Measurement Notes

Note 42

1 March 1993

Multiconductor-Transmission-Line Model of Balun and Inverter

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Abstract

This paper constructs a transmission-line model involving two conductors plus reference for a balun (coax to twin-line transition) and an inverter (coax to coax transition). This includes the effects of the external wave (or antenna mode) in a transmission-line approximation. A special diagonalization procedure is used for the product of the characteristic impedance matrix and the inverse of this matrix evaluated at a particular reference cross section. Using exponential interpolation for the eigenvalues to apply to the transition region, a closed-form solution to the wave propagation is obtained. Evaluation at high and low frequencies gives estimates of performance in terms of the geometric impedance factors involved.

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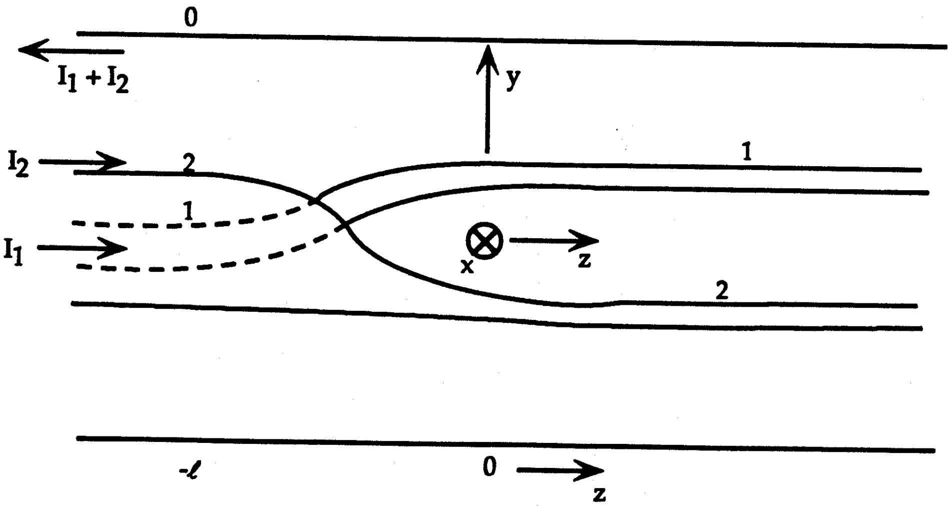
I. Introduction

A typical transition between a coaxial cable and a twinline is illustrated for $-\ell \leq z \leq 0$ in fig. 1.1A. This type of balun is used for converting between the two transmission lines over a broad band of frequencies. However, in converting the coax mode (between conductors 1 and 2) incident from the left both common and differential modes are launched to the right on the twin line. There is also a wave launched to the left outside the coax. While this is not strictly a transmission-line mode (and is sometimes referred to as an antenna mode) one can crudely model it as a transmission-line mode by inclusion of some outer reference conductor, here designated by 0 (with radius Ψ_0). Appropriate selection of Ψ_0 gives an impedance to this mode which can also be thought of as the common mode to the right. Now Ψ_0 can be chosen as something like a radian wavelength, which of course makes the impedance frequency dependent, but only logarithmically so. Furthermore, this model does not include the radiation losses. Nevertheless, this model can give some insight into the performance of this type of transition due to the analytical simplifications. In the limit of large external impedance ($\Psi_0 \to \infty$), then provided that the impedance for the remaining mode (coax $\to$ differential) is maintained independent of z, the balun works for all frequencies with no reflections (in the transmission-line approximation).

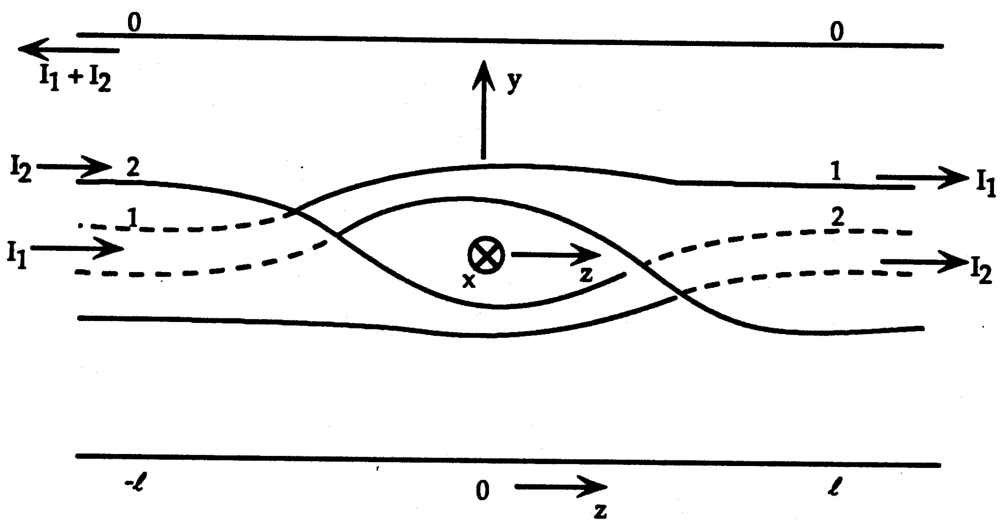
Extending the previous transition one can make a similar one for $0 \leq z \leq \ell$ which is like that to the left (rotation by π about the x axis), making an inverter as in fig. 1.1B. Now the idea is to transition the coaxial mode (for $z < -\ell$) to an inverted coaxial mode ($z > \ell$). Again if the external mode can be neglected ($\Psi_0 \to \infty$) this can be accomplished in an ideal sense (no reflections) provided the impedance of the remaining mode is made constant from one coax to the other (both coaxes with the same impedance).

Note that fig. 1.1 is schematic rather than a scaled drawing. In particular let us assume that the length (ℓ or 2ℓ) is much larger than the cross-section dimensions (which will be introduced later). So at any plane of constant z the conductors are approximately perpendicular to this plane. Then two-dimentional techniques can be used for the local per-unit-length inductance and capacitance matrices. The transmission-line formalism is accurate for radian wavelengths large compared to the cross section dimensions. (This restriction can be relaxed in special cases.) So one would like a range of frequencies where radian wavelengths are less than ℓ, but large compared to cross-section dimensions, for which the present results apply.

Coaxial cables are usually constructed with a dielectric insulator between inner and outer conductors, but air dielectric is sometimes used. For present purposes the outer medium is assumed the same as the inner one and is assumed lossless. Then as in [2, 3] both modes propagate at the same speed



A. Balun



B. Inverter

Fig. 1.1. Two Configurations of Transmission-Line Transitions

$$
\begin{aligned}
v &= [\mu\ \varepsilon]^{-\frac{1}{2}} \\
\mu &\equiv \text{permeability} \\
\varepsilon &\equiv \text{permittivity}
\end{aligned} \tag{1.1}
$$

which can also be the speed of light in vacuum in some cases. Note that the conductors are assumed perfect for the modes to have this speed. We then can define

$$
\begin{aligned}
\gamma &\equiv \frac{s}{v} \\
s &\equiv \text{Laplace transform variable (two-sided over time } t\text{) or complex frequency} \\
&\equiv \Omega + j\omega
\end{aligned} \tag{1.2}
$$

The resulting nonuniform multiconductor transmission line (NMTL) can be characterized by a 2 x 2 characteristic impedance matrix

$$
\begin{aligned}
\left(Z_{c_{n,m}}(z)\right) &= Z_w\left(f_{g_{n,m}}(z)\right) \equiv \left(Y_{c_{n,m}}(z)\right)^{-1} \\
Z_w &= \left[\frac{\mu}{\varepsilon}\right]^{\frac{1}{2}} \equiv \text{wave impedance of medium} \\
\left(f_{g_{n,m}}(z)\right) &= \left(f_{g_{n,m}}(z)\right)^T \equiv \text{geometric-factor matrix (real)}
\end{aligned} \tag{1.3}
$$

which is conveniently a function of only the coordinate z (frequency independent). Note that reciprocity assures that there are only three independent matrix elements. This is also related the per-unit-length inductance and capacitance matrices

$$
\begin{aligned}
\left(L'_{n,m}(z)\right) &= \mu\left(f_{g_{n,m}}(z)\right) \\
\left(C'_{n,m}(z)\right) &= \varepsilon\left(f_{g_{n,m}}(z)\right)^{-1}
\end{aligned} \tag{1.4}
$$

The propagation matrix assumes the simple form [2, 3]

$$\left(\tilde{\gamma}_{c_{n,m}}(s)\right) = \left[\left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(\tilde{Y}'_{n,m}(z,s)\right)\right]^{\frac{1}{2}} \text{ (positive real (p.r.) square root)}$$

$$= \left[\left(L'_{n,m}(z)\right) \cdot \left(C'_{n,m}(z)\right)\right]^{\frac{1}{2}} \qquad (1.5)$$

$$= \gamma\left(1_{n,m}\right)$$

Note that labelling of the conductors and the associated voltages $\left(\tilde{V}_n(z,s)\right)$ (with respect to the reference conductor labelled 0) and currents $\left(\tilde{I}_n(z,s)\right)$. As one progresses along the structure from left to right, 1 denotes the center conductor, but later the outer coaxial conductor of the inverter. Effectively the labels 1 and 2 exchange roles in going from $-\ell$ to ℓ. The coaxial mode is given by $\tilde{V}_1(-\ell,s) - \tilde{V}_2(-\ell,s)$ on the left, but $\tilde{V}_2(\ell,s) - \tilde{V}_1(\ell,s)$ on the right. This interchange of the roles of 1 and 2 can be incorporated in a symmetry constraint for the inverter in fig. 1.1B as

$$f_{g_{1,1}}(z) = f_{g_{2,2}}(-z)$$
$$f_{g_{1,2}}(z) = f_{g_{1,2}}(-z) \qquad (1.6)$$

Here the actual cross section as a function of z is not considered, but rather the geometric factor matrix is constructed in such a way that it is realizable, matches the conditions at $z = -\ell, 0, \ell$, and leads to analytic wave-propagation solutions.

For good transition performance we will need high external impedance. One way to achieve this is through external magnetic materials, say in place of an outer conductor at Ψ_0. A finite permeability of such a medium complicates the transmission-line analysis, but in the limit of infinite permeability one can consider a magnetic boundary at Ψ_0 along which no current can flow (giving $I_1 + I_2 = 0$) and reducing the problem to a single mode. This is equivalent to letting $\Psi_0 \to \infty$ in the present analysis. The use of such magnetic materials (chokes) can improve the performance of such transitions; this can be considered in other analyses.

II. Characteristic Impedance Matrices at z = -ℓ and z = +ℓ

As illustrated in fig. 2.1 the beginning (z = -ℓ) and end (z = +ℓ) of the transition has a triaxial structure with

$$0 < \Psi_a < \Psi_b < \Psi_o \leq \infty \tag{2.1}$$

Note the use of a and b subscripts which apply to 1 and 2 conductors respectively at z = -ℓ, but 2 and 1 conductors repsectively at z = +ℓ. Define, for use with the geometric-factor (and hence characteristic-impedance) matrix

$$f_g^{(in)} \equiv \frac{1}{2\pi} \ell n\left(\frac{\Psi_b}{\Psi_a}\right)$$

$$f_g^{(out)} \equiv \frac{1}{2\pi} \ell n\left(\frac{\Psi_o}{\Psi_b}\right) \tag{2.2}$$

$$f_g^{(in)} + f_g^{(out)} = \frac{1}{2\pi} \ell n\left(\frac{\Psi_o}{\Psi_a}\right)$$

Considering first the left end we have

$$\left(f_{g_{n,m}}(-\ell)\right) = \begin{pmatrix} f_g^{(in)} + f_g^{(out)} & f_g^{(out)} \\ f_g^{(out)} & f_g^{(out)} \end{pmatrix} \tag{2.3}$$

The coax mode incident between Ψ_a and Ψ_b on the transition (from the left) is characterized by $f_g^{(in)}$, but due to our definition of V_1 the $f_{g_{1,1}}$ parameter is referenced to Ψ_0. These parameters are readily generated by separately driving with the two currents and calculating the resulting voltages (two in each case). This matrix has the simple properties

$$\det\left(\left(f_{g_{n,m}}(-\ell)\right)\right) = f_g^{(in)} f_g^{(out)}$$

$$\mathrm{tr}\left(\left(f_{g_{n,m}}(-\ell)\right)\right) = f_g^{(in)} + 2 f_g^{(out)} \tag{2.4}$$

That this is a positive definite matrix (associated with a passive structure) is found by noting that it is Hermitian with real and positive eigenvalues.

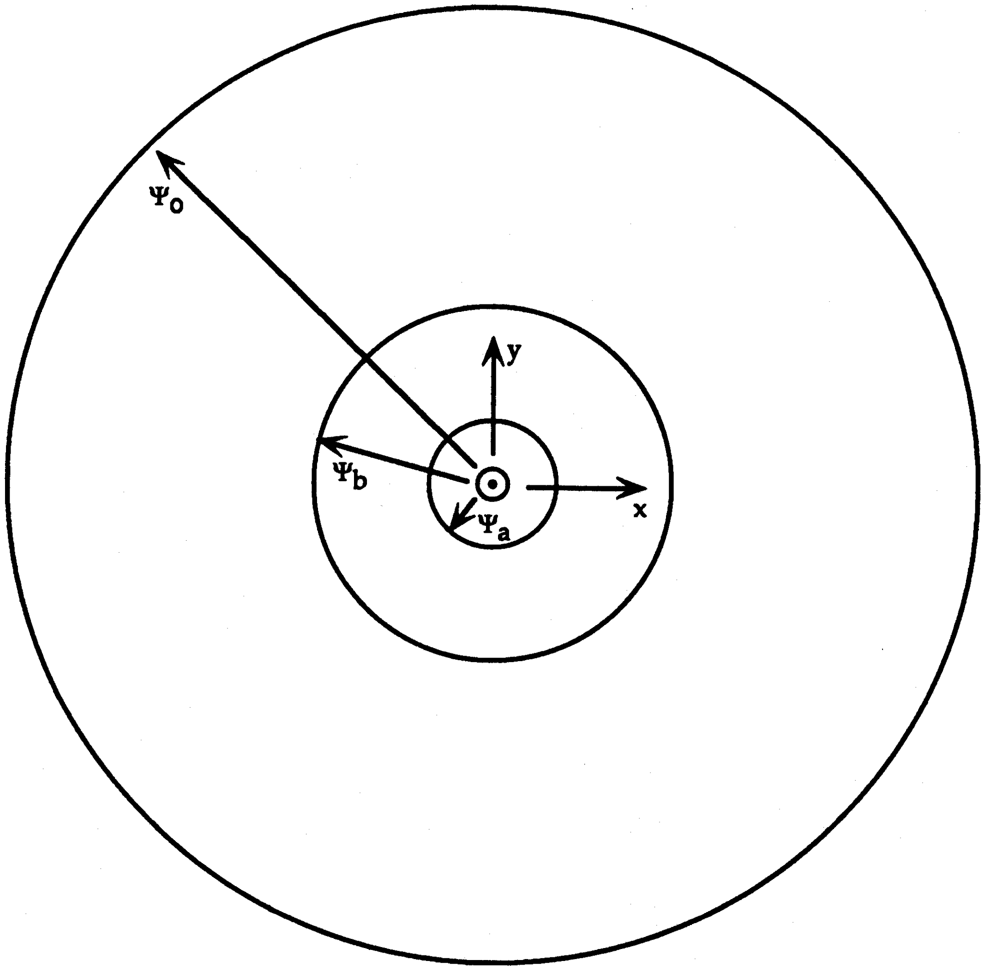


Fig. 2.1. Cross Section of Transition at $z = \pm \ell$

At the right end the corresponding matrix is

$$\begin{aligned}\left(f_{g_{n,m}}(\ell)\right) &= \begin{pmatrix} f_g^{(out)} & f_g^{(out)} \\ f_g^{(out)} & f_g^{(in)} + f_g^{(out)} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \left(f_{g_{n,m}}(-\ell)\right) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\end{aligned} \tag{2.5}$$

which is related to the left-end form by interchange of rows and interchange of columns, corresponding to reversal of the roles of the 1 and 2 indices. The determinant and trace are unchanged leaving this matrix positive definite like the previous one.

The special case of large external impedance is described by $\Psi_0 \to \infty$. At the left and right ends this corresponds to $f_g^{(out)} \to \infty$ with $f_g^{(in)}$ held constant.

III. Characteristic Impedance Matrix at z = 0: Differential and Common Modes

At z = 0 the transition assumes the symmetrical configuration shown in fig. 3.1. At this location the 1 and 2 voltages and currents are in identical conditions, giving an additional symmetry to the characteristic impedance matrix. This is treated in terms of differential and common modes for which the characteristic impedances are known to a good approximation [1, 7, 8]. Note that Ψ_0 can be allowed to vary as a function of z if desired, subject to the requirement $\Psi_0(-z) = \Psi_0(z)$ in the case of the inverter.

For the differential mode we define

$$\begin{aligned} \tilde{V}_d(z,s) &\equiv \tilde{V}_1(z,s) - \tilde{V}_2(z,s) \\ \tilde{I}_d(z,s) &\equiv \frac{1}{2}\left[\tilde{I}_1(z,s) - \tilde{I}_2(z,s)\right] \end{aligned} \tag{3.1}$$

which applies at $z = 0$ for the inverter and for $z \geq 0$ for the balun as in fig. 1.1. Under the same conditions the common mode is defined via

$$\begin{aligned} \tilde{V}_c(z,s) &\equiv \frac{1}{2}\left[\tilde{V}_1(z,s) + \tilde{V}_2(z,s)\right] \\ \tilde{I}_c(z,s) &\equiv \tilde{I}_1(z,s) + \tilde{I}_2(z,s) \end{aligned} \tag{3.2}$$

Note the factors of 1/2 used in the usual convention, related to the source driving current from conductor 2 into conductor 1 for the differential mode, and to the source driving current from conductor 0 into the parallel connection of conductors 1 and 2 in the common mode.

To convert between the d, c basis and the 1, 2 basis we have in matrix form

$$\begin{aligned} \begin{pmatrix} \tilde{V}_d(z,s) \\ \tilde{V}_c(z,s) \end{pmatrix} &= \begin{pmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \tilde{V}_1(z,s) \\ \tilde{V}_2(z,s) \end{pmatrix} \\ \begin{pmatrix} \tilde{I}_d(z,s) \\ \tilde{I}_c(z,s) \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \tilde{I}_1(z,s) \\ \tilde{I}_2(z,s) \end{pmatrix} \end{aligned} \tag{3.3}$$

Noting that these matrices have unity determinant we have the inverses in

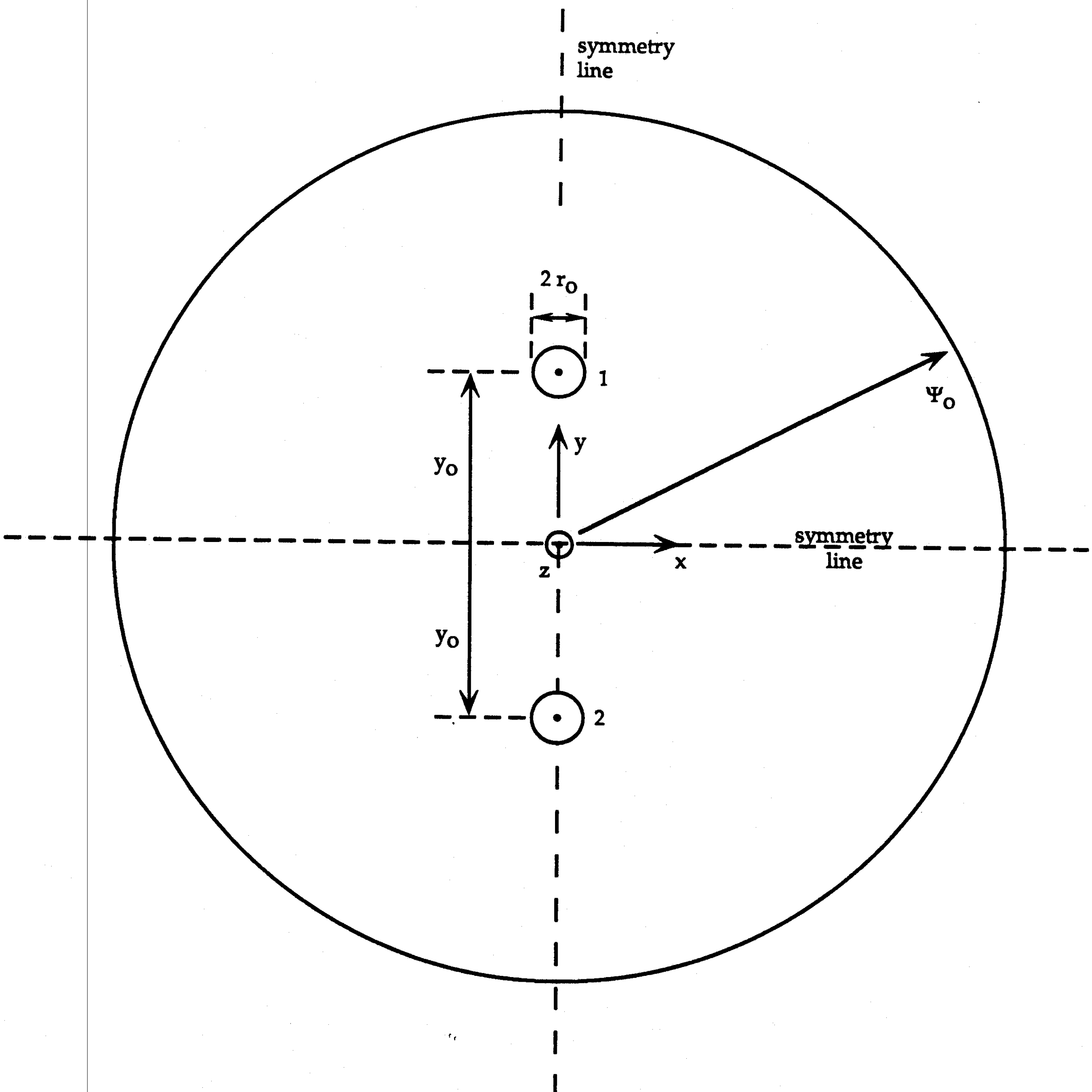


Fig. 3.1. Cross Section of Transition at z = 0

$$\begin{pmatrix}\tilde{V}_1(z,s)\\ \tilde{V}_2(z,s)\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & 1\\ -\frac{1}{2} & 1\end{pmatrix}\cdot\begin{pmatrix}\tilde{V}_d(z,s)\\ \tilde{V}_c(z,s)\end{pmatrix}$$

$$\begin{pmatrix}\tilde{I}_1(z,s)\\ \tilde{I}_2(z,s)\end{pmatrix} = \begin{pmatrix}1 & \frac{1}{2}\\ -1 & \frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}\tilde{I}_d(z,s)\\ \tilde{I}_c(z,s)\end{pmatrix} \tag{3.4}$$

The geometric-factor matrix appears in the per-unit-length inductance and capacitance matrices in (1.4) which appear in the telegrapher equations, as well as in the characteristic impedance matrix in (1.3). A convenient way to look at this is to consider a wave propagating only to the right (+z) on a uniform line such as for z > 0 in fig. 1.1A. In this case we have

$$\begin{pmatrix}\tilde{V}_1(z,s)\\ \tilde{V}_2(z,s)\end{pmatrix} = Z_w\begin{pmatrix}f_{g_{1,1}}(0) & f_{g_{1,2}}(0)\\ f_{g_{2,1}}(0) & f_{g_{2,2}}(0)\end{pmatrix}\cdot\begin{pmatrix}\tilde{I}_1(z,s)\\ \tilde{I}_2(z,s)\end{pmatrix}$$

$$\begin{pmatrix}\tilde{V}_d(z,s)\\ \tilde{V}_c(z,s)\end{pmatrix} = Z_w\begin{pmatrix}f_g^{(d)} & 0\\ 0 & f_g^{(c)}\end{pmatrix}\cdot\begin{pmatrix}\tilde{I}_d(z,s)\\ \tilde{I}_c(z,s)\end{pmatrix} \tag{3.5}$$

In the second of these matrix equations use (3.3) and (3.4) to convert from d, c to 1, 2 basis giving

$$\left(f_{g_{n,m}}(0)\right) = \begin{pmatrix}\frac{1}{2} & 1\\ -\frac{1}{2} & 1\end{pmatrix}\cdot\begin{pmatrix}f_g^{(d)} & 0\\ 0 & f_g^{(c)}\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ 1 & 1\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{4}f_g^{(d)} + f_g^{(c)} & -\frac{1}{4}f_g^{(d)} + f_g^{(c)}\\ -\frac{1}{4}f_g^{(d)} + f_g^{(c)} & \frac{1}{4}f_g^{(d)} + f_g^{(c)}\end{pmatrix} \tag{3.6}$$

which exhibits the symmetry on interchange of 1 and 2 indices (as well as reciprocity).

The diagonal form of $\left(f_{g_{n,m}}(0)\right)$ gives eigenvalues $f_g^{(d)}/2$ and $2f_g^{(c)}$ with corresponding normalized eigenvectors $2^{-\frac{1}{2}}(1,-1)$ and $2^{-\frac{1}{2}}(1,1)$ as one would expect from the symmetry. For later use we also have

$$\det\left(\left(f_{g_{n,m}}(0)\right)\right) = f_g^{(d)} f_g^{(c)}$$

$$\mathrm{tr}\left(\left(f_{g_{n,m}}(0)\right)\right) = \frac{1}{2} f_g^{(d)} + 2 f_g^{(c)}$$

$$\left(f_{g_{n,m}}(0)\right)^{-1} = \frac{1}{f_g^{(d)}\ f_g^{(c)}} \begin{pmatrix} \frac{1}{4} f_g^{(d)} + f_g^{(c)} & \frac{1}{4} f_g^{(d)} - f_g^{(c)} \\ \frac{1}{4} f_g^{(d)} - f_g^{(c)} & \frac{1}{4} f_g^{(d)} + f_g^{(c)} \end{pmatrix} \tag{3.7}$$

There is a constraint on the allowable values of $f_g^{(d)}$ and $f_g^{(c)}$. If we drive a current, say I_1, into one conductor and force $I_2 = 0$, then not only do we have a voltage V_1, but also a voltage V_2 of the same sign. The physical requirement has

$$\begin{aligned} f_{g_{1,1}}(0) &= f_{g_{2,2}}(0) \geq 0 \\ f_{g_{1,2}}(0) &= f_{g_{2,1}}(0) \geq 0 \end{aligned} \tag{3.8}$$

The second of these can also be viewed from a capacitance point of view. A positive charge on one conductor induces a positive potential on the second. From (3.6) we then have

$$\begin{aligned} \frac{1}{4} f_g^{(d)} + f_g^{(c)} &\geq 0 \\ -\frac{1}{4} f_g^{(d)} + f_g^{(c)} &\geq 0 \end{aligned} \tag{3.9}$$

The first of these comes from the realizability requirement of non-negative $f_g^{(d)}$ and $f_g^{(c)}$. The second imposes the additional realizability requirement

$$f_g^{(c)} \geq \frac{1}{4} f_g^{(d)} \geq 0 \tag{3.10}$$

As a limiting case consider the case of two identical coaxes of characteristic impedance $Z_c = Z_w f_g^{(0)}$ giving

$$f_g^{(d)} = 2 f_g^{(0)} \; , \; f_g^{(c)} = \frac{1}{2} f_g^{(0)}$$

$$f_g^{(c)} = \frac{1}{4} f_g^{(d)} \tag{3.11}$$

The special case of large external impedance can be interpreted here by large common-mode impedance. This corresponds to $f_g^{(c)} \rightarrow \infty$ with $f_g^{(d)}$ held constant.

