Physics Notes

Note 13

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The Combined Field in Quarternion Form

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Abstract

Quarternions (and biquaternions) are a classical subject. Here 2 x 2 dyadics are used as quaternion units to avoid hypercomplex numbers, and the usual matrix properties apply. The combined field with separation index $q = \pm 1$ is formulated in quaternion form yielding two equally valid forms which are used together.

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1. Introduction

The literature concerning quaternions is quite large. One can find a review in [4 (appendix IV)], and another set of references in a recent paper [3]. Some details of the application to the Maxwell equations are found in [2].

In [1, 5] the combined electric and magnetic field has been formulated with a separation index $q = \pm 1$, giving two equally valid formulations, from which the separate fields, currents, potentials, etc. can be reconstructed. (See Appendix A.) Note that in the frequency domain the electromagnetic fields, etc., are complex, in which case one does not separate by real and imaginary parts. However, by using $q = +1$ and $q = -1$, and taking sums and differences the parameters are still separated.

In [5] the 4-vector, 4 x 4 dyadic form of the Maxwell equations has been formulated by introducing an additional separation index $p = \pm 1$ corresponding to the two possible signs on the time variable in the 4-dimensional Minkowski space. This is summarized in Appendix B. Here we have found new forms of the field dyadic with special properties involving symmetries and the indices p and q combined as pq.

In the present paper the combined field with separation index q is formulated in quaternion (or, as some call biquaternion) form. This gives, as one should expect, two possible definitions, and both are carried throughout. Instead of the bicomplex form we use only one complex variable, j, and use 2 x 2 dyadics for the four coordinate units. These dyadics are related to the spin matrices and obey the multiplication rules of the "hypercomplex" numbers. The spatial quaternion units needed for cross product and curl are discussed in Appendix C. The main development in Section 2 is devoted to the choice of the temporal unit $\hat{1}_q$. (Note the use of $\wedge$ above a quantity to denote a quaternion.) This is followed by the development of the Maxwell equations (fields, currents, potential(s)) in this combined quaternion form. Many of the details concerning multiplication and operators are relegated to Appendix D where we find that both q and $-q$ are used.

2. The Electromagnetic Field in Quaternion Form

Begin from the combined field discussed in Appendix A. The Maxwell equations take on the form of a single 3-vector first-order partial differential equation

$$\left[\nabla\times \; - j\frac{q}{c}\frac{\partial}{\partial t}\right]\vec{E}_q(\vec{r},t) \;=\; j\,q\,Z\,\vec{J}(\vec{r},t) \tag{2.1}$$

In quaternion form the 3-vectors (Appendix C) are

$$\begin{aligned}
\hat{E}_q &= \left\{0,\; E_{x_q},\; E_{y_q},\; E_{z_q}\right\} = \left\{0,\; \vec{E}_q\right\} \\
\overset{\wedge(0)}{J}_q &= \left\{0,\; J_{x_q},\; J_{y_q},\; J_{z_q}\right\} = \left\{0,\; \vec{J}_q\right\} \\
\overset{\wedge}{\nabla} &= \left\{0,\; \frac{\partial}{\partial x},\; \frac{\partial}{\partial y},\; \frac{\partial}{\partial z}\right\} = \{0,\; \nabla\}
\end{aligned} \tag{2.2}$$

Substituting in (2.1) we have

$$\begin{aligned}
\overset{\wedge}{\nabla}\bullet\hat{E}_q + [\nabla\bullet\vec{E}_q]\overset{\wedge}{1} - j\frac{q}{c}\frac{\partial}{\partial t}\hat{E}_q &= jq\,Z\,\overset{\wedge(0)}{J}_q \\
\overset{\wedge}{\nabla}\bullet\hat{E}_q - j\frac{q}{c}\frac{\partial}{\partial t}\hat{E}_q &= -[\nabla\bullet\vec{E}_q]\overset{\wedge}{1} + jq\,Z\,\overset{\wedge(0)}{J}_q \\
&= -\frac{\rho_q}{\varepsilon}\overset{\wedge}{1} + jq\,Z\,\overset{\wedge(0)}{J}_q
\end{aligned} \tag{2.3}$$

In a source free region we have

$$\overset{\wedge}{\nabla}\bullet\hat{E}_q - j\frac{q}{c}\frac{\partial}{\partial t}\hat{E}_q = \overset{\wedge}{0} \tag{2.4}$$

which is suggestive of how to define a quaternion operator.

Writing a general quaternion as

$$\begin{aligned}
\hat{Q} &= \left\{Q_s,\; Q_x\; Q_y,\; Q_z\right\} = \left\{Q_s,\; \vec{Q}_V\right\} \\
&= Q_s\,\overset{\wedge}{1}_q + Q_x\,\overset{\wedge}{1}_x + Q_y\,\overset{\wedge}{1}_y + Q_z\,\overset{\wedge}{1}_z
\end{aligned} \tag{2.5}$$

we need to find an optimal definition of $\hat{1}_q$ (the rest being treated in Appendix C). For convenience we define

$$T \equiv ct \equiv \text{time in spatial units} \tag{2.6}$$

Let us try

$$\hat{1}_q = jq\,\hat{1} \tag{2.7}$$

Defining one form of a quaternion operator as

$$\begin{aligned} \square_q &\equiv \left\{-\frac{\partial}{\partial T}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\} = \left\{-\frac{\partial}{\partial T}, \nabla\right\} \\ &= -\hat{1}_q \frac{\partial}{\partial T} + \hat{1}_x \frac{\partial}{\partial x} + \hat{1}_y \frac{\partial}{\partial y} + \hat{1}_z \frac{\partial}{\partial z} \end{aligned} \tag{2.8}$$

then we have in a source free region

$$\hat{\square}_q \cdot \hat{E}_q = \hat{0} \tag{2.9}$$

provided

$$\begin{aligned} \hat{\square}_q &= \hat{\nabla} - \hat{1}_q \frac{\partial}{\partial T} = \left\{-\frac{\partial}{\partial T}, \nabla\right\} \\ \hat{1}_q &= jq\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{2.10}$$

