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# Axiomatics of classical electrodynamics and its relation to gauge field theory

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## Abstract

We give a concise axiomatic introduction into the fundamental structure of classical electrodynamics: It is based on electric charge conservation, the Lorentz force, magnetic flux conservation, and the existence of local and linear constitutive relations. The *inhomogeneous* Maxwell equations, expressed in terms of  $D^i$  and  $H_i$ , turn out to be a consequence of electric charge conservation, whereas the *homogeneous* Maxwell equations, expressed in terms of  $E_i$  and  $B^i$ , are derived from magnetic flux conservation and special relativity theory. The excitations  $D^i$  and  $H_i$ , by means of constitutive relations, are linked to the field strengths  $E_i$  and  $B^i$ . Eventually, we point out how this axiomatic approach is related to the framework of gauge field theory.

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# 1 Introduction

In nature one has, up to now, identified four fundamental interactions: Gravity, electromagnetism, weak interaction, and strong interaction. Gravity and electromagnetism manifest themselves on a macroscopic level. The weak and the strong interactions are generically microscopic in nature and require a quantum field theoretical description right from the beginning.

The four interactions can be modeled individually. Thereby it is recognized that electromagnetism has the simplest structure amongst these interactions. This simplicity is reflected in the Maxwell equations. They, together with a few additional assumptions, explain the electromagnetic phenomena that we observe in nature or in laboratories.

Without digressing to philosophy, one may wonder about the origin of the Maxwell equations. Should we believe in them as such and just study their consequences? Or should we rather derive them from some deeper lying structures? Certainly, there are already some answers known to the last question. The Maxwell equations rely on conservation laws and symmetry principles that are also known from elementary particle physics, see [1, 21]. In the framework of classical physics, authoritative accounts of electrodynamics are provided by [19, 23], e.g.. In this paper we would like to add some new insight into this subject.

We will provide a short layout of an axiomatic approach that allows to identify the *basic ingredients* that are necessary for formulating classical electrodynamics, see [4]. We believe that this axiomatic approach is not only characterized by simplicity and beauty, but is also of appreciable pedagogical value. The more clearly a structure is presented, the easier it is to memorize. Moreover, an understanding of how the fundamental electromagnetic quantities  $D^i$ ,  $H_i$ ,  $E_i$ ,  $B^i$  are related to each other may facilitate the formulation and solution of actual electromagnetic problems.

As it is appropriate for an axiomatic approach, we will start from as few prerequisites as possible. What we will need is some elementary mathematical background that comprises differentiation and integration in the framework of tensor analysis in three-dimensional space. In particular, the concept of integration is necessary for introducing electromagnetic objects as integrands in a natural way. To this end, we will use a tensor notation in which the components of mathematical quantities are explicitly indicated by means of upper (contravariant) or lower (covariant) indices [22]. The advantage of this notation is that it allows to represent geometric properties clearly. In this way, the electromagnetic objects become more transparent and can be discussed more easily. For the formalism of differential forms, which we recommend and which provides similar conceptual advantages, we refer to [10, 4].

We have compiled some mathematical material in the Appendix. Those who don't feel comfortable with some of the notation, may first want to have a look into the Appendix. Let us introduce the following conventions:

- Partial derivatives with respect to a spatial coordinate  $x^i$  (with  $i, j, \dots = 1, 2, 3$ ) or with respect to time  $t$  are abbreviated according to

$$\frac{\partial}{\partial x^i} \longrightarrow \partial_i, \quad \frac{\partial}{\partial t} \longrightarrow \partial_t. \quad (1)$$

- We use the “summation convention”. It states that a summation sign can be omitted if the same index occurs both in a lower and an upper position. That is, we have, for example, the correspondence

$$\sum_{i=1}^3 \alpha_i \beta^i \longleftrightarrow \alpha_i \beta^i. \quad (2)$$

- We define the Levi-Civita symbols  $\epsilon_{ijk}$  and  $\epsilon^{ijk}$ . They are antisymmetric with respect to all of their indices. Therefore, they vanish if two of their indices are equal. Their remaining components assume the values  $+1$  or  $-1$ , depending on whether  $ijk$  is an even or an odd permutation of 123:

$$\epsilon_{ijk} = \epsilon^{ijk} = \begin{cases} 1, & \text{for } ijk = 123, 312, 231, \\ -1, & \text{for } ijk = 213, 321, 132. \end{cases} \quad (3)$$

With these conventions we obtain for the *gradient* of a function  $f$  the expression  $\partial_i f$ . The *curl* of a (covariant) vector  $v_i$  is written according to  $\epsilon^{ijk} \partial_j v_k$  and the *divergence* of a (contravariant) vector (density)  $w^i$  is given by  $\partial_i w^i$ .

Now we are prepared to move on to the Maxwell theory.

## 2 Essential classical electrodynamics based on four axioms

In the next four subsections, we will base classical electrodynamics on electric charge conservation (axiom 1), the Lorentz force (axiom 2), magnetic flux conservation (axiom 3), and the existence of constitutive relations (axiom 4). This represents the core of classical electrodynamics: It results in the Maxwell equations together with the constitutive relations and the Lorentz force law.

In order to complete electrodynamics, one can require two more axioms, which we only mention shortly (see [4] for a detailed discussion). One can specify the energy-momentum distribution of the electromagnetic field (axiom 5) by means of its so-called energy-momentum tensor. This tensor yields the energy density  $(D^i E_i + H_i B^i)/2$  and the energy flux density  $\epsilon^{ijk} E_j H_k$  (the Poynting vector), inter alia. Moreover, if one treats electromagnetic problems of materials in macrophysics, one needs a further axiom by means of which the total electric charge (and the current) is split (axiom 6) in a bound or material charge (and current), which is also conserved, and in a free or external charge (and current). This completes classical electrodynamics.

### 2.1 Electric charge conservation (axiom 1) and the inhomogeneous Maxwell equations

In classical electrodynamics, the electric charge is characterized by its density  $\rho$ . From a geometric point of view, the charge density  $\rho$  constitutes an integrand of a volume integral.

This geometric identification is natural since, by definition, integration of  $\rho$  over a three-dimensional volume  $V$  yields the total charge  $Q$  enclosed in this volume

$$Q := \int_V \rho dv. \quad (4)$$

We note that, in the SI-system, electric charge is measured in units of ‘‘ampere times second’’ or coulomb,  $[Q] = \text{As} = \text{C}$ . Therefore the SI-unit of charge density  $\rho$  is  $[\rho] = \text{As}/\text{m}^3 = \text{C}/\text{m}^3$ .

It is instructive to invoke at this point the Poincaré lemma. There are different explicit versions of this lemma. We use the form (69) that is displayed in Appendix A. Then (if space fulfills suitable topological conditions) we can write the charge density  $\rho$  as the divergence of an integrand  $D^i$  of a surface integral. Thus,

$$\boxed{\partial_i D^i = \rho} \quad (\text{div } \mathcal{D} = \rho). \quad (5)$$

This result already constitutes one inhomogeneous Maxwell equation, the Coulomb-Gauss law. In parenthesis we put the symbolic form of this equation.