IV. General Form of Solution

The telegrapher equations without sources are

$$\frac{\partial}{\partial z}\left(\tilde{V}_n(z,s)\right) = -s\left(L'_{n,m}(z)\right)\cdot\left(\tilde{I}_n(z,s)\right)$$
$$\frac{\partial}{\partial z}\left(\tilde{I}_n(z,s)\right) = -s\left(C'_{n,m}(z)\right)\cdot\left(\tilde{V}_n(z,s)\right) \tag{4.1}$$

Then, as in [4], form

$$\begin{pmatrix}\left(\tilde{V}_n(z,s)\right)\\ \left(Z_{c_{n,m}}(0)\right)\cdot\left(\tilde{I}_n(z,s)\right)\end{pmatrix} = \left(\left(\tilde{a}_{n,m}(z,s)\right)_{\upsilon,\upsilon'}\right) \odot \begin{pmatrix}\left(\tilde{V}_n(z,s)\right)\\ \left(Z_{c_{n,m}}(0)\right)\cdot\left(\tilde{I}_n(z,s)\right)\end{pmatrix}$$

$$\left(\left(\tilde{a}_{n,m}(z,s)\right)_{\upsilon,\upsilon'}\right) = -\begin{pmatrix}\left(0_{n,m}\right) & s\left(L'_{n,m}(z)\right)\cdot\left(Z_{c_{n,m}}(0)\right)^{-1}\\ s\left(Z_{c_{n,m}}(0)\right)\cdot\left(C'_{n,m}(z)\right) & \left(0_{n,m}\right)\end{pmatrix}$$
$$= -\gamma\begin{pmatrix}\left(0_{n,m}\right) & \left(f_{g_{n,m}}(z)\right)\cdot\left(f_{g_{n,m}}(0)\right)^{-1}\\ \left(f_{g_{n,m}}(0)\right)\cdot\left(f_{g_{n,m}}(z)\right)^{-1} & \left(0_{n,m}\right)\end{pmatrix}$$
$$= \gamma\begin{pmatrix}\left(0_{n,m}\right) & \left(f_{n,m}(z)\right)\\ \left(f_{n,m}(z)\right)^{-1} & \left(0_{n,m}\right)\end{pmatrix} \tag{4.2}$$
$$\left(f_{n,m}(z)\right) \equiv \left(f_{g_{n,m}}(z)\right)\cdot\left(f_{g_{n,m}}(0)\right)^{-1}$$

Here the supervectors (divectors) have a total of 4 elements and the supermatrices (dimatrices) are 4 x 4. The normalizing impedance matrix (which needs to be independent of z) is chosen as $\left(Z_{c_{n,m}}(0)\right)$, due to the symmetry in this problem.

The solution of this equation is cast in the form of the supermatrix differential equation

$$\frac{\partial}{\partial z}\left(\left(\tilde{u}_{n,m}(z,z_o;s)\right)_{\upsilon,\upsilon'}\right) = \left(\left(\tilde{a}_{n,m}(z,s)\right)_{\upsilon,\upsilon'}\right) \odot \left(\left(\tilde{u}_{n,m}(z,z_o;s)\right)_{\upsilon,\upsilon'}\right)$$
$$\left(\left(\tilde{u}_{n,m}(z_o,z_o;s)\right)_{\upsilon,\upsilon'}\right) = \left(\left(1_{n,m}\right)_{\upsilon,\upsilon'}\right) \text{ (boundary condition)} \tag{4.3}$$

The solution of this differential equation is referred to as a matrizant (or supermatrizant if one prefers). The columns are independent vector solutions of (4.2). Given the supervector (voltage/current) at $z = z_o$ we have

$$\begin{pmatrix} \left(\tilde{V}_n(z,s)\right) \\ \left(Z_{c_{n,m}}(0)\right) \cdot \left(\tilde{I}_n(z,s)\right) \end{pmatrix} = \left(\left(\tilde{u}_{n,m}(z,z_o;s)\right)_{v,v'}\right) \odot \begin{pmatrix} \left(V_n(z_o,s)\right) \\ \left(Z_{c_{n,m}}(0)\right) \cdot \left(\tilde{I}_n(z_o,s)\right) \end{pmatrix} \tag{4.4}$$

Note that z can be greater than or less than z_o. The matrizant has some important properties

$$\begin{aligned} \left(\left(\tilde{u}_{n,m}(z,z_o)\right)_{v,v'}\right) &= \left(\left(\tilde{u}_{n,m}(z,z_1)\right)_{v,v'}\right) \odot \left(\left(\tilde{u}_{n,m}(z_1,z_o)\right)_{v,v'}\right) \\ \left(\left(\tilde{u}_{n,m}(z_o,z)\right)_{v,v'}\right) &= \left(\left(\tilde{u}_{n,m}(z,z_o)\right)_{v,v'}\right)^{-1} \end{aligned} \tag{4.5}$$

which allow one to express the solution as in (4.4) in various ways, including breaking up the matrizant into products of various numbers of matrizants, each representing a portion of the NMTL of interest.

Considering the section for $-\ell \le z \le 0$ we have

$$\begin{aligned} \begin{pmatrix} \left(\tilde{V}_n(0,s)\right) \\ \left(Z_{c_{n,m}}(0)\right) \cdot \left(\tilde{I}_n(0,s)\right) \end{pmatrix} &= \left(\left(\tilde{u}^{(-)}_{n,m}(0,-\ell;s)\right)_{v,v'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n(-\ell,s)\right) \\ \left(Z_{c_{n,m}}(0)\right) \cdot \left(\tilde{I}_n(-\ell,s)\right) \end{pmatrix} \\ \begin{pmatrix} \left(\tilde{V}_n(-\ell,s)\right) \\ \left(Z_{c_{n,m}}(0)\right) \cdot \left(\tilde{I}_n(-\ell,s)\right) \end{pmatrix} &= \left(\left(\tilde{u}^{(-)}_{n,m}(-\ell,0;s)\right)_{v,v'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n(0,s)\right) \\ \left(Z_{c_{n,m}}(0)\right) \cdot \left(\tilde{I}_n(0,s)\right) \end{pmatrix} \\ \left(\left(\tilde{u}^{(-)}_{n,m}(-\ell,0;s)\right)_{v,v'}\right) &= \left(\left(\tilde{u}^{(-)}_{n,m}(0,-\ell;s)\right)_{v,v'}\right)^{-1} \end{aligned} \tag{4.6}$$

Here a - (minus) superscript is used to designate this left section of the transition, this having a special form of the geometric factor matrix, distinct from that on the right. Similarly for $0 \le z \le \ell$ we have (with a + superscript)

$$\begin{pmatrix} (\tilde{V}_n(\ell,s)) \\ (Z_{c_{n,m}}(0)) \cdot (\tilde{I}_n(\ell,s)) \end{pmatrix} = \left((\tilde{u}^{(+)}_{n,m}(\ell,0;s))_{v,v'} \right) \odot \begin{pmatrix} (\tilde{V}_n(0,s)) \\ (Z_{c_{n,m}}(0)) \cdot (\tilde{I}_n(0,s)) \end{pmatrix}$$

$$\begin{pmatrix} (\tilde{V}_n(0,s)) \\ (Z_{c_{n,m}}(0)) \cdot (\tilde{I}_n(0,s)) \end{pmatrix} = \left((\tilde{u}^{(-)}_{n,m}(0,\ell;s))_{v,v'} \right) \odot \begin{pmatrix} (\tilde{V}_n(\ell,s)) \\ (Z_{c_{n,m}}(0)) \cdot (\tilde{I}_n(\ell,s)) \end{pmatrix} \tag{4.7}$$

$$\left((\tilde{u}^{(+)}_{n,m}(0,\ell;s))_{v,v'} \right) = \left((\tilde{u}^{(+)}_{n,m}(\ell,0;s))_{v,v'} \right)^{-1}$$

For the compound transition (balun) for $-\ell \le z \le \ell$ we have

$$\begin{pmatrix} (\tilde{V}_n(\ell,s)) \\ (Z_{c_{n,m}}(0)) \cdot (\tilde{I}_n(\ell,s)) \end{pmatrix} = \left((\tilde{u}_{n,m}(\ell,-\ell;s))_{v,v'} \right) \odot \begin{pmatrix} (\tilde{V}_n(-\ell,s)) \\ (Z_{c_{n,m}}(0)) \cdot (\tilde{I}_n(-\ell,s)) \end{pmatrix}$$

$$\begin{pmatrix} (\tilde{V}_n(-\ell,s)) \\ (Z_{c_{n,m}}(0)) \cdot (\tilde{I}_n(-\ell,s)) \end{pmatrix} = \left((\tilde{u}_{n,m}(-\ell,\ell;s))_{v,v'} \right) \odot \begin{pmatrix} (\tilde{V}_n(\ell,s)) \\ (Z_{c_{n,m}}(0)) \cdot (\tilde{I}(\ell,s)) \end{pmatrix}$$

$$\begin{aligned} \left((\tilde{u}_{n,m}(-\ell,\ell;s))_{v,v'} \right) &= \left((\tilde{u}_{n,m}(\ell,-\ell;s))_{v,v'} \right)^{-1} \\ &= \left((\tilde{u}^{(-)}_{n,m}(-\ell,0;s))_{v,v'} \right) \odot \left((\tilde{u}^{(+)}_{n,m}(0,\ell;s))_{v,v'} \right) \\ &= \left((\tilde{u}^{(-)}_{n,m}(0,-\ell;s))_{v,v'} \right)^{-1} \odot \left((\tilde{u}^{(+)}_{n,m}(\ell,0;s))_{v,v'} \right)^{-1} \end{aligned} \tag{4.8}$$

This is then described in terms of the matrizants for the two sections.

V. Interpolation Matrix for Negative z

From (4.2) we have to consider what to which one may refer as an interpolation matrix, in this case for $-\ell \le z \le 0$, as

$$\left(f_{n,m}^{(-)}(z)\right) \equiv \left(f_{g_{n,m}}(z)\right) \cdot \left(f_{g_{n,m}}(0)\right)^{-1} \tag{5.1}$$

Before considering the functional dependence on z, we have the constraint

$$\left(f_{n,m}^{(-)}(-\ell)\right) = \left(f_{g_{n,m}}(-\ell)\right) \cdot \left(f_{g_{n,m}}(0)\right)^{-1}$$

$$= \frac{1}{f_g^{(d)} f_g^{(c)}} \begin{pmatrix} f_g^{(in)} + f_g^{(out)} & f_g^{(out)} \\ f_g^{(out)} & f_g^{(out)} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{4} f_g^{(d)} + f_g^{(c)} & \frac{1}{4} f_g^{(d)} - f_g^{(c)} \\ \frac{1}{4} f_g^{(d)} - f_g^{(c)} & \frac{1}{4} f_g^{(d)} + f_g^{(c)} \end{pmatrix}$$

$$= \frac{1}{f_g^{(d)} f_g^{(c)}} \begin{pmatrix} \frac{1}{2} f_g^{(out)} f_g^{(d)} + f_g^{(in)}\left[\frac{1}{4} f_g^{(d)} + f_g^{(c)}\right] & \frac{1}{2} f_g^{(out)} f_g^{(d)} + f_g^{(in)}\left[\frac{1}{4} f_g^{(d)} - f_g^{(c)}\right] \\ \frac{1}{2} f_g^{(out)} f_g^{(d)} & \frac{1}{2} f_g^{(out)} f_g^{(d)} \end{pmatrix}$$

$$= \frac{f_g^{(out)}}{2 f_g^{(c)}} \begin{pmatrix} 1 + c_1 & 1 + c_2 \\ 1 & 1 \end{pmatrix}$$

$$c_1 \equiv \frac{f_g^{(in)}}{f_g^{(out)}} \left[\frac{1}{2} + 2 \frac{f_g^{(c)}}{f_g^{(d)}}\right]$$

$$c_2 \equiv \frac{f_g^{(in)}}{f_g^{(out)}} \left[\frac{1}{2} - 2 \frac{f_g^{(c)}}{f_g^{(d)}}\right]$$

$$\det\left(\left(f_{n,m}^{(-)}(-\ell)\right)\right) = \left[\frac{f_g^{(out)}}{2 f_g^{(c)}}\right]^2 [c_1 - c_2] = \frac{f_g^{(in)} f_g^{(out)}}{f_g^{(d)} f_g^{(c)}}$$

$$= f_1^{(-)}(-\ell) f_2^{(-)}(-\ell)$$

$$\mathrm{tr}\left(\left(f_{n,m}^{(-)}(-\ell)\right)\right) = \frac{f_g^{(out)}}{2 f_g^{(c)}}[2 + c_1] = f_1^{(-)}(-\ell) + f_2^{(-)}(-\ell) \tag{5.2}$$

$$f_\beta^{(-)}(-\ell) \equiv \text{eigenvalue of } \left(f_{n,m}^{(-)}(-\ell)\right) \text{ for } \beta = 1,\ 2$$

To diagonalize this matrix, first define normalized eigenvalues as

$$f_\beta^{(-)'}(z) = 2\frac{f_g^{(c)}}{f_g^{(out)}}\ f_\beta^{(-)}(z) \tag{5.3}$$

where our emphasis for the moment is $z = -\ell$. Then we have

$$\left[1 + c_1 - f_\beta^{(-)'}(-\ell)\right]\left[1 - f_\beta^{(-)'}(-\ell)\right] - 1 - c_2 = 0$$

$$f_\beta^{(-)'2}(-\ell) - [2 + c_1]\, f_\beta^{(-)'}(-\ell) + c_1 - c_2 = 0 \tag{5.4}$$

$$f_\beta^{(-)'}(-\ell) = 1 + \frac{c_1}{2} \pm c_3$$

$$c_3 \equiv \left[\left[1 + \frac{c_1}{2}\right]^2 - c_1 + c_2\right]^{\frac{1}{2}}$$

$$\pm \Rightarrow \beta = \begin{matrix} 1 \\ 2 \end{matrix} \text{ respectively}$$

Consider, by completing the square,

$$c_3^2 = \left[1 + \frac{c_1}{2}\right]^2 - c_1 + c_2 = 1 + \frac{c_1^2}{4} + c_2$$

$$= 1 + \left[\frac{f_g^{(in)}}{f_g^{(out)}}\right]^2 \left[\frac{1}{4} + \frac{f_g^{(c)}}{f_g^{(d)}}\right]^2 + \frac{f_g^{(in)}}{f_g^{(out)}}\left[\frac{1}{2} - 2\frac{f_g^{(c)}}{f_g^{(d)}}\right] \tag{5.5}$$

$$= \left\{1 + \left[\frac{1}{4} - \frac{f_g^{(c)}}{f_g^{(d)}}\right]\frac{f_g^{(in)}}{f_g^{(out)}}\right\}^2 + \frac{f_g^{(c)}}{f_g^{(d)}}\left[\frac{f_g^{(in)}}{f_g^{(out)}}\right]^2$$

which is evidently positive so that c3 can be taken as the positive square root of this expression, thereby assuring that the eigenvalues are real. Furthermore, from (3.10) we have

$$c_3 \geq \left[\frac{f_g^{(c)}}{f_g^{(d)}}\right]^{\frac{1}{2}} \frac{f_g^{(in)}}{f_g^{(out)}} \geq \frac{1}{2}\frac{f_g^{(in)}}{f_g^{(out)}} \tag{5.6}$$

so that c3 is strictly positive making the two eigenvalues distinct, and assuring the diagonalizability of the matrix. We also have

$$c_3^2 = \left[1 + \frac{c_1}{2}\right]^2 - c_1 + c_2 = \left[1 + \frac{c_1}{2}\right]^2 - 4\frac{f_g^{(in)} f_g^{(d)}}{f_g^{(out)} f_g^{(c)}}$$

$$c_3 < 1 + \frac{c_1}{2} \tag{5.7}$$

$$f_\beta^{(-)'}(-\ell) = 1 + \frac{c_1}{2} \pm c_3 > 0$$

so that the eigenvalues (both) are strictly positive.

The right eigenvectors are now defined by

$$\begin{pmatrix} 1+c_1 & 1+c_2 \\ 1 & 1 \end{pmatrix} \cdot \left(g_n^{(-)}\right)_\beta = f_\beta^{(-)'}(-\ell)\left(g_n^{(-)}\right)_\beta$$

$$\left[1 + c_1 - f_\beta^{(-)'}(-\ell)\right] g_{1;\beta}^{(-)} + [1 + c_2]\, g_{2;\beta}^{(-)} = 0$$

$$g_{1;\beta}^{(-)} + \left[1 - f_\beta^{(-)'}(-\ell)\right] g_{2;\beta}^{(-)} = 0$$

$$\frac{g_{2;\beta}^{(-)}}{g_{1;\beta}^{(-)}} = \left[f_\beta^{(-)'}(-\ell) - 1\right]^{-1} = [1 + c_2]^{-1}\left[f_\beta^{(-)'}(-\ell) - 1 - c_1\right]$$

$$g_n^{(-)}{}_\beta = p_\beta^{(-)} \begin{pmatrix} \frac{c_1}{2} \pm c_3 \\ 1 \end{pmatrix}$$

$$p_\beta^{(-)} \equiv \text{normalization constant} \tag{5.8}$$

Similarly the left eigenvectors are

$$\left(h_n^{(-)}\right)_\beta \cdot \begin{pmatrix} 1+c_1 & 1+c_2 \\ 1 & 1 \end{pmatrix} = f_\beta^{(-)'}(-\ell) \left(h_n^{(-)}\right)_\beta$$

$$\left[1 + c_1 - f_\beta^{(-)'}(-\ell)\right] h_{1;\beta}^{(-)} + h_{2;\beta}^{(-)} = 0$$

$$[1 + c_2] h_{1;\beta}^{(-)} + \left[1 - f_\beta^{(-)'}(-\ell)\right] h_{2;\beta}^{(-)} = 0$$

$$\frac{h_{2;\beta}^{(-)}}{h_{1;\beta}^{(-)}} = f_\beta^{(-)'}(-\ell) - 1 - c_1 = [1 + c_2]\left[f_\beta^{(-)'}(-\ell) - 1\right]^{-1} \tag{5.9}$$

$$\left(h_n^{(-)}\right)_\beta = q_\beta^{(-)} \begin{pmatrix} 1 \\ -\frac{c_1}{2} \pm c_3 \end{pmatrix}$$

$$q_\beta^{(-)} \equiv \text{normalization constant}$$

The biorthonormal property has

$$\left(h_n^{(-)}\right)_{\beta_1} \cdot \left(g_n^{(-)}\right)_{\beta_2} = 1_{\beta_1,\beta_2}$$

$$p_\beta^{(-)} q_\beta^{(-)} = \pm \frac{1}{2c_3} \tag{5.10}$$

The eigendyad can now be written as

$$\left(g_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta = \pm \frac{1}{2c_3} \begin{pmatrix} \frac{c_1}{2} \pm c_3 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{c_1}{2} \pm c_3 \end{pmatrix} \tag{5.11}$$

which is purely real. The diagonal representation of the matrix can now be written in dyadic form as

$$\left(f_{n,m}^{(-)}(-\ell)\right) = \sum_{\beta=1}^{2} f_\beta^{(-)}(-\ell)\left(g_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta$$

$$f_\beta^{(-)}(-\ell) = 2\frac{f_g^{(c)}}{f_g^{(out)}} f_\beta^{(-)'}(-\ell) = 2\frac{f_g^{(c)}}{f_g^{(out)}}\left[1 + \frac{c_1}{2} \pm c_3\right] > 0 \tag{5.12}$$

At z = 0 we have (from (5.1)

$$\left(f_{n,m}^{(-)}(0)\right) = \left(1_{n,m}\right) = \sum_{\beta=1}^{2}\left(g_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta$$

$$f_\beta^{(-)}(0) = 1 \tag{5.13}$$

Then we can select for all $-\ell \le z \le 0$

$$\left(f_{n,m}^{(-)}(z)\right) = \sum_{\beta=1}^{2} f_\beta^{(-)}(z)\left(g_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta \tag{5.14}$$

where the eigenvalues $f_\beta^{(-)}(z)$ can be selected to match the values at z = 0, $-\ell$ as above. This allows us to constrain the $f_\beta^{(-)}(z)$ as purely real and positive. For convenience we can also make these monotonic functions of z over the range of interest.

VI. Interpolation Matrix for Positive z

For positive z ($0 \leq z \leq \ell$) we have another interpolation matrix as

$$\left(f_{n,m}^{(+)}(z)\right) \equiv \left(f_{g_{n,m}}(z)\right) \cdot \left(f_{g_{n,m}}(0)\right)^{-1} \tag{6.1}$$

The constraint at $z = \ell$ gives

$$\left(f_{n,m}^{(+)}(\ell)\right) = \left(f_{g_{n,m}}(\ell)\right) \cdot \left(f_{g_{n,m}}(0)\right)^{-1}$$

$$= \frac{1}{f_g^{(d)} f_g^{(c)}} \begin{pmatrix} f_g^{(out)} & f_g^{(out)} \\ f_g^{(out)} & f_g^{(in)} + f_g^{(out)} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{4} f_g^{(d)} + f_g^{(c)} & \frac{1}{4} f_g^{(d)} - f_g^{(c)} \\ \frac{1}{4} f_g^{(d)} - f_g^{(c)} & \frac{1}{4} f_g^{(d)} + f_g^{(c)} \end{pmatrix}$$

$$= \frac{1}{f_g^{(d)}\ f_g^{(c)}} \begin{pmatrix} \frac{1}{2} f_g^{(out)}\ f_g^{(d)} & \frac{1}{2} f_g^{(out)}\ f_g^{(d)} \\ \frac{1}{2} f_g^{(out)}\ f_g^{(d)} + f_g^{(in)}\left[\frac{1}{4} f_g^{(d)} - f_g^{(c)}\right] & \frac{1}{2} f_g^{(out)}\ f_g^{(d)} + f_g^{(in)}\left[\frac{1}{4} f_g^{(d)} + f_g^{(c)}\right] \end{pmatrix}$$

$$= \frac{f_g^{(out)}}{2\, f_g^{(c)}} \begin{pmatrix} 1 & 1 \\ 1 + c_2 & 1 + c_1 \end{pmatrix}$$

$$\det\left(\left(f_{n,m}^{(+)}(\ell)\right)\right) = \left[\frac{f_g^{(out)}}{2\, f_g^{(c)}}\right]^2 \left[c_1 - c_2\right] = \frac{f_g^{(in)} f_g^{(out)}}{f_g^{(d)} f_g^{(c)}}$$

$$= f_1^{(+)}(\ell)\, f_2^{(+)}(\ell)$$

$$\mathrm{tr}\left(\left(f_{n,m}^{(+)}(\ell)\right)\right) = \frac{f_g^{(out)}}{2\, f_g^{(c)}}\left[2 + c_1\right] = f_1^{(+)}(\ell) + f_2^{(+)}(\ell) \tag{6.2}$$

$$f_\beta^{(+)}(\ell) \equiv \text{eigenvalue of } \left(f_{n,m}^{(+)}(\ell)\right) \text{ for } \beta = 1, 2$$

where the c_ns are the same as in the previous section. The eigenvalues are evidently the same as before which allows us to set

$$f_\beta^{(+)}(\ell) = f_\beta^{(-)}(-\ell) = 2\frac{f_g^{(c)}}{f_g^{(out)}}\, f_\beta^{(+)'}(\ell) > 0$$

$$f_\beta^{(+)}(\ell) = f_\beta^{(-)}(-\ell) = 1+\frac{c_1}{2} \pm c_3 > 0 \tag{6.3}$$

$$\pm \Rightarrow \beta = \begin{matrix} 1 \\ 2 \end{matrix} \quad \text{respectively (as before)}$$

The right eigenvectors are now

$$\begin{pmatrix} 1 & 1 \\ 1+c_2 & 1+c_1 \end{pmatrix} \cdot \left(g_n^{(+)}\right)_\beta = f_\beta^{(+)'}(\ell)\left(g_n^{(+)}\right)_\beta \tag{6.4}$$

Instead of going through the details (as in (5.8)) we can observe that the matrix is as before except that rows and columns are interchanged. This is the same as interchanging the two vector elements, giving

$$\left(g_n^{(+)}\right)_\beta = p_\beta^{(+)}\begin{pmatrix} 1 \\ \frac{c_1}{2} \pm c_3 \end{pmatrix} = \begin{pmatrix} g_1^{(+)} \\ g_2^{(+)} \end{pmatrix}_\beta = \begin{pmatrix} g_2^{(-)} \\ g_1^{(-)} \end{pmatrix}_\beta$$

$$p_\beta^{(+)} = p_\beta^{(-)} \tag{6.5}$$

Similarly the left eigenvectors are

$$\left(h_n^{(+)}\right)_\beta \cdot \begin{pmatrix} 1 & 1 \\ 1+c_2 & 1+c_1 \end{pmatrix} = f_\beta^{(+)'}(\ell)\left(h_n^{(+)}\right)_\beta$$

$$\left(h_n^{(+)}\right)_\beta = q_\beta^{(+)}\begin{pmatrix} -\frac{c_1}{2} \pm c_3 \\ 1 \end{pmatrix} = \begin{pmatrix} h_1^{(+)} \\ h_2^{(+)} \end{pmatrix} = \begin{pmatrix} h_2^{(-)} \\ h_1^{(-)} \end{pmatrix} \tag{6.6}$$

$$q_\beta^{(+)} = q_\beta^{(-)}$$

As before the biorthonormal property is

$$\left(h_n^{(+)}\right)_{\beta_1} \cdot \left(g_n^{(+)}\right)_{\beta_2} = 1_{\beta_1,\beta_2}$$

$$p_\beta^{(+)}\, q_\beta^{(+)} = \pm\frac{1}{2\,c_3} \tag{6.7}$$

The eigendyad is now

$$\left(g_n^{(+)}\right)_\beta \left(h_n^{(+)}\right)_\beta = \pm\frac{1}{2\,c_3}\begin{pmatrix} 1 \\ \frac{c_1}{2} \pm c_3 \end{pmatrix}\begin{pmatrix} -\frac{c_1}{2} \pm c_3 \\ 1 \end{pmatrix} \tag{6.8}$$

which is again purely real. The diagonal representation of the matrix in dyadic form is

$$\left(f_{n,m}^{(+)}(\ell)\right) = \sum_{\beta=1}^{2} f_\beta^{(+)}(\ell)\left(g_n^{(+)}\right)_\beta \left(h_n^{(+)}\right)_\beta$$

$$f_\beta^{(+)}(\ell) = 2\frac{f_g^{(c)}}{f_g^{(out)}}\, f_\beta^{(+)}(\ell) = 2\frac{f_g^{(c)}}{f_g^{(out)}}\left[1+\frac{c_1}{2}\pm c_3\right] > 0 \tag{6.9}$$

At z = 0 we have (from (6.1))

$$\left(f_{n,m}^{(+)}(0)\right) = \left(1_{n,m}\right) = \sum_{\beta=1}^{2}\left(g_n^{(+)}\right)_\beta \left(h_n^{(+)}\right)_\beta$$

$$f_\beta^{(+)}(0) = 1 \tag{6.10}$$

Then we can select for all $0 \le z \le \ell$

$$\left(f_{n,m}^{(+)}(z)\right) = \sum_{\beta=1}^{2} f_\beta^{(+)}(z)\,\left(g_n^{(+)}\right)_\beta\,\left(h_n^{(+)}\right)_\beta \tag{6.11}$$

These eigenvalues can be chosen real and positive as before. Preserving symmetry let us set

$$f_\beta^{(+)}(z) \equiv f_\beta^{(-)}(-z) \tag{6.12}$$

and both sets of eigenvalues have the same desirable properties.

