Physics Notes

Note 4

Concerning an Analogy Between Quantum and Classical Electrodynamics

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Abstract

This paper considers an alternate way of viewing the role of the vector potential in quantum electrodynamics (QED). The closed-path line integral of the vector potential can be related to the same integral of the time integral of the electric field. Alternately, by looking at a perfectly conducting loop on this path we can have a static current proportional to the vector potential or associated magnetic flux (i.e. no additional time integral).

I. Potentials and Quantum Phase Shift Around a Closed Path

Recapping from [1] the phase change for a particle (such as an electron) of charge q is

$$\phi = \frac{q}{\hbar}\int_{p}\left[\vec{A}(\vec{r},t) - \int_{-\infty}^{t}\nabla\Phi(\vec{r},t')dt'\right]\cdot d\vec{\ell} \qquad (1.1)$$

in terms of the vector potential $\vec{A}$ and scalar potential Φ. This changes the quantum wave function ψ from the form prior to introduction of the potentials as

$$\psi \to e^{j\phi}\psi \qquad (1.2)$$

For reference we have [2]

$$\begin{aligned}\hbar &= \frac{h}{2\pi} = 1.05443 \times 10^{-34}\ \textit{Joule seconds}\\ q_e &\equiv \textit{electron charge}\\ &\simeq -1.60206 \times 10^{-19}\end{aligned} \qquad (1.3)$$

As indicated in Fig. 1, if we have two paths P_1 and P_2 from $\vec{r}_a$ to $\vec{r}_b$, there are in general two different phase changes given by (1.1). For the same wave function ψ at $\vec{r}_a$ for particles traversing both paths we have at $\vec{r}_b$ wave functions

$$\psi_n = e^{j\phi_n}\psi \text{ for path } P_n \qquad (1.4)$$

The difference in phase is the closed-path integral

$$\begin{aligned}\phi_1 - \phi_2 &= \frac{q}{\hbar}\int_{C}\vec{A}(\vec{r},t)\cdot d\vec{\ell}\\ &= \frac{q}{\hbar}\int_{S}[\nabla\times\vec{A}(\vec{r},t)]\cdot d\vec{\ell}_S\, dS\\ &= \frac{q}{\hbar}\int_{S}\vec{B}(\vec{r},t)\cdot\vec{\ell}_S\, dS\\ &= \frac{q}{\hbar}\Phi_m(t)\end{aligned} \qquad (1.5)$$