Electric charges often move. We represent this motion by a material velocity field  $u^i$ , that is, we assign locally a velocity to each portion of charge in space. The product of electric charge density  $\rho$  and material velocity  $u^i$  defines the electric current density  $J^i$ ,

$$J^i = \rho u^i. \quad (6)$$

Geometrically, the electric current density constitutes an integrand of surface integrals since integration of  $J^i$  over a two-dimensional surface  $S$  yields the total electric current  $I$  that crosses this surface,

$$I = \int_S J^i da_i. \quad (7)$$

We have, in SI-units,  $[I] = \text{A}$  and  $[J^i] = \text{A}/\text{m}^2$ .

We now turn to electric charge conservation, the first axiom of our axiomatic approach. To this end we have to determine how individual packets of charge change in time as they move with velocity  $u^i$  through space. A convenient way to describe this change is provided by the material derivative  $D/Dt$  which also is often called convective derivative [20]. It allows to calculate the change of a physical quantity as it appears to an observer or a probe that follows this quantity. Then electric charge conservation can be expressed as

$$\frac{DQ}{Dt} = 0, \quad (8)$$

where the material derivative is taken with respect to the velocity field  $u^i$ . It can be rewritten in the following way [20],

$$\frac{DQ}{Dt} = \frac{D}{Dt} \int_{V(t)} \rho dV$$

$$\begin{aligned}
&= \int_{V(t)} \frac{\partial \rho}{\partial t} dV + \oint_{\partial V(t)} \rho u^i da_i \\
&= \int_{V(t)} \left( \frac{\partial \rho}{\partial t} + \partial_i(\rho u^i) \right) dV.
\end{aligned} \tag{9}$$

Here we used in the last line the Stokes theorem in the form of (70). The volume  $V(t)$  that is integrated over depends in general on time since it moves together with the electric charge that it contains. By means of (6), (8), and (9) we obtain the axiom of electric charge conservation in the local form as continuity equation,

$$\partial_t \rho + \partial_i J^i = 0. \tag{10}$$

Now we use the inhomogeneous Maxwell equation (5) in order to replace within the continuity equation (10) the charge density by the divergence of  $D^i$ . This yields

$$\partial_i \left( \partial_t D^i + J^i \right) = 0. \tag{11}$$

Again we invoke the Poincaré lemma, now in the form (68), and write the sum  $\partial_t D^i + J^i$  as the curl of the integrand of a line integral which we denote by  $H_i$ . This yields

$$\boxed{\epsilon^{ijk} \partial_j H_k - \partial_t D^i = J^i} \quad (\text{curl } H - \dot{D} = \mathcal{J}). \tag{12}$$

Equation (12) constitutes the remaining inhomogeneous Maxwell equation, the Ampère-Maxwell law, which, in this way, is derived from the axiom of charge conservation. The fields  $D^i$  and  $H_i$  are called electric excitation (historically: electric displacement) and magnetic excitation (historically: magnetic field), respectively. From (5) and (12) it follows that their SI-units are  $[D^i] = \text{As/m}^2$  and  $[H_i] = \text{A/m}$ .

Some remarks are appropriate now: We first note that we obtain the excitations  $D^i$  and  $H_i$  from the Poincaré lemma and charge conservation, respectively, without introducing the concept of force. This is in contrast to other approaches that rely on the Coulomb and the Lorentz force laws [2]. Furthermore, since electric charge conservation is valid not only on macroscopic scales but also in micropysics, the inhomogeneous Maxwell equations (5) and (12) are microphysical equations as long as the source terms  $\rho$  and  $J^i$  are microscopically formulated as well. The same is valid for the excitations  $D^i$  and  $H_i$ . They are microphysical quantities — in contrast to what is often stated in textbooks, see [7], for example. We finally remark that the inhomogeneous Maxwell equations (5) and (12) can be straightforwardly put into a relativistically invariant form. This is not self-evident but suggested by electric charge conservation in the form of the continuity equation (10) since this fundamental equation can also be shown to be relativistically invariant.

## 2.2 Lorentz force (axiom 2) and merging of electric and magnetic field strengths

During the discovery of the electromagnetic field, the concept of force has played a major role. Electric and magnetic forces are directly accessible to experimental observation. Experimental

evidence shows that, in general, an electric charge is subject to a force if an electromagnetic field acts on it. For a point charge  $q$  at position  $x_q^i$ , we have  $\rho(x^i) = q\delta(x^i - x_q^i)$ . If it has the velocity  $u^i$ , we postulate the Lorentz force

$$F_i = q(E_i + \epsilon_{ijk}u^j B^k) \quad (13)$$

as second axiom. It introduces the electric field strength  $E_i$  and the magnetic field strength  $B^i$ . The Lorentz force already yields a prescription of how to measure  $E_i$  and  $B^i$  by means of the force that is experienced by an infinitesimally small test charge  $q$  which is either at rest or moving with velocity  $u^i$ . Turning to the dimensions, we introduce voltage as “work per charge”. In SI, it is measured in volt (V). Then  $[F_i]=VC/m$  and, according to (13),  $[E^i] = V/m$  and  $[B_i] = Vs/m^2 = Wb/m^2 = T$ , with Wb as abbreviation for weber and T for tesla.

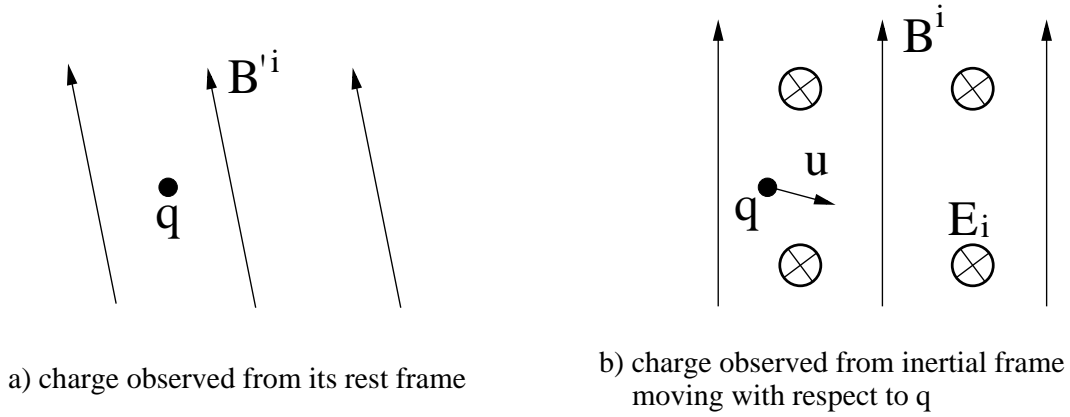


Figure 1: A charge that is, in some inertial frame, at rest and is immersed in a purely magnetic field experiences no Lorentz force, see Fig.1a. The fact that there is no Lorentz force should be independent of the choice of the inertial system that is used to observe the charge. Therefore, a compensating electric field accompanies the magnetic field if viewed from an inertial laboratory system which is in relative motion to the charge, see Fig.1b.

From the axiom of the Lorentz force (13), we can draw the conclusion that the electric and the magnetic field strengths are not independent of each other. The corresponding argument is based on the special relativity principle: According to the special relativity principle, the laws of physics are independent of the choice of an inertial system [2]. Different inertial systems move with constant velocities  $v^i$  relative to each other. The outcome of a physical experiment, as expressed by an empirical law, has to be independent of the inertial system where the experiment takes place.