VII. Eigenvector Matrices

Using the eigenvectors for the left section ($-\ell \leq z \leq 0$) we can form

$$\left(G_{n,m}^{(-)}\right) \equiv \left(\left(g_n^{(-)}\right)_1, \left(g_n^{(-)}\right)_2\right)$$

$$\left(H_{n,m}^{(-)}\right) \equiv \left(\left(h_n^{(-)}\right)_1, \left(h_n^{(-)}\right)_2\right) \tag{7.1}$$

where the eigenvectors are taken as columns here. Then the biorthonormality relation becomes

$$\left(H_{n,m}^{(-)}\right)^T \cdot \left(G_{n,m}^{(-)}\right) = \left(1_{n,m}\right) \tag{7.2}$$

which in turn implies

$$\left(G_{n,m}^{(-)}\right)^{-1} = \left(H_{n,m}^{(-)}\right)^T, \quad \left(H_{n,m}^{(-)}\right)^{-1} = \left(G_{n,m}^{(-)}\right)^T \tag{7.3}$$

We also have

$$\left(G_{n,m}^{(-)}\right) \cdot \left(H_{n,m}^{(-)}\right)^T = \left(1_{n,m}\right)$$

$$\begin{pmatrix} \sum_{\beta=1}^{2} g_{1;\beta}^{(-)} h_{1;\beta}^{(-)} & \sum_{\beta=1}^{2} g_{1;\beta}^{(-)} h_{2;\beta}^{(-)} \\ \sum_{\beta=1}^{2} g_{2;\beta}^{(-)} h_{1;\beta}^{(-)} & \sum_{\beta=1}^{2} g_{2;\beta}^{(-)} h_{2;\beta}^{(-)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{7.4}$$

thereby giving some additional orthonormality relations.

The eigenvector defining equations now become

$$\left(f_{n,m}^{(-)}(-\ell)\right) \cdot \left(G_{n,m}^{(-)}\right) = \left(G_{n,m}^{(-)}\right) \cdot \begin{pmatrix} f_1^{(-)}(-\ell) & 0 \\ 0 & f_2^{(-)}(-\ell) \end{pmatrix}$$

$$\left(H_{n,m}^{(-)}\right)^T \cdot \left(f_{n,m}^{(-)}(-\ell)\right) = \begin{pmatrix} f_1^{(-)}(-\ell) & 0 \\ 0 & f_2^{(-)}(-\ell) \end{pmatrix} \cdot \left(H_{n,m}^{(-)}\right)^T$$

$$\left(H_{n,m}^{(-)}\right)^T \cdot \left(f_{n,m}^{(-)}(-\ell)\right) \cdot \left(G_{n,m}^{(-)}\right) = \begin{pmatrix} f_1^{(-)}(-\ell) & 0 \\ 0 & f_2^{(-)}(-\ell) \end{pmatrix} \tag{7.5}$$

$$\left(f_{n,m}^{(-)}(-\ell)\right) = \left(G_{n,m}^{(-)}\right) \cdot \begin{pmatrix} f_1^{(-)}(-\ell) & 0 \\ 0 & f_2^{(-)}(-\ell) \end{pmatrix} \cdot \left(H_{n,m}^{(-)}\right)^T$$

The foregoing can be reinterpreted for the right section ($0 \le z \le \ell$) by replacing - → + for superscripts and $-\ell \to \ell$, and for the special case of the inverter (to be introduced later) by use of 2 for the superscripts.

VIII. Geometric-Factor Matrix

Choosing first the left section ($-\ell \le z \le 0$) we begin with

$$\left(f_{n,m}^{(-)}(-\ell)\right) = \left(f_{g_{n,m}}(-\ell)\right) \cdot \left(f_{g_{n,m}}(0)\right)^{-1} = \sum_{\beta=1}^{2} f_{\beta}^{(-)}(-\ell)\left(g_n^{(-)}\right)_{\beta}\left(h_n^{(-)}\right)_{\beta}$$

$$\left(1_{n,m}\right) = \sum_{\beta=1}^{2}\left(g_n^{(-)}\right)_{\beta}\left(h_n^{(-)}\right)_{\beta} = \sum_{\beta=1}^{2}\left(h_n^{(-)}\right)_{\beta}\left(g_n^{(-)}\right)_{\beta} \tag{8.1}$$

Then, noting that the two geometric-factor matrices are symmetric, we have (dot multiplying on the right by the identity)

$$\begin{aligned}\left(f_{g_{n,m}}(-\ell)\right) &= \sum_{\beta=1}^{2} f_{\beta}^{(-)}(-\ell)\left(g_n^{(-)}\right)_{\beta}\left(h_n^{(-)}\right)_{\beta}\cdot\left(f_{g_{n,m}}(0)\right)\\ &= \sum_{\beta=1}^{2} f_{\beta}^{(-)}(-\ell)\left(g_n^{(-)}\right)_{\beta}\left(h_n^{(-)}\right)_{\beta}\cdot\left(f_{g_{n,m}}(0)\right)\cdot\left(h_n^{(-)}\right)_{\beta}\left(g_n^{(-)}\right)_{\beta}\\ &\quad + f_1^{(-)}(-\ell)\left(g_n^{(-)}\right)_1\left(h_n^{(-)}\right)_1\cdot\left(f_{g_{n,m}}(0)\right)\cdot\left(h_n^{(-)}\right)_2\left(g_n^{(-)}\right)_2\\ &\quad + f_2^{(-)}(-\ell)\left(g_n^{(-)}\right)_2\left(h_n^{(-)}\right)_2\cdot\left(f_{g_{n,m}}(0)\right)\cdot\left(h_n^{(-)}\right)_1\left(g_n^{(-)}\right)_1\end{aligned} \tag{8.2}$$

Setting this equal to its transpose and subtracting common terms gives

$$\begin{aligned}&f_1^{(-)}(-\ell)\left(g_n^{(-)}\right)_1\left(h_n^{(-)}\right)_1\cdot\left(f_{g_{n,m}}(0)\right)\cdot\left(h_n^{(-)}\right)_2\left(g_n^{(-)}\right)_2\\ &+ f_2^{(-)}(-\ell)\left(g_n^{(-)}\right)_2\left(h_n^{(-)}\right)_2\cdot\left(f_{g_{n,m}}(0)\right)\cdot\left(h_n^{(-)}\right)_1\left(g_n^{(-)}\right)_1\\ &= f_1^{(-)}(-\ell)\left(g_n^{(-)}\right)_2\left(h_n^{(-)}\right)_2\cdot\left(f_{g_{n,m}}(0)\right)\cdot\left(h_n^{(-)}\right)_1\left(g_n^{(-)}\right)_1\\ &+ f_2^{(-)}(-\ell)\left(g_n^{(-)}\right)_1\left(h_n^{(-)}\right)_1\cdot\left(f_{g_{n,m}}(0)\right)\cdot\left(h_n^{(-)}\right)_2\left(g_n^{(-)}\right)_2\end{aligned} \tag{8.3}$$

Dot multiply on the left by $\left(h_n^{(-)}\right)_1$ and on the right by $\left(h_n^{(-)}\right)_2$ to give

$$f_1^{(-)}(-\ell)\,\left(h_n^{(-)}\right)_1 \cdot \left(f_{g_{n,m}}(0)\right) \cdot \left(h_n^{(-)}\right)_2$$

$$= f_2^{(-)}(-\ell)\,\left(h_n^{(-)}\right)_1 \cdot \left(f_{g_{n,m}}(0)\right) \cdot \left(h_n^{(-)}\right)_2 \tag{8.4}$$

$$0 = \left[f_1^{(-)}(-\ell) - f_2^{(-)}(-\ell)\right]\left(h_n^{(-)}\right)_1 \cdot \left(f_{g_{n,m}}(0)\right) \cdot \left(h_n^{(-)}\right)_2$$

Then since the eigenvalues are distinct we have

$$\left(h_n^{(-)}\right)_1 \cdot \left(f_{g_{n,m}}(0)\right) \cdot \left(h_n^{(-)}\right)_2 = 0 = \left(h_n^{(-)}\right)_2 \cdot \left(f_{g_{n,m}}(0)\right) \cdot \left(h_n^{(-)}\right)_1 \tag{8.5}$$

Now in (8.2) dot multiply on the left by $\left(h_n^{(-)}\right)_\beta$ giving

$$\left(h_n^{(-)}\right)_\beta \cdot \left(f_{g_{n,m}}(-\ell)\right) = f_\beta^{(-)}(-\ell)\left(h_n^{(-)}\right)_\beta \cdot \left(f_{g_{n,m}}(0)\right)$$

$$= f_\beta^{(-)}(-\ell)\left(h_n^{(-)}\right)_\beta \cdot \left(f_{g_{n,m}}(0)\right) \cdot \left(h_n^{(-)}\right)_\beta \left(g_n^{(-)}\right)_\beta \tag{8.6}$$

$$\left(g_n^{(-)}\right)_\beta = \left[\left(h_n^{(-)}\right)_\beta \cdot \left(f_{g_{n,m}}(0)\right) \cdot \left(h_n^{(-)}\right)_\beta\right]^{-1} \left(h_n^{(-)}\right)_\beta \cdot \left(f_{g_{n,m}}(0)\right)$$

Thus we have two vectors related by a constant coefficient. Recalling the normalization condition in (5.10), we still can choose one of the eigenvector coefficients $\left(p_\beta^{(-)} \text{ or } q_\beta^{(-)}\right)$ at will. So we set, as another convenient normalization condition,

$$\left(g_n^{(-)}\right)_\beta = \left(h_n^{(-)}\right)_\beta \cdot \left(f_{g_{n,m}}(0)\right) = \left(f_{g_{n,m}}(0)\right) \cdot \left(h_n^{(-)}\right)_\beta$$

$$\left(h_n^{(-)}\right)_\beta \cdot \left(f_{g_{n,m}}(0)\right) \cdot \left(h_n^{(-)}\right)_\beta = 1 \tag{8.7}$$

which constrains $q_\beta^{(-)}$ and thereby $p_\beta^{(-)}$. In terms of previously defined quantities we have from the first of (8.7) (vector equation)

$$\left(g_n^{(-)}\right)_\beta = p_\beta^{(-)}\begin{pmatrix}\frac{c_1}{2}\pm c_3\\ 1\end{pmatrix} = \left(f_{g_{n,m}}(0)\right)\cdot\left(h_n^{(-)}\right)_\beta$$

$$= q_\beta^{(-)}\begin{pmatrix}\frac{1}{4}f_g^{(d)}+f_g^{(c)} & -\frac{1}{4}f_g^{(d)}+f_g^{(c)}\\ -\frac{1}{4}f_g^{(d)}+f_g^{(c)} & \frac{1}{4}f_g^{(d)}+f_g^{(c)}\end{pmatrix}\cdot\begin{pmatrix}1\\ -\frac{c_1}{2}\pm c_3\end{pmatrix}$$

$$= q_\beta^{(-)}\begin{pmatrix}\frac{1}{4}f_g^{(d)}\left[1+\frac{c_1}{2}\mp c_3\right] + f_g^{(c)}\left[1-\frac{c_1}{2}\pm c_3\right]\\ \frac{1}{4}f_g^{(d)}\left[-1-\frac{c_1}{2}\pm c_3\right] + f_g^{(c)}\left[1-\frac{c_1}{2}\pm c_3\right]\end{pmatrix}$$

$$\frac{p_\beta^{(-)}}{q_\beta^{(-)}} = \left[\frac{c_1}{2}\pm c_3\right]^{-1}\left[\frac{1}{4}f_g^{(d)}\left[1+\frac{c_1}{2}\mp c_3\right] + f_g^{(c)}\left[1-\frac{c_1}{2}\pm c_3\right]\right] \tag{8.8}$$

$$= \frac{1}{4}f_g^{(d)}\left[-1-\frac{c_1}{2}\pm c_3\right] + f_g^{(c)}\left[1-\frac{c_1}{2}\pm c_3\right]$$

With (5.10) this gives

$$p_\beta^{(-)2} = \pm\frac{1}{2c_3}\left[\frac{c_1}{2}\pm c_3\right]^{-1}\left[\frac{1}{4}f_g^{(d)}\left[1+\frac{c_1}{2}\mp c_3\right] + f_g^{(c)}\left[1-\frac{c_1}{2}\pm c_3\right]\right]$$

$$= \pm\frac{1}{2c_3}\left[\frac{1}{4}f_g^{(d)}\left[-1-\frac{c_1}{2}\pm c_3\right] + f_g^{(c)}\left[1-\frac{c_1}{2}\pm c_3\right]\right] \tag{8.9}$$

$$q_\beta^{(-)} = \pm\frac{1}{2c_3\,p_\beta^{(-)}}$$

Another form can be found from the second of (8.7) (scalar equation) giving

$$q_\beta^{(-)2} = \left[\frac{1}{4}f_g^{(d)}\left[1+\frac{c_1}{2}\mp c_3\right]^2 + f_g^{(c)}\left[1-\frac{c_1}{2}\pm c_3\right]^2\right]^{-1} > 0$$

(8.10)

$$p_\beta^{(-)} = \pm\frac{1}{2c_3\,q_\beta^{(-)}}$$

Note that $q_\beta^{(-)}$ is real, and with sign chosen from the square root then $p_\beta^{(-)}$ is consistently chosen and is also real.

Then from (8.2) we have

$$\left(f_{g_{n,m}}(-\ell)\right) = \sum_{\beta=1}^{2} f_\beta^{(-)}(-\ell)\left(g_n^{(-)}\right)_\beta \left(g_n^{(-)}\right)_\beta$$

$$\left(h_n^{(-)}\right)_\beta \cdot \left(f_{g_{n,m}}(-\ell)\right) \cdot \left(h_n^{(-)}\right)_\beta = f_\beta^{(-)}(-\ell) \tag{8.11}$$

and from (8.1) we have

$$\left(f_{g_{n,m}}(0)\right) = \left(f_{n,m}^{(-)}(-\ell)\right)^{-1} \cdot \left(f_{g_{n,m}}(-\ell)\right) = \sum_{\beta=1}^{2} \left(g_n^{(-)}\right)_\beta \left(g_n^{(-)}\right)_\beta \tag{8.12}$$

Similarly the inverse geometric-factor matrices can be constructed as

$$\left(f_{g_{n,m}}(0)\right)^{-1} = \sum_{\beta=1}^{2} \left(h_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta$$

$$\left(g_n^{(-)}\right)_\beta \cdot \left(f_{g_{n,m}}(0)\right)^{-1} \cdot \left(g_n^{(-)}\right)_\beta = 1$$

$$\left(f_{g_{n,m}}(-\ell)\right)^{-1} = \sum_{\beta=1}^{2} f_\beta^{(-)^{-1}}(-\ell)\left(h_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta \tag{8.13}$$

$$\left(g_n^{(-)}\right)_\beta \cdot \left(f_{g_{n,m}}(-\ell)\right)^{-1} \cdot \left(g_n\right)_\beta = f_\beta^{(-)^{-1}}(-\ell)$$

which can be verified by multiplying to produce the identity.

The geometric factor matrix can now be extended to all $-\ell \leq z \leq 0$ from (5.1) and (5.14) as

$$\left(f_{g_{n,m}}(z)\right) = \left(f_{n,m}^{(-)}(z)\right) \cdot \left(f_{g_{n,m}}(0)\right)$$

$$= \sum_{\beta=1}^{2} f_{\beta}^{(-)}(z) \left(g_n^{(-)}\right)_{\beta} \left(h_n^{(-)}\right)_{\beta} \cdot \left(f_{g_{n,m}}(0)\right)$$

$$= \sum_{\beta=1}^{2} f_{\beta}^{(-)}(z) \left(g_n^{(-)}\right)_{\beta} \left(g_n^{(-)}\right)_{\beta} \tag{8.14}$$

$$\left(f_{g_{n,m}}(z)\right)^{-1} = \sum_{\beta=1}^{2} f_{\beta}^{(-)}(z) \left(h_n^{(-)}\right)_{\beta} \left(h_n^{(-)}\right)_{\beta}$$

This generalizes the foregoing results and reduces directly to them at $z = 0, -\ell$. The eigenvalues now can be expressed as

$$f_{\beta}^{(-)}(z) = \left(h_n^{(-)}\right)_{\beta} \cdot \left(f_{g_{n,m}}(z)\right) \cdot \left(h_n^{(-)}\right)_{\beta}$$

$$= \left[\left(g_n^{(-)}\right)_{\beta} \cdot \left(f_{g_{n,m}}(z)\right)^{-1} \cdot \left(g_n^{(-)}\right)_{\beta}\right]^{-1} \tag{8.15}$$

Per the discussion in Sections II and III the geometric factor matrix is realizable at $z = 0, -\ell$. The general form in (8.14) is evidently symmetric, i.e.

$$\left(f_{g_{n,m}}(z)\right)^{T} = \left(f_{g_{n,m}}(z)\right) \tag{8.16}$$

thereby satisfying the reciprocity requirement. Furthermore, as discussed in Section V the $\left(g_n^{(-)}\right)_{\beta}$ and $\left(h_n^{(-)}\right)_{\beta}$ are real and the $f_{\beta}^{(-)}(z)$ can be selected real and positive over the entire interval. Hence the geometric factor matrix is real (as required). While an impedance or admittance matrix (passive) needs to be positive real (p. r.) this reduces to the requirement of non-negative definite for the geometric factor matrix. While the vectors in (8.15) are not eigenvectors (in general) of $\left(f_{g_{n,m}}(z)\right)$ they can be used to construct any two-dimensional vector (span the space). Using the $\left(h_n^{(-)}\right)_{\beta}$ form (with real coefficients to give a real vector)

$$\left[\xi_1\left(h_n^{(-)}\right)_1 + \xi_2\left(h_n^{(-)}\right)_1\right] \cdot \left(f_{g_{n,m}}(z)\right) \cdot \left[\xi_1\left(h_n^{(-)}\right)_1 + \xi_2\left(h_n^{(-)}\right)_2\right]$$

$$= \sum_{\beta=1}^{2} f_\beta^{(-)}(z)\,\xi_\beta^2 > 0 \text{ for at least one } \xi_\beta \neq 0 \tag{8.17}$$

since the $f_\beta^{(-)}(z)$ are both strictly positive. Hence the geometric-factor matrix is positive definite.

Another physical requirement has been introduced in (3.8). In our problem this means that if a current is injected in one wire (for an incremental length of line appropriately terminated) the voltage in this wire will exceed that in the second wire, which though retains the same sign. This means that

$$\text{lesser of } \left[f_{g_{1,1}}(z), f_{g_{2,2}}(z)\right] \geq f_{g_{1,2}}(z) = f_{g_{2,1}}(z) \geq 0 \tag{8.18}$$

The geometric-factor matrix is written out as

$$\left(f_{g_{n,m}}(z)\right) = \sum_{\beta=1}^{2} f_\beta^{(-)}(z)\left(g_n^{(-)}\right)_\beta \left(g_n^{(-)}\right)_\beta$$

$$= f_1^{(-)}(z)\,p_1^{(-)^2}\begin{pmatrix}\left[\frac{c_1}{2}+c_3\right]^2 & \frac{c_1}{2}+c_3 \\ \frac{c_1}{2}+c_3 & 1\end{pmatrix} + f_2^{(-)}(z)\,p_2^{(-)^2}\begin{pmatrix}\left[\frac{c_1}{2}-c_3\right]^2 & \frac{c_1}{2}-c_3 \\ \frac{c_1}{2}-c_3 & 1\end{pmatrix} \tag{8.19}$$

Since the $p_\beta^{(-)^2}$ and $f_\beta^{(-)}(z)$ are all positive, then we have

$$f_{g_{1,1}}(z) > 0 \ , \ f_{g_{2,2}}(z) > 0 \tag{8.20}$$

as required.

The off-diagonal term is

$$f_{g_{1,2}}(z) = f_{g_{2,1}}(z) = f_1^{(-)}(z)\,p_1^{(-)^2}\left[\frac{c_1}{2}+c_3\right] + f_2^{(-)}(z)\,p_2^{(-)^2}\left[\frac{c_1}{2}+c_3\right] \tag{8.21}$$

At $z = 0$ this is non-negative (per Section III), so we have

$$0 \le 2\frac{f_g^{(c)}}{f_g^{(out)}}\left\{p_1^{(-)2}\left[\frac{c_1}{2}+c_3\right]+p_2^{(-)2}\left[\frac{c_1}{2}-c_3\right]\right\}$$

$$p_2^{(-)2}\left[\frac{c_1}{2}-c_3\right] \ge -p_1^{(-)2}\left[\frac{c_1}{2}+c_3\right] \tag{8.22}$$

which can be substituted into (8.21) (noting positive eigenvalues) to give

$$f_{g_{1,2}}(z) \ge \left[f_1^{(-)}(z) - f_2^{(-)}(z)\right] p_1^{(-)2}\left[\frac{c_1}{2}+c_3\right] \tag{8.23}$$

This is non-negative provided

$$f_1^{(-)}(z) - f_2^{(-)}(z) \ge 0 \tag{8.24}$$

which we now make a constraint. Note that this is satisfied at $z = 0, -\ell$ as discussed in Section V. So let us require that the interpolated eigenvalues satisfy this constraint for all $-\ell \le z \le 0$.

Similarly we have

$$f_{g_{2,2}}(z) - f_{g_{1,2}}(z) = f_1^{(-)}(z)\,p_1^{(-)2}\left[1-\frac{c_1}{2}-c_3\right]+f_2^{(-)}(z)\,p_2^{(-)2}\left[1-\frac{c_1}{2}+c_3\right] \tag{8.25}$$

At $z = -\ell$ this is zero (per Section II) giving

$$0 = f_1^{(-)}(-\ell)\,p_1^{(-)2}\left[1-\frac{c_1}{2}-c_3\right]+f_2^{(-)}(-\ell)\,p_2^{(-)2}\left[1-\frac{c_1}{2}+c_3\right] \tag{8.26}$$

From (5.5) we have

$$c_3^2 = 1+\frac{c_1^2}{4}+c_2 = \left[1-\frac{c_1}{2}\right]^2 + c_1+c_2$$

$$= \left[1-\frac{c_1}{2}\right]^2 + \frac{f_g^{(in)}}{f_g^{(out)}} > \left[1-\frac{c_1}{2}\right]^2$$

$$c_3 > \left|1-\frac{c_1}{2}\right| \text{ (positive square root)}$$

$$1-\frac{c_1}{2}+c_3 > 0$$

$$1-\frac{c_1}{2}-c_3 > 0 \tag{8.27}$$

Substituting from (8.26) in (8.25) gives

$$f_{g_{2,2}}(z) - f_{g_{1,2}}(z) = \left[-\frac{f_2^{(-)}(-\ell)}{f_1^{(-)}(-\ell)} f_1^{(-)}(z) + f_2^{(-)}(z)\right] p_2^{(-)^2}\left[1-\frac{c_1}{2}+c_3\right] \tag{8.28}$$

Noting the positive coefficient (from (8.27)) this is positive (as required) provided

$$-\frac{f_2^{(-)}(-\ell)}{f_1^{(-)}(-\ell)} f_1^{(-)}(z) + f_2^{(-)}(z) \geq 0 \tag{8.29}$$

Noting that (from (5.4) and following)

$$0 < \frac{f_2^{(-)}(-\ell)}{f_1^{(-)}(-\ell)} < 1 \tag{8.30}$$

then (8.29) is satisfied at $z = 0, -\ell$. Accepting (8.29) as a constraint it can be combined with (8.24) to give

$$1 \geq \frac{f_2^{(-)}(z)}{f_1^{(-)}(z)} \geq \frac{f_2^{(-)}(-\ell)}{f_1^{(-)}(-\ell)} \tag{8.31}$$

Thereby giving both upper and lower bounds for the ratio of acceptable interpolated eigenvalues.