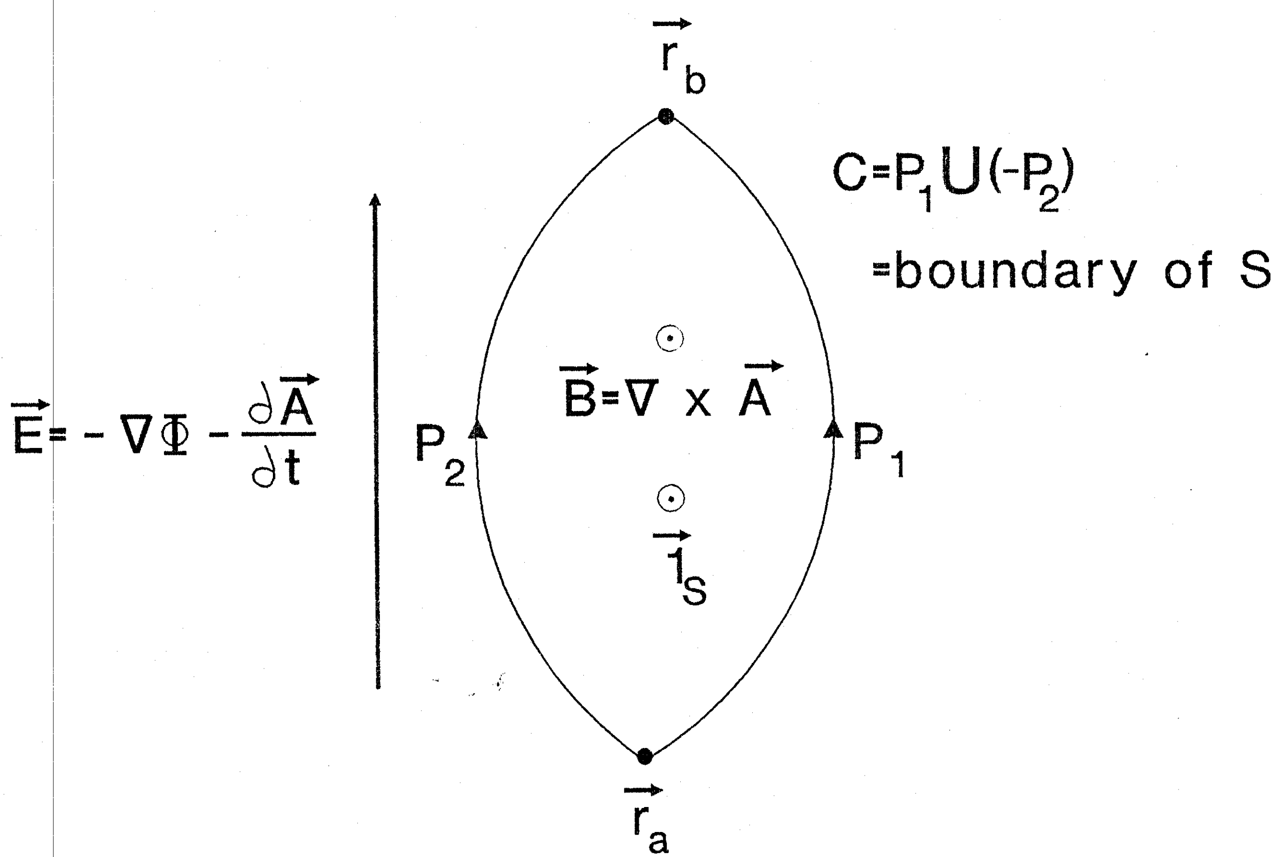

FIG. 1. Paths For Charged Particle

$\Phi_m(t) \equiv$ magnetic flux through S (with boundary C)

$\vec{1}_S \equiv$ unit normal to S

Note that neither Φ nor $\nabla \cdot \vec{A}$ enter into this result which depends only on $\nabla \times \vec{A}$ which is equivalently $\vec{B}$. The gauge invariance of QED is associated with the fact that $\nabla \times \vec{A}$ is the same in every gauge.

Note that quantum phase is not a physical observable (at least in current formulations). The observable is $\psi\psi^*$ or $|\psi|^2$. If, however, we have two (or more) quantum wave functions, such as ψ_1 and ψ_2 at $\vec{r}_b$ we obtain interference at $\vec{r}_b$ from the relative phase with appropriate normalization via

$$|\psi_1 + \psi_2|^2 = |e^{j\Phi_1} + e^{j\Phi_2}|^2 \, |\psi|^2 = 4\cos^2(\Phi_1 - \Phi_2)|\psi|^2 \qquad (1.6)$$

with the phase difference as in (1.5) being the operative parameter. Note that if P_1 and P_2 become the same then $\Phi_m = 0$ and there is no phase difference. The absolute phase at $\vec{r}_b$ can be made whatever one wishes by a gauge transformation involving an arbitrary scalar potential, but this has no effect on the phase difference.

It is also possible to have this phase difference with negligible electric and magnetic fields on C. It is essential that there be a magnetic field and associated flux Φ_m passing through S. A solenoid (with current) or magnetic materials (permanent magnet) can be used to confine the magnetic fields away from C. However, as discussed in [1], in setting up such a field we have

$$\Phi_m(t) = -\oint_C \left[\int_{-\infty}^{t} \vec{E}(\vec{r}, t')\, dt'\right] \cdot d\vec{\ell} = \oint_C \vec{A}(\vec{r}, t) \cdot d\vec{\ell}$$

$$\Phi_m(-\infty) = 0 \ (\textit{initial condition}) \qquad (1.7)$$

Thus, the phase difference is related to the time integral of the electric field (the electric impulse) on the contour. At some time t the electric field (or its contour integral) can be zero, while the corresponding impulse (time integral) is non-zero.

II. Magnetic-Field Measurement by Integration of Electric Field Around a Closed Path

The basic way to measure a magnetic field is a loop as indicated in Fig. 2. For simplicity, let this be a thin conductor on the contour C as indicated with a port (at the loop gap) where we can define voltage and current with some load taken as a resistance R. The basic performance for wavelengths large compared to the loop (electrically small loop) is [4]

$$
\begin{aligned}
V_{o.c.}(t) &= \vec{A}_{h_{eq}} \cdot \frac{\partial \vec{B}^{(inc)}(t)}{\partial t} \text{ (open circuit voltage)} \\
I_{s.c.}(t) &= \vec{l}_{h_{eq}} \cdot \vec{H}^{(inc)}(t) \text{ (short circuit current)} \\
\vec{B}^{(inc)}(t) &= \mu_o \vec{H}^{(inc)}(t) \text{ (incident magnetic field)} \\
\mu_o \vec{A}_{h_{eq}} &= L\vec{\ell}_{h_{eq}} \\
L &\equiv \text{loop inductance} \\
\vec{A}_{h_{eq}} &\equiv \text{equivalent area} \\
\vec{\ell}_{h_{eq}} &\equiv \text{equivalent length}
\end{aligned}
\tag{2.1}
$$