Let us suppose a point charge  $q$  with a certain mass moves with velocity  $u^i$  in an electromagnetic field  $E_i$  and  $B^i$ . The velocity and the electromagnetic field are measured in an inertial laboratory frame. The point charge can also be observed from its instantaneous inertial *rest frame*. If we denote quantities that are measured with respect to this rest frame by a

prime, i.e., by  $u'^i$ ,  $E'_i$ , and  $B'^i$ , then we have  $u'^i = 0$ . In the absence of an electric field in the laboratory system, i.e., if additionally  $E'_i = 0$ , the charge experiences no Lorentz force and therefore no acceleration,

$$F'_i = q(E'_i + \epsilon_{ijk}u'^j B'^k) = 0. \quad (14)$$

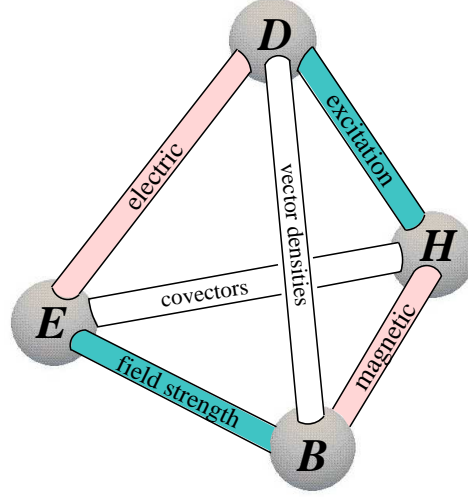


Figure 2: The tetrahedron of the electromagnetic field. The electric and the magnetic excitations  $D^i$ ,  $H_i$  and the electric and the magnetic field strengths  $E_i$ ,  $B^i$  build up 4-dimensional quantities in spacetime. These four fields describe the electromagnetic field completely. Of electric nature are  $D^i$  and  $E_i$ , of magnetic nature  $H_i$  and  $B^i$ . The electric and the magnetic excitations  $D^i$ ,  $H_i$  are extensivities, also called quantities (how much?), the electric and the magnetic field strengths  $E_i$ ,  $B^i$  are intensivities, also called forces (how strong?).

The fact that the charge experiences no acceleration is also true in the laboratory frame. This is a consequence of the special relativity principle or, more precisely, of the fact that the square of the acceleration can be shown to form a relativistic invariant. Consequently,

$$F_i = q(E_i + \epsilon_{ijk}u^j B^k) = 0. \quad (15)$$

Thus, in the laboratory frame, electric and magnetic field are related by

$$E_i = -\epsilon_{ijk}u^j B^k. \quad (16)$$

This situation is depicted in Fig.1. Accordingly, we find that electric and magnetic field strength cannot be viewed as independent quantities. They are connected to each other by transformations between different inertial systems.

Let us pause for a moment and summarize: So far we have introduced the four electromagnetic field quantities  $D^i$ ,  $H_i$  and  $E_i$ ,  $B^i$ . These four quantities are interrelated by physical and mathematical properties. This is illustrated in Fig.2 by the “tetrahedron of the electromagnetic field”.



### 2.3 Magnetic flux conservation (axiom 3) and the homogeneous Maxwell equations

We digress for a moment and turn to hydrodynamics. Helmholtz was one of the first who studied rotational or “vortex” motion in hydrodynamics, see [8]. He derived theorems for vortex lines. An important consequence of his work was the conclusion that vortex lines are conserved. They may move or change orientation but they are never spontaneously created nor annihilated. The vortex lines that pierce through a two-dimensional surface can be integrated over and yield a scalar quantity that is called circulation. The circulation in a perfect fluid, which satisfies certain conditions, is constant provided the loop enclosing the surface moves with the fluid [8].

There are certainly fundamental differences between electromagnetism and hydrodynamics. But some suggestive analogies exist. A vortex line in hydrodynamics seems analogous to a magnetic flux line. The magnetic flux  $\Phi$  is determined from magnetic flux lines, represented by the magnetic field strength  $B^i$ , that pierce through a two-dimensional surface  $S$ ,

$$\Phi := \int_S B^i da_i. \quad (17)$$

As the circulation in a perfect fluid is conserved, we can guess that, in a similar way, the magnetic flux may be conserved. Of course, the consequences of such an axiom have to be borne out by experiment.

At first sight, one may find vortex lines of a fluid easier to visualize than magnetic flux lines. However, on a microscopic level, magnetic flux can occur in quanta. The corresponding magnetic flux unit is called flux quantum or fluxon and it carries  $\Phi_0 = h/(2e) \approx 2,07 \cdot 10^{-15}$  Wb, with  $h$  as Planck constant and  $e$  as elementary charge. Single quantized magnetic flux lines have been observed in the interior of type II superconductors if exposed to a sufficiently strong magnetic field, see [4], p.131. They even can be counted. The corresponding experiments provide good evidence that magnetic flux is a conserved quantity.

But how can we formulate magnetic flux conservation mathematically? It is at this point instructive to reconsider the notion of the electric charge

$$Q = \int_V \rho dv \quad (18)$$

together with its corresponding conservation law

$$\partial_t Q + \int_{\partial V} J^i da_i = 0. \quad (19)$$

The rate of change of the electric charge within a specified volume  $V$  is balanced by the out- or inflowing charge across the surface  $\partial V$ . This charge transport is described by the electric charge *current*  $J^i$  that is integrated over the enveloping surface  $\partial V$ . By means of the Stokes theorem in the form (70), equation (19) yields the local continuity equation

$$\partial_t \rho + \partial_i J^i = 0. \quad (20)$$

Let us follow the same pattern in formulating magnetic flux conservation: Starting with the definition (17) of the magnetic flux, the corresponding conservation law, in analogy to (19), reads

$$\partial_t \Phi + \int_{\partial S} J_i^\Phi dc^i = 0, \quad (21)$$

where we introduced the magnetic flux *current*  $J_i^\Phi$ . Geometrically, this is a covariant vector that is integrated along a line  $\partial S$ , that is, along the curve bordering the 2-dimensional surface  $S$ . The conservation law (21) tells us that the rate of change of the magnetic flux within a specified area  $S$  is balanced by the magnetic flux current  $J_i^\Phi$  that is integrated along the boundary  $\partial S$ . Then the Stokes theorem in the form (71) yields the local continuity equation

$$\partial_t B^i + \epsilon^{ijk} \partial_j J_k^\Phi = 0. \quad (22)$$

One interesting consequence is the following: The divergence of (22) reads

$$\partial_i (\partial_t B^i) = 0 \quad \implies \quad \partial_i B^i = \rho_{\text{mag}}, \quad \partial_t \rho_{\text{mag}} = 0. \quad (23)$$

Thus, we find a time-*independent* term  $\rho_{\text{mag}}$ , which acquires tentatively the meaning of a magnetic charge density. Let us choose a specific reference system in which  $\rho_{\text{mag}}$  is constant in time, i.e.,  $\partial_t \rho_{\text{mag}} = 0$ . Now we go over to an arbitrary reference system with time coordinate  $t'$  and spatial coordinates  $x^{i'}$ . Clearly, in general  $\partial_{t'} \rho_{\text{mag}} \neq 0$ . The only way to evade a contradiction to (23) is to require  $\rho_{\text{mag}} = 0$ , that is, the magnetic field strength  $B^i$  has no sources, its divergence vanishes:

$$\boxed{\partial_i B^i = 0} \quad (\text{div } \mathcal{B} = 0). \quad (24)$$

This is recognized as one of the homogeneous Maxwell equations. Note that our derivation of (24) was done under the assumption of magnetic flux conservation (21). Under this condition we find  $\rho_{\text{mag}} = 0$ .