We also need to consider

$$f_{g_{1,1}}(z) - f_{g_{2,2}}(z) = f_1^{(-)}(z)\, p_1^{(-)^2}\left[\left[\frac{c_1}{2}+c_3\right]^2 - 1\right] + f_2^{(-)}(z)\, p_2^{(-)^2}\left[\left[\frac{c_1}{2}-c_3\right]^2 - 1\right] \tag{8.32}$$

Using the results of (8.27) we have

$$\left[\frac{c_1}{2}-c_3\right]^2 - 1 = \left[\frac{c_1}{2}-c_3+1\right]\left[\frac{c_1}{2}-c_3-1\right]$$

$$= f_2^{(-)'}(-\ell)\left[\frac{c_1}{2}-c_3-1\right] < 0$$

$$\left[\frac{c_1}{2}+c_3\right]^2 - 1 = \left[\frac{c_1}{2}+c_3+1\right]\left[\frac{c_1}{2}+c_3-1\right] \tag{8.33}$$

$$= f_1^{(-)'}(-\ell)\left[\frac{c_1}{2}+c_3-1\right] > 0$$

Then (8.32) becomes, with the help of (8.26),

$$f_{g_{1,1}}(z) - f_{g_{2,2}}(z) = \left[f_1^{(-)}(z) - f_2^{(-)}(z)\right] p_1^{(-)^2} \left[\left[\frac{c_1}{2}+c_3\right]^2 - 1\right] \tag{8.34}$$

With the coefficient positive then

$$f_{g_{1,1}}(z) - f_{g_{2,2}}(z) \geq 0 \tag{8.35}$$

provided the constraint of (8.24) is met. So provided (8.31) is met we have (for $-\ell \leq z \leq 0$)

$$f_{g_{1,1}}(z) \geq f_{g_{2,2}}(z) \geq f_{g_{1,2}}(z) = f_{g_{2,1}}(z) \geq 0 \tag{8.36}$$

which is sufficient for realizability of the transition.

The right section ($0 \leq z \leq \ell$) is now also realizable, all the results of this section applying, provided we replace the superscripts $- \rightarrow +$ and replace $-\ell \rightarrow \ell$. Together with the symmetry condition in (6.12) the constraint of (8.31) is satisfied. Note in (8.36) the indices 1 and 2 need to be interchanged, consistent with the discussion in Section VI.

IX. Exponential Interpolation

We are now in a position to select an appropriate form for the eigenvalues matching the required values at the ends of the sections of the transition. Various forms are possible as discussed in [4]. A convenient form for present purposes is exponential as

$$
\begin{aligned}
f_\beta^{(-)}(z) &= e^{2\alpha_\beta^{(-)}z} \\
\alpha_\beta^{(-)} &= -\frac{1}{2\ell}\,\ell n\left(f_\beta^{(-)}(-\ell)\right) \\
f_\beta^{(+)}(z) &= e^{2\alpha_\beta^{(+)}z} = f_\beta^{(-)}(-z) \\
\alpha_\beta^{(+)} &= \frac{1}{2\ell}\,\ell n\left(f_\beta^{(+)}(\ell)\right) = -\alpha_\beta^{(-)}
\end{aligned}
\tag{9.1}
$$

where symmetry between left and right sections is enforced. The constraint of (8.31) is now satisfied since (for $-\ell \le z \le 0$)

$$
\begin{aligned}
1 &\ge \frac{f_2^{(-)}(z)}{f_1^{(-)}(z)} = e^{2\left[\alpha_2^{(-)}-\alpha_1^{(+)}\right]z} \ge \frac{f_2^{(-)}(\ell)}{f_1^{(-)}(-\ell)} \\
\alpha_2^{(-)} - \alpha_1^{(-)} &= -\frac{1}{2\ell}\,\ell n\left(\frac{f_2^{(-)}(-\ell)}{f_1^{(-)}(-\ell)}\right) > 0 \\
z\left(\alpha_2^{(-)} - \alpha_1^{(-)}\right) &\le 0
\end{aligned}
\tag{9.2}
$$

and similarly for the right section.

From [4] with $z_0 = 0$ for our reference coordinate we have the general form of the matrizant as

$$\left(\left(\tilde{u}_{n,m}(z,0;s)\right)_{v,v'}\right) =$$

$$
\begin{pmatrix}
\sum_{\beta=1}^{2} e^{\alpha_\beta z}\left[\cosh\left(\left[\alpha_\beta^2+\gamma^2\right]^{\frac{1}{2}} z\right)-\frac{\alpha_\beta}{\left[\alpha_\beta^2+\gamma^2\right]^{\frac{1}{2}}}\sinh\left(\left[\alpha_\beta^2+\gamma^2\right]^{\frac{1}{2}} z\right)\right](g_n)_\beta\,(h_n)_\beta & -\sum_{\beta=1}^{2} e^{\alpha_\beta z}\frac{\gamma}{\left[\alpha_\beta^2+\gamma^2\right]^{\frac{1}{2}}}\sinh\left(\left[\alpha_\beta^2+\gamma^2\right]^{\frac{1}{2}} z\right)(g_n)_\beta\,(h_n)_\beta \\
-\sum_{\beta=1}^{2} e^{-\alpha_\beta z}\frac{\gamma}{\left[\alpha_\beta^2+\gamma^2\right]^{\frac{1}{2}}}\sinh\left(\left[\alpha_\beta^2+\gamma^2\right]^{\frac{1}{2}} z\right)(g_n)_\beta\,(h_n)_\beta & \sum_{\beta=1}^{2} e^{-\alpha_\beta z}\left[\cosh\left(\left[\alpha_\beta^2+\gamma^2\right]^{\frac{1}{2}} z\right)+\frac{\alpha_\beta}{\left[\alpha_\beta^2+\gamma^2\right]^{\frac{1}{2}}}\sinh\left(\left[\alpha_\beta^2+\gamma^2\right]^{\frac{1}{2}} z\right)\right](g_n)_\beta\,(h_n)_\beta
\end{pmatrix}
\tag{9.3}
$$

Here z can be positive or negative, so one has to interpret this result in terms of the -/+ superscripts depending on negative or positive z. To simplify the above for later use define

$$
A_\beta \equiv -\alpha_\beta^{(-)}\ell = \alpha_\beta^{(+)}\ell
$$

$$
\tilde{\Gamma}_\beta \equiv \left[\alpha_\beta^{(-)^2}+\gamma^2\right]^{\frac{1}{2}}\ell = \left[\alpha_\beta^{(-)^2}+\gamma^2\right]^{\frac{1}{2}}\ell \tag{9.4}
$$

$$
= \left[A_\beta^2+[\gamma\ell]^2\right]^{\frac{1}{2}}
$$

It is important to note that (9.3) applies to the case that the normalizing geometric-factor matrix is taken at z = 0. If another reference coordinate is chosen another set of eigenvalues etc. can be developed to handle the case. Nevertheless, by appropriate manipulations (9.3) can be manipulated into forms where other starting positions (e.g. $z = \pm\,\ell$) can be used, such as via (4.5).

Considering negative z first, we have

$$\left(\left(\tilde{u}^{(-)}_{n,m}(-\ell,0;s)\right)_{v,v'}\right) =$$

$$\begin{pmatrix} \sum_{\beta=1}^{2} e^{A_\beta}\left[\cosh\left(\tilde{\Gamma}_\beta\right)-\frac{A_\beta}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\right]\left(g_n^{(-)}\right)_\beta\left(h_n^{(-)}\right)_\beta & \sum_{\beta=1}^{2} e^{A_\beta}\frac{\gamma\ell}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\left(g_n^{(-)}\right)_\beta\left(h_n^{(-)}\right)_\beta \\ \sum_{\beta=1}^{2} e^{-A_\beta}\frac{\gamma\ell}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\left(g_n^{(-)}\right)_\beta\left(h_n^{(-)}\right)_\beta & \sum_{\beta=1}^{2} e^{-A_\beta}\left[\cosh\left(\tilde{\Gamma}_\beta\right)+\frac{A_\beta}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\right]\left(g_n^{(-)}\right)_\beta\left(h_n^{(-)}\right)_\beta \end{pmatrix} \tag{9.5}$$

Define an impedance-renormalization supermatrix (or geometric-factor-renormalization supermatrix) as

$$\left(\left(\zeta^{(-)}_{n,m}\right)_{v,v'}\right)$$

$$\equiv \begin{pmatrix} (1_{n,m}) & (0_{n,m}) \\ (0_{n,m}) & \sum_{\beta=1}^{2} e^{2A_\beta}\left(g_n^{(-)}\right)_\beta\left(h_n^{(-)}\right)_\beta \end{pmatrix}$$

$$\equiv \begin{pmatrix} (1_{n,m}) & (0_{n,m}) \\ (0_{n,m}) & \sum_{\beta=1}^{2} f_\beta^{(-)}(-\ell)\left(g_n^{(-)}\right)_\beta\left(h_n^{(-)}\right)_\beta \end{pmatrix} \tag{9.6}$$

Then we have

$$
\left(\left(\zeta^{(-)}_{n,m}\right)_{\upsilon,\upsilon'}\right) \odot \left(\left(\tilde{u}^{(-)}_{n,m}(-\ell,0;s)\right)_{\upsilon,\upsilon'}\right) =
$$

$$
\begin{pmatrix}
\sum_{\beta=1}^{2} e^{A_\beta}\left[\cosh\left(\tilde{\Gamma}_\beta\right)-\frac{A_\beta}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\right]\left(g_n^{(-)}\right)_\beta\left(h_n^{(-)}\right)_\beta & \sum_{\beta=1}^{2} e^{A_\beta}\frac{\gamma\ell}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\left(g_n^{(-)}\right)_\beta\left(h_n^{(-)}\right)_\beta \\
\sum_{\beta=1}^{2} e^{A_\beta}\frac{\gamma\ell}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\left(g_n^{(-)}\right)_\beta\left(h_n^{(-)}\right)_\beta & \sum_{\beta=1}^{2} e^{A_\beta}\left[\cosh\left(\tilde{\Gamma}_\beta\right)+\frac{A_\beta}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\right]\left(g_n^{(-)}\right)_\beta\left(h_n^{(-)}\right)_\beta
\end{pmatrix}
$$

(9.7)

$$
\left[\left(\left(\zeta^{(-)}_{n,m}\right)_{\upsilon,\upsilon'}\right) \odot \left(\left(\tilde{u}^{(-)}_{n,m}(-\ell,0;s)\right)_{\upsilon,\upsilon'}\right)\right]^{-1} =
$$

$$
\begin{pmatrix}
\sum_{\beta=1}^{2} e^{-A_\beta}\left[\cosh\left(\tilde{\Gamma}_\beta\right)+\frac{A_\beta}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\right]\left(g_n^{(-)}\right)_\beta\left(h_n^{(-)}\right)_\beta & -\sum_{\beta=1}^{2} e^{-A_\beta}\frac{\gamma\ell}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\left(g_n^{(-)}\right)_\beta\left(h_n^{(-)}\right)_\beta \\
-\sum_{\beta=1}^{2} e^{-A_\beta}\frac{\gamma\ell}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\left(g_n^{(-)}\right)_\beta\left(h_n^{(-)}\right)_\beta & \sum_{\beta=1}^{2} e^{-A_\beta}\left[\cosh\left(\tilde{\Gamma}_\beta\right)-\frac{A_\beta}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\right]\left(g_n^{(-)}\right)_\beta\left(h_n^{(-)}\right)_\beta
\end{pmatrix}
$$

The inverse can be checked by multiplying the two supermatrices, using the biorthonormality of the eigenvectors, the definitions in (9.4), and an identity for the hyperbolic functions, thereby obtaining the supermatrix identity. Using (4.5) gives

$$
\left(\left(\tilde{u}^{(-)}_{n,m}(0,-\ell;s)\right)_{\upsilon,\upsilon'}\right) = \left(\left(\tilde{u}^{(-)}_{n,m}(-\ell,0;s)\right)_{\upsilon,\upsilon'}\right)^{-1}
$$

$$
=\left[\left(\left(\zeta^{(-)}_{n,m}\right)_{\upsilon,\upsilon'}\right)^{-1} \odot \left(\left(\zeta^{(-)}_{n,m}\right)_{\upsilon,\upsilon'}\right) \odot \left(\left(\tilde{u}^{(-)}_{n,m}(-\ell,0;s)\right)_{\upsilon,\upsilon'}\right)\right]^{-1}
$$

$$
=\left[\left(\left(\zeta^{(-)}_{n,m}\right)_{\upsilon,\upsilon'} \odot \left(\left(\tilde{u}^{(-)}_{n,m}(-\ell,0;s)\right)_{\upsilon,\upsilon'}\right)\right)\right]^{-1} \odot \left(\left(\zeta^{(-)}_{n,m}\right)_{\upsilon,\upsilon'}\right) =
$$

$$
\begin{pmatrix}
\sum_{\beta=1}^{2} e^{-A_\beta}\left[\cosh\left(\tilde{\Gamma}_\beta\right)+\frac{A_\beta}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\right]\left(g_n^{(-)}\right)_\beta\left(h_n^{(-)}\right)_\beta & -\sum_{\beta=1}^{2} e^{A_\beta}\frac{\gamma\ell}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\left(g_n^{(-)}\right)_\beta\left(h_n^{(-)}\right)_\beta \\
-\sum_{\beta=1}^{2} e^{-A_\beta}\frac{\gamma\ell}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\left(g_n^{(-)}\right)_\beta\left(h_n^{(-)}\right)_\beta & \sum_{\beta=1}^{2} e^{A_\beta}\left[\cosh\left(\tilde{\Gamma}_\beta\right)-\frac{A_\beta}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\right]\left(g_n^{(-)}\right)_\beta\left(h_n^{(-)}\right)_\beta
\end{pmatrix}
\quad (9.8)
$$

Considering positive z second, we have

$$\left(\left(\tilde{u}^{(+)}_{n,m}(\ell,0;s)\right)_{v,v'}\right) =$$

$$\begin{pmatrix} \sum_{\beta=1}^{2} e^{A_\beta}\left[\cosh\left(\tilde{\Gamma}_\beta\right) - \frac{A_\beta}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\right]\left(g_n^{(+)}\right)_\beta \left(h_n^{(+)}\right)_\beta & -\sum_{\beta=1}^{2} e^{A_\beta}\frac{\gamma\ell}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\left(g_n^{(+)}\right)_\beta \left(h_n^{(+)}\right)_\beta \\ -\sum_{\beta=1}^{2} e^{-A_\beta}\frac{\gamma\ell}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\left(g_n^{(+)}\right)_\beta \left(h_n^{(+)}\right)_\beta & \sum_{\beta=1}^{2} e^{-A_\beta}\left[\cosh\left(\tilde{\Gamma}_\beta\right) + \frac{A_\beta}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\right]\left(g_n^{(+)}\right)_\beta \left(h_n^{(+)}\right)_\beta \end{pmatrix} \quad (9.9)$$

This is the same as (9.5) except for the change in eigenvectors from - to + type (which as discussed in Section VI corresponds merely to an interchange of the 1 and 2 elements of the eigenvectors), and the change of some signs. Carrying this through we have

$$\left(\left(\zeta^{(+)}_{n,m}\right)_{v,v'}\right)$$

$$\equiv \begin{pmatrix} \left(1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \sum_{\beta=1}^{2} e^{2A_\beta}\left(g_n^{(+)}\right)_\beta \left(h_n^{(+)}\right)_\beta \end{pmatrix}$$

$$\equiv \begin{pmatrix} \left(1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \sum_{\beta=1}^{2} f_\beta^{(+)}(\ell)\left(g_n^{(+)}\right)_\beta \left(h_n^{(+)}\right)_\beta \end{pmatrix}$$

$$\left(\left(\tilde{u}^{(+)}_{n,m}(0,\ell;s)\right)_{v,v'}\right) = \left(\left(\tilde{u}^{(+)}_{n,m}(\ell,0;s)\right)_{v,v'}\right)^{-1} =$$

$$\begin{pmatrix} \sum_{\beta=1}^{2} e^{-A_\beta}\left[\cosh\left(\tilde{\Gamma}_\beta\right) + \frac{A_\beta}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\right]\left(g_n^{(+)}\right)_\beta \left(h_n^{(+)}\right)_\beta & \sum_{\beta=1}^{2} e^{A_\beta}\frac{\gamma\ell}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\left(g_n^{(+)}\right)_\beta \left(h_n^{(+)}\right)_\beta \\ \sum_{\beta=1}^{2} e^{-A_\beta}\frac{\gamma\ell}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\left(g_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta & \sum_{\beta=1}^{2} e^{A_\beta}\left[\cosh\left(\tilde{\Gamma}_\beta\right) - \frac{A_\beta}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\right]\left(g_n^{(+)}\right)_\beta \left(h_n^{(+)}\right)_\beta \end{pmatrix} \quad (9.10)$$

In (4.8) the matrizant for the compound transition, involving both left and right sections, is introduced. This involves products of the separate matrizants already exhibited in this section. The representation is straightforward, but unfortunately the eigenvectors are different in each case (due to the 1, 2 interchange of the conductors), and they are not in general mutually orthogonal. So products of left and right eigenvectors appear, and instead of a single summation over β, a double summation (over say $\beta,\ \beta'$) appears.

Consider first the scalar products for the same βs, for which we have

$$
\begin{aligned}
\left(h_n^{(-)}\right)_\beta \cdot \left(g_n^{(+)}\right)_\beta &= \left(h_n^{(+)}\right)_\beta \cdot \left(g_n^{(-)}\right)_\beta \\
&= p_\beta^{(+)}\, q_\beta^{(-)} \begin{pmatrix} 1 \\ -\frac{c_1}{2} \pm c_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -\frac{c_1}{2} \pm c_3 \end{pmatrix} \\
&= p_\beta^{(+)}\, q_\beta^{(+)} \left[1 - \frac{c_1^2}{4} + c_3^2\right] \qquad (9.11) \\
&= p_\beta^{(+)}\, q_\beta^{(+)} \left[2 + c_2\right] = \pm\frac{1}{2c_3}\left[2 + c_2\right]
\end{aligned}
$$

These pairs of vectors can be orthogonal provided

$$
\begin{aligned}
0 &= 2 + c_2 = 2 + \frac{f_g^{(in)}}{f_g^{(out)}}\left[\frac{1}{2} - 2\frac{f_g^{(c)}}{f_g^{(d)}}\right] \\
\frac{f_g^{(out)}}{f_g^{(in)}} &= \frac{f_g^{(c)}}{f_g^{(d)}} - \frac{1}{4}
\end{aligned}
\qquad (9.12)
$$

i.e. for a certain relation among the geometric factors. Let this relation define a special case for the balun. For this special case the roles of the two βs are interchanged on passing through z = 0 and one has a common set of eigenvectors for the entire balun. Note that the two vectors above can never be parallel since c_1 and c_3 are both positive.

For the scalar products with different βs define

$$\beta_c \equiv 3-\beta \quad \text{(complementary eigenindex)}$$

$$\beta = \begin{cases}1\\2\end{cases} \Leftrightarrow \beta_c = \begin{cases}2\\1\end{cases} \tag{9.13}$$

so that we now have

$$\left(h_n^{(-)}\right)_\beta \cdot \left(g_n^{(+)}\right)_{\beta_c} = p_{\beta_c}^{(+)} q_\beta^{(-)} \begin{pmatrix} 1 \\ \\ -\frac{c_1}{2} \pm c_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \\ \frac{c_1}{2} \mp c_3 \end{pmatrix}$$

$$= p_{\beta_c}^{(+)} q_\beta^{(-)} \left[1 - \left[\frac{c_1}{2} \pm c_3\right]^2\right]$$

$$\left(h_n^{(+)}\right)_\beta \cdot \left(g_n^{(-)}\right)_{\beta_c} = p_{\beta_c}^{(-)} q_\beta^{(+)} \begin{pmatrix} -\frac{c_1}{2} \pm c_3 \\ \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{c_1}{2} \pm c_3 \\ \\ 1 \end{pmatrix} \tag{9.14}$$

$$= p_{\beta_c}^{(-)} q_\beta^{(+)} \left[1 - \left[\frac{c_1}{2} \pm c_3\right]^2\right]$$

where the coefficients are found in (8.9) and (8.10). The common factor is

$$1 - \left[\frac{c_1}{2} \pm c_3\right]^2 = \left[1 + \frac{c_1}{2} \pm c_3\right]\left[1 - \frac{c_1}{2} \mp c_3\right] \tag{9.15}$$

The first factor is non zero (and strictly positive) from (5.7). Furthermore

$$c_3 = \left[\left[1 - \frac{c_1}{2}\right]^2 + c_1 + c_2\right]^{\frac{1}{2}} \neq 1 - \frac{c_1}{2} \tag{9.16}$$

since from (5.2)

$$c_1 + c_2 = \frac{f_g^{(in)}}{f_g^{(out)}} > 0 \tag{9.17}$$

Hence the products in (9.14) are non zero.

X. Special Case of Inverter

A matrizant can be constructed for the entire inverter ($-\ell \le z \le \ell$) as a single section through the construction of an interpolation matrix via

$$
\left(f^{(2)}_{n,m}(\ell)\right) = \left(f_{g_{n,m}}(\ell)\right) \cdot \left(f_{g_{n,m}}(-\ell)\right)^{-1}
$$

$$
= \frac{1}{f_g^{(in)}\, f_g^{(out)}} \begin{pmatrix} f_g^{(out)} & f_g^{(out)} \\ f_g^{(out)} & f_g^{(in)} + f_g^{(out)} \end{pmatrix} \cdot \begin{pmatrix} f_g^{(out)} & -f_g^{(out)} \\ -f_g^{(out)} & f_g^{(in)} + f_g^{(out)} \end{pmatrix}
$$

$$
= \frac{1}{f_g^{(in)}\, f_g^{(out)}} \begin{pmatrix} 0 & f_g^{(in)} f_g^{(out)} \\ -f_g^{(in)}\, f_g^{(out)} & f_g^{(in)2} + 2 f_g^{(in)}\, f_g^{(out)} \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & 1 \\ -1 & 2c_4 \end{pmatrix}
$$

$$
c_4 \equiv \frac{f_g^{(in)}}{2 f_g^{(out)}} + 1 \equiv \cosh(c_5)
$$

$$
\det\left(\left(f^{(2)}_{n,m}(\ell)\right)\right) = 1 = f_1^{(2)}(\ell)\; f_2^{(2)}(\ell) \tag{10.1}
$$

$$
\mathrm{tr}\left(\left(f^{(2)}_{n,m}(\ell)\right)\right) = 2c_4 = f_1^{(2)}(\ell) + f_2^{(2)}(\ell)
$$

where a superscript 2 is used to distinguish this special case. The eigenvalues are

$$
f_\beta^{(2)^2}(\ell) - 2c_4\, f_\beta^{(2)}(\ell) + 1 = 0
$$

$$
f_\beta^{(2)}(\ell) = c_4 \pm \sqrt{c_4^2 - 1} = e^{\pm c_5} \tag{10.2}
$$

$$
\frac{f_1^{(2)}(\ell)}{f_2^{(2)}(\ell)} = e^{2c_5}
$$

with both evidently strictly positive. Exponential interpolation of these eigenvalues gives

$$f_{\beta}^{(2)}(z) = e^{2\alpha_{\beta}^{(2)}}\left[z+\ell\right]$$

$$\alpha_{\beta}^{(2)} = \frac{1}{4\ell}\,\ell n\left(f_{\beta}^{(2)}(\ell)\right) = \pm\frac{1}{4\ell}c_5 \qquad (10.3)$$

$$f_{\beta}^{(2)}(-\ell) = 1$$

which are also strictly positive. Note now that $z = -\ell$ is taken as the reference coordinate, although $z = +\ell$ would do as well. The form at $z = 0$ is constrained (not specified a priori) by the interpolation.

The right eigenvectors are

$$\begin{pmatrix} 0 & 1 \\ -1 & 2c_4 \end{pmatrix} \cdot \left(g_n^{(2)}\right)_\beta = f_{\beta}^{(2)}(\ell)\left(g_n^{(2)}\right)_\beta$$

$$-\ f_{\beta}^{(2)}(\ell)\, g_{1;\beta}^{(2)} + g_{2;\beta}^{(2)} = 0$$

$$-\ g_{1;\beta}^{(2)} + \left[2c_4 - f_{\beta}^{(2)}(\ell)\right] g_{2;\beta}^{(2)} = 0$$

$$\frac{g_{2;\beta}^{(2)}}{g_{1;\beta}^{(2)}} = f_{\beta}^{(2)}(\ell) = \left[2c_4 - f_{\beta}^{(2)}(\ell)\right]^{-1}$$

$$\left(g_n^{(2)}\right)_\beta = p_{\beta}^{(2)}\begin{pmatrix} 1 \\ f_{\beta}^{(2)}(\ell) \end{pmatrix} = p_{\beta}^{(2)}\begin{pmatrix} 1 \\ e^{\pm c_5} \end{pmatrix} \qquad (10.4)$$

$p_{\beta}^{(2)} \equiv$ normalization constant .