One could also choose the negative of the above. Compare this to the two choices for such a separation index as ± 1 in the case of the 4-vector, 4 x 4-tensor formulation (Appendix B). We still retain q as a separation index = ± 1, from which one can reconstruct the usual (uncombined) electromagnetic parameters by sum and difference. This shows that there are two acceptable ways to construct quaternions (or biquaternions if one prefers). Note now that

$$\hat{1}_q^2 = \hat{1}_q \cdot \hat{1}_q = -\hat{1}\,, \quad \hat{1}_q^{-1} = -\hat{1}_q = \hat{1}_q^{\dagger} \tag{2.11}$$

so that it is also a square root of the negative identity, like the spatial units.

See Appendix D for more concerning this quaternion operator $\hat{\Box}_q$. There we also have the quaternion coordinate

$$\hat{T} = \hat{T}_q = \{T, x, y, z\} = \{T, \vec{r}\} \tag{2.12}$$

We can now regard our quaternions as functions of this quaternion variable, i.e.

$$\begin{aligned} \vec{E}_q(\vec{r},t) &\rightarrow \hat{E}_q(\hat{T}_q) \\ \vec{J}_q^{(0)}(\vec{r},t) &\rightarrow \hat{J}_q^{(0)}(\hat{T}_q) \end{aligned} \tag{2.13}$$

etc., and need not explicitly exhibit this dependence.

Now let us redefine the quaternion current so that

$$\begin{aligned} \hat{J}_q &= \hat{J}_q^{(0)} + jqc\rho_q \hat{1} = \hat{J}_q^{(0)} + c\rho_q \hat{1}_q \\ &= \left\{c\rho_q, J_{x_q}, J_{y_q}, J_{z_q}\right\} \equiv \left\{c\rho_q, \vec{J}_q\right\} \\ \hat{\Box}_q \bullet \hat{E}_q &= jqZ\hat{J}_q \quad \text{(Maxwell equations)} \end{aligned} \tag{2.14}$$

So here we have constructed a form of the Maxwell equations with one separation index, instead of the two in 4-vector form (Appendix B).

From (D.9)

$$\begin{aligned} \hat{\Box}_{-q} \bullet \hat{\Box}_q &= \hat{\Box}^2 \hat{1} \\ \hat{\Box}^2 &= \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial T^2} - \nabla^2 \end{aligned} \tag{2.15}$$

Then from (D.12) we operate on the quaternion current to find

$$\hat{\Box}^2 \hat{E}_q = jqZ \hat{\Box}_{-q} \cdot \hat{J}_q$$
$$= jqZ \left\{ jq \left[\frac{\partial [c\rho_q]}{\partial T} + \nabla \cdot \vec{J}_q \right], \ jq \left[\frac{\partial \vec{J}_q}{\partial T} + \nabla [c\rho_q] \right] + \nabla \times \vec{J}_q \right\} \tag{2.16}$$
$$= jqZ \left\{ 0 \ , \ jq \left[\frac{\partial \vec{J}_q}{\partial T} + \nabla [c\rho_q] \right] + \nabla \times \vec{J}_q \right\}$$

owing to the equation of continuity

$$\frac{\partial [c\rho_q]}{\partial T} + \nabla \cdot \vec{J}_q = 0 \tag{2.17}$$

Thus, as we expect, only the vector part of the quaternions appear in (2.16) with separate (decoupled) equations for each vector component in the wave equation.

Extending to the potentials we begin with

$$\begin{aligned} \vec{E}_q(\vec{r},t) &= -\nabla \Phi_q(\vec{r},t) + jqc \left[\nabla \times + \frac{jq}{c} \frac{\partial}{\partial T} \right] \vec{A}_q(\vec{r},t) \\ \nabla \cdot \vec{A}_q(\vec{r},t) &+ \frac{1}{c^2} \frac{\partial}{\partial T} \Phi_q(\vec{r},t) = 0 \quad \text{(Lorentz gauge condition)} \end{aligned} \tag{2.18}$$

Defining

$$\hat{A}_q \equiv \left\{ \frac{1}{c} \Phi_q \, , \, \vec{A} \right\} \tag{2.19}$$

we have

$$\hat{\Box}_{-q} \cdot \hat{A}_q$$
$$= \left\{ jq \left[\frac{1}{c} \frac{\partial \Phi_q}{\partial T} + \nabla \cdot \vec{A}_q \right], \ jq \left[\frac{\partial \vec{A}_q}{\partial T} + \frac{1}{c} \nabla \Phi_q \right] + \nabla \times \vec{A}_q \right\}$$

$$= \left\{ 0 \, , \; jq \left[\frac{\partial \vec{A}_q}{\partial T} + \frac{1}{c} \nabla \Phi_q \right] + \nabla \times \vec{A}_q \right\} \tag{2.20}$$

$$\hat{E}_q = jqc \hat{\Box}_{-q} \bullet \hat{A}_q$$

agreeing with (2.18)

