,dxdy = \left(I_{00} - \frac{2I_{20}}{R^2}\right) p(0,0)$$

$$+ \frac{I_{20}}{2R^2}\left[p(\mu,\nu) + p(-\mu,-\nu) + p(\nu,-\mu) + p(-\nu,\mu)\right]$$

for any $R > 0$, $\mu, \nu \geq 0$ such that $\mu^2 + \nu^2 = R^2$

Proof: Consider the points and weights

weight	A_2	A_1	A_1	A_1	A_1
x-coord	0	μ	$-\mu$	ν	$-\nu$
y-coord	0	ν	$-\nu$	μ	μ

Equation 1 yields

$$I_{ij} = 0 = \sum_{k=1}^{5} w_k x_k^i y_k^j$$

if i or j is odd, and

$$4A_1 + A_2 = I_{00}$$

$$2A_1 (\mu^2 + \nu^2) = I_{20}$$

We have two equations in four unknowns. There are several ways to solve these equations. All of the possible solutions can be described if we define $R^2 \equiv \mu^2 + \nu^2$ and solve for A_1 and A_2. This yields

$$A_1 = I_{20}/2R^2$$

$$A_2 = I_{00} - 2I_{20}/R^2$$

The user now has several choices:

(1) He can choose any one point, except (0,0), in the plane as an evaluation point. Once this point is chosen, the positions of the other 3 points, the weights, and R are determined.

(2) He can make $A_2 = 0$ and have a 4-point formula. In this case $0 = A_2 = I_{00} - 2I_{20}/R^2$ would yield $R^2 = 2I_{20}/I_{00}$. He is still free to place one of the evaluation points on a circle of radius R about (0,0).

(3) He can make all weights equal. In this case

$$I_{20}/2R^2 = A_1 = A_2 = I_{00} - 2I_{20}/R^2$$

would yield

$$I_{20}(1/2 + 2)/R^2 = I_{00}$$

and

$$R^2 = 5I_{20}/2I_{00}$$

(4) He can avoid negative weights. $A_1 > 0$ is obvious. By definition

$$A_2 = I_{00} - 2I_{20}/R^2 > 0$$

which is equivalent to

$$I_{00} > 2I_{20}/R^2$$

which is equivalent to

$$R^2 > 2I_{20}/I_{00}$$

(5) He can use the formula in several different ways to obtain a variety of approximations: rotate the points, change the radius of the circle, etc.

Negative weights cause problems. For example, consider

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\ dxdy$$

where the error in f(x,y) is always less than or equal to Δ. If the previous 4-point formula is used

$$\sum_{k=1}^{4} A\ f(x_k,y_k) \qquad \text{where } A = 1$$

approximates the integral and the maximum error is 4Δ. But if R = 1/3, we have $A_1 = 6$ and $A_2 = -20$. The formula becomes

$$-20f(0,0) + \sum_{i=1}^{4} 6f(x_i,y_i)$$

which has a maximum error of 44Δ. This is 11 times the previous estimate.

In tables III, IV, and V, we give the results of applications of the given formulas to the calculation of two integrals. In table V, we show the results of applying quadrature formulas in a different way. The square with corners at $(\pm 1,\pm 1)$ is divided up into two fully symmetric regions having equal area. One region is a smaller square about (0,0). The second region is the squared doughnut that results from removing the smaller square from the larger square. Variations of the given quadrature formula are applied to both regions and summed to obtain an approximation of the integral.

Table III

APPLICATION OF THIRD-DEGREE FORMULAS

No. of Pts	Formula Parameters	$\int_{-1}^{1}\int_{-1}^{1}(\lvert x\rvert+\lvert y\rvert)\,dxdy$	Error
4	$\nu = 0$	3. 265 986 323	0. 734 013 677
4	$\nu = \mu$	4. 218 802 152	0. 218 802 152
4	$\nu = 2/9$	4. 031 585 694	0. 031 585 694
5	$\nu = 0$ $R = 1$	2. 666 666 666	1. 333 333 333
5	$\nu = \mu$ $R = 1$	3. 771 236 166	0. 228 763 834
5	$\nu = 9/10$ $R = 95/100$	3. 765 138 522	0. 234 861 478