The left eigenvectors are

$$\left(h_n^{(2)}\right)_\beta \cdot \begin{pmatrix} 0 & 1 \\ -1 & 2c_4 \end{pmatrix} = f_\beta^{(2)}(\ell)\left(h_n^{(2)}\right)_\beta$$

$$-\ f_\beta^{(2)}(\ell)\ h_{1;\beta}^{(2)} - h_{2;\beta}^{(2)} = 0$$

$$h_{1;\beta}^{(2)} + \left[2c_4 - f_\beta^{(2)}(\ell)\right] h_{2;\beta}^{(2)} = 0$$

$$\frac{h_{2;\beta}^{(2)}}{h_{1;\beta}^{(2)}} = -\, f_\beta^{(2)}(\ell) = -\left[2c_4 - f_\beta^{(2)}(\ell)\right]^{-1}$$

$$\left(h_n^{(2)}\right)_\beta = q_\beta^{(2)} \begin{pmatrix} 1 \\ -f_\beta^{(2)}(\ell) \end{pmatrix} = q_\beta^{(2)} \begin{pmatrix} 1 \\ -e^{\pm c_5} \end{pmatrix} \qquad (10.5)$$

$q_\beta^{(2)}$ ≡ normalization constant

Biorthonormalization gives

$$\left(h_n^{(2)}\right)_{\beta_1} \cdot \left(g_n^{(2)}\right)_{\beta_2} = 1_{\beta_1,\beta_2}$$

$$\left(h_n^{(2)}\right)_\beta \cdot \left(g_n^{(2)}\right)_\beta = p_\beta^{(2)}\, q_\beta^{(2)} \left[1 - f_\beta^{(2)^2}(\ell)\right] = 1$$

$$p_\beta^{(2)}\, q_\beta^{(2)} = \left[1 - f_\beta^{(2)^2}(\ell)\right]^{-1} = \left[1 - e^{\pm 2c_5}\right]^{-1}$$

$$= \mp\ 2e^{\mp c_5} \sinh^{-1}(c_5) \qquad (10.6)$$

$$\sinh(c_5) = \left[c_4^2 - 1\right]^{\frac{1}{2}}$$

Normalization via the geometric-factor matrix gives

$$\left(g_n^{(2)}\right)_\beta = \left(h_n^{(2)}\right)_\beta \cdot \left(f_{g_{n,m}}(-\ell)\right) = \left(f_{g_{n,m}}(-\ell)\right) \cdot \left(h_n^{(2)}\right)_\beta$$

$$\left(g_n^{(2)}\right)_\beta = p_\beta^{(2)} \begin{pmatrix} 1 \\ f_\beta^{(2)}(\ell) \end{pmatrix} = \left(f_{g_{n,m}}(-\ell)\right) \cdot \left(h_n^{(2)}\right)_\beta$$

$$= q_\beta^{(2)} \begin{pmatrix} f_g^{(in)} + f_g^{(out)} & f_g^{(out)} \\ f_g^{(out)} & f_g^{(out)} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -f_\beta^{(2)}(\ell) \end{pmatrix}$$

$$\frac{p_\beta^{(2)}}{q_\beta^{(2)}} = f_g^{(in)} + f_g^{(out)} \left[1 - f_\beta^{(2)}(\ell)\right]$$

$$= \frac{f_g^{(out)}}{f_\beta^{(2)}(\ell)} \left[1 - f_\beta^{(2)}(\ell)\right]$$

$$p_\beta^{(2)^2} = \frac{f_g^{(out)}}{f_\beta^{(2)}(\ell)\left[1 + f_\beta^{(2)}(\ell)\right]} > 0 \qquad (10.7)$$

$$q_\beta^{(2)^2} = \frac{f_\beta^{(2)}(\ell)}{f_g^{(out)}} \left[1 - f_\beta^{(2)}(\ell)\right]^{-2} \left[1 + f_\beta^{(2)}(\ell)\right]^{-1} > 0$$

So both normalization coefficients are real as are the eigenvectors.

As before the interpolated geometric-factor matrix is constructed as

$$\left(f_{g_{n,m}}(z)\right) = \sum_{\beta=1}^{2} f_\beta^{(2)}(z) \quad \left(g_n^{(2)}\right)_\beta \left(g_n^{(2)}\right)_\beta$$

$$= \sum_{\beta=1}^{2} p_\beta^{(2)^2} \; f_\beta^{(2)}(z) \begin{pmatrix} 1 \\ f_\beta \end{pmatrix} \begin{pmatrix} 1 \\ f_\beta \end{pmatrix}$$

$$= \sum_{\beta=1}^{2} \frac{f_g^{(out)} \, f_\beta^{(2)}(z)}{f_\beta^{(2)}(\ell)\left[1 + f_\beta^{(2)}(\ell)\right]} \begin{pmatrix} 1 & f_\beta^{(2)}(\ell) \\ f_\beta^{(2)}(\ell) & f_\beta^{(2)^2}(\ell) \end{pmatrix}$$

$$= f_g^{(out)} \sum_{\beta=1}^{2} \frac{f_\beta^{(2)}(z)}{1 + f_\beta^{(2)}(\ell)} \begin{pmatrix} f_\beta^{(2)^{-1}}(\ell) & 1 \\ 1 & f_\beta^{(2)}(\ell) \end{pmatrix}$$

$$= f_g^{(out)} \sum_{\beta=1}^{2} \frac{e^{\pm\alpha_\beta^{(2)} z}}{2\cosh\left(\frac{c_5}{2}\right)} \begin{pmatrix} e^{\mp c_5} & 1 \\ 1 & e^{\pm c_5} \end{pmatrix} \tag{10.8}$$

The terms in the $\beta = 1$ matrix evaluated at z equal the corresponding terms in the $\beta = 2$ matrix evaluated at -z. Summing these two matrices gives an even function, i.e.

$$\left(f_{g_{n,m}}(-z)\right) = \left(f_{g_{n,m}}(z)\right) \tag{10.9}$$

consistent with the previous two-section balun.

Considering the realizability conditions, besides the positive eigenvalues we have all positive elements. Furthermore we have

$$f_{g_{1,1}}(z) - f_{g_{1,2}}(z) = \frac{f_g^{(out)}}{2\cosh\left(\frac{c_5}{2}\right)} \sum_{\beta=1}^{2} e^{\pm\left[\alpha_\beta z - \frac{c_5}{2}\right]} \left[e^{\mp\frac{c_5}{2}} - e^{\pm\frac{c_5}{2}}\right]$$

$$= \frac{f_g^{(out)}}{\cosh\left(\frac{c_5}{2}\right)} \sum_{\beta=1}^{2} \mp e^{\pm\left[\alpha_\beta z - \frac{c_5}{2}\right]} \sinh\left(\frac{c_5}{2}\right)$$

$$= f_g^{(out)} \tanh\left(\frac{c_5}{2}\right) \sinh\left(\frac{c_5}{2} - \alpha_\beta z\right) \tag{10.10}$$

$$\geq 0$$

$$\left|\alpha_\beta z\right| \leq \frac{c_5}{2} \text{ for } -\ell \leq z \leq \ell$$

with a similar result giving

$$f_{g_{2,2}}(z) - f_{g_{1,2}}(z) \geq 0 \tag{10.11}$$

so this interpolated geometric-factor matrix meets the realizability conditions discussed in Section V.

At z = 0 the geometric factor matrix is

$$\left(f_{g_{n,m}}(0)\right) = \frac{f_g^{(out)}}{2\cosh\left(\frac{c_5}{2}\right)} \sum_{\beta=1}^{2} \begin{pmatrix} e^{\mp c_5} & 1 \\ 1 & e^{\pm c_5} \end{pmatrix}$$

$$= \frac{f_g^{(out)}}{\cosh\left(\frac{c_5}{2}\right)} \begin{pmatrix} \cosh(c_5) & 1 \\ 1 & \cosh(c_5) \end{pmatrix} \tag{10.12}$$

This has equal 1,1 and 2,2 elements as required by the symmetry in (3.6). In addition we have the ratio of elements which can be related in each case to give

$$\frac{f_{g_{1,1}}(0)}{f_{g_{1,2}}(0)} = \cosh(c_5) = \frac{f_g^{(in)}}{2f_g^{(out)}} + 1 = \frac{\frac{f_g^{(c)}}{f_g^{(d)}} + \frac{1}{4}}{\frac{f_g^{(c)}}{f_g^{(d)}} - \frac{1}{4}}$$

$$\frac{f_g^{(out)}}{f_g^{(in)}} = \frac{f_g^{(c)}}{f_g^{(d)}} - \frac{1}{4} \tag{10.13}$$

which is precisely the relationship in (9.12) for the special case of common eigenvectors in the left and right sections of the inverter. This is not enough by itself to uniquely specify $f_g^{(d)}$ and $f_g^{(c)}$ in terms of $f_g^{(in)}$ and $f_g^{(out)}$. These need to be scaled so that the exponential interpolation (for the entire $-\ell \le z \le \ell$) gives the parameters at z = 0. Alternately one can use two different interpolations for the two sections of the balun to obtain another degree of freedom.

To complete the specification of the parameters at z = 0 for the single exponential interpolation we have

$$\cosh(c_5) = c_4 = \frac{f_g^{(in)}}{2f_g^{(out)}} + 1$$

$$\cosh\left(\frac{c_5}{2}\right) = \left[\frac{\cosh(c_5) + 1}{2}\right]^{\frac{1}{2}} = \left[\frac{c_4 + 1}{2}\right]^{\frac{1}{2}} \qquad (10.14)$$

$$= \left[\frac{f_g^{(in)}}{2f_g^{(out)}} + 1\right]^{\frac{1}{2}}$$

Comparing (10.12) to (3.6) we have

$$f_g^{(c)} = \frac{1}{2}\left[f_{g_{1,1}}(0) + f_{g_{1,2}}(0)\right] = \frac{1}{2} f_g^{(out)} \frac{\cosh(c_5)+1}{\cosh\left(\frac{c_5}{2}\right)}$$

$$= f_g^{(out)}\left[\frac{f_g^{(in)}}{4\,f_g^{(out)}} + 1\right]^{\frac{1}{2}}$$

$$f_g^{(d)} = 2\left[f_{g_{1,1}}(0) - f_{g_{1,2}}(0)\right] = 2f_g^{(out)} \frac{\cosh(c_5)+1}{\cosh\left(\frac{c_5}{2}\right)} \qquad (10.15)$$

$$= f_g^{(in)}\left[\frac{f_g^{(in)}}{4\,f_g^{(out)}} + 1\right]^{-\frac{1}{2}}$$

So the beginning and ending conditions ($z = \mp \ell$) constrain $f_g^{(c)}$ and $f_g^{(d)}$ at $z = 0$ for the exponential interpolation for the entire inverter to apply. For large ratio of outside to inside impedance we have

$$f_g^{(c)} \to f_g^{(out)} \;,\; f_g^{(d)} \to f_g^{(in)} \;\; as \;\; \frac{f_g^{(in)}}{f_g^{(out)}} \to 0 \qquad (10.16)$$

which is a kind of constant-impedance condition through the inverter.

With this exponential interpolation as in (10.3) the matrizant for the inverter can be written from (9.3) as

$$A_\beta^{(2)} \equiv 2\alpha_\beta^{(2)}\ell = \pm\frac{c_5}{2}, \; A_1^{(2)} + A_2^{(2)} = 0$$

$$\tilde{\Gamma}_\beta^{(2)} \equiv \left[\left[2\alpha_\beta^{(2)}\ell\right]^2 + \left[2\gamma\ell\right]^2\right]^{\frac{1}{2}} = \left[A_\beta^{(2)2} + \left[2\gamma\ell\right]^2\right]^{\frac{1}{2}}$$

$$\left(\left(\tilde{u}_{n,m}^{(2)}(\ell,-\ell;s)\right)_{v,v'}\right) =$$

$$\begin{pmatrix} \sum_{\beta=1}^{2} e^{A_\beta^{(2)}}\left[\cosh\left(\tilde{\Gamma}_\beta^{(2)}\right) - \frac{A_\beta^{(2)}}{\tilde{\Gamma}_\beta^{(2)}}\sinh\left(\tilde{\Gamma}_\beta^{(2)}\right)\right]\left(g_n^{(2)}\right)_\beta \left(h_n^{(2)}\right)_\beta & -\sum_{\beta=1}^{2} e^{A_\beta^{(2)}}\frac{2\gamma\ell}{\tilde{\Gamma}_\beta^{(2)}}\sinh\left(\tilde{\Gamma}_\beta^{(2)}\right)\left(g_n^{(2)}\right)_\beta \left(h_n^{(2)}\right)_\beta \\ -\sum_{\beta=1}^{2} e^{-A_\beta^{(2)}}\frac{2\gamma\ell}{\tilde{\Gamma}_\beta^{(2)}}\sinh\left(\tilde{\Gamma}_\beta^{(2)}\right)\left(g_n^{(2)}\right)_\beta \left(h_n^{(2)}\right)_\beta & \sum_{\beta=1}^{2} e^{-A_\beta^{(2)}}\left[\cosh\left(\tilde{\Gamma}_\beta^{(2)}\right) + \frac{A_\beta^{(2)}}{\tilde{\Gamma}_\beta^{(2)}}\sinh\left(\tilde{\Gamma}_\beta^{(2)}\right)\right]\left(g_n^{(2)}\right)_\beta \left(h_n^{(2)}\right)_\beta \end{pmatrix}$$

(10.17)

where the starting point (boundary condition) is taken at z = $-\ell$ instead of z = 0, a simple shift in (9.3). As in Section IX define an impedance-renormalization supermatrix as

$$\left(\left(\zeta_{n,m}^{(2)}\right)_{v,v'}\right)$$

$$= \begin{pmatrix} \left(1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \sum_{\beta=1}^{2} e^{2A_\beta^{(2)}}\left(g_n^{(2)}\right)_\beta \left(h_n^{(2)}\right)_\beta \end{pmatrix}$$

$$= \begin{pmatrix} \left(1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \sum_{\beta=1}^{2} f_\beta^{(2)}(\ell)\left(g_n^{(2)}\right)_\beta \left(h_n^{(2)}\right)_\beta \end{pmatrix} \quad (10.18)$$

from which we obtain

$$\left(\left(\tilde{u}^{(2)}_{n,m}(-\ell,\ell;s)\right)_{v,v'}\right) = \left(\left(\tilde{u}^{(2)}_{n,m}(\ell,-\ell;s)\right)_{v,v'}\right)^{-1} =$$

$$\begin{pmatrix} \sum_{\beta=1}^{2} e^{-A^{(2)}_\beta}\left[\cosh\left(\tilde{\Gamma}^{(2)}_\beta\right)+\frac{A^{(2)}_\beta}{\tilde{\Gamma}^{(2)}_\beta}\sinh\left(\tilde{\Gamma}^{(2)}_\beta\right)\right]\left(g^{(2)}_n\right)_\beta\left(h^{(2)}_n\right)_\beta & \sum_{\beta=1}^{2} e^{A^{(2)}_\beta}\frac{2\gamma\ell}{\tilde{\Gamma}^{(2)}_\beta}\sinh\left(\tilde{\Gamma}^{(2)}_\beta\right)\left(g^{(2)}_n\right)_\beta\left(h^{(2)}_n\right)_\beta \\ \sum_{\beta=1}^{2} e^{-A^{(2)}_\beta}\frac{2\gamma\ell}{\tilde{\Gamma}^{(2)}_\beta}\sinh\left(\tilde{\Gamma}^{(2)}_\beta\right)\left(g^{(2)}_n\right)_\beta\left(h^{(2)}_n\right)_\beta & \sum_{\beta=1}^{2} e^{A^{(2)}_\beta}\left[\cosh\left(\tilde{\Gamma}^{(2)}_\beta\right)-\frac{A^{(2)}_\beta}{\tilde{\Gamma}^{(2)}_\beta}\sinh\left(\tilde{\Gamma}^{(2)}_\beta\right)\right]\left(g^{(2)}_n\right)_\beta\left(h^{(2)}_n\right)_\beta \end{pmatrix}$$

(10.19)

XI. Boundary Condition at z = -ℓ

At the input to the transition section (z = -ℓ), one can define an input-impedance matrix (2 x 2) via

$$\left(\tilde{V}_n(-\ell,s)\right) = \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n(-\ell,s)\right) \tag{11.1}$$

with current convention to the right (increasing z) as indicated in fig. 1.1. As the particular form of this impedance matrix depends on the details of the transition section (to the right), the superscript is here left blank, to be specified later by the particular case under consideration. In this section the same convention is used for other related parameters corresponding to the transition section and load to the right.

Consider a wave incident from the left ($z \leq -\ell$) as

$$\begin{aligned}\left(\tilde{V}_n^{(inc)}(z,s)\right) &= Z_w\left(f_{g_{n,m}}(-\ell)\right) \cdot \left(\tilde{I}_n^{(inc)}(z,s)\right) \\ &= \left(\tilde{V}^{(inc)}(0,s)\right) e^{-\gamma z} \\ \left(\tilde{I}_n^{(inc)}(z,s)\right) &= \left(\tilde{I}_n^{(inc)}(0,s)\right) e^{-\gamma z}\end{aligned} \tag{11.2}$$

Here the excitation takes the form of a wave in the inner coaxial region ($\Psi_a \leq \Psi \leq \Psi_b$ in fig. 2.1) as

$$\tilde{V}^{(inc)}(z,s) = V_o e^{-\gamma z} = Z_w\, f_g^{(in)}\, \tilde{I}^{(inc)}(z,s) \tag{11.3}$$

The current is interpreted in the sense of I_1, and voltage in the sense of $V_1 - V_2$, i.e.

$$\begin{aligned}\tilde{V}^{(inc)}(z,s) &= \tilde{V}_1^{(inc)}(z,s) - \tilde{V}_2^{(inc)}(z,s) \\ \tilde{I}^{(inc)}(z,s) &= \tilde{I}_1^{(inc)}(z,s) = -\tilde{I}_2^{(inc)}(z,s)\end{aligned} \tag{11.4}$$

and since there is no incident wave in the outer region

$$\tilde{V}_2^{(inc)}(z,s) = 0\,,\ \tilde{I}_1^{(inc)}(z,s) + \tilde{I}_2^{(inc)}(z,s) = 0 \tag{11.5}$$

In vector form this is

$$\tilde{V}_n^{(inc)}(z,s) = V_o r^{-\gamma z}\begin{pmatrix}1\\0\end{pmatrix}$$

$$\left(\tilde{I}_n^{(inc)}(z,s)\right) = \frac{1}{Z_w}\left(f_{g_{n,m}}(-\ell)\right)^{-1}\cdot\left(\tilde{V}_n^{(inc)}(z,s)\right) = \frac{V_o r^{-\gamma z}}{Z_w\, f_g^{(in)}}\begin{pmatrix}1\\-1\end{pmatrix} \tag{11.6}$$

At the input location ($z = -\ell$) there is a scattered wave going to the left given by

$$\tilde{V}_n^{(sc)}(-\ell,s) \equiv \left(\tilde{S}_{n,m}(s)\right)\cdot\left(\tilde{V}_n^{(inc)}(-\ell,s)\right)$$

$$\left(\tilde{S}_{n,m}(s)\right) \equiv \text{scattering matrix at input} \tag{11.7}$$

$$\left(\tilde{V}_n^{(sc)}(z,s)\right) = -Z_w\left(f_{g_{n,m}}(-\ell)\right)\cdot\left(\tilde{I}_n^{(sc)}(z,s)\right)$$

The total voltages and currents at the input are

$$\begin{aligned}
\left(\tilde{V}_n(z,s)\right) &= \left(\tilde{V}_n^{(inc)}(-\ell,s)\right) + \left(\tilde{V}^{(sc)}(-\ell,s)\right) = \\
&= \left[\left(1_{n,m}\right) + \left(S_{n,m}(s)\right)\right]\cdot\left(\tilde{V}_n^{(inc)}(-\ell,s)\right) \\
&= \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{I}_n(-\ell,s)\right) \\
&= \left(\tilde{Z}_{n,m}(s)\right)\cdot\left[\left(\tilde{I}_n^{(inc)}(-\ell,s)\right) + \left(\tilde{I}_n^{(sc)}(-\ell,s)\right)\right] \\
&= \left(\tilde{Z}_{n,m}(s)\right)\cdot\left[\left(1_{n,m}\right) - \left(f_{g_{n,m}}(-\ell)\right)^{-1}\cdot\left(S_{n,m}(s)\right)\cdot\left(f_{g_{n,m}}(-\ell)\right)\right]\cdot\left(\tilde{I}_n^{(inc)}(-\ell,s)\right)
\end{aligned} \tag{11.8}$$

The scattering matrix is related to the input-impedance matrix in the usual way [2] as

$$\left(\tilde{S}_{n,m}(s)\right) = \left[\left(\tilde{Z}_{n,m}(s)\right) \cdot \left(Z_{c_{n,m}}(-\ell)\right)^{-1} + \left(1_{n,m}\right)\right]^{-1}$$

$$\cdot \left[\left(\tilde{Z}_{n,m}(s)\right) \cdot \left(Z_{c_{n,m}}(-\ell)\right)^{-1} - \left(1_{n,m}\right)\right]$$

$$= \left[\left(1_{n,m}\right) + \left(Z_{c_{n,m}}(-\ell)\right) \cdot \left(\tilde{Z}_{n,m}(s)\right)^{-1}\right]^{-1} \tag{11.9}$$

$$\cdot \left[\left(1_{n,m}\right) - \left(Z_{c_{n,m}}(-\ell)\right) \cdot \left(\tilde{Z}_{n,m}(s)\right)^{-1}\right]$$

Noting the commutation of the "numerator" and "denominator" matrices, the above factors can be taken in any order in (11.9). This can be rewritten as

$$\left(\tilde{Z}_{n,m}(s)\right) \cdot \left(Z_{c_{n,m}}(-\ell)\right)^{-1} = \left[\left(1_{n,m}\right) - \left(\tilde{S}_{n,m}(s)\right)\right]^{-1} \cdot \left[\left(1_{n,m}\right) + \left(\tilde{S}_{n,m}(s)\right)\right] \tag{11.10}$$

with the right-side terms commuting as before.

Included in the scattered wave propagating to the left is a wave in the outer region $(\Psi_b \leq \Psi \leq \Psi_o)$. This propagates into a resistive load $Z_w\ f_g^{(out)}$, and is the only wave (i.e. no incident wave) in this region. The effect of this wave is included in the scattering matrix introduced above.

A scattering parameter of interest concerns the wave to the left in the inner coaxial region $(\Psi_a \leq \Psi \leq \Psi_b)$ as

$$\tilde{V}^{(sc)}(-\ell,s) \equiv \tilde{V}_1^{(sc)}(-\ell,s) - \tilde{V}_2^{(sc)}(-\ell,s)$$

$$= \tilde{S}(s)\ \tilde{V}^{(inc)}(-\ell,s)$$

$$= \tilde{S}(s)\left[\tilde{V}_1^{(inc)}(-\ell,s) - \tilde{V}_2^{(inc)}(-\ell,s)\right]$$

$$\tilde{I}^{(sc)}(-\ell,s) = \tilde{I}_1^{(sc)}(-\ell,s) \tag{11.11}$$

$$= -\tilde{S}(s)\ \tilde{I}^{(inc)}(-\ell,s)$$

$$= -\tilde{S}(s)\left[\tilde{I}_1^{(inc)}(-\ell,s)\right]$$

This is the scattering directly back into the source which is driving the transition. At the input then we also have for the inner coaxial region

$$\tilde{V}^{(in)}(-\ell,s) = \tilde{V}_1(-\ell,s) - \tilde{V}_2(-\ell,s)$$

$$= \tilde{Z}_{in}(s)\,\tilde{I}^{(in)}(-\ell,s) = \tilde{Z}_{in}(s)\;\tilde{I}_1(-\ell,s) \tag{11.12}$$

$$\tilde{Z}_{in}(s) \equiv \text{input impedance at coaxial port}$$

These can be related to the incident wave in the coax as

$$\tilde{V}^{(in)}(-\ell,s) = \left[1+\tilde{S}(s)\right]\;\tilde{V}^{(inc)}(-\ell,s)$$

$$\tilde{I}^{(in)}(-\ell,s) = \left[1-\tilde{S}(s)\right]\;\tilde{I}^{(inc)}(-\ell,s)$$

$$\tilde{S}(s) = \frac{\tilde{Z}_{in}(s) - Z_w\,f_g^{(in)}}{\tilde{Z}_{in}(s) + Z_w\,f_g^{(in)}}$$

$$\equiv \text{reflection (scattering) coeffient back into input coax} \tag{11.13}$$

$$\tilde{Z}_{in}(s) = Z_w\,f_g^{(in)}\;\frac{1+\tilde{S}(s)}{1-\tilde{S}(s)}$$

Having the scattering matrix in (11.9) one can obtain the scattering parameter for the coax in (11.11) via

$$\left(\tilde{V}_n^{(sc)}(-\ell,s)\right) = \left(\tilde{S}_{n,m}(s)\right)\cdot\left(\tilde{V}_n^{(inc)}(-\ell,s)\right)$$

$$= V_o e^{\gamma\ell}\left(\tilde{S}_{n,m}(s)\right)\cdot\begin{pmatrix}1\\0\end{pmatrix}$$

$$= V_o e^{\gamma\ell}\begin{pmatrix}\tilde{S}_{1,1}(s)\\ \\ \tilde{S}_{2,1}(s)\end{pmatrix}$$

$$\tilde{V}^{(sc)}(-\ell,s) = \tilde{V}_1^{(sc)}(-\ell,s) - \tilde{V}_2^{(sc)}(-\ell,s)$$

$$= V_o e^{\gamma\ell}\left[\tilde{S}_{1,1}(s) - \tilde{S}_{2,1}(s)\right]$$

$$= \tilde{S}(s)\ \tilde{V}^{(inc)}(-\ell,s) \qquad (11.14)$$

$$= \tilde{S}(s)\, V_o e^{\gamma\ell}$$

$$\tilde{S}(s) = \tilde{S}_{1,1}(s) - \tilde{S}_{2,1}(s)$$

From (11.9) the scattering matrix is

$$\left(S_{n,m}(s)\right) = \left[\left(\tilde{Z}_{n,m}(s)\right) \cdot \left(Z_{c_{n,m}}(-\ell)\right)^{-1} - \left(1_{n,m}\right)\right]$$

$$\cdot \left[\left(\tilde{Z}_{n,m}(s)\right) \cdot \left(Z_{c_{n,m}}(-\ell)\right)^{-1} + \left(1_{n,m}\right)\right]^{-1}$$

$$= \left[\left(\tilde{Z}_{n,m}(s)\right) - \left(Z_{c_{n,m}}(-\ell)\right)\right] \cdot \left[\left(\tilde{Z}_{n,m}(s)\right) + \left(Z_{c_{n,m}}(-\ell)\right)\right]^{-1}$$

$$= \left[\left(\tilde{Z}_{n,m}(s)\right) + \left(Z_{c_{n,m}}(-\ell)\right) - 2\left(Z_{c_{n,m}}(-\ell)\right)\right]$$

$$\cdot \left[\left(\tilde{Z}_{n,m}(s)\right) + \left(Z_{c_{n,m}}(-\ell)\right)\right]^{-1} \qquad (11.15)$$

$$= \left(1_{n,m}\right) - 2\left(Z_{c_{n,m}}(-\ell)\right) \cdot \left[\left(\tilde{Z}_{n,m}(s)\right) + \left(Z_{c_{n,m}}(-\ell)\right)\right]^{-1}$$

Writing out

$$\left(Z_{c_{n,m}}(-\ell)\right) = Z_w\left[f_g^{(in)}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + f_g^{(out)}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\right] \qquad (11.16)$$

then we have (including reciprocity)

$$\left(\tilde{S}_{n,m}(s)\right) = \left(1_{n,m}\right) - \frac{2}{\det\left(\left(\tilde{Z}_{n,m}(s)\right) + \left(Z_{c_{n,m}}(-\ell)\right)\right)}\left(Z_{c_{n,m}}(-\ell)\right) \cdot \left[\begin{pmatrix} \tilde{Z}_{2,2}(s) & -\tilde{Z}_{1,2}(s) \\ -\tilde{Z}_{1,2}(s) & Z_{1,1}(s) \end{pmatrix} + Z_w\, f_g^{(in)} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + Z_w\, f_g^{(out)} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right]$$

$$= \left(1_{n,m}\right) - \frac{2\,Z_w}{\det\left(\left(\tilde{Z}_{n,m}(s)\right) + \left(Z_{c_{n,m}}(-\ell)\right)\right)}\left[f_g^{(in)} \begin{pmatrix} \tilde{Z}_{2,2}(s) & -\tilde{Z}_{1,2}(s) \\ 0 & 0 \end{pmatrix} + f_g^{(out)} \begin{pmatrix} \tilde{Z}_{2,2}(s) - \tilde{Z}_{1,2}(s) & \tilde{Z}_{1,1}(s) - Z_{1,2}(s) \\ \tilde{Z}_{2,2}(s) - \tilde{Z}_{1,2}(s) & \tilde{Z}_{1,1}(s) - \tilde{Z}_{1,2}(s) \end{pmatrix} + Z_w\, f_g^{(in)}\, f_g^{(out)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \tag{11.17}$$

This gives the scattering coefficient from (11.14) as

$$\tilde{S}(s) = 1 - \frac{2\,Z_w\, f_g^{(in)}}{\det\left(\left(\tilde{Z}_{n,m}(s)\right) + \left(Z_{c_{n,m}}(-\ell)\right)\right)}\left[\tilde{Z}_{2,2} + Z_w\, f_g^{(out)}\right] \tag{11.18}$$

and the input impedance

$$\tilde{Z}_{in}(s) = Z_w\, f_g^{(in)}\, \frac{1+\tilde{S}(s)}{1-\tilde{S}(s)} = Z_w\, f_g^{(in)} \left[-1 + \frac{2}{1-\tilde{S}(s)}\right]$$
$$= Z_w\, f_g^{(in)} \left[-1 + \frac{\det\left(\left(\tilde{Z}_{n,m}(s)\right) + \left(Z_{c_{n,m}}(-\ell)\right)\right)}{Z_w\, f_g \left[\tilde{Z}_{2,2}(s) + Z_w\, f_g^{(out)}\right]}\right] \tag{11.19}$$

Noting that

$$\det\left(\left(\tilde{Z}_{n,m}(s) \right) + Z_w\, f_g^{(in)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + Z_w\, f_g^{(out)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right)$$

$$= \det\left(\left(\tilde{Z}_{n,m}(s) \right) + Z_w\, f_g^{(out)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) + Z_w\, f_g^{(in)} \left[\tilde{Z}_{2,2}(s) + Z_w\, f_g^{(out)} \right] \tag{11.20}$$

we have an alternate form for the input impedance as

$$Z_{in}(s) = \frac{\det\left(\left(\tilde{Z}_{n,m}(s) \right) + Z_w\, f_g^{(out)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right)}{\tilde{Z}_{2,2}(s) + Z_w\, f_g^{(out)}} \tag{11.21}$$

Note that this is just the reciprocal of the 1,1 element of the inverse of the matrix of which the determinant is being taken.