In order to understand better the magnetic flux current, we note that  $J_i^\Phi$ , as a covariant vector, has the same geometric properties as the electric field strength  $E_i$ . Additionally,  $J_i^\Phi$  and  $E_i$  share the same physical dimension voltage/length, i.e., in SI, V/m. Accordingly, it is plausible to identify both quantities,

$$J_i^\Phi \equiv E_i. \quad (25)$$

That also the sign chosen is the appropriate one (consistent with the Lenz rule) was discussed in [6]. Then the local continuity equation (22) assumes the form

$$\boxed{\partial_t B^i + \epsilon^{ijk} \partial_j E_k = 0} \quad (\dot{\mathcal{B}} + \text{curl } E = 0). \quad (26)$$

This equation reflects magnetic flux conservation, the third axiom of our axiomatic approach. It also constitutes the remaining homogeneous Maxwell equation, that is, Faraday's induction law.

At this point one might wonder to what extent the identification (25) is mandatory. It turns out that it is special relativity that dictates this identification. We illustrate this circumstance

as follows: In the rest frame of a magnetic flux line  $B'^i$  the magnetic flux current vanishes,  $J'^{\Phi}_i = 0$ . The rest frame is also defined via the Lorentz force: In the absence of an electric field,  $E'_i = 0$ , a test charge  $q$  is not accelerated by  $B'^i$ . Then a Lorentz transformation, together with (24), yields an equation that relates  $B^i$  and  $J^{\Phi}_i$  in a laboratory frame according to

$$J^{\Phi}_i = -\epsilon_{ijk} u^j B^k. \quad (27)$$

A comparison with (16), which was obtained by an analogous transformation of a magnetic flux line from its rest frame to a laboratory frame, shows that the identification (25) needs to be valid, indeed. However, one should be aware that our simple argument requires  $E'_i = 0$  in the rest frame of the considered magnetic flux line.

## 2.4 Constitutive relations (axiom 4) and the properties of spacetime

So far we have introduced  $4 \times 3 = 12$  unknown electromagnetic field components  $D^i$ ,  $H_i$ ,  $E_i$ , and  $B^i$ . These components have to fulfill the Maxwell equations (5), (12), (24), and (26), which represent  $1 + 3 + 1 + 3 = 8$  partial differential equations. In fact, among the Maxwell equations, only (12) and (26) contain time derivatives and are dynamical. The remaining equations, (5) and (24), are so-called “constraints”. They are, by virtue of the dynamical Maxwell equations, fulfilled at all times if fulfilled at one time. It follows that they don’t contain information on the time evolution of the electromagnetic field. Therefore, we arrive at only 6 dynamical equations for 12 unknown field components. To make the Maxwell equations a determined set of partial differential equations, we still have to introduce additionally the so-called “constitutive relations” between the excitations  $D^i$ ,  $H_i$  and the field strengths  $E_i$ ,  $B^i$ .

The simplest case to begin with is to find constitutive relations for the case of electromagnetic fields in vacuum. There are guiding principles that limit their structure. We demand that constitutive relations in vacuum are invariant under translation and rotation, furthermore they should be local and linear, i.e., they should connect fields at the same position and at the same time. Finally, in vacuum the constitutive relations should not mix electric and magnetic properties. These features characterize the vacuum and not the electromagnetic field itself. We will not be able to prove them but postulate them as fourth axiom.

If we want to relate the field strengths and the excitations we have to remind ourselves that  $E_i$ ,  $H_i$  are natural integrands of *line* integrals and  $D^i$ ,  $B^i$  are natural integrands of *surface* integrals. Therefore,  $E_i$ ,  $H_i$  transform under a change of coordinates as covariant vectors while  $D^i$ ,  $B^i$  transform as contravariant vector densities. To compensate these differences we will have to introduce a symmetric metric field  $g_{ij} = g_{ji}$ . The metric tensor determines spatial distances and introduces the notion of orthogonality. The determinant of the metric is denoted by  $g$ . It follows that  $\sqrt{g}g^{ij}$  transforms like a density and maps a covariant vector into a contravariant vector density. We then take as fourth axiom the constitutive equations for vacuum,

$$\boxed{D^i = \epsilon_0 \sqrt{g} g^{ij} E_j}, \quad (28)$$

$$\boxed{H_i = (\mu_0 \sqrt{g})^{-1} g_{ij} B^j}. \quad (29)$$

In flat spacetime and in cartesian coordinates, we have  $g = 1$ ,  $g^{ii} = 1$ , and  $g^{ij} = 0$  for  $i \neq j$ . We recognize the familiar vacuum relations between field strengths and excitations. The electric constant  $\varepsilon_0$  and the magnetic constant  $\mu_0$  characterize the vacuum. They acquire the SI-units  $[\varepsilon_0] = \text{As/Vm}$  and  $[\mu_0] = \text{Vs/Am}$ .

What seems to be conceptually important about the constitutive equations (28), (29) is that they not only provide relations between the excitations  $D^i$ ,  $H_i$  and the field strengths  $E_i$ ,  $B^i$ , but also connect the electromagnetic field to the structure of spacetime, which here is represented by the metric tensor  $g_{ij}$ . The formulation of the first three axioms that were presented in the previous sections does not require information on this metric structure. The connection between the electromagnetic field and spacetime, as expressed by the constitutive equations, indicates that physical fields and spacetime are not independent of each other. The constitutive equations might suggest the point of view that the structure of spacetime determines the structure of the electromagnetic field. However, one should be aware that the opposite conclusion has a better truth value: It can be shown that the propagation properties of the electromagnetic field determine the metric structure of spacetime [4, 9].

Constitutive equations in matter usually assume a more complicated form than (28), (29). In this case it would be appropriate to derive the constitutive equations, after an averaging procedure, from a microscopic model of matter. Such procedures are the subject of solid state or plasma physics, for example. A discussion of these subjects is out of the scope of this paper but, without going into details, we quote the constitutive relations of a general linear *magnetolectric* medium:

$$D^i = (\boldsymbol{\varepsilon}^{ij} - \varepsilon^{ijk} n_k) E_j + (\boldsymbol{\gamma}^i_j + \tilde{s}_j^i) B^j + (\boldsymbol{\alpha} - s) B^i, \quad (30)$$

$$H_i = (\boldsymbol{\mu}_{ij}^{-1} - \varepsilon_{ijk} m^k) B^j + (-\boldsymbol{\gamma}^j_i + \tilde{s}_i^j) E_j - (\boldsymbol{\alpha} + s) E_i. \quad (31)$$

This formulation is due to Hehl & Obukhov [4, 5, 15], an equivalent formulation of a “bianisotropic medium” — this is the same as what we call general linear medium — was given by Lindell & Olyslager [17, 10]. Both matrices  $\varepsilon^{ij}$  and  $\mu_{ij}^{-1}$  are symmetric and possess 6 independent components each,  $\varepsilon^{ij}$  is called *permittivity* tensor and  $\mu_{ij}^{-1}$  *impermeability* tensor (reciprocal permeability tensor). The magnetolectric cross-term  $\boldsymbol{\gamma}^i_j$ , which is tracefree,  $\boldsymbol{\gamma}^k_k = 0$ , has 8 independent components. It is related to the Fresnel-Fizeau effects.