Substituting from (2.20) in (2.14) gives

$$\hat{\Box}^2 \hat{A}_q = \mu \hat{J}_q \tag{2.21}$$

as the wave equation for the quaternion potential. Note the separation of the four components from each other. From (A.5) we can then write

$$\hat{A}_q = \mu \int_V \frac{\hat{J}_q(\hat{T}')}{4\pi \, | \vec{r} - \vec{r}\,' \, |} \, dV' \tag{2.22}$$

$$\hat{T}' = \left\{ T - | \vec{r} - \vec{r}\,' \, | , \; \vec{r}\,' \right\}$$

If one wishes $| \vec{r} - \vec{r}\,' \, |$ can also be written in quaternion form from (C.15) as

$$[\hat{r} - \hat{r}'] \bullet [\hat{r} - \hat{r}'] = - | \vec{r} - \vec{r}\,' \, |^2 \, \hat{1} \tag{2.23}$$

3. Concluding Remarks

Now we have placed the combined field in quaternion form, complementing the previous results in 4-vector, 4 x 4 dyadic form. Here we have a single separation index, $q = \pm 1$. Both choices are equally valid, and both forms are used in conjunction with each other.

Appendix A. The Combined Field

Summarizing from [1, 5], we have the combined field

$$\vec{E}_q(\vec{r},t) = \vec{E}(\vec{r},t) + jqZ\,\vec{H}(\vec{r},t)$$
$$q = \pm 1 \equiv \text{separation index}$$
$$Z = \left[\frac{\mu}{\varepsilon}\right]^{1/2} \equiv \text{wave impedance of uniform isotropic medium characterized by frequency-independent (real) permeability } \mu \text{ and permittivity } \varepsilon \tag{A.1}$$
$$c = [\mu\,\varepsilon]^{-1/2} \equiv \text{wave propagation speed in medium (light speed in free space)}$$

The combined current density is

$$\vec{J}_q(\vec{r},t) = \vec{J}(\vec{r},t) + j\frac{q}{Z}\vec{J}_m(\vec{r},t)$$
$$\vec{J}_m \equiv \text{magnetic-current density, in general fictitious, but useful in various cases.} \tag{A.2}$$

The combined Maxwell equations then are written as

$$\left[\nabla\times - j\frac{q}{c}\frac{\partial}{\partial t}\right]\ \vec{E}_q(\vec{r},t) = jqZ\,\vec{J}_q(\vec{r},t) \tag{A.3}$$

The combined charge density is

$$\rho_q(\vec{r},t) \equiv \rho(\vec{r},t) + j\frac{q}{Z_0}\rho_m(\vec{r},t)$$
$$\nabla \bullet \vec{E}_q(\vec{r},t) = \frac{1}{\varepsilon}\rho_q(\vec{r},t) \tag{A.4}$$
$$\nabla \bullet \vec{J}_q(\vec{r},t) = -\varepsilon\frac{\partial}{\partial t}\nabla \bullet \vec{E}_q(\vec{r},t) = -\frac{\partial}{\partial t}\rho_q(\vec{r},t) \quad \text{(continuity equation)}$$

This is extended to the potentials

$$\vec{A}_q(\vec{r},t) \equiv \vec{A}(\vec{r},t) + jqZ\,\vec{A}_m(\vec{r},t)$$
$$\Phi_q(\vec{r},t) \equiv \Phi(\vec{r},t) + jqZ\Phi_m(\vec{r},t)$$

$$\vec{E}_q(\vec{r},t) = -\nabla\Phi_q(\vec{r},t) + \left[jqc\nabla\times - \frac{\partial}{\partial t}\right]\vec{A}_q(\vec{r},t)$$

$$= -\nabla\Phi_q(\vec{r},t) + jqc\left[\nabla\times + \frac{jq}{c}\frac{\partial}{\partial t}\right]\vec{A}_q(\vec{r},t)$$

$$\vec{A}_q(\vec{r},t) = \mu\int_V \frac{\vec{J}_q\left(\vec{r}\,',t-\frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{4\pi|\vec{r}-\vec{r}\,'|}dV' \tag{A.5}$$

$$\Phi_q(\vec{r},t) = \frac{1}{\varepsilon}\int_V \frac{\rho_q\left(\vec{r}\,',t-\frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{4\pi|\vec{r}-\vec{r}\,'|}dV'$$

These are related by the Lorentz gauge condition

$$\nabla\cdot\vec{A}_q(\vec{r},t) + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi_q(\vec{r},t) = 0 \tag{A.6}$$

Appendix B. Four-Vectors and Dyadics

Also summarizing from [5] we have the reformulation of the usual four-vectors and dyadics using the combined field, leading to interesting simplifications. For this purpose the position vector for four-dimensional Minkowski space takes the form (with an overbar indicating a four-vector)

$$
\begin{aligned}
\bar{r}_p &\equiv (\vec{r}, T_p) = (x, y, z, T_p) \\
&= x\,\bar{1}_x + y\,\bar{1}_y + z\,\bar{1}_z + T_p\,\bar{1}_{T_p} \\
T_p &\equiv jpct\,,\ p = \pm 1
\end{aligned}
\tag{B.1}
$$

giving two choices for the new index on the time coordinate (imaginary). The del operator and Laplacian become

$$
\begin{aligned}
\Box_p &= \left(\nabla, \frac{\partial}{\partial T_p}\right) \\
\Box_p^2 &= \Box_{-p}^2 = \nabla^2 + \frac{\partial^2}{\partial T_p^2} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}
\end{aligned}
\tag{B.2}
$$