Complementing the scattering coefficient back into the inner coax, there is also a wave launched back into the outer region $(\Psi_b \leq \Psi \leq \Psi_o)$ as

$$\tilde{S}_{out}(s) \equiv \frac{\tilde{V}_2(-\ell, s)}{V_o e^{\gamma \ell}} = \tilde{S}_{2,1}(s)$$

$$= -\frac{2 Z_w\, f_g^{(out)} \left[\tilde{Z}_{2,2}(s) - \tilde{Z}_{1,2}(s) \right]}{\det\left(\left(\tilde{Z}_{n,m}(s) \right) + \left(Z_{c_{n,m}}(-\ell) \right) \right)} \tag{11.22}$$

Note that the characteristic impedance for this outer region is in general different from that of the inner coax. So this formula has to be interpreted in the sense of voltage; and power will include the different impedances.

XII. Solution for Single-Ended-to-Differential Balun

Specialize now to the case in fig. 1.1A in which the transition region covers only $-\ell \le z \le 0$ with the symmetrical two-wire transmission line (as in fig. 3.1) continued to the right. Since the wave for $z \le 0$ propagates only to the right with a constant characteristic-impedance matrix we have

$$\left(\tilde{V}_n(0,s)\right) = \left(Z_{c_{n,m}}(0)\right) \cdot \left(\tilde{I}_n(0,s)\right) \tag{12.1}$$

As discussed in Sections III, V, and VIII this matrix can be written as

$$\begin{aligned}\left(Z_{c_{n,m}}(0)\right) &= Z_w \left(f_{g_{n,m}}(0)\right) \\ &= \frac{Z_w\, f_g^{(d)}}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1\end{pmatrix} + Z_w\, f_g^{(c)}\begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix} \\ &= Z_w \sum_{\beta=1}^{2} \left(g_n^{(-)}\right)_\beta \left(g_n^{(-)}\right)_\beta\end{aligned} \tag{12.2}$$

Note the "–" superscript corresponding to the diagonalization of the left transition section.

From Section IV we have the voltage and current at $z = -\ell$ as

$$\begin{aligned}\begin{pmatrix} \left(\tilde{V}_n(-\ell,s)\right) \\ \left(Z_{c_{n,m}}(0)\right)\cdot\left(\tilde{I}_n(-\ell,s)\right)\end{pmatrix} &= \left(\left(\tilde{u}_{n,m}^{(-)}(-\ell,0;s)\right)_{v,v'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n(0,s)\right) \\ \left(Z_{c_{n,m}}(0)\right)\cdot\left(\tilde{I}_n(0,s)\right)\end{pmatrix} \\ &= \left(\left(\tilde{u}_{n,m}^{(-)}(-\ell,0;s)\right)_{v,v'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n(0,s)\right) \\ \left(\tilde{V}_n(0,s)\right)\end{pmatrix}\end{aligned} \tag{12.3}$$

Writing this out in terms of the matrix blocks and eliminating $\left(\tilde{V}_n(0,s)\right)$ gives

$$\left(\tilde{V}_n(-\ell,s)\right) = \left(\tilde{Z}^{(1)}_{n,m}(s)\right) \cdot \left(\tilde{I}_n(-\ell,s)\right)$$

$$\left(\tilde{Z}^{(1)}_{n,m}(s)\right) = \left[\left(\tilde{u}^{(-)}_{n,m}(-\ell,0;s)\right)_{1,1} + \left(\tilde{u}^{(-)}_{n,m}(-\ell,0;s)\right)_{1,2}\right]$$

$$\cdot \left[\left(\tilde{u}^{(-)}_{n,m}(-\ell,0;s)\right)_{2,1} + \left(\tilde{u}^{(-)}_{n,m}(-\ell,0;s)\right)_{2,2}\right]^{-1} \tag{12.4}$$

$$\cdot \left(Z_{c_{n,m}}(0)\right)$$

where the matrizant blocks all have common eigenvectors so that these terms commute. Note the superscript "1" to designate the input-impedance matrix for this case for use with the results of Section XI.

From (9.5) we now have the impedance matrix as

$$\left(\tilde{Z}^{(1)}_{n,m}(s)\right)$$

$$= \left\{ \sum_{\beta=1}^{2} e^{2A_\beta} \frac{\cosh\left(\tilde{\Gamma}_\beta\right) + \frac{\gamma\ell - A_\beta}{\tilde{\Gamma}_\beta} \sinh\left(\tilde{\Gamma}_\beta\right)}{\cosh\left(\tilde{\Gamma}_\beta\right) + \frac{\gamma\ell + A_\beta}{\tilde{\Gamma}_\beta} \sinh\left(\Gamma_\beta\right)} \left(g_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta \right\} \cdot \left(Z_{c_{n,m}}(0)\right)$$

$$= Z_w \sum_{\beta=1}^{2} e^{2A_\beta} \frac{\cosh\left(\tilde{\Gamma}_\beta\right) + \frac{\gamma\ell - A_\beta}{\tilde{\Gamma}_\beta} \sinh\left(\tilde{\Gamma}_\beta\right)}{\cosh\left(\tilde{\Gamma}_\beta\right) + \frac{\gamma\ell + A_\beta}{\tilde{\Gamma}_\beta} \sinh\left(\tilde{\Gamma}_\beta\right)} \left(g_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta \tag{12.5}$$

The transfer matrix through the balun is now found via

$$\left(\tilde{V}_n(-\ell,s)\right) = \left[\left(1_{n,m}\right) + \left(\tilde{S}^{(1)}_{n,m}(s)\right)\right] \cdot \left(\tilde{V}_n^{(inc)}(-\ell,s)\right)$$

$$\left(\tilde{V}_n(0,s)\right) \equiv \left(\tilde{T}^{(1)}_{n,m}(s)\right) \cdot \left(\tilde{V}_n^{(inc)}(-\ell,s)\right)$$

$$= \left(\tilde{T}^{(1)}_{n,m}(s)\right) \cdot \left[\left(1_{n,m}\right) + \left(S^{(1)}_{n,m}(s)\right)\right]^{-1} \cdot \left(\tilde{V}_n(-\ell,s)\right) \tag{12.6}$$

$$\left(\tilde{T}^{(1)}_{n,m}(s)\right) = \text{transfer matrix}$$

From Section IV we have

$$\begin{pmatrix} \left(\tilde{V}_n(0,s)\right) \\ \left(Z_{c_{n,m}}(0)\right) \cdot \left(\tilde{I}_n(0,s)\right) \end{pmatrix} = \left(\left(\tilde{u}_{n,m}^{(-)}(0,-\ell;s)\right)_{v,v'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n(-\ell,s)\right) \\ \left(Z_{c_{n,m}}(0)\right) \cdot \left(\tilde{I}_n(-\ell,s)\right) \end{pmatrix}$$

$$\left(\tilde{V}_n(0,s)\right) = \left(\tilde{u}_{n,m}^{(-)}(0,-\ell;s)\right)_{1,1} \cdot \left(\tilde{V}_n(-\ell,s)\right) + \left(\tilde{u}_{n,m}^{(-)}(0,-\ell;s)\right)_{1,2} \cdot \left(Z_{c_{n,m}}(0)\right) \cdot \left(\tilde{I}_n(-\ell,s)\right) \tag{12.7}$$

$$= \left[\left(\tilde{u}_{n,m}^{(-)}(0,-\ell;s)\right)_{1,1} + \left(\tilde{u}_{n,m}^{(-)}(0,-\ell;s)\right)_{1,2} \cdot \left(Z_{c_{n,m}}(0)\right) \cdot \left(\tilde{Z}_{n,m}^{(1)}(s)\right)^{-1}\right] \cdot \left(\tilde{V}_n(-\ell,s)\right)$$

which gives the transfer matrix as

$$\begin{aligned} \left(\tilde{T}_{n,m}^{(1)}(s)\right) &= \left[\left(\tilde{u}_{n,m}^{(-)}(0,-\ell;s)\right)_{1,1} + \left(\tilde{u}_{n,m}^{(-)}(0,-\ell;s)\right)_{1,2} \cdot \left(Z_{c_{n,m}}(0)\right) \cdot \left(\tilde{Z}_{n,m}^{(1)}(s)\right)^{-1}\right] \\ &\quad \cdot \left[\left(1_{n,m}\right) + \left(\tilde{S}_{n,m}^{(1)}(s)\right)\right] \end{aligned} \tag{12.8}$$

In computing the scattering matrix it is convenient to use the eigenvectors for which we have

$$\begin{aligned} \left(Z_{c_{n,m}}(-\ell)\right) \cdot \left(Z_{c_{n,m}}(0)\right)^{-1} = \left(f_{n,m}^{(-)}(-\ell)\right) &= \sum_{\beta=1}^{2} f_\beta^{(-)}(-\ell) \left(g_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta \\ &= \sum_{\beta=1}^{2} e^{2A_\beta} \left(g_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta \end{aligned} \tag{12.9}$$

From which we obtain

$$\left(\tilde{Z}^{(1)}_{n,m}(s)\right) \cdot \left(Z_{c_{n,m}}(-\ell)\right)^{-1} = \left[\left(\tilde{u}^{(-)}_{n,m}(-\ell,0;s)\right)_{1,1} + \left(\tilde{u}^{(-)}_{n,m}(-\ell,0;s)\right)_{1,2}\right]$$

$$\cdot \left[\left(\tilde{u}^{(-)}_{n,m}(-\ell,0;s)\right)_{2,1} + \left(\tilde{u}^{(-)}_{n,m}(-\ell,0;s)\right)_{2,2}\right]^{-1} \cdot \left(Z_{c_{n,m}}(0)\right) \cdot \left(Z_{c_{n,m}}(-\ell)\right)^{-1}$$

$$= \sum_{\beta=1}^{2} \frac{\cosh\left(\tilde{\Gamma}_\beta\right) + \dfrac{\gamma\ell - A_\beta}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)}{\cosh\left(\tilde{\Gamma}_\beta\right) + \dfrac{\gamma\ell + A_\beta}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)} \left(g^{(-)}_n\right)_\beta \left(h^{(-)}_n\right)_\beta \tag{12.10}$$

Then from (11.9) we have

$$\left(\tilde{S}^{(1)}_{n,m}(s)\right) = \sum_{\beta=1}^{2} - \frac{\dfrac{A_\beta}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)}{\cosh\left(\tilde{\Gamma}_\beta\right) + \dfrac{\gamma\ell}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)} \left(g^{(-)}_n\right)_\beta \left(h^{(-)}_n\right)_\beta$$

$$\left(1_{n,m}\right) + \left(\tilde{S}^{(1)}_{n,m}(s)\right) = 2\left(\tilde{Z}^{(1)}_{n,m}(s)\right) \cdot \left(Z_{c_{n,m}}(-\ell)\right)^{-1}$$

$$\cdot \left[\left(\tilde{Z}_{n,m}(s)\right) \cdot \left(Z_{c_{n,m}}(-\ell)\right)^{-1} + \left(1_{n,m}\right)\right]^{-1}$$

$$= \sum_{\beta=1}^{2} \frac{\cosh\left(\tilde{\Gamma}_\beta\right) + \dfrac{\gamma\ell - A_\beta}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)}{\cosh\left(\tilde{\Gamma}_\beta\right) + \dfrac{\gamma\ell}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)} \left(g^{(-)}_n\right)_\beta \left(h^{(-)}_n\right)_\beta \tag{12.11}$$

Recalling the matrix blocks from (9.8) we now have

$$\left(\tilde{T}^{(1)}_{n,m}(s)\right) = \left\{\sum_{\beta=1}^{2} e^{-A_\beta}\left[\cosh\left(\tilde{\Gamma}_\beta\right) + \frac{A_\beta}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\right]\left(g_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta\right.$$

$$\left. - \left[\sum_{\beta=1}^{2} e^{A_\beta}\frac{\gamma\ell}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\left(g_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta\right]\cdot\left(Z_{c_{n,m}}(0)\right)\cdot\left(\tilde{Z}^{(1)}_{n,m}(s)\right)^{-1}\right\}$$

$$\cdot\left[\left(1_{n,m}\right) + \left(\tilde{S}^{(1)}_{n,m}(s)\right)\right]$$

$$= \left\{\sum_{\beta=1}^{2} e^{-A_\beta}\left[\cosh\left(\tilde{\Gamma}_\beta\right) + \frac{\gamma\ell - A_\beta}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\right]^{-1}\left(g_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta\right\}\cdot\left[\left(1_{n,m}\right) + \left(\tilde{S}^{(1)}_{n,m}(s)\right)\right] \tag{12.12}$$

$$= \sum_{\beta=1}^{2} e^{-A_\beta}\left[\cosh\left(\tilde{\Gamma}_\beta\right) + \frac{\gamma\ell}{\tilde{\Gamma}_\beta}\sinh\left(\tilde{\Gamma}_\beta\right)\right]^{-1}\left(g_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta$$

From Section XI we have the scattering parameters (to the left) and input impedance as

$$\tilde{S}^{(1)}(s) = \tilde{S}^{(1)}_{1,1}(s) - \tilde{S}^{(1)}_{2,1}(s)\ ,\ \tilde{S}^{(1)}_{out}(s) = \tilde{S}^{(1)}_{2,1}(s)$$

$$\tilde{Z}^{(1)}_{in}(s) = \frac{1 + \tilde{S}^{(1)}(s)}{1 - \tilde{S}^{(1)}(s)} \tag{12.13}$$

In addition we need transmission parameters to the differential and common modes (to the right) respectively as

$$\tilde{T}^{(1)}_{d}(s) \equiv \frac{\tilde{V}_1(0,s) - \tilde{V}_2(0,s)}{e^{-\gamma\ell}\left[\tilde{V}_1^{(inc)}(-\ell,s) - \tilde{V}_2^{(inc)}(-\ell,s)\right]} = \frac{\tilde{V}_1(0,s) - \tilde{V}_2(0,s)}{V_0}$$

$$= \left[\tilde{T}^{(1)}_{1,1}(s) - \tilde{T}^{(1)}_{2,1}(s)\right] e^{\gamma\ell}$$

$$\tilde{T}^{(1)}_{c}(s) \equiv \frac{1}{2}\frac{\tilde{V}_1(0,s) + \tilde{V}_2(0,s)}{e^{-\gamma\ell}\left[\tilde{V}_1^{(inc)}(-\ell,s) - \tilde{V}_2^{(inc)}(-\ell,s)\right]} = \frac{\tilde{V}_1(0,s) + \tilde{V}_2(0,s)}{2\,V_0} \tag{12.14}$$

$$= \frac{1}{2}\left[\tilde{T}^{(1)}_{1,1}(s) + \tilde{T}^{(1)}_{2,1}(s)\right] e^{\gamma\ell}$$

Note the factor of $e^{\gamma\ell}$ to remove the delay through the transition (for convenience). Also remember that the impedances for these two modes, as discussed in Section III, are in general different from the source (input coax) so the above coefficients correspond to voltage, not current or power.

Consider the behavior of these various parameters for low frequencies. This is aided by the equivalent circuit in Fig. 12.1. The output twinline is represented by a tee network, but a pi network is also possible. Note that, by separately considering the differential and common modes, the correct respective impedances are obtained. By the usual circuit manipulations we find

$$\tilde{Z}_{in}^{(1)}(0) = Z_w \left\{ \frac{1}{2} f_g^{(d)} + \left[\frac{2}{f_g^{(d)}} + \frac{1}{f_g^{(out)} + f_g^{(c)} - \frac{1}{4} f_g^{(d)}} \right]^{-1} \right\}$$

$$= Z_w \, f_g^{(d)} \left\{ 1 + \frac{1}{4} \left[\frac{f_g^{(out)}}{f_g^{(d)}} + \frac{f_g^{(c)}}{f_g^{(d)}} \right]^{-1} \right\}^{-1}$$

$$1 + \tilde{S}^{(1)}(0) = \frac{2\,\tilde{Z}_{in}^{(1)}(0)}{\tilde{Z}_{in}^{(1)}(0) + Z_w \, f_g^{(in)}} = 2 \left\{ 1 + \frac{f_g^{(in)}}{f_g^{(d)}} + \frac{1}{4} \left[\frac{f_g^{(out)}}{f_g^{(in)}} + \frac{f_g^{(c)}}{f_g^{(in)}} \right]^{-1} \right\}^{-1}$$

$$\tilde{S}_{out}^{(1)}(0) = \frac{\tilde{V}_2(-\ell,0)}{\tilde{V}_1(-\ell,0) - \tilde{V}_2(-\ell,0)} \left[1 + \tilde{S}^{(1)}(0) \right]$$

$$= -\frac{1}{2} \frac{f_g^{(out)}}{f_g^{(out)} + f_g^{(c)}} \left[1 + \tilde{S}^{(1)}(0) \right]$$

$$\tilde{T}_d^{(1)}(0) = 1 + \tilde{S}^{(1)}(0)$$

$$\tilde{T}_c^{(1)}(0) = \frac{1}{2} \frac{\tilde{V}_1(0,s) + \tilde{V}_2(0,s)}{\tilde{V}_1(0,s) - \tilde{V}_2(0,s)} \left[1 + \tilde{S}^{(1)}(0) \right] \qquad (12.15)$$

$$= \frac{1}{2} \frac{f_g^{(c)}}{f_g^{(out)} + f_g^{(c)}} \left[1 + \tilde{S}^{(1)}(0) \right]$$

From our eigenvector expansions, for low frequencies we also have

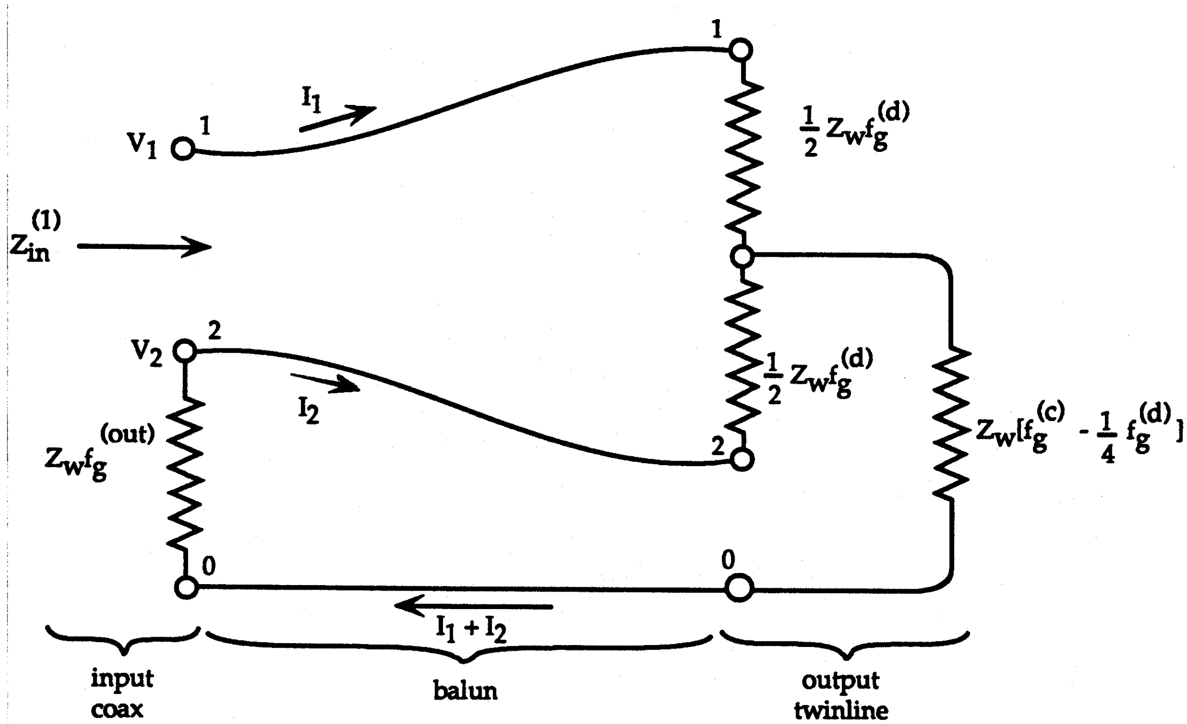


Fig. 12.1.Low-Frequency Equivalent Circuit of Balun

$$f_\beta^{(-)}(-\ell) = \frac{f_g^{(out)}}{2\, f_g^{(c)}}\left[1 + \frac{c_1}{2} \pm c_3\right] = e^{2A_\beta}$$

$$\tilde{\Gamma}_\beta = \left[A_\beta^2 + [\gamma\ell]^2\right]^{\frac{1}{2}} = A_\beta\left[1 + O(s^2)\right] \text{ as } s \to 0$$

$$\left(\tilde{Z}_{n,m}^{(1)}(s)\right) \cdot \left(Z_{c_{n,m}}(-\ell)\right)^{-1} = \sum_{\beta=1}^{2} e^{-2A_\beta}\left[1 + O(s)\right]\left(g_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta$$

$$= \sum_{\beta=1}^{2} f_\beta^{(-)^{-1}}(-\ell)\, \left(g_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta + O(s)$$

$$= \left(f_{n,m}^{(-)}(-\ell)\right)^{-1} + O(s)$$

$$= \left(f_{g_{n,m}}(0)\right) \cdot \left(f_{g_{n,m}}(-\ell)\right)^{-1} + O(s) \text{ as } s \to 0 \tag{12.16}$$

$$\left(\tilde{Z}_{n,m}^{(1)}(s)\right) = Z_w\left(f_{g_{n,m}}(0)\right) + O(s) = \left(Z_{c_{n,m}}(0)\right) + O(s) \text{ as } s \to 0$$

so the low-frequency impedance matrix is just that of the transmission line to the right (the load) as one would expect. The scattering matrix becomes

$$\left(\tilde{S}_{n,m}^{(1)}(s)\right) = \sum_{\beta=1}^{2} -\tanh\left(A_\beta\right) \left(g_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta + O(s)$$

$$= \sum_{\beta=1}^{2} \frac{1 - f_\beta^{(-)}(-\ell)}{1 + f_\beta^{(-)}(-\ell)} \left(g_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta + O(s) \text{ as } s \to 0 \tag{12.17}$$

$$\left(1_{n,m}\right) + \left(\tilde{S}_{n,m}^{(1)}(s)\right) = \sum_{\beta=1}^{2} \frac{2}{1 + f_\beta^{(-)}(-\ell)} \left(g_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta + O(s) \text{ as } s \to 0$$

This can also be represented by inserting the impedance matrices directly in (11.15). The transfer matrix becomes

$$\left(\tilde{T}_{n,m}^{(1)}(s)\right) = \left[\left(1_{n,m}\right) + \left(\tilde{S}_{n,m}^{(1)}(s)\right)\right]\left[\left(1_{n,m}\right) + O(s)\right]$$

$$= \sum_{\beta=1}^{2} \frac{2}{1 + f_\beta^{(-)}(-\ell)} \left(g_n^{(-)}\right)_\beta \left(h_n^{(-)}\right)_\beta + O(s) \text{ as } s \to 0 \tag{12.18}$$

This is physically reasonable in that at low frequencies the voltages at $z = -\ell$ and $z = 0$ are the same.