Table IV

SECOND APPLICATION OF THIRD-DEGREE FORMULAS

No. of Pts	Formula Parameters	$\int_{-1}^{1}\int_{-1}^{1}\exp(x+y)\,dxdy$	Error
4	$\nu = 0$	5. 409 073 240	0. 115 318 142
4	$\nu = \mu$	5. 488 224 960	0. 036 466 422
4	$\nu = 2/9$	5. 430 784 341	0. 093 607 041
5	$\nu = 0$ $R = 1$	4. 954 375 769	0. 570 015 613
5	$\nu = \mu$ $R = 1$	5. 570 911 408	0. 046 520 026
5	$\nu = 9/10$ $R = 95/100$	5. 506 711 524	0. 017 679 858

Table V

DOUBLE APPLICATION OF THIRD-DEGREE FORMULAS

No. of Pts	Formula Parameters	$\int_{-1}^{1}\int_{-1}^{1} \exp(x+y)dxdy$	Error
8	$\nu = 0/\nu = 0$	5. 428 857 356	0. 095 534 026
8	$\nu = 0/\nu = \mu$	5. 520 879 643	0. 003 511 739
8	$\nu = 0/\nu = \mu\sqrt{3}$	5. 497 864 241	0. 026 527 141
8	$\nu = \mu/\nu = 0$	5. 438 429 579	0. 085 961 803
8	$\nu = \mu/\nu = \mu$	5. 530 451 866	0. 006 060 484
8	$\nu = \mu/\nu = \mu\sqrt{3}$	5. 507 436 464	0. 016 954 918
8	$\nu = \mu\sqrt{3}/\nu = 0$	5. 436 036 406	0. 088 354 976
8	$\nu = \mu\sqrt{3}/\nu = \mu$	5. 528 058 693	0. 003 667 311
8	$\nu = \mu\sqrt{3}/\nu = \mu\sqrt{3}$	5. 505 043 291	0. 019 348 091

SECTION VI

FORMULAS OF FIFTH-DEGREE ACCURACY

Two formulas are discussed in this section. The first was published by Radon (ref. 2) in 1948. The second is a near-minimum point formula from AFWL-TR-71-162.

<u>Theorem</u>: Fifth-degree, 7-point formula (Radon). Let $p(x,y) \in P_5(x,y)$. If D is fully symmetric and if for every i and j such that $0 \le i + j \le 5$ implies $I_{ij} < \infty$, then

$$\int_D p(x,y)\,dxdy = A_1 \sum p(\pm\lambda,0) + A_2 \sum p(\pm\mu,\pm\nu) + A_3 p(0,0)$$

where all $A_i > 0$, and

$$\mu = (I_{22}/I_{20})^{1/2},\ \nu = (I_{40}/I_{20})^{1/2}$$

$$\lambda = \left([I_{40} + I_{22}]/I_{20}\right)^{1/2}$$

$$A_2 = I_{20}^2/4I_{40}$$

$$A_1 = I_{20}^2(I_{40} - I_{22})/2I_{40}(I_{40} + I_{22})$$

$$A = I_{00} - 2I_{20}^2/(I_{40} + I_{22})$$

Consider the points

weight	A_1	A_1	A_2	A_2	A_2	A_2	A_3
x-coord	λ	$-\lambda$	μ	μ	$-\mu$	$-\mu$	0
y-coord	0	0	ν	$-\nu$	ν	$-\nu$	0