We also have

$$
\bar{r}_p^2 = \bar{r}_p \bullet \bar{r}_p = \vec{r} \bullet \vec{r} - c^2 t^2 = |\vec{r}|^2 - c^2 t^2
\tag{B.3}
$$

Then various electromagnetic quantities take the forms

$$
\begin{aligned}
&\bar{J}_{p,q}(\bar{r}_p) = (\vec{J}_q(\vec{r},t),\ jpc\rho_q(\vec{r},t)) \text{ (4-current density)} \\
&\Box_p \bullet \bar{J}_{p,q}(\bar{r}_p) = 0 = \nabla \bullet \vec{J}_q(\vec{r},t) + \frac{\partial}{\partial t}\rho_q(\vec{r},t)) \text{ (continuity)} \\
&\bar{A}_{p,q}(\bar{r}_p) = \left(\vec{A}_q(\vec{r},t),\ j\frac{p}{c}\Phi_q(\vec{r},t)\right) \text{ (4-potential)} \\
&\Box_p \bullet \bar{A}_{p,q}(\bar{r}_p) = 0 = \nabla \bullet \vec{A}_q(\vec{r},t) + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi_q(\vec{r},t) \text{ (Lorentz gauge)}
\end{aligned}
\tag{B.4}
$$

$$\Box_p^2 \bullet \overline{A}_{p,q}(\overline{r}_p) = -\mu \, \overline{J}_{p,q}(\overline{r}_p)$$

$$\overline{A}_{p,q}(\overline{r}_p) = \mu \int_V \frac{\overline{J}_{p,q}(\overline{r}_p'')}{4\pi \,|\, \overrightarrow{r} - \overrightarrow{r}' \,|} \, dV' \; , \quad \overline{r}_p'' = (\overrightarrow{r}', jp[ct - |\, \overrightarrow{r} - \overrightarrow{r}' \,|])$$

The combined Maxwell equations are now written as

$$\Box_p \bullet \overline{\overline{E}}_{p,q}(\overline{r}_p) = jq\, Z\, \overline{J}_{p,q}(\overline{r}_p)$$

$$\overline{\overline{E}}_{p,q}(\overline{r}_p) \equiv \begin{pmatrix} 0 & -E_{z_q} & E_{y_q} & -pqE_{x_q} \\ E_{z_q} & 0 & -E_{x_q} & -pqE_{y_q} \\ -E_{y_q} & E_{x_q} & 0 & -pqE_{z_q} \\ pqE_{x_q} & pqE_{y_q} & -pqE_{z_q} & 0 \end{pmatrix} \qquad \text{(B.5)}$$

$$= -\overline{\overline{E}}_{p,q}^{\,T}(\overline{r}_p) \quad \text{(skew symmetric)}$$

This 4 x 4 dadic is much more symmetric (involving basically only three vector components) than the traditional field tensor(s). This is also self dual in the sense discussed in [5] and can be derived from $\overline{A}_{p,q}$ using $\Box_p$ and the dual operator.

Note also that the field tensor has an inverse as

$$\overline{\overline{E}}_{p,q}^{-1}(\overline{r}_p) = \left[\overrightarrow{E}_q(\overrightarrow{r},t) \bullet \overrightarrow{E}_q(\overrightarrow{r},t) \right]^{-1} \overline{\overline{E}}_{p,q}(\overline{r}_p) \qquad \text{(B.6)}$$

so that, except for a constant multiplier, it is its own inverse. While a vector does not have an inverse, the electromagnetic field in the above form does have an inverse provided

$$\begin{aligned} & \overrightarrow{E}_q(\overrightarrow{r},t) \bullet \overrightarrow{E}_q(\overrightarrow{r},t) \\ &= \overrightarrow{E}(\overrightarrow{r},t) \bullet \overrightarrow{E}(\overrightarrow{r},t) - Z_0^2\, \overrightarrow{H}(\overrightarrow{r},t) \bullet \overrightarrow{H}(\overrightarrow{r},t) + j2q\, \overrightarrow{E}(\overrightarrow{r},t) \bullet \overrightarrow{H}(\overrightarrow{r},t) \qquad \text{(B.7)} \\ &\neq 0 \end{aligned}$$

Note for a plane wave that this is zero, and the inverse does not exist.

Appendix C. The Spatial Quaternion Units

Quaternions have taken various forms, depending on the various authors. Here we treat quaternions as complex 2 x 2 dyadics, rather than as hypercomplex numbers. In this form the usual rules of matrix algebra apply. Let us use an overhead ^ to symbolize such a quaternion for which we immediately write

$$\hat{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \text{identity quaternion}$$

$$\hat{0} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \equiv \text{zero quaternion} \tag{C.1}$$

We write the four quaternion units as

$$\hat{1}_q\,,\ \hat{1}_x\,,\ \hat{1}_y\,,\ \hat{1}_z \tag{C.2}$$

The first is time- and scalar-like while the remaining three are space- and vector-like. In this form we can write a general quaternion as

$$\hat{Q} \equiv Q_s\,\hat{1}_q + Q_x\,\hat{1}_x + Q_y\,\hat{1}_y + Q\,\hat{1}_z \equiv \{Q_s\,,\ Q_x\,,\ Q_y\,,\ Q_z\} = \left\{Q_s\,,\ \vec{Q}_{\text{v}}\right\} \tag{C.3}$$

Here we concentrate on the second through fourth units, leaving the first for the main text.