Recalling

$$\left(g_n^{(-)}\right)_\beta \; \left(h_n^{(-)}\right)_\beta = \pm \frac{1}{2c_3}\begin{pmatrix} \frac{c_1}{2} \pm c_3 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ \overline{-\frac{c_1}{2} \pm c_3} \end{pmatrix}$$

$$f_\beta^{(-)}(-\ell) = \frac{f_g^{(out)}}{2\, f_g^{(c)}}\left[1 + \frac{c_1}{2} \pm c_3\right] = e^{2A_\beta} \tag{12.19}$$

the various matrix elements can be computed. Common combinations include

$$f_1^{(-)}(-\ell)\; f_2^{(-)}(-\ell) = \frac{f_g^{(in)}\, f_g^{(out)}}{f_g^{(d)}\, f_g^{(c)}}$$

$$f_1^{(-)}(-\ell) + f_2^{(-)}(-\ell) = \frac{f_g^{(out)}}{f_g^{(c)}}\left[1 + \frac{f_g^{(in)}}{f_g^{(out)}}\left[\frac{f_g^{(c)}}{f_g^{(d)}} + \frac{1}{4}\right]\right]$$

$$= \frac{f_g^{(out)}}{f_g^{(c)}} + \frac{f_g^{(in)}}{f_g^{(d)}} + \frac{f_g^{(in)}}{4\, f_g^{(c)}}$$

$$\left[1 + f_1^{(-)}(-\ell)\right]\left[1 + f_2^{(-)}(-\ell)\right] = 1 + \left[f_1^{(-)}(-\ell) + f_2^{(-)}(-\ell)\right] + f_1^{(-)}(-\ell)\, f_2^{(-)}(-\ell) \tag{12.20}$$

$$= 1 + \frac{f_g^{(out)}}{f_g^{(in)}} + \frac{f_g^{(in)}}{f_g^{(d)}} + \frac{f_g^{(in)}}{4\, f_g^{(c)}} + \frac{f_g^{(in)}\, f_g^{(out)}}{f_g^{(d)}\, f_g^{(c)}}$$

Then we have for low frequencies

$$\tilde{T}_d^{(1)}(s) = 1 + \tilde{S}^{(1)}(s) + O(s) = 1 + \tilde{S}_{1,1}^{(1)}(s) - \tilde{S}_{2,1}^{(1)}(s) + O(s)$$

$$= \left[\tilde{T}_{1,1}^{(1)}(s) - \tilde{T}_{2,1}^{(1)}(s)\right] e^{\varkappa} = \sum_{\beta=1}^{2} \pm \frac{1}{1 + f_\beta^{(-)}(-\ell)} \frac{\frac{c_1}{2} \pm c_3 - 1}{c_3} + O(s)$$

$$= \left\{\left[1 + f_1^{(-)}(-\ell)\right]\left[1 + f_2^{(-)}(-\ell)\right]\right\}^{-1} \left\{\frac{f_1^{(-)}(-\ell) - f_2^{(-)}(-\ell)}{c_3}\left[1 - \frac{c_1}{2}\right]\right.$$

$$\left. + 2 + f_1^{(-)}(-\ell) + f_2^{(-)}(-\ell)\right\} + O(s)$$

$$= \left\{\left[1 + f_1^{(-)}(-\ell)\right]\left[1 + f_2^{(-)}(-\ell)\right]\right\}^{-1} \left\{\frac{f_g^{(out)}}{f_g^{(c)}}\left[1 - \frac{f_g^{(in)}}{f_g^{(out)}}\left[\frac{1}{4} + \frac{f_g^{(c)}}{f_g^{(out)}}\right]\right]\right.$$

$$\left. + 2\ f_1^{(-)}(-\ell) + f_2^{(-)}(-\ell)\right\} + O(s) \qquad (12.21)$$

$$= 2\left\{1 + \frac{f_g^{(out)}}{f_g^{(c)}}\right\} \left\{\left[1 + f_1^{(-)}(-\ell)\right]\left[1 + f_2^{(-)}(-\ell)\right]\right\}^{-1} + O(s) \text{ as } s \to 0$$

which can be shown to agree with (12.15). We also have

$$\tilde{S}_{out}^{(1)}(s) = \tilde{S}_{2,1}^{(1)}(s) = \sum_{\beta=1}^{2} \pm \frac{1 - f_\beta^{(-)}(-\ell)}{1 + f_\beta^{(-)}(-\ell)} \frac{1}{2c_3} + O(s)$$

$$= \left\{\left[1 + f_1^{(-)}(-\ell)\right]\left[1 - f_2^{(-)}(-\ell)\right]\right\}^{-1} \frac{\left[1 - f_1^{(-)}(-\ell)\right]\left[1 + f_2^{(-)}(-\ell)\right] - \left[1 - f_2^{(-)}(-\ell)\right]\left[1 + f_1^{(-)}(-\ell)\right]}{2\ c_3}$$

$$+ O(s)$$

$$= \left\{\left[1 + f_1^{(-)}(-\ell)\right]\left[1 + f_2^{(-)}(-\ell)\right]\right\}^{-1} \frac{f_2^{(-)}(-\ell) - f_1^{(-)}(-\ell)}{c_3} + O(s)$$

$$= -\frac{f_g^{(out)}}{f_g^{(c)}} \left\{ \left[1 + f_1^{(-)}(-\ell)\right] \left[1 + f_2^{(-)}(-\ell)\right] \right\}^{-1} + O(s) \text{ as } s \to 0$$

$$\tilde{T}_c^{(1)}(s) = \frac{1}{2}\left[\tilde{T}_{1,1}^{(1)}(s) + \tilde{T}_{2,1}^{(1)}(s)\right] + O(s) = \frac{1}{2}\left[\tilde{T}_d^{(1)}(s) + 2\,\tilde{S}_{2,1}^{(1)}(s)\right] + O(s) \tag{12.22}$$

$$= \left\{ \left[1 + f_1^{(-)}(-\ell)\right] \left[1 + f_2^{(-)}(-\ell)\right] \right\}^{-1} + O(s) \text{ as } s \to 0$$

completing the low-frequency expansions.

Turning now to high frequencies, consider first the scattering from the input port of the balun for which we have for $s \to \infty$ with $Re[s] \geq 0$ (i.e. in right half plane including $j\omega$ axis)

$$\tilde{\Gamma}_\beta = \gamma\ell\left[1 + O\left(s^{-2}\right)\right] \text{ as } s \to \infty$$

$$\tilde{S}^{(1)}(s) = \tilde{S}_{1,1}^{(1)}(s) - \tilde{S}_{2,1}^{(1)}(s) = \sum_{\beta=1}^{2} \mp \frac{A_\beta}{\gamma\ell} \frac{1 - e^{-2\gamma\ell}}{2} \frac{\frac{c_1}{2} \pm c_3 - 1}{2\,c_3} + O\left(s^{-2}\right)$$

$$= \left\{ -\left[A_1 + A_2\right] + \left[A_1 - A_2\right]\left[1 - \frac{c_1}{2}\right] \right\} \frac{1 - e^{-2\gamma\ell}}{4\,\gamma\ell} + O\left(s^{-2}\right)$$

$$= \left\{ -\ell n\left(f_1^{(-)}(-\ell)\ f_2^{(-)}(-\ell)\right) + \ell n\left(\frac{f_1^{(-)}(-\ell)}{f_2^{(-)}(-\ell)}\right) \left[1 - \frac{f_g^{(in)}}{f_g^{(out)}} \left[\frac{1}{4} + \frac{f_g^{(c)}}{f_g^{(d)}}\right]\right] \right\}$$

$$\frac{1 - e^{-2\gamma\ell}}{8\,\gamma\ell} + O\left(s^{-2}\right) \text{ as } s \to \infty$$

$$\tilde{S}_{out}^{(1)}(s) = \tilde{S}_{2,1}^{(1)}(s) = \sum_{\beta=1}^{2} \mp \frac{A_\beta}{\gamma\ell} \frac{1 - e^{-2\gamma\ell}}{2} \frac{1}{2\,c_3} + O\left(s^{-2}\right)$$

$$= \frac{A_2 - A_1}{c_3} \frac{1 - e^{-2\gamma\ell}}{4\,\gamma\ell} + O\left(s^{-2}\right) \tag{12.23}$$

$$= \frac{1}{c_3} \ell n\left(\frac{f_2^{(-)}(-\ell)}{f_1^{(-)}(-\ell)}\right) \frac{1 - e^{-2\gamma\ell}}{8\,\gamma\ell} + O\left(s^{-2}\right) \text{ as } s \to \infty$$

So the reflection from the balun, both inside and outside the coax, behaves like $(\gamma\ell)^{-1}$ times some coefficients which are only functions of the geometric impedance factors. Unlike the low-frequency results, these involve more complicated functions of these factors. Being a smooth taper it is to be expected that for $|\gamma\ell| \geq 1$ the reflection should be small and $\rightarrow 0$ as $s \rightarrow \infty$. Note also the factor $1 - e^{-2\gamma\ell}$, the second term of which represents the reflection from the output at the balun (z = 0).

Next the high-frequency transfer function through the balun for the differential mode is given by

$$\tilde{T}_d^{(1)}(s) = \left[\tilde{T}_{1,1}^{(1)}(s) - \tilde{T}_{2,1}^{(1)}(s)\right] e^{\gamma\ell} = \sum_{\beta=1}^{2} \pm e^{-A_\beta} \frac{\frac{c_1}{2} \pm c_3 - 1}{2\, c_3} + O\left(s^{-1}\right)$$

$$= \frac{1}{2\, c_3} \left\{ \left[-2 + \frac{2\, f_g^{(c)}}{f_g^{(out)}} f_1^{(-)}(-\ell) \right] f_1^{(-)^{-\frac{1}{2}}}(-\ell) - \left[-2 + \frac{2 f_g^{(c)}}{f_g^{(out)}} f_2^{(-)}(-\ell) \right] f_2^{(-)^{-\frac{1}{2}}}(-\ell) \right\}$$

$$= \frac{1}{c_3} \left\{ \frac{f_g^{(c)}}{f_g^{(out)}} \left[f_1^{(-)^{\frac{1}{2}}}(-\ell) - f_2^{(-)^{\frac{1}{2}}}(-\ell) \right] + f_2^{(-)^{-\frac{1}{2}}}(-\ell) - f_1^{(-)^{-\frac{1}{2}}}(-\ell) \right\} + O\left(s^{-1}\right)$$

$$= \frac{1}{c_3} \left[\frac{f_g^{(c)}}{f_g^{(out)}} + \left[f_1^{(-)}(-\ell)\, f_2^{(-)}(-\ell) \right]^{-\frac{1}{2}} \right] \left[f_1^{(-)^{\frac{1}{2}}}(-\ell) - f_2^{(-)^{\frac{1}{2}}}(-\ell) \right] + O\left(s^{-1}\right)$$

$$= \frac{1}{c_3} \left[\frac{f_g^{(c)}}{f_g^{(out)}} + \left[f_1^{(-)}(-\ell)\, f_2^{(-)}(-\ell) \right]^{-\frac{1}{2}} \right] \frac{f_1^{(-)}(-\ell) - f_2^{(-)}(-\ell)}{f_1^{(-)^{\frac{1}{2}}}(-\ell) + f_2^{(-)^{\frac{1}{2}}}(-\ell)} + O\left(s^{-1}\right)$$

$$= \frac{f_g^{(out)}}{f_g^{(c)}} \left[\frac{f_g^{(c)}}{f_g^{(out)}} + \left[f_1^{(-)}(-\ell)\, f_2^{(-)}(-\ell) \right]^{-\frac{1}{2}} \right] \left[f_1^{(-)}(-\ell) + f_2^{(-)}(-\ell) + 2 \left[f_1^{(-)}(-\ell)\, f_2^{(-)}(-\ell) \right]^{\frac{1}{2}} \right]^{-\frac{1}{2}} + O\left(s^{-1}\right)$$

$$= \left[1 + \left[\frac{f_g^{(out)}\, f_g^{(d)}}{f_g^{(c)}\, f_g^{(in)}} \right]^{\frac{1}{2}} \right] \left[\frac{f_g^{(out)}}{f_g^{(c)}} \left[1 + \frac{f_g^{(in)}}{4\, f_g^{(out)}} + \frac{f_g^{(c)}\, f_g^{(in)}}{f_g^{(out)}\, f_g^{(d)}} \right] + 2 \left[\frac{f_g^{(out)}\, f_g^{(d)}}{f_g^{(c)}\, f_g^{(in)}} \right]^{\frac{1}{2}} \right]^{-\frac{1}{2}} + O\left(s^{-1}\right)$$

$$= \left[\frac{f_g^{(d)}}{f_g^{(in)}}\right]^{\frac{1}{2}} \left[1 + \left[\frac{f_g^{(out)}\, f_g^{(d)}}{f_g^{(c)}\, f_g^{(in)}}\right]^{\frac{1}{2}}\right] \left[\frac{f_g^{(out)}\, f_g^{(d)}}{f_g^{(c)}\, f_g^{(in)}} + \frac{f_g^{(d)}}{2\, f_g^{(c)}} + 1 + 2\left[\frac{f_g^{(out)}\, f_g^{(d)}}{f_g^{(c)}\, f_g^{(in)}}\right]^{\frac{1}{2}}\right]^{-\frac{1}{2}} + O\left(s^{-1}\right)$$

$$= \left[\frac{f_g^{(d)}}{f_g^{(in)}}\right]^{\frac{1}{2}} \left[1 + \left[\frac{f_g^{(out)}\, f_g^{(d)}}{f_g^{(c)}\, f_g^{(in)}}\right]^{\frac{1}{2}}\right] \left[\left[1 + \left[\frac{f_g^{(out)}\, f_g^{(d)}}{f_g^{(c)}\, f_g^{(in)}}\right]^{\frac{1}{2}}\right]^2 + \frac{f_g^{(d)}}{2\, f_g^{(c)}}\right]^{-\frac{1}{2}} + O\left(s^{-1}\right)$$

$$= \left[\frac{f_g^{(d)}}{f_g^{(in)}}\right]^{\frac{1}{2}} \left[1 + \left[\frac{f_g^{(d)}}{2\, f_g^{(c)}}\left[1 + \left[\frac{f_g^{(out)}\, f_g^{(d)}}{f_g^{(c)}\, f_g^{(in)}}\right]^{\frac{1}{2}}\right]\right]^{-2}\right]^{-\frac{1}{2}} + O\left(s^{-1}\right) \text{ as } s \to \infty \qquad (12.24)$$

where we have used the product and sum of the eigenvalues from (5.2) (determinant and trace) to obtain this explicit result in terms of the geometric factors. A case of interest has

$$f_g^{(out)} >> f_g^{(d)} \;,\; f_g^{(c)} >> f_g^{(in)}$$

$$\lim_{s \to \infty} \tilde{T}_d^{(1)}(s) \to \left[\frac{f_g^{(d)}}{f_g^{(in)}}\right]^{\frac{1}{2}} \text{ as } f_g^{(out)} \to \infty \text{ and } f_g^{(c)} \to \infty \qquad (12.25)$$

$$\text{with } \frac{f_g^{(out)}}{f_g^{(c)}} \text{ constant}$$

This is just the result for a tapered-transmission-line transformer, i.e. the square root of the impedance ratio (at high frequencies). Also from (12.24), noting that the impedance factors are all positive, we have the general result

$$0 < \lim_{s \to \infty} \tilde{T}_d^{(1)}(s) < \left[\frac{f_g^{(d)}}{f_g^{(in)}}\right]^{\frac{1}{2}} \qquad (12.26)$$

As we have already seen, the reflection at the input (both inside and outside) is negligible at high freqencies. By (12.24) not all the power goes into the differential mode; the remainder must go into the common mode. The high-frequency transfer function for the common mode is given by

$$\tilde{T}_c^{(1)}(s) = \frac{1}{2}\left[\tilde{T}_{1,1}^{(1)}(s) + \tilde{T}_{2,1}^{(1)}(s)\right]e^{\gamma\ell} = \sum_{\beta=1}^{2} \pm \frac{e^{-A_\beta}}{2} \frac{\frac{c_1}{2} \pm c_3 + 1}{2\,c_3} + O\left(s^{-1}\right)$$

$$= \frac{1}{4\,c_3} \frac{2\,f_g^{(c)}}{f_g^{(out)}} \left[f_1^{(-)^{\frac{1}{2}}}(-\ell) - f_2^{(-)^{\frac{1}{2}}}(-\ell)\right] + O\left(s^{-1}\right)$$

$$= \frac{f_g^{(c)}}{2c_3\,f_g^{(out)}} \frac{f_1^{(-)}(-\ell) - f_2^{(-)}(-\ell)}{f_1^{(-)^{\frac{1}{2}}}(-\ell) + f_2^{(-)^{\frac{1}{2}}}(-\ell)} + O\left(s^{-1}\right)$$

$$= \frac{1}{2}\left[f_1^{(-)^{\frac{1}{2}}}(-\ell) + f_2^{(-)^{\frac{1}{2}}}(-\ell)\right]^{-1} + O\left(s^{-1}\right)$$

$$= \frac{1}{2}\left[f_1^{(-)}(-\ell) + f_2^{(-)}(-\ell) + 2\left[f_1^{(-)}(-\ell)\,f_2^{(-)}(-\ell)\right]^{\frac{1}{2}}\right]^{-\frac{1}{2}} + O\left(s^{-1}\right)$$

$$= \frac{1}{2}\left[\frac{f_g^{(d)}}{f_g^{(in)}}\right]^{\frac{1}{2}} \left[\left[1 + \left[\frac{f_g^{(out)}\,f_g^{(d)}}{f_g^{(c)}\,f_g^{(in)}}\right]^{\frac{1}{2}}\right]^2 + \frac{f_g^{(d)}}{2\,f_g^{(c)}}\right]^{-\frac{1}{2}} + O\left(s^{-1}\right) \text{ as } s \to \infty \tag{12.27}$$

where the last several steps follow in the same way as in (12.24). In the same limiting sense as in (12.25) we have

$$\lim_{s\to\infty} \tilde{T}_c^{(1)}(s) \to \frac{1}{2}\left[\frac{f_g^{(d)}}{f_g^{(in)}}\right]^{\frac{1}{2}} \left[1 + \left[\frac{f_g^{(out)}\,f_g^{(d)}}{f_g^{(c)}\,f_g^{(in)}}\right]^{\frac{1}{2}}\right]^{-1} \text{ as } f_g^{(out)} \to \infty,\ f_g^{(c)} \to \infty \tag{12.28}$$

$$\text{with } \frac{f_g^{(out)}}{f_g^{(c)}} \text{ constant}$$

Noting the orthogonality of the differential and common modes on the twin line let us add the normalized powers as

$$
\begin{aligned}
&\frac{1}{f_g^{(d)}}\left[\lim_{s\to\infty} \tilde{T}_d^{(1)}(s)\right]^2 + \frac{1}{f_g^{(c)}}\left[\lim_{s\to\infty} \tilde{T}_d^{(1)}(s)\right]^2 \\
&= \frac{1}{f_g^{(in)}}\left[1+\left[\frac{f_g^{(out)}\, f_g^{(d)}}{f_g^{(c)}\, f_g^{(in)}}\right]^{\frac{1}{2}}\right]^2 \left[\left[1+\left[\frac{f_g^{(out)}\, f_g^{(d)}}{f_g^{(c)}\, f_g^{(in)}}\right]^{\frac{1}{2}}\right]^2 + \frac{f_g^{(d)}}{2\, f_g^{(c)}}\right]^{-\frac{1}{2}} \\
&+ \frac{f_g^{(d)}}{2\, f_g^{(in)}\, f_g^{(c)}}\left[\left[1+\left[\frac{f_g^{(out)}\, f_g^{(d)}}{f_g^{(c)}\, f_g^{(in)}}\right]^{\frac{1}{2}}\right]^2 + \frac{f_g^{(d)}}{2\, f_g^{(c)}}\right]^{-\frac{1}{2}} \qquad (12.29) \\
&= \frac{1}{f_g^{(in)}}
\end{aligned}
$$

which is the normalized power incident from the coax.

XIII. Solution for Special Case of Inverter

In Section IX the eigenvalues were constructed using exponential interpolation for the left ($-\ell \leq z \leq 0$) and right ($0 \leq z \leq \ell$) sections of the inverter (transition section). There it was observed that one could construct a solution by multiplying certain matrices which had different eigenvectors, thereby introducing various additional cross terms. If, however, a constraint as in (9.12) is imposed the eigenvectors are common to both halves of the transition, thereby simplifying the results. Furthermore, in Section X it was shown that an exponential interpolation for the entire transition can be constructed, further simplifying the formulas. This constrains both $f_g^{(d)}$ and $f_g^{(c)}$ at $z = 0$ via (10.15) in terms of $f_g^{(in)}$ and $f_g^{(out)}$ which characterize the structure outside the transition region. Let us now consider the solution for this special case for the geometry in fig. 1.1B.

The wave leaving the inverter to the right ($z \geq \ell$) is characterized by a characteristic-impedance matrix as

$$\left(\tilde{V}_n(\ell,s)\right) = \left(Z_{c_{n,m}}(\ell)\right) \cdot \left(\tilde{I}_n(\ell,s)\right) \tag{13.1}$$

As discussed in Sections II, VI, VIII, and X this matrix can be written as

$$\begin{aligned}\left(Z_{c_{n,m}}(\ell)\right) &= Z_w\left(f_{g_{n,m}}(\ell)\right) \\ &= Z_w\, f_g^{(out)}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + Z_w\, f_g^{(in)}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \\ &= Z_w \sum_{\beta=1}^{2} e^{2A_\beta^{(2)}}\left(g_n^{(2)}\right)_\beta \left(g_n^{(2)}\right)_\beta\end{aligned} \tag{13.2}$$

Here a superscript "2" is used to designate parameters peculiar to this case. We also have

$$\left(f_{n,m}^{(2)}(\ell)\right) = \left(f_{g_{n,m}}(\ell)\right) \cdot \left(f_{g_{n,m}}(-\ell)\right)^{-1} = \sum_{\beta=1}^{2} e^{2A_\beta^{(2)}}\left(g_n^{(2)}\right)_\beta \left(g_n^{(2)}\right)_\beta \tag{13.3}$$

so that the normalization is taken at the left end of the inverter.

From Section IV the voltage and current at $z = -\ell$ is

$$\begin{pmatrix} \left(\tilde{V}_n(-\ell,s)\right) \\ \left(Z_{c_{n,m}}(-\ell)\right)\cdot\left(\tilde{I}_n(-\ell,s)\right) \end{pmatrix} = \left(\left(\tilde{u}_{n,m}^{(2)}(-\ell,\ell;s)\right)_{\upsilon,\upsilon'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n(\ell,s)\right) \\ \left(Z_{c_{n,m}}(-\ell)\right)\cdot\left(\tilde{I}_n(\ell,s)\right) \end{pmatrix}$$

$$= \left(\left(\tilde{u}_{n,m}^{(2)}(-\ell,\ell;s)\right)_{\upsilon,\upsilon'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n(\ell,s)\right) \\ \left(Z_{c_{n,m}}(-\ell)\right)\cdot\left(Z_{c_{n,m}}(\ell)\right)^{-1}\cdot\left(\tilde{V}_n(\ell,s)\right) \end{pmatrix} \tag{13.4}$$

In terms of the matrix blocks we now have

$$\left(\tilde{V}_n(-\ell,s)\right) = \left(\tilde{Z}_{n,m}^{(2)}(s)\right)\cdot\left(\tilde{I}_n(-\ell,s)\right)$$

$$\left(\tilde{Z}_{n,m}^{(2)}(s)\right) = \left[\left(\tilde{u}_{n,m}^{(2)}(-\ell,\ell;s)\right)_{1,1} + \left(\tilde{u}_{n,m}^{(2)}(-\ell,\ell;s)\right)_{1,2}\cdot\left(f_{n,m}(\ell)\right)^{-1}\right]$$

$$\cdot\left[\left(\tilde{u}_{n,m}^{(2)}(-\ell,\ell;s)\right)_{2,1} + \left(\tilde{u}_{n,m}^{(2)}(-\ell,\ell;s)\right)_{2,2}\cdot\left(f_{n,m}(\ell)\right)^{-1}\right]^{-1}$$

$$\cdot\left(Z_{c_{n,m}}(-\ell)\right) \tag{13.5}$$

$$\left(Z_{c_{n,m}}(-\ell)\right) = Z_w \sum_{\beta=1}^{2}\left(g_n^{(2)}\right)_\beta\left(g_n^{(2)}\right)_\beta$$

From (10.19) the impedance matrix is

$$\left(\tilde{Z}_{n,m}^{(2)}(s)\right)$$

$$= \left\{\sum_{\beta=1}^{2}\frac{\cosh\left(\tilde{\Gamma}_\beta^{(2)}\right)+\frac{2\gamma\ell+A_\beta^{(2)}}{\tilde{\Gamma}_\beta^{(2)}}\sinh\left(\tilde{\Gamma}_\beta^{(2)}\right)}{\cosh\left(\tilde{\Gamma}_\beta^{(2)}\right)+\frac{2\gamma\ell-A_\beta^{(2)}}{\tilde{\Gamma}_\beta^{(2)}}\sinh\left(\tilde{\Gamma}_\beta^{(2)}\right)}\left(g_n^{(2)}\right)_\beta\left(h_n^{(2)}\right)_\beta\right\}\cdot\left(\tilde{Z}_{c_{n,m}}(-\ell)\right)$$

$$= Z_w \sum_{\beta=1}^{2}\frac{\cosh\left(\tilde{\Gamma}_\beta^{(2)}\right)+\frac{2\gamma\ell+A_\beta^{(2)}}{\tilde{\Gamma}_\beta^{(2)}}\sinh\left(\tilde{\Gamma}_\beta^{(2)}\right)}{\cosh\left(\tilde{\Gamma}_\beta^{(2)}\right)+\frac{2\gamma\ell-A_\beta^{(2)}}{\tilde{\Gamma}_\beta^{(2)}}\sinh\left(\tilde{\Gamma}_\beta^{(2)}\right)}\left(g_n^{(2)}\right)_\beta\left(g_n^{(2)}\right)_\beta \tag{13.6}$$