Applying equation (1) we find the following equations need to be satisfied:

$$2A_1 + 4A_2 + A_3 = I_{00}$$

$$2A_1\lambda^2 + 4A_2\mu^2 = I_{20}$$

$$4A_2\nu^2 = I_{20}$$

$$2A_1\lambda^4 + 4A_2\mu^4 = I_{40}$$

$$4A_2\nu^4 = I_{40}$$

$$4A_2\mu^2\nu^2 = I_{22}$$

By division $\mu^2 = I_{22}/I_{20}$, and $\mu^2/\nu^2 = I_{22}/I_{40}$, and thus $\nu^2 = I_{40}/I_{20}$. From the third equation, $A_2 = I_{20}^2/4I_{40} > 0$. From the middle four equations,

$$2A_1\lambda^4 = 4A_2(\mu^2 + \nu^2)\ (\mu^2 - \nu^2)$$

$$2A_1\lambda^2 = 4A_2(\mu^2 - \nu^2)$$

Thus,

$$\lambda^2 = \mu^2 + \nu^2 = (I_{40} + I_{22})/I_{20}$$

Now we have $4A_2\mu^2 = I_{20}I_{22}/I_{40}$. Solving the second equation for A_1 we obtain

$$A_1 = I_{20}^2(I_{40} - I_{22})/2I_{40}(I_{40} + I_{22})$$

By the first corollary of section III we have $I_{40} > I_{22}$, and thus $A_1 > 0$. In a similar way we can obtain A_3. The first equation is solved for A_3. Known expressions for A_1 and A_2 are substituted into the resulting expression to obtain the claimed expression for A_3. To prove that $A_3 > 0$, we must show that

$$2I_{20}^2 < I_{00}(I_{40} + I_{22})$$

We start with

$$2I_{20}^2 = 2\left[\int_D x^2 dxdy\right]^2 = (1/2)\left[\int_D 2x^2 dxdy\right]^2$$

$$= (1/2)\left[\int_D (x^2 + y^2) dxdy\right]^2 \quad \text{by } I_{20} = I_{02}$$

$$< (1/2)\int_D 1^2 dxdy \int_D (x^2+y^2)^2 dxdy \quad \text{by Hölder's inequality}$$

$$< I_{00}(I_{40} + I_{22}) \quad \text{by expanding } (x^2 + y^2)^2$$

As examples in applying Radons formula we have

	Error
$\int_{-1}^{1}\int_{-1}^{1}(\lvert x\rvert+\lvert y\rvert)\,dxdy \approx 3.004\ 326\ 529$	0. 995 673 471
$\int_{-1}^{1}\int_{-1}^{1}\exp(x+y)\,dxdy \approx 5.521\ 576\ 981$	0. 002 814 401

Problem: Find another fifth-degree, 7-point formula, due to Radon, by using the previous method with the following points: $(0,\pm\lambda)$, $(\pm\mu,\pm\nu)$, $(0,0)$.

Theorem: Fifth-degree, 9-point formula. Let $p(x,y) \in P_5(x,y)$. If D is fully symmetric and $I_{ij} < \infty$ for each i and j such that $0 \leq i + j \leq 5$ then

$$\int_D p(x,y)dxdy = A_1 \sum p(\pm R,\pm R) + A_3 p(0,0)$$

$$+ A_2 \sum [p(\pm r,0) + p(0,\pm r)]$$

for

$$R^2 > I_{22}/I_{20}$$

where

$$A_1 = I_{22}/4R^4$$

$$A_2 = (I_{20} - I_{22}/R^2)\ /2(I_{40} - I_{22})$$

$$A_3 = I_{00} - \frac{2I_{22}}{I_{40}-I_{22}}\left[\frac{I_{20}^2}{I_{22}} - \frac{2I_{20}}{R^2} + \frac{I_{40}+I_{22}}{2R^4}\right]$$

$$r = \left[(I_{40} - I_{22})/(I_{20} - I_{22}/R^2)\right]^{1/2}$$

Consider the points and weights

weight	A_1	A_1	A_1	A_1	A_2	A_2	A_2	A_2	A_3
x-coord	R	R	-R	-R	r	-r	0	0	0
y-coord	R	-R	R	-R	0	0	r	-r	0

Equation (1) assumes the form:

$$4A_1 + 4A_2 + A_3 = I_{00}$$

$$4A_1R^2 + 2A_2r^2 = I_{20}$$

$$4A_1R^4 + 2A_2r^4 = I_{40}$$

$$4A_1R^4 = I_{22}$$

As we have four equations in five variables, solving will be done in terms of R. An expression for A_1 in terms of R is immediate from the last equation. With this we can obtain expressions for $2A_2r^2$ and $2A_2r^4$ from the middle two equations. Division yields

$$r^2 = (I_{40} - I_{22})/(I_{20} - I_{22}/R^2)$$

We need $r^2 > 0$. That is we need $R^2 > I_{22}/I_{20}$. This is a restriction on R. The second equation can be solved for A_2, and the first equation can be solved for A_3.