The 4-dimensional pseudo-scalar  $\alpha$ , we call it axion piece [4], represents one component. It corresponds to the perfect electromagnetic conductor (PEMC) of Lindell & Sihvola [11], a Tellegen type structure [24, 25].

Accordingly, these pieces altogether, which we printed in (30) and (31) in boldface for better visibility, add up to  $6 + 6 + 8 + 1 = 20 + 1 = 21$  independent components. The situation with 20 components is described in Post [18] (he required  $\alpha = 0$  without a real proof), that with 21 components in O’Dell [16].

We can have 15 more components related to dissipation, which *cannot* be derived from a Lagrangian, the so-called skewon piece (see [14] and the literature given), namely  $3 + 3$  components of  $n_k$  and  $m^k$  (electric and magnetic Faraday effects), 8 components from the matrix  $\tilde{s}_i^j$  (optical activity), which is traceless  $\tilde{s}_k^k = 0$ , and 1 component from the 3-dimensional

scalar  $s$  (spatially isotropic optical activity). This scalar was introduced by Nieves & Pal [13]. It has also been discussed in electromagnetic materials as chiral parameter, see Lindell et al. [12]. Note that  $s$ , in contrast to the 4-dimensional scalar  $\alpha$ , is only a 3D scalar. We end then up with the *general linear medium* with  $20 + 1 + 15 = 36$  components.

With the introduction of constitutive equations the axiomatic approach to classical electrodynamics is completed. We will see in the next Section 3 how this approach relates to the framework of gauge theory.

### 3 On the relation between the axiomatics and the gauge approach

Modern descriptions of the fundamental interactions heavily rely on symmetry principles. In particular, this is true for the electromagnetic interaction which can be formulated as a gauge field theory that is based on a corresponding gauge symmetry. In a recent article this approach towards electromagnetism has been explained in some detail [3]. The main steps were the following:

- Accept the fact that physical matter fields (which represent electrons, for example) are described microscopically by complex wave functions.
- Recognize that the absolute phase of these wave functions has no physical relevance. This arbitrariness of the absolute phase constitutes a one-dimensional rotational type symmetry  $U(1)$  (the circle group) that is the gauge symmetry of electromagnetism.
- To derive observable physical quantities from the wave functions requires to define derivatives of wave functions in a way that is invariant under the gauge symmetry. The construction of such “gauge covariant” derivatives requires the introduction of gauge potentials. One gauge potential, the scalar potential  $\phi$ , defines a gauge covariant derivative  $D_t^\phi$  with respect to time, while another gauge potential, the vector potential  $A_i$ , defines gauge covariant derivatives  $D_i^A$  with respect to the three independent directions of space.
- Finally, the gauge potentials  $\phi$  and  $A_i$  describe an electrodynamically non-trivial situation, if their corresponding electric and magnetic field strengths

$$E_i = -\partial_i\phi - \partial_t A_i, \quad (32)$$

$$B^i = \varepsilon^{ijk}\partial_j A_k, \quad (33)$$

are non-vanishing.

In the following we want to comment on the interrelation between the previously presented axiomatic approach and the gauge approach. It is interesting to see how the axioms find their proper place within the gauge approach.

### 3.1 Noether theorem and electric charge conservation

In field theory there is a famous result which connects symmetries of laws of nature to conserved quantities. This is the Noether theorem which has been proven to be useful in both classical and quantum contexts. It is, in particular, discussed in books on classical electrodynamics, see [19, 23], for example.

Laws of nature, like in electrodynamics, e.g., can often (but not always) be characterized concisely by a Lagrangian density  $\mathcal{L} = \mathcal{L}(\Psi, \partial_i \Psi, \partial_t \Psi)$  which, in the standard case, is a function of the fields  $\Psi$  of the theory and their first derivatives. Integration of the Lagrangian density  $\mathcal{L}$  over space yields the Lagrangian  $L$ ,

$$L = \int \mathcal{L}(\Psi, \partial_i \Psi, \partial_t \Psi) dV, \quad (34)$$

and further integration over time yields the action  $S$ ,

$$S = \int L dt. \quad (35)$$

There are guiding principles that tell us how to obtain an appropriate Lagrangian density for a given theory. Once we have an appropriate Lagrangian density, we can derive conveniently the properties of the fields  $\Psi$ . For example, the equations of motion which determine the dynamics of  $\Psi$  follow from extremization of the action  $S$  with respect to variations of  $\Psi$ ,

$$\delta_\Psi S = 0 \quad \Longrightarrow \quad \text{equations of motion for } \Psi. \quad (36)$$

Now we turn to the Noether theorem which connects the symmetry of a Lagrangian density  $\mathcal{L}(\Psi, \partial_i \Psi, \partial_t \Psi)$  to conserved quantities. Suppose that  $\mathcal{L}$  is invariant under time translations  $\delta_t$ . In daily life this assumption makes sense since we do not expect that the laws of nature change in time. Then the Noether theorem implies a local conservation law which expresses the conservation of energy. Similarly, invariance under translations  $\delta_{x^i}$  in space implies conservation of momentum, while invariance under rotations  $\delta_{\omega_i^j}$  yields the conservation of angular momentum,

$$\delta_t \mathcal{L} = 0 \quad \Longrightarrow \quad \text{conservation of energy}, \quad (37)$$

$$\delta_{x^i} \mathcal{L} = 0 \quad \Longrightarrow \quad \text{conservation of momentum}, \quad (38)$$

$$\delta_{\omega_i^j} \mathcal{L} = 0 \quad \Longrightarrow \quad \text{conservation of angular momentum}. \quad (39)$$

These symmetries of spacetime are called external symmetries. But the Noether theorem also works for other types of symmetries, so-called internal ones — especially gauge symmetries. In this case, gauge invariance of the Lagrangian implies a conserved current with an associated charge. That is, if we denote a gauge transformation by  $\delta_\epsilon$  we conclude

$$\delta_\epsilon \mathcal{L} = 0 \quad \Longrightarrow \quad \text{charge conservation}. \quad (40)$$

If we apply this conclusion to electrodynamics, we have to specify the Lagrangian density to be the one of matter fields that represent electrically charged particles. Then invariance of this Lagrangian density under the gauge symmetry of electrodynamics yields the conservation of electric charge. Thus, if we accept the validity of the Lagrangian formalism, then we can arrive at electric charge conservation from gauge invariance via the Noether theorem.

### 3.2 Minimal coupling and the Lorentz force

We already have mentioned that, according to (36), we can derive the equations of motion of a physical theory from a Lagrangian density and its associated action. We can use this scheme to derive the equations of motion of electrically charged particles. In this case, the corresponding Lagrangian density (that of the electrically charged particles) has to be gauge invariant.