If the loop is approximately planar, encompassing an area A_h with $\vec{1}_S$ perpendicular to this plane, then we have

$$
\begin{aligned}
\vec{A}_{h_{eq}} &= A_h \vec{\ell}_S \\
\vec{\ell}_h &= \ell_h \vec{1}_S
\end{aligned}
\tag{2.2}
$$

For general complex frequencies (still electrically small) we have

$$
\tilde{V}(s) = R\tilde{I}(s) = \frac{sR}{R + sL} \vec{A}_{h_{eq}} \cdot \vec{B}^{(inc)}(s) \tag{2.3}
$$

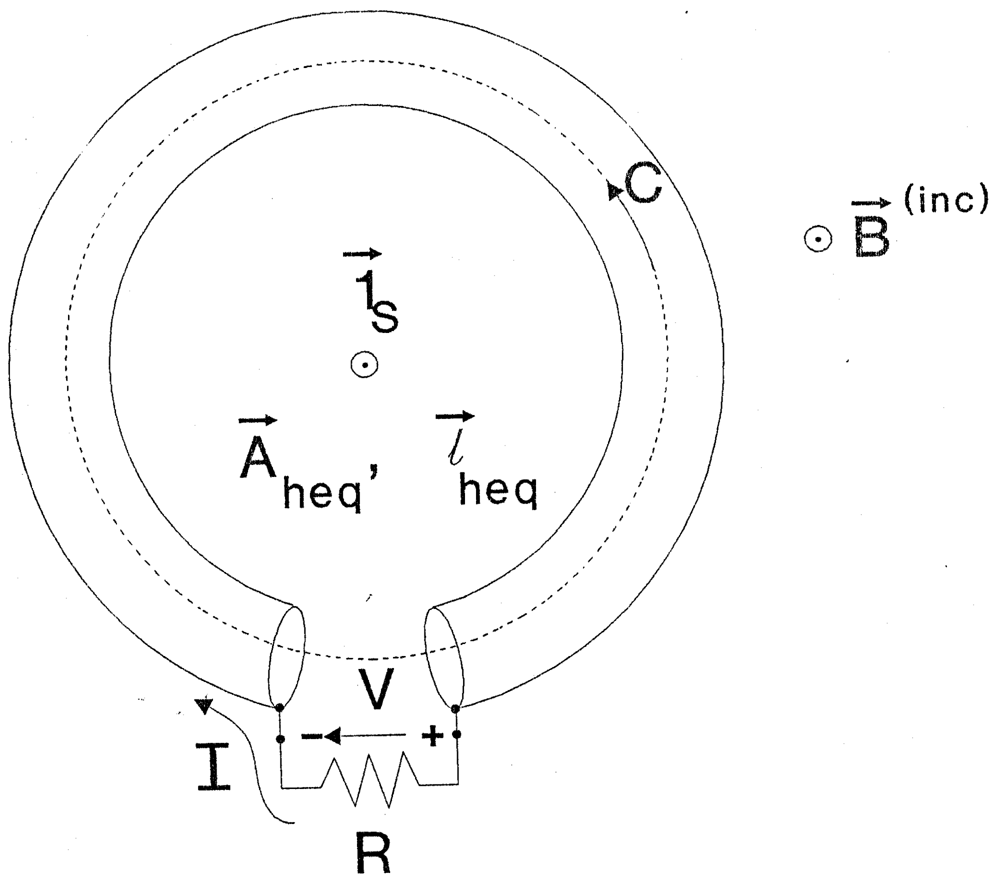

FIG. 2. Basic Magnetic-Field Sensor

Looking at the open-circuit voltage we have

$$V_{o.c.}(t) = \frac{d}{dt}\Phi_m^{(inc)}(t) = -\oint_C \vec{E}^{(inc)}(t) \cdot d\vec{\ell} \tag{2.4}$$

$$\Phi_m^{(inc)}(t) = A_h \vec{1}_S \cdot \vec{B}^{(inc)}(t)$$

So the open-circuit voltage here looks like the phase difference in (1.5) except for a time derivative. The short-circuit current takes the form

$$I_{s.c.}(t) = \frac{1}{L}\vec{A}_{h_{eq}} \cdot \vec{B}^{(inc)}(t) = \frac{\Phi_m^{(inc)}(t)}{L} \tag{2.5}$$

Now we have a loop parameter, the short-circuit current, which is proportional to a magnetic flux without a time derivative, like the phase difference in (1.5). Of course, this is now an incident flux which has been excluded by the closed (perfectly conducting) loop.