The user now has several options with this near-minimum point formula. He can make $A_3 = 0$ or $A_1 = A_2$. He can have $R = r$ or $2R^2 = r^2$. As examples of applications of this formula we have

Error

$$\int_{-1}^{1}\int_{-1}^{1}(|x|+|y|)\, dxdy \approx 3.377\ 663\ 441 \qquad 0.\ 622\ 336\ 559$$

$$\int_{-1}^{1}\int_{-1}^{1}\exp(x+y)\, dxdy \approx 5.521\ 568\ 727 \qquad 0.\ 002\ 822\ 655$$

Having two fifth-degree quadrature formulas, we again divide the square with corners at $(\pm1,\pm1)$ into two fully symmetric regions and apply each formula to each region to calculate

$$\int_{-1}^{1}\int_{-1}^{1}(|x|+|y|)\, dxdy$$

Table VI

DOUBLE APPLICATION OF FIFTH-DEGREE FORMULAS

Formula Used		Value	Error
Outside Region	Inside Region		
first	first	3. 809 600 373	0. 190 399 627
first	second	3. 882 538 348	0. 117 461 652
second	first	3. 856 171 924	0. 143 828 076
second	second	3. 929 109 899	0. 070 890 101

The following problems apply to the previously discussed 9-point formula.

Problem: Are there formulas where $R < r$?
The answer is yes. Find one.

Problem: From the equations obtained by applying equation (1) to the given points and weights, derive expressions for A_2 and A_3.

Problem: Find the R for which $A_1 = A_2$.

Problem: Find R, if $R = r$.
The answer is I_{40}/I_{20}.

Problem: If $2R = r$, all of the points, except (0,0) can be put on what type of geometrical figure?
The answer is: Circle, $A_3 = I_{00} - 2I_{20}^2/(I_{40} + I_{22})$.

Problem: Approximate the integral of the x^7 over the square with corners at $(\pm 1, \pm 1)$. Why is the answer zero? Would you always get zero if x^7 was replaced by $x^i y^j$, where i or j was odd?
Answer: 0. Symmetry of points, Yes

Problem: With two variations of the formula, approximate the integrals of x^6 and x sin y where D is the square with corners at $(\pm 1, \pm 1)$.

Problem: A scientist needs a double integral to fifth-degree polygonal accuracy over a disc of radius 4. The cost of obtaining integrand data over the disc is $2r^3 - 15r^2 + 36r + 55$ dollars per point for any point a distance r from the center. What is the cost of using Radon's formula given in this section? Is there a cheaper way using the 9-point formula given in this section? What is the minimum cost?

Problem: An engineer needs to calculate a double integral over a square. The square is 100 units by 100 units and integrand values can only be obtained at coordinates on the square whose x and y components are integral multiples of one unit. Find a formula of fifth-degree accuracy for the whole square. Can the engineer use the previously given 9-point formula if the square is divided up into four 50-by-50 squares?

Readers who are interested in formulas of seventh-degree accuracy may read AFWL-TR-71-162 (ref. 4) or Franke (ref. 7).

SECTION VII

A DIFFERENT BASIS FOR ACCURACY

Thus far all of our formulas have been accurate for polynomials of given degrees. It is instructive to consider other bases for numerical integration formulas. In this section different bases are considered formally and in problems. If a formula user knows that his integrand is better approximated with exponentials or trigonometric polynomials, he may wish to use a formula with such an accuracy basis.

<u>Theorem</u>: Consider any polynomial p of the form

$$c_0 + c_{10}x + c_{01}y + c_{11}xy + c_{30}x^3 + c_{21}x^2y + c_{12}xy^2 + c_{03}y^3 + c_{22}x^2y^2$$

Then for D fully symmetric such that

$$\int_D p(x,y)\,dxdy < \infty$$

$$\int_D p(x,y)\,dxdy = (I_{00}/4)\left[p(\mu,\nu) + p(-\mu-\nu) + p(-\nu,\mu) + p(\nu,-\mu)\right]$$

where $\mu^2\nu^2 = I_{22}/I_{00}$.