If electrically charged particles are represented by their wave functions, the corresponding Lagrangian density will contain derivatives with respect to time and space. It follows that the Lagrangian density will be gauge invariant if we pass from partial derivatives to gauge covariant derivatives according to

$$\partial_t \quad \longrightarrow \quad D_t^\phi := \partial_t - j \frac{q}{\hbar} \phi, \quad (41)$$

$$\partial_i \quad \longrightarrow \quad D_i^A := \partial_i + j \frac{q}{\hbar} A_i, \quad (42)$$

with  $q$  the electric charge of a particle,  $\hbar = h/(2\pi)$  with  $h$  as the Planck constant and  $\phi$ ,  $A_i$  as electromagnetic potentials [3]. This enforcement of gauge invariance has a classical analogue. If electrically charged particles are represented by point particles, rather than by wave functions, we have to replace within the Lagrangian density the energy  $E$  and the momentum  $p_i$  of each particle according to [23]

$$E \quad \longrightarrow \quad E - q\phi, \quad (43)$$

$$p_i \quad \longrightarrow \quad p_i - qA_i. \quad (44)$$

The substitutions (41), (42) or (43), (44) constitute the simplest way to ensure gauge invariance of the Lagrangian density of electrically charged particles. They constitute what commonly is called “minimal coupling”. Due to minimal coupling, we relate electrically charged particles and the electromagnetic field in a natural way that is dictated by the requirement of gauge invariance.

Having ensured gauge invariance of the action  $S$ , we can derive equations of motion by extremization, compare (36). It then turns out that these equations of motion contain the Lorentz force law (13). Therefore the Lorentz force is a consequence of the minimal coupling procedure which couples electrically charged particles to the electromagnetic potentials and makes the Lagrangian gauge invariant.

### 3.3 Bianchi identity and magnetic flux conservation

The electromagnetic gauge potentials  $\phi$  and  $A_i$  are often introduced as mathematical tools to facilitate the integration of the Maxwell equations. Indeed, if we put the relations (32) and (33) into the homogeneous Maxwell equations (24) and (26), we recognize that the homogeneous Maxwell equations are fulfilled automatically. They become mere mathematical identities. This is an interesting observation since within the gauge approach the gauge potentials are fundamental physical quantities and are not only the outcome of a mathematical trick. Thus

we can state that the mathematical structure of the gauge potentials already implies the homogeneous Maxwell equations and, in turn, magnetic flux conservation. In this light, magnetic flux conservation, within the gauge approach, appears as the consequence of a geometric identity. This is in contrast to electric charge conservation that can be viewed as the consequence of gauge invariance, i.e., as the consequence of a physical symmetry.

The mathematical identity that is reflected in the homogeneous Maxwell equations is a special case of a ‘‘Bianchi identity’’. Bianchi identities are the result of differentiating a potential twice. For example, in electrostatics the electric field strength  $E_i$  can be derived from a scalar potential  $\phi$  according to

$$E_i = \partial_i \phi. \quad (45)$$

Differentiation reveals that the curl of  $E_i$  vanishes,

$$\epsilon^{ijk} \partial_j E_k = \epsilon^{ijk} \partial_j \partial_k \phi = 0, \quad (46)$$

which is due to the antisymmetry of  $\epsilon^{ijk}$ . Again, this equation is a mathematical identity, a simple example of a Bianchi identity.

### 3.4 Gauge approach and constitutive relations

The gauge approach towards electrodynamics deals with the properties of gauge fields, which represent the electromagnetic field, and with matter fields. It does not reflect properties of spacetime. In contrast to this, the constitutive equations do reflect properties of spacetime, as can be already seen from the constitutive equations of vacuum that involve the metric  $g_{ij}$ , compare (28) and (29). Thus, also in the gauge approach the constitutive equations have to be postulated as an axiom in some way. One should note that, according to (32), (33), the gauge potentials are directly related to the field strengths  $E_i$  and  $B^i$ . The excitations  $D^i$  and  $H_i$  are part of the inhomogeneous Maxwell equations which, within the gauge approach, are derived as equations of motion from an action principle, compare (36). Since the action itself involves the gauge potentials, one might wonder how it is possible to obtain equations of motion for the excitations rather than for the field strengths. The answer is that during the construction of the action from the gauge potentials the constitutive equations are already used, at least implicitly.

## 4 Conclusion

We have presented an axiomatic approach to classical electrodynamics in which the Maxwell equations are derived from the conservation of electric *charge* and magnetic *flux*. In the context of the derivation of the inhomogeneous Maxwell equations, one introduces the electric and the magnetic excitation  $D^i$  and  $H_i$ , respectively. The explicit calculation is rather simple because the continuity equation for electric charge is already relativistically invariant such that for the derivation of the inhomogeneous Maxwell equations no additional ingredients from special relativity are necessary. The situation is slightly more complicated for the derivation of the



homogeneous Maxwell equations from magnetic flux conservation since it is not immediately clear of how to formulate magnetic flux conservation in a relativistic invariant way. It should be mentioned that if the complete framework of relativity were available, the derivation of the axiomatic approach could be done with considerable more ease and elegance [4].

Finally, we would like to comment on a question that sometimes leads to controversial discussions, as summarized in [20], for example. This is the question of how the quantities  $E_i$ ,  $D^i$ ,  $B^i$ , and  $H_i$  should be grouped in pairs, i.e., the question of “which quantities belong together?”. Some people like to form the pairs  $(E_i, B^i)$ ,  $(D^i, H_i)$ , while others prefer to build  $(E_i, H_i)$ ,  $(D^i, B^i)$ . Already from a dimensional point of view, the answer to this question is obvious. Both,  $E_i$  and  $B^i$  are *voltage*-related quantities, that is, related to the notions of force and work: In SI, we have  $[E_i] = \text{V/m}$ ,  $[B^i] = \text{T} = \text{Vs/m}^2$ , or  $[B^i] = [E_i]/\text{velocity}$ . Consequently, they belong together. Analogously,  $D^i$  and  $H_i$  are *current*-related quantities:  $[D^i] = \text{C/m}^2 = \text{As/m}^2$ ,  $[H_i] = \text{A/m}$ , or  $[D^i] = [H_i]/\text{velocity}$ . Thermodynamically speaking,  $(E_i, B^i)$  are intensities (answer to the question: how strong?) and  $(D^i, H_i)$  extensities (how much?)

These conclusions are made irrefutable by relativity theory. Classical electrodynamics is a relativistic invariant theory and the implications of relativity have been proven to be correct on macro- and microscopic scales over and over again. And relativity tells us that the electromagnetic field strengths  $E_i$ ,  $B^i$  are inseparably intertwined by relativistic transformations, and the same is true for the electromagnetic excitations  $D^i$ ,  $H_i$ . In the spacetime of relativity theory, the pair  $(E_i, B^i)$  forms one single quantity, the tensor of electromagnetic field strength, while the pair  $(D^i, H_i)$  forms another single quantity, the tensor of electromagnetic excitations. If compared to these facts, arguments in favor of the pairs  $(E_i, H_i)$ , namely that both are covectors, and  $(D^i, B^i)$ , both are vector densities (see the tetrahedron in Fig.2), turn out to be of secondary nature. Accordingly, there is no danger that the couples  $(E_i, B^i)$  and  $(D^i, H_i)$  ever get divorced.

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## A Mathematical Background

Within a theoretical formulation physical quantities are modeled as mathematical objects. The understanding and application of appropriate mathematics yields, in turn, the properties of physical quantities. In the development of the axiomatic approach, we made repeated use of integration, of the Poincaré lemma, and of the Stokes theorem. It is with these mathematical concepts that it is straightforward to derive the basics of electromagnetism from a small number of axioms.

## A.1 Integration

Integration is an operation that yields coordinate independent values. It requires an integration measure, the dimension of which depends on the type of region that is integrated over. We want to integrate over one-dimensional curves, two-dimensional surfaces, or three-dimensional volumes that are embedded in three-dimensional space. Therefore, we have to define line-, surface-, and volume-elements as integration measures. Then we can think of suitable objects as integrands that can be integrated over to yield coordinate independent physical quantities.