Mimicing the cross product

$$\begin{aligned} \vec{1}_x \times \vec{1}_y &= \vec{1}_z = -\vec{1}_y \times \vec{1}_x \\ \vec{1}_y \times \vec{1}_z &= \vec{1}_x = -\vec{1}_z \times \vec{1}_y \\ \vec{1}_z \times \vec{1}_x &= \vec{1}_y = -\vec{1}_x \times \vec{1}_z \end{aligned} \tag{C.4}$$

we have the usual form of the corresponding quaternion units, replacing cross products by dot products, as

$$\hat{1}_x \bullet \hat{1}_y = \hat{1}_z = -\hat{1}_y \bullet \hat{1}_x$$
$$\hat{1}_y \bullet \hat{1}_z = \hat{1}_x = -\hat{1}_z \bullet \hat{1}_y \qquad \text{(C.5)}$$
$$\hat{1}_z \bullet \hat{1}_x = \hat{1}_y = -\hat{1}_x \bullet \hat{1}_z$$

If we enforce

$$\hat{1}_\zeta^2 \equiv \hat{1}_\zeta \bullet \hat{1}_\zeta = -\hat{1} \quad , \quad \zeta = x, y, z \qquad \text{(C.6)}$$

analogous to the square root of -1, then the right sides of (C.6) (anticommuting) are immediately derivable from the left sides. We also have

$$\hat{1}_x \bullet \hat{1}_y \bullet \hat{1}_z = -\hat{1} \text{ (and all cyclic permutations)}$$
$$\hat{1}_z \bullet \hat{1}_y \bullet \hat{1}_x = +\hat{1} \text{ (and all cyclic permutations)}$$
$$\hat{1}_\zeta^{-1} \equiv -\hat{1}_\zeta \qquad \text{(C.7)}$$
$$\det(\hat{1}_\zeta) = \pm 1 \text{ (determinant)}$$

Let us adopt a convention for all ζ (for symmetry)

$$\det(\hat{1}_\zeta) = +1 \qquad \text{(C.8)}$$

which is one choice consistent with (C.6) (Note that minus-sign coefficients go to plus when taking the determinant of an N x N matrix for N = even.)

From (C.6) we have that the eigenvalues of $\hat{1}_\zeta^2$ are both -1. Hence the eigenvalues of $\hat{1}_\zeta$ are $\pm j$. For unit determinant, then one eigenvalue is $+j$ and the other $-j$, giving

$$tr(\hat{1}_\zeta) = 0 \text{ (trace)}$$
$$\lambda_{\frac{1}{2}}(\hat{1}_\zeta) = \pm j \text{ (eigenvalues)} \qquad \text{(C.9)}$$

for all the spatial quaternion units. We can note that the (C.9) properties also apply to the transpose, adjoints, and conjugates since

$$-\hat{1} = \hat{1}_\zeta^* \cdot \hat{1}_\zeta^* = \hat{1}_\zeta^T \cdot \hat{1}_\zeta^T = \hat{1}_\zeta^\dagger \equiv \hat{1}_\zeta^\dagger \tag{C.10}$$

One particular choice of these spatial quaternion units is

$$\begin{gathered} \hat{1}_x = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix} , \quad \hat{1}_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad \hat{1}_z = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \\ \hat{1}_\zeta^\dagger = -\hat{1}_\zeta = \hat{1}_\zeta^{-1} \end{gathered} \tag{C.11}$$

This choice is not unique since anything of the form

$$\hat{X} \cdot \hat{1}_\zeta \cdot \hat{X}^{-1} \quad , \quad \zeta = x, y, z \tag{C.12}$$

also satisfies (C.5) through (C.8). If, in addition, $\hat{X}$ is unitary we have

$$\begin{gathered} \hat{X}^{-1} = \hat{X}^\dagger \quad \text{(unitary)} \\ \left[\hat{X} \cdot \hat{1}_\zeta \cdot \hat{X}^\dagger \right]^\dagger = \hat{X} \cdot \hat{1}_\zeta^\dagger \cdot \hat{X}^\dagger = -\hat{X} \cdot \hat{1}_\zeta \cdot \hat{X}^\dagger \end{gathered} \tag{C.13}$$

with the same property of inverse equals negative adjoint.

We can write these spatial quaternion units in dyadic form as

$$\begin{aligned} \hat{1}_x &= j\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - j\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \hat{1}_y &= j\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ j \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -j \end{pmatrix} - j\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -j \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ j \end{pmatrix} \\ \hat{1}_z &= j\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - j\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned} \tag{C.14}$$

where the eigenvectors are exhibited, including their normalizations, to give biorthonormal eigenvectors.