<u>Proof</u>: Consider the weights and points:

weight	w	w	w	w
x-coord	μ	$-\mu$	$-\nu$	ν
y-coord	ν	$-\nu$	μ	$-\mu$

By matching components of p(x,y) from both sides of the formula, we have that zero equals zero for seven of the components. For the remaining two we have

$$4w = I_{00}$$

$$4w\mu^2\nu^2 = I_{22}$$

As we have two equations in three variables, one of the variables can be arbitrary. Obviously $w = I_{00}/4$, and we must have μ and ν such that $\mu\nu = I_{22}/I_{00}$.

Problem: Redo the above theorem when x^2 and y^2 components are added to p.

Problem: Let p be a polynomial made up of any combination of $x^i y^j$ such that $0 \le i + j \le 5$, excluding x^4 and y^4. Derive the following formula for D fully symmetric such that

$$\int_D p(x,y)\, dxdy < \infty$$

$$\int_D p(x,y)\, dxdy = (I_{00} - I_{20}^2/I_{22})p(0,0)$$

$$+ (I_{20}^2/4I_{22}) \sum p\left[\pm(I_{22}/I_{20})^{1/2}, \pm (I_{22}/I_{20})^{1/2}\right]$$

Is the first weight always positive?
The answer is no.

Problem: Find a region where $I_{22}I_{00} > I_{20}^2$. Fine a region where $I_{22}I_{00} < I_{20}^2$.
Hint: try the square first.

Applying these two formulas to

$$\int_{-1}^{1}\int_{-1}^{1} \left(|x|+|y|\right) dxdy$$

we have the following values:

Formula	Value	Error
Theorem formula	4.90032	0.90032
Second formula	4.61879	0.61879

Problem: Develop a formula for any fifth-degree polynomial p that does not have a x^2y^2 component. Again D is fully symmetric.
Answer: If ($\pm r$, $\pm r$) and (0,0) are used as points, the formula is like the formula in the second problem of this section, where I_{22} is replaced by I_{40}.

Problem: Develop an eight-point formula for any polynomial p made up of components that are of degree 7 or less, excluding x^n and y^n for even n, which is accurate over fully symmetric regions.
Answer:

$$\int_D p(x,y)\, dxdy = \frac{I_{00}}{8}\left[\sum p(\pm\mu,\pm\nu) + \sum p(\pm\nu,\pm\mu)\right]$$

where

$$\mu,\ \nu = \left[\frac{I_{42}}{I_{22}} \pm \left(\frac{I_{42}^2}{I_{22}^2} - \frac{I_{22}}{I_{00}}\right)^{1/2}\right]^{1/2}$$

if ($\pm\mu$,$\pm\nu$), ($\pm\nu$,$\pm\mu$) are used as the points.

Problem: Let D be fully symmetric. Develop a quadrature formula if the basis of accuracy is

a. sin x, sin y, cos x, cos y

b. sin x, sin 2x, sin 3x

c. sin x, sin y, cos x, cos y, sin 2x, sin 2y, cos 2x, cos 2y

d. e^x, e^y, 1

e. log x, x, e^x

f. $x^{1/2}$, $y^{1/2}$, $(xy)^{1/2}$

g. $x^{1/2}$, x, $x^{3/2}$, x^2

h. Make up your own basis

Answer for d: If D is the square with corners at (± 1,± 1), and our points are equally weighted and are at ($\pm a$,0) and (0,$\pm a$) then w = 1 and a = + $\cosh^{-1}$ (e -1 -1/e).

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Quadrature formulas for double integrals are described. Examples are used to give an idea of what should be proven about quadrature formulas. Proofs are simple and cover formulas of third- and fifth-degree polygonal accuracy. Several problems are provided and should be worked to gain some idea of why quadrature formulas work, and how they can be created or altered to suit ones' need.

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