### A.1.1 Integration over a curve and covariant vectors as line integrands

We consider a one-dimensional curve  $\mathbf{c} = \mathbf{c}(t)$  in three-dimensional space. In a specific coordinate system  $x^i$ , with indices  $i = 1, 2, 3$ , a parametrization of  $\mathbf{c}$  is given by the vector

$$\mathbf{c}(t) = (c^1(t), c^2(t), c^3(t)). \quad (47)$$

The functions  $c^i(t)$  define the shape of the curve. For small changes of the parameter  $t$ , with  $t \rightarrow t + \Delta t$ , the difference vector between  $\mathbf{c}(t + \Delta t)$  and  $\mathbf{c}(t)$  is given by

$$\Delta \mathbf{c}(t) = \left( \frac{\Delta c^1}{\Delta t}, \frac{\Delta c^2}{\Delta t}, \frac{\Delta c^3}{\Delta t} \right) \Delta t, \quad (48)$$

compare Fig.3. In the limit where  $\Delta t$  becomes infinitesimally we obtain the line element

$$\begin{aligned} d\mathbf{c}(t) &= (dc^1(t), dc^2(t), dc^3(t)) \\ &:= \left( \frac{\partial c^1(t)}{\partial t}, \frac{\partial c^2(t)}{\partial t}, \frac{\partial c^3(t)}{\partial t} \right) dt. \end{aligned} \quad (49)$$

It is characterized by an infinitesimal length and an orientation.

We now construct objects that we can integrate over the curve  $\mathbf{c}$  in order to obtain a coordinate invariant scalar. The line element  $d\mathbf{c}$  contains three independent components  $dc^i$ . If we shift from old coordinates  $x^i$  to new coordinates  $y^{j'} = y^{j'}(x^i)$  these components transform according to

$$dc^{j'} = \frac{\partial y^{j'}}{\partial x^i} dc^i. \quad (50)$$

Therefore we can form an invariant expression if we introduce objects  $\alpha = \alpha(x^i)$ , with three independent components  $\alpha_i$ , that transform in the opposite way,

$$\alpha_{j'} = \frac{\partial x^i}{\partial y^{j'}} \alpha_i. \quad (51)$$

This transformation behavior characterizes a vector or, more precisely, a covariant vector (a 1-form). It follows that the expression

$$\alpha_i dc^i = \alpha_{j'} dc^{j'} \quad (52)$$

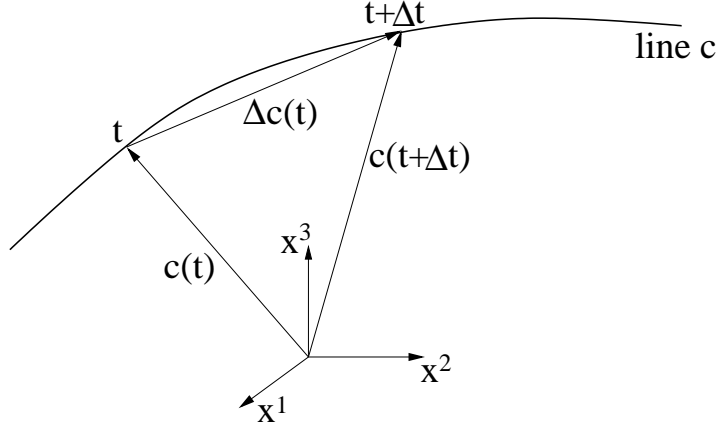


Figure 3: Parametrization of a curve  $c(t)$ . The difference vector  $\Delta c(t)$  between  $c(t + \Delta t)$  and  $c(t)$  yields, in the limit  $\Delta t \rightarrow 0$ , the line element  $dc(t)$ .

yields the same value in each coordinate system.

Thus, we can now immediately define integration over a curve by the expression

$$\begin{aligned} \int \alpha_i dc^i &= \int \alpha_1 dc^1 + \alpha_2 dc^2 + \alpha_3 dc^3 \\ &= \int \left( \alpha_1 \frac{\partial c^1}{\partial t} + \alpha_2 \frac{\partial c^2}{\partial t} + \alpha_3 \frac{\partial c^3}{\partial t} \right) dt. \end{aligned} \quad (53)$$

The last line shows how to carry out explicitly the integration since  $\alpha_i$  and  $c^i$  are functions of the parameter  $t$ .

### A.1.2 Integration over a surface and contravariant vector densities as surface integrands

Now we consider a two-dimensional surface  $\mathbf{a} = \mathbf{a}(t, s)$ . Within a specific coordinate system  $x^i$ , a parametrization of  $\mathbf{a}$  is of the form

$$\mathbf{a}(t, s) = (a^1(t, s), a^2(t, s), a^3(t, s)) \quad (54)$$

with parameters  $t, s$  and functions  $a^i(t, s)$  that define the shape of the surface.

An elementary surface element is bound by lines  $t = \text{const}$ ,  $t + dt = \text{const}$ ,  $s = \text{const}$ , and  $s + ds = \text{const}$ , compare Fig.4. It is characterized by the two edges  $\frac{\partial a^i}{\partial t} dt$  and  $\frac{\partial a^i}{\partial s} ds$ . These edges span an infinitesimal surface, the area and orientation of which is characterized by a covariant vector  $da_i$  that points normal to the infinitesimal surface. The vector  $da_i$  is given by the vector product of  $\frac{\partial a^i}{\partial t} dt$  and  $\frac{\partial a^i}{\partial s} ds$ ,

$$da_i = \epsilon_{ijk} \frac{\partial a^j}{\partial t} \frac{\partial a^k}{\partial s} dt ds. \quad (55)$$

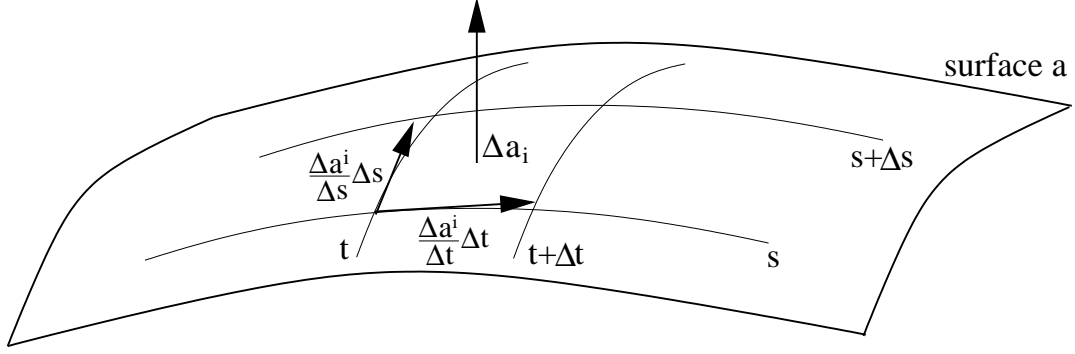


Figure 4: Parametrization of a surface  $\mathbf{a}(t, s)$ . The lines  $t = \text{const}$ ,  $t + \Delta t = \text{const}$ ,  $s = \text{const}$ , and  $s + \Delta s = \text{const}$  circumscribe a surface  $\Delta a_i$  that is spanned by the edges  $\frac{\Delta a_i}{\Delta t} dt$  and  $\frac{\Delta a_i}{\Delta s} ds$ . In the limit  $\Delta t \rightarrow dt$ ,  $\Delta s \rightarrow ds$ , it becomes an elementary surface element  $da_i$ .