The discussion above is in terms of a shorted loop excluding a magnetic flux. Remaining in a classical context, it is also possible to have such a flux in a loop (passing through S) in the absence of an incident field by impressing the current from some source and then shorting the loop gap. For a perfectly conducting loop, the resulting fields on the contour C in Fig. 2 can be zero in such a steady state condition; the loop current flows on the surface of the conductor.

Suppose we take a long solenoid (with current flowing) or a permanent magnet and place it inside the loop in Fig. 2. The resulting magnetic field on C (after placement) due to the solenoid can be made quite small if the solenoid is long compared to the loop diameter. This is what one does in a QED sense to establish the flux between the two paths in Fig. 1. In so doing, one establishes a current in the perfectly conducting loop. So, phase shift around C in a quantum sense is like establishing a current in a perfectly conducting loop on C.

III. Quantization of Magnetic Flux Enclosed by a Superconducting Path

Returning to a quantum view, the short-circuit current in a loop needs to be quantized in the normal form for a superconducting loop [3]. Basically, one just makes the phase difference in (1.5) around a superconducting loop as in Fig. 3 take the form

$$\phi_1 - \phi_2 - \frac{q}{\hbar}\Phi_m = 2\pi n = \textit{integer} \qquad (3.1)$$

so that the wave function is continuous around the loop. It has also been observed that the appropriate charge is

$$q = 2q_e \qquad (3.2)$$

since the superconducting electrons (which are ordinarily Fermi particles) apparently form bound pairs which act as Bose particles. Noting the negative electron charge we have [2].

$$\Delta\Phi_m = -\frac{\pi\hbar}{q_e} = -\frac{h}{2q_e} \simeq 2.0677 x\, 10^{-15} \;\textit{Webers} \qquad (3.3)$$

as the separation of the quantized flux levels. For typical areas this can represent a rather small magnetic field, e.g.

$$A_h = 10^{-4}\ m^2 \qquad (3.4)$$

$$\Delta B \simeq 2 \times 10^{-11}\ \text{Teslas}\ (\simeq 2 \times 10^{-7}\ \text{of earth's magnetic field})$$

Larger areas correspond to even smaller magnetic-field increments.

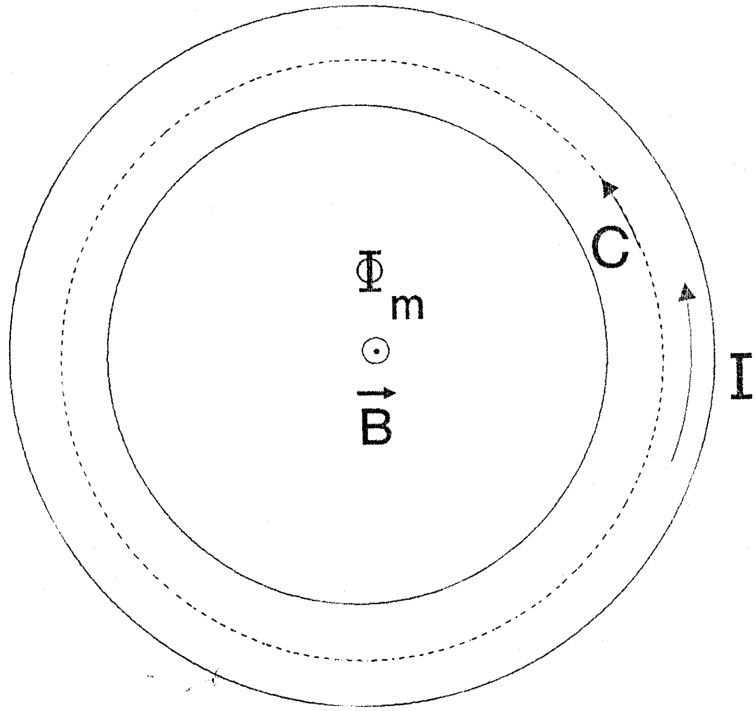

FIG. 3. Flux in Superconducting Loop