The transfer matrix through the inverter is now found via

$$\left(\tilde{V}_n(-\ell,s)\right) = \left[\left(1_{n,m}\right)+\left(\tilde{S}^{(2)}_{n,m}(s)\right)\right] \cdot \left(\tilde{V}^{(inc)}_n(-\ell,s)\right)$$

$$\left(\tilde{V}_n(\ell,s)\right) \equiv \left(\tilde{T}^{(2)}_{n,m}(s)\right) \cdot \left[\left(1_{n,m}\right)+\left(\tilde{S}^{(2)}_{n,m}(s)\right)\right]^{-1} \cdot \left(\tilde{V}_n(-\ell,s)\right) \tag{13.7}$$

$$\left(\tilde{T}^{(2)}_{n,m}(s)\right) \equiv \text{transfer matrix}$$

From Section X we have

$$\begin{pmatrix} \left(\tilde{V}_n(\ell,s)\right) \\ \left(Z_{c_{n,m}}(-\ell)\right) \cdot \left(\tilde{I}_n(\ell,s)\right) \end{pmatrix} = \left(\left(\tilde{u}^{(2)}_{n,m}(\ell,-\ell;s)\right)_{\upsilon,\upsilon'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n(-\ell,s)\right) \\ \left(Z_{c_{n,m}}(-\ell)\right) \cdot \left(\tilde{I}_n(-\ell,s)\right) \end{pmatrix}$$

$$\left(\tilde{V}_n(\ell,s)\right) = \left(\tilde{u}^{(2)}_{n,m}(\ell,-\ell;s)\right)_{1,1} \cdot \left(\tilde{V}_n(-\ell,s)\right) + \left(\tilde{u}^{(2)}_{n,m}(\ell,-\ell;s)\right)_{1,2} \cdot \left(Z_{c_{n,m}}(-\ell)\right) \cdot \left(\tilde{I}_n(-\ell,s)\right) \tag{13.8}$$

$$= \left[\left(\tilde{u}^{(2)}_{n,m}(\ell,-\ell;s)\right)_{1,1} + \left(\tilde{u}^{(2)}_{n,m}(\ell,-\ell;s)\right)_{1,2} \cdot \left(Z_{c_{n,m}}(-\ell)\right) \cdot \left(\tilde{Z}^{(2)}_{n,m}(s)\right)^{-1}\right] \cdot \left(\tilde{V}_n(-\ell,s)\right)$$

so the transfer matrix is

$$\left(\tilde{T}^{(2)}_{n,m}(s)\right) = \left[\left(\tilde{u}^{(2)}_{n,m}(\ell,-\ell;s)\right)_{1,1} + \left(\tilde{u}^{(2)}_{n,m}(\ell,-\ell;s)\right)_{1,2} \cdot \left(Z_{c_{n,m}}(-\ell)\right) \cdot \left(\tilde{Z}^{(2)}_{n,m}(s)\right)^{-1}\right]$$
$$\cdot \left[\left(1_{n,m}\right) + \left(\tilde{S}^{(2)}_{n,m}(s)\right)\right] \tag{13.9}$$

From (11.9) and (13.6) the scattering matrix is

$$\left(\tilde{S}_{n,m}^{(2)}(s)\right) = \sum_{\beta=1}^{2} \frac{\frac{A_\beta^{(2)}}{\tilde{\Gamma}_\beta^{(2)}} \sinh\left(\tilde{\Gamma}_\beta^{(2)}\right)}{\cosh\left(\tilde{\Gamma}_\beta^{(2)}\right) + \frac{2\gamma\ell}{\tilde{\Gamma}_\beta^{(2)}} \sinh\left(\tilde{\Gamma}_\beta^{(2)}\right)} \left(g_n^{(2)}\right)_\beta \left(h_n^{(2)}\right)_\beta$$

$$\left(1_{n,m}\right) + \left(S_{n,m}^{(2)}(s)\right) = \sum_{\beta=1}^{2} \frac{\cosh\left(\tilde{\Gamma}_\beta^{(2)}\right) + \frac{2\gamma\ell + A_\beta^{(2)}}{\tilde{\Gamma}_\beta^{(2)}} \sinh\left(\tilde{\Gamma}_\beta^{(2)}\right)}{\cosh\left(\tilde{\Gamma}_\beta^{(2)}\right) + \frac{2\gamma\ell}{\tilde{\Gamma}_\beta^{(2)}} \sinh\left(\tilde{\Gamma}_\beta^{(2)}\right)} \left(g_n^{(2)}\right)_\beta \left(h_n\right)_\beta \tag{13.10}$$

The transfer matrix then is

$$\begin{aligned}
\left(\tilde{T}_{n,m}^{(2)}(s)\right) = &\left\{\left[\sum_{\beta=1}^{2} e^{A_\beta^{(2)}} \left[\cosh\left(\tilde{\Gamma}_\beta^{(2)}\right) - \frac{A_\beta^{(2)}}{\tilde{\Gamma}_\beta^{(2)}} \sinh\left(\tilde{\Gamma}_\beta^{(2)}\right)\right] \left(g_n^{(2)}\right)_\beta \left(h_n^{(2)}\right)_\beta\right.\right. \\
&- \left[\sum_{\beta=1}^{2} e^{A_\beta^{(2)}} \frac{2\gamma\ell}{\tilde{\Gamma}_\beta^{(2)}} \sinh\left(\tilde{\Gamma}_\beta^{(2)}\right) \left(g_n^{(2)}\right)_\beta \left(h_n^{(2)}\right)_\beta\right] \cdot \left(Z_{c_{n,m}}(-\ell)\right) \cdot \left(\tilde{Z}_{n,m}^{(2)}(s)\right)^{-1} \Bigg\} \\
&\cdot \left[\left(1_{n,m}\right) + \left(\tilde{S}_{n,m}^{(2)}(s)\right)\right] \\
= &\left\{\sum_{\beta=1}^{2} e^{A_\beta^{(2)}} \left[\cosh\left(\tilde{\Gamma}_\beta^{(2)}\right) + \frac{2\gamma\ell + A_\beta^{(2)}}{\tilde{\Gamma}_\beta^{(2)}} \sinh\left(\tilde{\Gamma}_\beta^{(2)}\right)\right]^{-1} \left(g_n^{(2)}\right)_\beta \left(h_n^{(2)}\right)_\beta\right\} \cdot \left[\left(1_{n,m}\right) + \left(\tilde{S}_{n,m}^{(2)}(s)\right)\right] \\
= &\sum_{\beta=1}^{2} e^{A_\beta^{(2)}} \left[\cosh\left(\tilde{\Gamma}_\beta^{(2)}\right) + \frac{2\gamma\ell}{\tilde{\Gamma}_\beta^{(2)}} \sinh\left(\tilde{\Gamma}_\beta^{(2)}\right)\right]^{-1} \left(g_n^{(2)}\right)_\beta \left(h_n^{(2)}\right)_\beta
\end{aligned} \tag{13.11}$$

As before the scattering parameters (to the left) and input impedance are

$$\tilde{S}^{(2)}(s) = \tilde{S}_{1,1}^{(2)}(s) - \tilde{S}_{2,1}^{(2)}(s) \quad , \quad \tilde{S}_{out}^{(2)}(s) = \tilde{S}_{2,1}^{(2)}(s)$$

$$\tilde{Z}_{in}^{(2)}(s) = \frac{1 + \tilde{S}^{(2)}(s)}{1 - \tilde{S}^{(2)}(s)} \tag{13.12}$$

For the transmission parameters we have

$$\tilde{T}_{in}^{(2)}(s) \equiv \frac{\tilde{V}_2(\ell,s) - \tilde{V}_1(\ell,s)}{e^{-2\gamma\ell}\left[\tilde{V}_1^{(inc)}(-\ell,s) - \tilde{V}_2^{(inc)}(-\ell,s)\right]} = \frac{\tilde{V}_2(\ell,s) - \tilde{V}_1(\ell,s)}{V_o} e^{\gamma\ell}$$

$$= -\left[\tilde{T}_{1,1}^{(2)}(s) - \tilde{T}_{2,1}^{(2)}(s)\right] e^{2\gamma\ell}$$

$\equiv$ transmission to inside of output coax

$$\tilde{T}_{out}^{(1)}(s) = \frac{\tilde{V}_1(\ell,s)}{e^{-2\gamma\ell}\left[\tilde{V}_1^{(inc)}(-\ell,s) - \tilde{V}_2^{(inc)}(-\ell,s)\right]} = \frac{\tilde{V}_1(\ell,s)}{V_o} e^{\gamma\ell} \qquad (13.13)$$

$$= \tilde{T}_{1,1}^{(2)}(s)\ e^{2\gamma\ell}$$

$\equiv$ transmission to outside of output coax

Note the minus sign for $\tilde{T}_{in}^{(2)}(s)$, associated with the use of the center conductor (conductor 2) of the output coax to define the output voltage; this gives the meaning of "inverter". The factor $e^{2\gamma\ell}$ is used to again remove the delay through the transition (for convenience).

The equivalent circuit in Fig. 13.1 can be used to aid in the analysis of the low-frequency behavior of these various parameters, for which we find

$$\tilde{Z}_{in}^{(2)}(0) = Z_w\left[\frac{1}{f_g^{(in)}} + \frac{1}{2\,f_g^{(out)}}\right]^{-1}$$

$$= Z_w\, f_g^{(in)}\left[1 + \frac{f_g^{(in)}}{2f_g^{(out)}}\right]^{-1}$$

$$1 + \tilde{S}^{(2)}(0) = \frac{2\,\tilde{Z}_{in}^{(2)}(0)}{\tilde{Z}_{in}^{(2)}(0) + \tilde{Z}_w\, f_g^{(in)}} = \left[1 + \frac{f_g^{(in)}}{4f_g^{(out)}}\right]^{-1}$$

$$\tilde{S}_{out}^{(2)}(0) = \frac{\tilde{V}_2(-\ell,0)}{\tilde{V}_1(-\ell,0) - \tilde{V}_2(-\ell,0)}\left[1 + \tilde{S}^{(2)}(0)\right]$$

$$= -\frac{1}{2}\left[1 + \tilde{S}^{(2)}(0)\right]$$

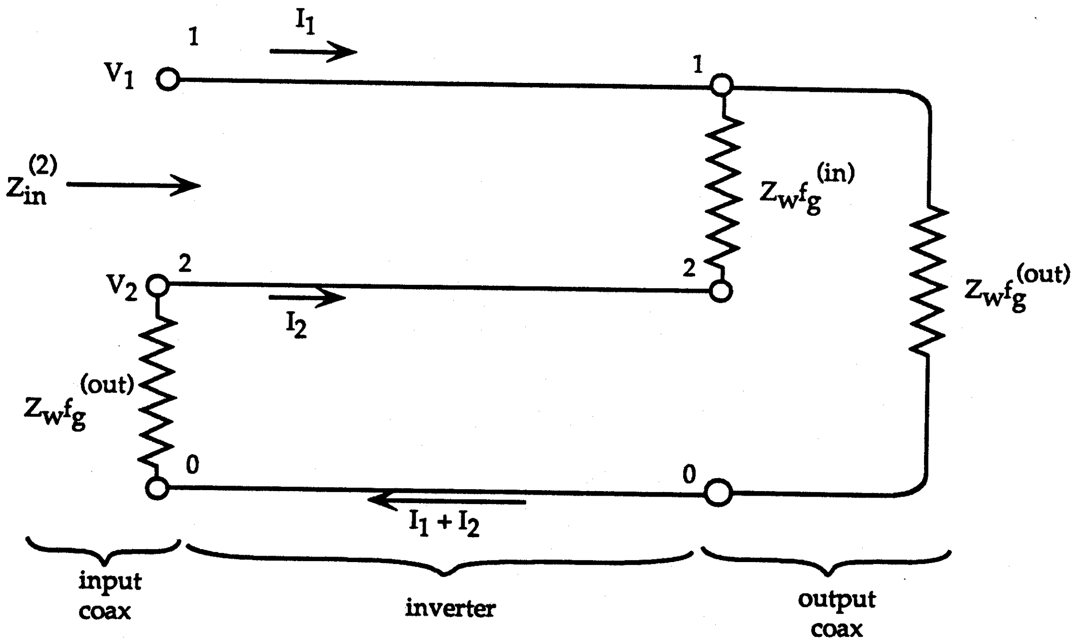


Fig. 13.1. Low-Frequency Equivalent Circuit of Inverter

$$\tilde{T}_{in}^{(2)}(0) = -\left[1 + \tilde{S}^{(2)}(0)\right]$$

$$\tilde{T}_{out}^{(2)}(0) = \tilde{S}_{out}^{(2)}(0) = -\frac{1}{2}\left[1 + \tilde{S}^{(2)}(0)\right] \tag{13.14}$$

From our eigenvector expansions, for low frequencies we also have

$$f_\beta^{(2)}(\ell) = e^{\pm c_s} = e^{2A_\beta^{(2)}}$$

$$\tilde{\Gamma}_\beta^{(2)} = \left[A_\beta^{(2)^2} + [2\gamma\ell]^2\right]^{\frac{1}{2}} = A_\beta^{(2)}\left[1 + O\left(s^2\right)\right] \text{ as } s \to 0$$

$$\left(\tilde{Z}_{n,m}^{(2)}(s)\right) \cdot \left(Z_{c_{n,m}}(-\ell)\right)^{-1} = \sum_{\beta=1}^{2} e^{2A_\beta}\left[1 + O(s)\right]\left(g_n^{(2)}\right)_\beta \left(h_n^{(2)}\right)_\beta$$

$$= \sum_{\beta=1}^{2} f_\beta^{(2)}(\ell)\left(g_n^{(2)}\right)_\beta \left(h_n^{(2)}\right)_\beta = O(s)$$

$$= \left(f_{n,m}^{(2)}(\ell)\right) + O(s) = \left(f_{g_{n,m}}(\ell)\right) \cdot \left(f_{g_{n,m}}(-\ell)\right) + O(s) \tag{13.15}$$

$$\left(\tilde{Z}_{n,m}^{(2)}(s)\right) = Z_w\left(f_{g_{n,m}}(\ell)\right) + O(s) = \left(\tilde{Z}_{c_{n,m}}'(\ell)\right) + O(s) \text{ as } s \to 0$$

Again the low-frequency impedance matrix is that of the transmission line to the right (the load). The scattering matrix is

$$\left(\tilde{S}_{n,m}^{(2)}(s)\right) = \sum_{\beta=1}^{2} \tanh\left(A_\beta^{(2)}\right)\left(g_n^{(2)}\right)_\beta \left(h_n^{(2)}\right)_\beta + O(s)$$

$$= \sum_{\beta=1}^{2} \frac{f_\beta^{(2)}(\ell) - 1}{f_\beta^{(2)}(\ell) + 1}\left(g_n^{(2)}\right)_\beta \left(h_n^{(2)}\right)_\beta + O(s) \text{ as } s \to 0 \tag{13.16}$$

$$\left(1_{n,m}\right) + \left(\tilde{S}_{n,m}^{(2)}(s)\right) = \sum_{\beta=1}^{2} \frac{2f_\beta^{(2)}(\ell)}{f_\beta^{(2)}(\ell) + 1}\left(g_n^{(2)}\right)_\beta \left(h_n^{(2)}\right)_\beta + O(s) \text{ as } s \to 0$$

The transfer matrix is

$$\left(\tilde{T}^{(2)}_{n,m}(s)\right) = \sum_{\beta=1}^{2} \frac{2 f^{(2)}_{\beta}(\ell)}{f^{(2)}_{\beta}(\ell)+1} \left(g^{(2)}_{n}\right)_{\beta} \left(h^{(2)}_{n}\right)_{\beta} + O(s) \text{ as } s \to 0 \tag{13.17}$$

Recalling

$$\left(g^{(2)}_{n}\right)_{\beta} \left(h^{(2)}_{n}\right)_{\beta} = \frac{1}{1-f^{(2)^2}_{\beta}(\ell)} \left(\frac{1}{f^{(2)}_{\beta}(\ell)}\right)\left(\frac{1}{-f^{(2)}_{\beta}(\ell)}\right)$$

$$= \frac{1}{1-e^{4A^{(2)}_{\beta}}} \left(\frac{1}{e^{2A^{(2)}_{\beta}}}\right)\left(\frac{1}{-e^{2A^{(2)}_{\beta}}}\right) \tag{13.18}$$

the various matrix elements can be computed. Then for low frequencies we have

$$\tilde{T}^{(2)}_{in}(s) = -\left[1 + \tilde{S}^{(2)}(s)\right] + O(s) = 1 + \tilde{S}^{(2)}_{1,1}(s) - \tilde{S}^{(2)}_{2,1}(s) + O(s)$$

$$= -\left[\tilde{T}^{(2)}_{1,1}(s) - \tilde{T}^{(2)}_{2,1}(s)\right] e^{2\gamma\ell}$$

$$= -\sum_{\beta=1}^{2} \frac{2 f^{(2)}_{\beta}(\ell)}{\left[f^{(2)}_{\beta}(\ell) + 1\right]^2} + O(s) = -\frac{2e^{2A^{(2)}_{1}}}{\left[e^{2A^{(2)}_{1}} + 1\right]^2} - \frac{2e^{-2A^{(2)}_{1}}}{\left[e^{-2A^{(2)}_{1}} + 1\right]^2} + O(s)$$

$$= -4\left[e^{A^{(2)}_{1}} + e^{-A^{(2)}_{1}}\right]^{-2} + O(s) = -2\left[1 + \cosh(c_5)\right] + O(s)$$

$$= -\frac{2}{1+c_4} + O(s) = -\left[1 + \frac{f^{(in)}_{g}}{4 f^{(out)}_{g}}\right]^{-1} + O(s) \text{ as } s \to 0$$

$$\tilde{S}^{(2)}_{out}(s) = \tilde{S}^{(2)}_{2,1}(s) = -\sum_{\beta=1}^{2} \frac{f^{(2)}_{\beta}(\ell)}{\left[f^{(2)}_{\beta}(\ell) + 1\right]^2} + O(s)$$

$$= -\frac{1}{2}\tilde{T}^{(2)}_{in}(s) + O(s) = -\frac{1}{2}\left[1 + \tilde{S}^{(2)}(0)\right] + O(s) \text{ as } s \to 0$$

$$\tilde{T}^{(2)}_{out}(s) = \tilde{T}^{(2)}_{1,1}(s)\, e^{2\gamma\ell} = \tilde{S}^{(2)}_{2,1}(s) + O(s) \tag{13.19}$$

$$= -\frac{1}{2}\left[1 + \tilde{S}^{(2)}(0)\right] + O(s) \text{ as } s \to 0$$

completing the low-frequency expansions and showing the agreement with the previous equivalent-circuit results.

Turning now to high frequencies, consider first from the input port (back to the left). For $s \to \infty$ with $Re[s] \geq 0$ we have

$$\tilde{\Gamma}_\beta^{(2)} = 2\gamma\ell\left[1 + O\left(s^{-2}\right)\right] \text{ as } s \to \infty$$

$$\tilde{S}^{(2)}(s) = \tilde{S}_{1,1}(s) - \tilde{S}_{2,1}(s) = \sum_{\beta=1}^{2} \frac{A_\beta}{2\gamma\ell}\left[1 - e^{-4\gamma\ell}\right]\frac{1}{f_\beta^{(2)}(\ell) + 1} + O\left(s^{-2}\right)$$

$$= \frac{1 - e^{-4\gamma\ell}}{2\gamma\ell} A_1 \left[\frac{1}{e^{2A_1^{(2)}} + 1} - \frac{1}{1 + e^{-2A_1^{(2)}}}\right] + O\left(s^{-2}\right)$$

$$= \frac{1 - e^{-4\gamma\ell}}{2\gamma\ell} \frac{e^{-2A_1^{(2)}} - e^{2A_1^{(2)}}}{1 + \cosh\left(2A_1^{(2)}\right)} \frac{A_1}{2} + O\left(s^{-2}\right)$$

$$= \frac{1 - e^{-4\gamma\ell}}{2\gamma\ell} \frac{e^{-c_5} - e^{c_5}}{1 + c_4} c_5 + O\left(s^{-2}\right)$$

$$= -\frac{1 - e^{-4\gamma\ell}}{\gamma\ell} \frac{\sqrt{c_4^2 - 1}}{c_4 + 1} \operatorname{arccosh}(c_4) + O\left(s^{-2}\right)$$

$$= -\frac{1 - e^{-4\gamma\ell}}{\gamma\ell}\left[1 + 4\frac{f_g^{(out)}}{f_g^{(in)}}\right]^{-\frac{1}{2}} \operatorname{arccosh}\left(1 + \frac{f_g^{(in)}}{2 f_g^{(out)}}\right) + O\left(s^{-2}\right) \text{ as } s \to \infty$$

$$\tilde{S}_{out}^{(2)}(s) = \tilde{S}_{2,1}^{(2)}(s) = \sum_{\beta=1}^{2} \frac{A_\beta^{(2)}}{2\gamma\ell}\left[1 - e^{-4\gamma\ell}\right]\frac{f_\beta^{(2)}(\ell)}{1 - f_\beta^{(2)^2}(\ell)} + O\left(s^{-2}\right)$$

$$= \frac{1 - e^{-4\gamma\ell}}{2\gamma\ell} A_1^{(2)} \left[\frac{e^{2A_1^{(2)}}}{1 - e^{4A_1^{(2)}}} - \frac{e^{-2A_1^{(2)}}}{1 - e^{-4A_1^{(2)}}}\right] + O\left(s^{-2}\right)$$

$$= -\frac{1-e^{-4\gamma\ell}}{\gamma\ell} A_1^{(2)} \left[e^{c_5} - e^{-c_5}\right]^{-1} + O\left(s^{-2}\right)$$

$$= -\frac{1-e^{-4\gamma\ell}}{4\gamma\ell} \left[c_4^2 - 1\right]^{-\frac{1}{2}} \operatorname{arccosh}\left(c_4\right) + O\left(s^{-2}\right)$$

$$= -\frac{1-e^{-4\gamma\ell}}{4\gamma\ell} \left[\frac{f_g^{(in)}}{f_g^{(out)}} + \frac{1}{4}\left[\frac{f_g^{(in)}}{f_g^{(out)}}\right]^2\right]^{-\frac{1}{2}} \operatorname{arccosh}\left(1 + \frac{f_g^{(in)}}{2 f_g^{(out)}}\right) + O\left(s^{-2}\right) \text{ as } s \to \infty \tag{13.20}$$

Again the reflection from the transition, both inside and outside the coax, behaves like $(\gamma\ell)^{-1}$.

The high-frequency transfer function through the inverter (into the coax) is given by

$$\tilde{T}_{in}^{(2)}(s) = -\left[\tilde{T}_{1,1}^{(2)}(s) - \tilde{T}_{2,1}^{(2)}(s)\right] e^{2\gamma\ell} = -\sum_{\beta=1}^{2} \frac{e^{A_\beta^{(2)}}}{f_\beta^{(2)}(\ell)+1} + O\left(s^{-1}\right)$$

$$= -2\left[e^{A_1^{(2)}} + e^{-A_1^{(2)}}\right]^{-1} + O\left(s^{-1}\right) = -2\left[e^{c_5} + e^{-c_5} + 2\right]^{-\frac{1}{2}} + O\left(s^{-1}\right) \tag{13.21}$$

$$= -2\left[2c_4 + 2\right]^{-\frac{1}{2}} + O\left(s^{-1}\right) = -\left[1 + \frac{f_g^{(in)}}{4 f_g^{(out)}}\right]^{-\frac{1}{2}} + O\left(s^{-1}\right) \text{ as } s \to \infty$$

Note that

$$\lim_{s\to\infty} \tilde{T}_{in}^{(2)}(s) \to -1 \text{ as } \frac{f_g^{(out)}}{f_g^{(in)}} \to \infty$$

$$0 > \lim_{s\to\infty} \tilde{T}_{in}^{(2)}(s) > -1 \tag{13.22}$$

On the outside we have

$$\tilde{T}_{out}^{(2)}(s) = \tilde{T}_{1,1}^{(2)}(s)e^{2\gamma\ell} = \sum_{\beta=1}^{2} \frac{e^{A_\beta^{(2)}}}{1 - f_\beta^{(2)^2}(\ell)} + O(s^{-1})$$

$$= \left[e^{A_1^{(2)}} + e^{-A_1^{(2)}}\right]^{-1} + O(s^{-1}) = -\frac{1}{2}\tilde{T}_{in}^{(2)}(s) + O(s^{-1}) \tag{13.23}$$

$$= \frac{1}{2}\left[1 + \frac{f_g^{(in)}}{4 f_g^{(out)}}\right]^{-\frac{1}{2}} + O(s^{-1}) \text{ as } s \to \infty$$

With the orthogonality of the inside and outside fields, the normalized powers can be added as

$$\frac{1}{f_g^{(in)}}\left[\lim_{s\to\infty} \tilde{T}_{in}^{(2)}(s)\right]^2 + \frac{1}{f_g^{(out)}}\left[\lim_{s\to\infty} \tilde{T}_{out}^{(2)}(s)\right]^2$$

$$= \left[1 + \frac{f_g^{(in)}}{4 f_g^{(out)}}\right]^{-1}\left[\frac{1}{f_g^{(in)}} + \frac{1}{4 f_g^{(out)}}\right] \tag{13.24}$$

$$= \frac{1}{f_g^{(in)}}$$

which is the normalized power incident from the left coax.

XIV. Concluding Remarks

The solutions here of the NMTL equations can be used to construct transitions (baluns and inverters) based on an exponential interpolation of the eigenvalues. Note that the medium in which the perfect conductors are embedded is constrained to be uniform for the present solution procedure to apply. This gives a special kind of spatial variation of the characteristic impedance matrix. One is now in a position to realize the particular geometric-factor matrices. However, the actual cross-section geometry is not in general unique for realizing a particular geometric-factor matrix. This gives some extra degrees of freedom that one can use to optimize the transition design, e.g. by introducing considerations from high-frequency wave propagation which include three-dimensional waves instead of only the one-dimentional transmission-line theory.

Besides the explicit forms given for both high- and low-frequency behavior of the transitions, there are more general expressions applicable to all frequencies, and via the Laplace/Fourier transform to time-domain waveforms as well. The high frequency results are directly applicable to the early-time performance, say for the transport of a step-function incident wave through the transition. One can consider successive terms in the step response to see the decay of the response to the late-time (or low-frequency) response in times proportional to the transit time through (or length of) the transition.

In going through the details of constructing the interpolated geometric-factor matrix the results of [4] have been extended somewhat. In particular it has been shown that this matrix is symmetric, consistent with reciprocity. Furthermore, for the cases considered, it has been shown that the elements are all positive, and that the off-diagonal elements are less than or equal to all the diagonal elements, a condition necessary to the realizability of the cross-section geometry. Viewed another way, what has been constructed is a way to diagonalize the product of two symmetric matrices and use the resulting eigenvectors and eigenvalues to express both original symmetric matrices.

As the high-frequency formulas shown, better performance of baluns and inverters is obtained with large external impedances, the outside wave (sometimes referred to as an antenna mode) undesirably loading the device. In the present paper the model used is applicable for uniform media. One way to achieve a larger external impedance is by use of magnetic chokes such as ferrite cores or related (special media [5, 6]. However, a different model is appropriate to this case.

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