In order to know how the components  $da_i$  transform under coordinate transformations  $y^{j'} = y^{j'}(x^i)$ , we have to know the transformation behavior of the symbol  $\epsilon_{ijk}$ . Since in any coordinate system,  $\epsilon_{ijk}$  assumes the values 0, 1, or -1 by definition, it is obvious that in general

$$\epsilon_{i'j'k'} \neq \frac{\partial x^i}{\partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}} \epsilon_{ijk}. \quad (56)$$

This is because the determinant of the transformation matrix, i.e.,

$$\det(\partial x / \partial y) = \epsilon_{ijk} \frac{\partial x^i}{\partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}}, \quad (57)$$

is, in general, not equal to one. But it follows from (57) that the correct transformation rule for  $\epsilon_{ijk}$  is given by

$$\begin{aligned} \epsilon_{i'j'k'} &= \frac{1}{\det(\partial x / \partial y)} \frac{\partial x^i}{\partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}} \epsilon_{ijk} \\ &= \det(\partial y / \partial x) \frac{\partial x^i}{\partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}} \epsilon_{ijk}. \end{aligned} \quad (58)$$

With (55) this yields the transformation rule for the components  $da_i$ ,

$$da_{j'} = \det(\partial y / \partial x) \frac{\partial x^i}{\partial y^{j'}} da_i. \quad (59)$$

Now we construct quantities that can be integrated over a surface. Since a surface element is determined from three independent components  $da_i$  we introduce an integrand with three independent components  $\beta^i$  that transform according to

$$\beta^{j'} = \frac{1}{\det(\partial y / \partial x)} \frac{\partial y^{j'}}{\partial x^i} \beta^i. \quad (60)$$

Transformation rules that involve the determinant of the transformation matrix characterize so-called densities. Densities are sensitive towards changes of the scale of elementary volumes. In physics they represent additive quantities, also called extensities, that describe how much of a quantity is distributed within a volume or over the surface of a volume. This is in contrast to intensities. The covariant vectors that we introduced as natural line integrals are intensive quantities that represent the strength of a physical field.

The transformation behavior (60) of the components  $\beta^i$  characterizes a contravariant vector density. With this transformation behavior the surface integral

$$\int \beta^i da_i = \int \beta^i \epsilon_{ijk} \frac{\partial a^j}{\partial t} \frac{\partial a^k}{\partial s} dt ds \quad (61)$$

yields a scalar value that is coordinate independent.

### A.1.3 Integration over a volume and scalar densities as volume integrands

We finally consider integration over a three-dimensional volume  $v$  in three-dimensional space. Again we choose a specific coordinate system  $x^i$  and specify a parametrization of  $v$  by

$$\mathbf{v}(t, s, r) = (v^1(t, s, r), v^2(t, s, r), v^3(t, s, r)), \quad (62)$$

with three parameters  $t$ ,  $s$ , and  $r$ .

An elementary volume element  $dv$  is characterized by three edges  $\frac{\partial v^i}{\partial t} dt$ ,  $\frac{\partial v^i}{\partial s} ds$ , and  $\frac{\partial v^i}{\partial r} dr$ . The volume, which is spanned by these edges, is given by the determinant

$$\begin{aligned} dv &= \det \left( \frac{\partial v^i}{\partial t} dt, \frac{\partial v^i}{\partial s} ds, \frac{\partial v^i}{\partial r} dr \right) \\ &= \epsilon_{ijk} \frac{\partial v^i}{\partial t} \frac{\partial v^j}{\partial s} \frac{\partial v^k}{\partial r} dt ds dr. \end{aligned} \quad (63)$$

It is not coordinate invariant but transforms under coordinate transformations  $y^{j'} = y^{j'}(x^i)$  according to

$$dv' = \det(\partial y / \partial x) dv. \quad (64)$$

Since the volume element  $dv$  constitutes one independent component, a natural object to integrate over a volume has one independent component as well. We denote such an integrand by  $\gamma$ . It transforms according to

$$\gamma' = \frac{1}{\det(\partial y / \partial x)} \gamma. \quad (65)$$

This transformation rule characterizes a scalar density and yields

$$\int \gamma dv = \int \gamma \epsilon_{ijk} \frac{\partial v^i}{\partial t} \frac{\partial v^j}{\partial s} \frac{\partial v^k}{\partial r} dt ds dr \quad (66)$$

as a coordinate independent value.

## A.2 Poincaré Lemma

The axiomatic approach takes advantage of the Poincaré lemma. The Poincaré lemma states under which conditions a mathematical object can be expressed in terms of a derivative, i.e., in terms of a potential.

We consider integrands  $\alpha_i$ ,  $\beta^i$ , and  $\gamma$  of line-, surface-, and volume integrals, respectively, and assume that they are defined in an open and simply connected region of three-dimensional space. Then the Poincaré lemma yields the following conclusions:

1. If  $\alpha_i$  is curl free, it can be written as the gradient of a scalar function  $f$ ,

$$\epsilon^{ijk} \partial_j \alpha_k = 0 \quad \implies \quad \alpha_i = \partial_i f. \quad (67)$$

2. If  $\beta^i$  is divergence free, it can be written as the curl of the integrand  $\alpha_i$  of a line integral,

$$\partial_i \beta^i = 0 \quad \implies \quad \beta^i = \epsilon^{ijk} \partial_j \alpha_k. \quad (68)$$

3. The integrand  $\gamma$  of a volume integral can be written as the divergence of an integrand  $\beta^i$  of a surface integral,

$$\gamma \text{ is a volume integrand} \quad \implies \quad \gamma = \partial_i \beta^i. \quad (69)$$

While conclusions (67), (68) are familiar from elementary vector calculus, this might not be the case for conclusion (69). However, (69) is rather trivial since, in cartesian coordinates  $x, y, z$ , for a given volume integrand  $\gamma = \gamma(x, y, z)$  the vector  $\beta^i$  with components  $\beta^x = \int_0^x \gamma(t, y, z)/3 dt$ ,  $\beta^y = \int_0^y \gamma(x, t, z)/3 dt$ , and  $\beta^z = \int_0^z \gamma(x, y, t)/3 dt$  fulfills (69). Of course, the vector  $\beta^i$  is not uniquely determined from  $\gamma$  since any divergence free vector field can be added to  $\beta^i$  without changing  $\gamma$ . We further note that  $\gamma$ , as a volume integrand, constitutes a scalar density. It can be integrated as above to yield the components of  $\beta^i$  as components of a contravariant vector density. Therefore the integration does not yield a coordinate invariant scalar such that  $\gamma$  cannot be considered as a natural integrand of a line integral.

## A.3 Stokes Theorem

In our notation Stokes theorem, if applied to line integrands  $\alpha_i$  or surface integrands  $\beta^i$ , yields the identities:

$$\int_V \partial_i \beta^i dv = \int_{\partial V} \beta^i da_i, \quad (70)$$

$$\int_S \epsilon^{ijk} \partial_j \alpha_k da_i = \int_{\partial S} \alpha_i dc^i, \quad (71)$$

where  $\partial V$  denotes the two-dimensional boundary of a simply connected volume  $V$  and  $\partial S$  denotes the one-dimensional boundary of a simply connected surface  $S$ .

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