The position vector takes the form

$$\hat{r} \equiv x\,\hat{1}_x + y\,\hat{1}_y + z\,\hat{1}_z = \{0, \vec{r}\}$$
$$\hat{r} \bullet \hat{r} = -\left[x^2 + y^2 + z^2\right]\hat{1} = -r^2\,\hat{1} \tag{C.15}$$

Note that if the quaternion units are changed by a unitary transformation (C.13), this corresponds to a rotation of the coordinate axes. An alternate form for (C.15) has

$$\hat{r}^{\dagger} = -\hat{r}$$
$$\hat{r} \bullet \hat{r}^{\dagger} = \hat{r}^{\dagger} \bullet \hat{r} = r^2\,\hat{1} \tag{C.16}$$

for a "positive" result. (Note that $\vec{r}$ is real.)

While vectors do not have inverses, quaternions (being 2 x 2 matrices) do, here for the vector part, in the form

$$\hat{f}^{-1} \bullet \hat{f} = \hat{1}$$
$$\hat{f}^{-1} \bullet \hat{f} \bullet \hat{f} = \hat{f} = -\left[\vec{f} \bullet \vec{f}\right]\hat{f}^{-1} \tag{C.17}$$
$$\hat{f}^{-1} = -\frac{\hat{f}}{\vec{f} \bullet \vec{f}} \quad \text{for} \quad \vec{f} \bullet \vec{f} \neq 0$$

One can interpret this as reversing the direction and taking one over the magnitude. This is the negative of inversion. The above is fine for real $\vec{f}$. For complex $\vec{f}$ one might prefer to use $\hat{f}^{\dagger}$, but this gives the complicated form

$$\hat{f} \bullet \hat{f}^{\dagger} = \vec{f} \bullet \vec{f}^{*}\,\hat{1} + \hat{1}_x\left[f_y f_z^* - f_z f_y^*\right]$$
$$+ \hat{1}_y\left[f_z f_x^* - f_x f_z^*\right] + \hat{1}_z\left[f_x f_y^* - f_y f_x^*\right] \tag{C.18}$$
$$\vec{f} \bullet \vec{f}^{*} = |\vec{f}|^2 \neq 0 \text{ unless } \vec{f} = \vec{0}$$

While the complex magnitude is present, it is only one of the terms.

At this point we can consider the cross products of space vectors in quaternion form. Let

$$\vec{f}^{(1)} \times \vec{f}^{(2)} = \vec{f}^{(3)} \tag{C.19}$$

Then we have after some algebra

$$\begin{aligned} \hat{f}^{(3)} &= f_x^{(3)} \hat{1}_x + f_y^{(3)} \hat{1}_y + f_z^{(3)} \hat{1}_z = \{0, \vec{f}^{(3)}\} \\ &= \hat{f}^{(1)} \bullet \hat{f}^{(2)} - \vec{f}^{(1)} \bullet \vec{f}^{(2)} \hat{1} \end{aligned} \tag{C.20}$$

showing the mixture of the cross and dot products in quaternion form.

Consider the spatial part of a quaternion operator

$$\begin{aligned} \hat{\nabla} &\equiv \hat{1}_x \frac{\partial}{\partial x} + \hat{1}_y \frac{\partial}{\partial y} + \hat{1}_z \frac{\partial}{\partial z} = \{0, \nabla\} \\ \vec{g} &\equiv \nabla \times \vec{f} \\ \hat{g} &= \hat{\nabla} \bullet \hat{f} + [\nabla \bullet \vec{f}]\hat{1} \\ \hat{g} &= \hat{1}_x \left[\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right] + \hat{1}_y \left[\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right] = \hat{1}_z \left[\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right] \\ &= \{0, \vec{g}\} = \{0, \nabla \times \vec{f}\} \end{aligned} \tag{C.21}$$

As we can see $\hat{\nabla}$ includes both divergence and curl.

Appendix D. More Concerning Quaternions

Having established the first quaternion unit in (2.7) as

$$\hat{1}_q = jq\,\hat{1} = jq\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = -\hat{1}_q^{\dagger} = -\hat{1}_q^{-1} \tag{D.1}$$

we can proceed to other aspects of our form of the quaternions. Write a general quaternion (notation) as

$$\begin{aligned}
\hat{Q} &= Q_s\,\hat{1}_q + \hat{Q}_{\rm v} = \{Q_s, \vec{Q}_{\rm v}\} \equiv \{Q_s, Q_x, Q_y, Q_z\}\\
\hat{Q}_{\rm v} &= \{0, \vec{Q}_{\rm v}\} = \{0, Q_x, Q_y, Q_z\}\\
&= Q_x\,\hat{1}_x + Q_y\,\hat{1}_y + Q_z\,\hat{1}_z \equiv \text{vector part}\\
\vec{Q}_{\rm v} &\equiv Q_x\,\vec{1}_x + Q_y\,\vec{1}_y + Q_z\,\vec{1}_z \equiv \text{associated vector}\\
Q_s &\equiv \text{scalar part}
\end{aligned} \tag{D.2}$$

Two quaternions then multiply as

$$\begin{aligned}
\hat{Q}^{(1)} \bullet \hat{Q}^{(2)} &= -Q_s^{(1)}\,Q_s^{(2)}\,\hat{1} + jq\;Q_s^{(1)}\,\hat{Q}_{\rm v}^{(2)} + jq\;Q_s^{(2)}\,\hat{Q}_{\rm v}^{(1)} + \hat{Q}_{\rm v}^{(1)} \bullet \hat{Q}_{\rm v}^{(2)}\\
\hat{Q}_{\rm v}^{(1)} \bullet \hat{Q}_{\rm v}^{(2)} &= -\left[Q_x^{(1)}\,Q_x^{(2)} + Q_y^{(1)}\,Q_y^{(2)} + Q_z^{(1)}\,Q_z^{(2)}\right]\hat{1}\\
&\quad + \left[Q_y^{(1)}Q_z^{(2)} - Q_z^{(1)}Q_y^{(2)}\right]\hat{1}_x + \left[Q_z^{(1)}Q_x^{(2)} - Q_x^{(1)}Q_z^{(2)}\right]\hat{1}_y\\
&\quad + \left[Q_x^{(1)}Q_y^{(2)} - Q_y^{(1)}Q_x^{(2)}\right]\hat{1}_z\\
&= -\left[\vec{Q}_{\rm v}^{(1)} \bullet \vec{Q}_{\rm v}^{(2)}\right]\hat{1}\\
&\quad + \left[\vec{Q}_{\rm v}^{(1)} \times \vec{Q}_{\rm v}^{(2)}\right]_x \hat{1}_x + \left[\vec{Q}_{\rm v}^{(1)} \times \vec{Q}_{\rm v}^{(2)}\right]_y \hat{1}_y + \left[\vec{Q}_{\rm v}^{(1)} \times \vec{Q}_{\rm v}^{(2)}\right]_z \hat{1}_z\\
&= \left\{ jq\,\vec{Q}_{\rm v}^{(1)} \bullet \vec{Q}_{\rm v}^{(2)} \;,\; \vec{Q}_{\rm v}^{(1)} \times \vec{Q}_{\rm v}^{(2)} \right\}\\
\hat{Q}^{(1)} \bullet \hat{Q}^{(2)} &\\
&= \left\{ jq\left[Q_s^{(1)}\,Q_s^{(2)} + \vec{Q}_{\rm v}^{(1)} \bullet \vec{Q}_{\rm v}^{(2)}\right],\; jq\left[Q_s^{(1)}\,\vec{Q}_{\rm v}^{(2)} + Q_s^{(2)} \bullet \vec{Q}_{\rm v}^{(1)}\right] + \vec{Q}_{\rm v}^{(1)} \times \vec{Q}_{\rm v}^{(2)} \right\}
\end{aligned} \tag{D.3}$$

If we subscript by q and $-q$ so that

$$\hat{Q}_q = \left\{ Q_s \, , \, \vec{Q}_{\text{V}} \right\} = Q_s \, \hat{1}_q + \hat{Q}_{\text{V}}$$
$$\hat{Q}_{-q} = \left\{ -Q_s \, , \, \vec{Q}_{\text{V}} \right\} = -Q_s \, \hat{1}_q + \hat{Q}_{\text{V}} \tag{D.4}$$

then we have

$$\hat{Q}_q^{(1)} \cdot \hat{Q}_{-q}^{(2)} = Q_s^{(1)} Q_s^{(2)} \, \hat{1} + jq Q_s^{(1)} Q_{\text{V}}^{(2)} - jq Q_s^{(2)} Q_{\text{V}}^{(1)} + \hat{Q}_{\text{V}}^{(1)} \cdot \hat{Q}_{\text{V}}^{(2)}$$
$$\hat{Q}_q^{(1)} \cdot \hat{Q}_{-q}^{(2)} = \left[Q_s^2 + \vec{Q}_{\text{V}} \cdot \vec{Q}_{\text{V}} \right] \hat{1} = \hat{Q}_{-q} \cdot \hat{Q}_q \quad \text{(commute)} \tag{D.5}$$

this being valid for general complex components. Notationally we have

$$\hat{Q}_q = \hat{Q} \tag{D.6}$$

i.e., a quaternion with first unit $\hat{1}_q$ need not be subscripted by q.

We can now consider a quaternion inverse as

$$\hat{Q}_q^{-1} \cdot \hat{Q}_q = \hat{1}$$
$$\hat{Q}_q^{-1} \cdot \hat{Q}_q \cdot \hat{Q}_{-q} = \hat{Q}_{-q}$$
$$\hat{Q}_q^{-1} = \left[\hat{Q}_q \cdot \hat{Q}_{-q} \right]^{-1} \cdot \hat{Q}_{-q} \tag{D.6}$$
$$= \left[Q_s^2 + \vec{Q}_{\text{V}} \cdot \vec{Q}_{\text{V}} \right]^{-1} \hat{Q}_{-q}$$

valid for complex components, provided $\hat{Q}$ is not singular, i.e.,

$$Q_s^2 + Q_x^2 + Q_y^2 + Q_z^2 \neq 0 \tag{D.7}$$

We can now define a quaternion coordinate as

$$\hat{T}_q \equiv \hat{T} \equiv \{T, \vec{r}\} \quad , \quad \hat{T}_{-q} = \{-T, \vec{r}\}$$
$$\hat{T}_q \cdot \hat{T}_{-q} = \left[T^2 - r^2\right]\hat{1} = \hat{T}_{-q} \cdot \hat{T}_q \quad \text{(independent of } q\text{)} \tag{D.8}$$

which gives the usual light cone when set to zero. Similarly we have the quaternion operator

$$\begin{aligned}
\hat{\Box}_{-q} \cdot \hat{\Box}_q &= \left[\hat{\nabla} - \hat{1}_q \frac{\partial}{\partial T}\right] \cdot \left[\hat{\nabla} + \hat{1}_q \frac{\partial}{\partial T}\right] = \hat{\nabla}^2 + \frac{\partial^2}{\partial T^2} \\
&= \left[\frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right]\hat{1} = \hat{\Box}_q \cdot \hat{\Box}_{-q} \\
&\equiv \hat{\Box}^2 \hat{1} \\
\hat{\Box}^2 &\equiv \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \quad \text{(independent of } q\text{)}
\end{aligned} \tag{D.9}$$

which is the usual wave-equation operator.

Now we can regard our quaternions as functions of $\hat{T}$ instead of $\vec{r}$ and t separately. So we have

$$\hat{Q}(\vec{r}, t) = \hat{Q}(\hat{T}) = \hat{Q} \tag{D.10}$$

The foregoing then gives the analytic properties of quaternion functions of the quaternion variable $\hat{T}$.

Operating on $\hat{Q}$ we have

$$\hat{\Box}_q \cdot \hat{Q} = \left[-\hat{1}_q \frac{\partial}{\partial T} + \hat{\nabla}\right] \cdot \left[Q_s \hat{1}_q + \hat{Q}_v\right]$$

$$-\hat{1}_q \frac{\partial}{\partial T}\left[Q_s \hat{1}_q + \hat{Q}_v\right] = -jq \frac{\partial \hat{Q}}{\partial T} = -jq \frac{\partial}{\partial T}\left\{Q_s, \vec{Q}_v\right\}$$

$$\hat{\nabla} \cdot \left[Q_s \hat{1}_q\right] = jq \hat{\nabla} Q_s = jq\{0, \nabla Q_s\}$$

$$\hat{\nabla} \cdot \hat{Q}_v = -\left[\nabla \cdot \vec{Q}_v\right]\hat{1} + \left[\nabla \times \vec{Q}_v\right]_x \hat{1}_x + \left[\nabla \times \vec{Q}_v\right]_y \hat{1}_y + \left[\nabla \times \vec{Q}_v\right]_z \hat{1}_z$$

$$= \left\{ jq \nabla \bullet \vec{Q}_{\mathrm{V}} ,\ \ \nabla \times \vec{Q}_{\mathrm{V}} \right\}$$

$$\hat{\Box}_q \bullet \hat{Q} = \left\{ jq \left[-\frac{\partial Q_s}{\partial T} + \nabla \bullet \vec{Q}_{\mathrm{V}} \right] ,\ \ jq \left[-\frac{\partial \vec{Q}_{\mathrm{V}}}{\partial T} + \nabla Q_s \right] + \nabla \times \vec{Q}_{\mathrm{V}} \right\} \tag{D.11}$$

Using $\hat{\Box}_{-q}$ we have

$$\begin{aligned} \hat{\Box}_{-q} \bullet \hat{Q} &= \left[\hat{1}_q \frac{\partial}{\partial T} + \hat{\nabla} \right] \bullet \left[Q_s \hat{1}_q + \hat{Q}_{\mathrm{V}} \right] \\ &= \left\{ jq \left[\frac{\partial Q_s}{\partial T} + \nabla \bullet \hat{Q}_{\mathrm{V}} \right] ,\ \ jq \left[\frac{\partial \vec{Q}_{\mathrm{V}}}{\partial T} + \nabla Q_s \right] + \nabla \times \vec{Q}_{\mathrm{V}} \right\} \end{aligned} \tag{D.12}$$

Now we can form

$$\hat{\Box}_{-q} \bullet \hat{T} = \left\{ jq\ 4 ,\ \vec{0} \right\} = -4\,\hat{1} \tag{D.13}$$

corresponding to the 4-dimensional spacetime. For completeness one can note that

$$\hat{\Box}_q \bullet \hat{T} = \{ jq\ 2 ,\ 0 \} = -2\,\hat{1} \tag{D.14}$$

We can also note that

$$\begin{aligned} \frac{1}{2} \left[\hat{\Box}_q + \hat{\Box}_{-q} \right] &= \hat{\nabla} \\ \frac{1}{2} \left[\hat{\Box}_q - \hat{\Box}_{-q} \right] &= -\vec{1}_q \frac{\partial}{\partial T} \end{aligned} \tag{D.15}$$

from which the foregoing can also be constructed.

References

1. B. K. Singaraju and C. E. Baum, "The Combined Field: Boundary Conditions, Integral Representation, Dyadic Green's Functions and Some Theorems and Concepts", Mathematics Note 38, September 1974.

2. Z. A. Typaldos and R. J. Pogorzelski, "Quaternion Calculus and the Solution of Maxwell's Equations", Mathematics Note 43, December 1975.

3. H. T. Anastassiu, P. E. Atlamazoglou, and D. I. Kaklamani, "Application of Bicomplex (Quaternion) Algebra to Fundamental Electromagnetics: A Lower Order Alternative to the Helmholtz Equation", IEEE Trans. Antennas and Propagation, 2001, pp. 2130-2136.

4. S. Cornbleet, *Microwave and Geometrical Optics*, Academic Press, 1994.

5. C. E. Baum and H. N. Kritikos, "Symmetry in Electromagnetics", ch. 1, pp. 1-90, in C. E. Baum and H. N. Kritikos (eds.), *Electromagnetic Symmetry*, Taylor & Francis, 1995.