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CURRENT FLOW IN SPACE-CHARGE LIMITED CYLINDER-PLANE DIODES

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1 INTRODUCTION

Studies of high voltage, high current, field emission diodes usually include some attempt to relate total current flow, I , or current density, J , to inter-electrode voltage, V_a , and diode geometry (spacing, d , area, A , etc.) via a space-charge limited flow hypothesis. All such studies to date have been made by approximating the diode geometry to an infinite parallel-plane situation, enabling the use of the simple (non-relativistic) Langmuir-Child relation:

$$J = \frac{4}{9} \varepsilon_o \sqrt{\frac{e}{m}} \cdot \frac{V_a^{3/2}}{d^2} \quad \text{(MKS)} \qquad \ldots\ldots (1)$$

where ε_o , e and m have usual meanings.

This simplification is justifiable in the majority of cases, where macroscopic parallel-plane geometry (as in, for example, "plasma" and multi-needle cathodes) is modified only by small edge effects and where the overall accuracy aimed at is, in any case, not high.

There are situations, however, in which a parallel-plane solution is probably too far from the truth and where even an approximate solution of a more realistic geometry would be of value. This has recently arisen in connection with the investigation of the effects of plasma growth in highly divergent electric-field diodes.

This note describes one approach to the particular case of cylinder-plane geometry, where a single edge emitter is assumed to be covered by an expanding plasma cylinder of zero work function.

2 OUTLINE OF APPROACH

A cross-section of the system under study is shown in Fig 1, where an infinitely long uniform cylindrical emitter of radius a has its centre at distance d from an infinite plane anode. For Laplacian fields, at least, the associated two-dimensional potential distribution is transformable to that resulting from two concentric cylinders by a conformal transformation.[1] According to Weber[2], it is also transformable for Poissonian fields arising from superimposed space charge. We are thus able to calculate exactly the field distribution arising from applied potential and known charge density distribution in the system of Fig. 1. We can obtain an approximation to current density distribution by making a fundamental assumption: for an electron velocity vector $\bar{v}$, $\nabla V \times \bar{v} = 0$, i.e. all electrons move

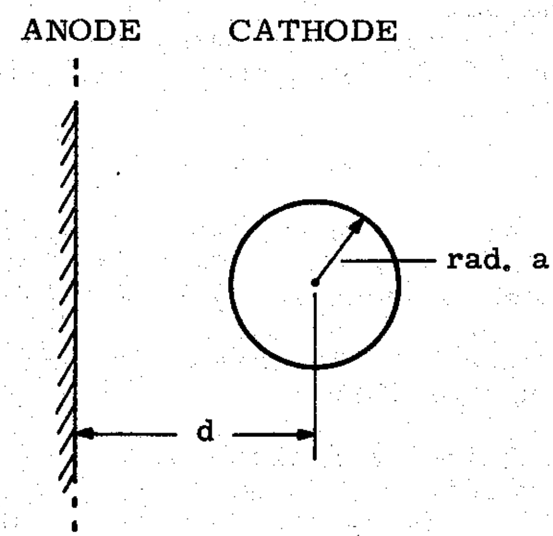


Figure 1

along field lines. We are, in effect, considering zero "transverse mass" charge carriers, so that transverse inertial effects are neglected. This may be looked at alternatively as an assumption that the form of the equipotential surfaces are unchanged by the presence of moving charges; only the value of potential assigned to them is changed.

The error introduced by this assumption is dependent on field line curvature, which in a cylinder-plane system is not large. The error is reduced by self-magnetic field effects of the current flow; for an electron with instantaneous radius of curvature R, mv^2/R forces are offset to a greater or smaller extent by $\bar{v} \times B_\theta$ forces.

Two further simplifications have been made in this approach. Relativistic effects have been ignored and self-magnetic field forces neglected. A rough idea of relativistic correction can be obtained by noting that, for the parallel plane situation, the current density is decreased by ~ 9% at 0.5 MeV and by 14% at 1 MeV. The neglect of magnetic forces obviously restricts application of the results to cases where violent pinching is not evident.

The analytical scheme is this:

(i) Transformation of real (z-plane) co-ordinate system to w-plane by a conformal transformation which makes electrodes concentric cylinders.

(ii) Calculation of current density distribution in w-plane by use of Langmuir's solution of $\nabla^2 r = -\rho/\varepsilon_o$

(iii) Inversion to z-plane by use of transformation ratio which is valid for charge density (ρ) and hence for current density (with preceding assumptions): - $J = \rho v$, $v = f(V)$ only and V invariant;

i.e. we are saying[2] :

$$J_{(z)}\, dx\, dy = J_{(z)} \cdot \frac{du\, dv}{\left|\frac{dw}{dz}\right|^2} = J_{(w)}\, du\, dv \qquad \ldots. (2)$$

where $J_{(z)}$ and $J_{(w)}$ are current density in Z and W planes respectively, and $|dw/dz|^2$ is the appropriate scale ratio.

(iv) Integration of $J_{(z)}$ over appropriate contour to obtain current per unit length I/l .

3 CALCULATION OF CURRENT DENSITY AND TOTAL CURRENT

(1) TRANSFORMATION

We take the origin in the Z-plane as p units behind the anode plane (Fig 2). p will be subsequently chosen to make the transformed circles of anode and cathode surfaces in the w-plane concentric.

The appropriate transformation in this case is the inversion transformation[1] defined by

$$w = \frac{1}{z} \qquad \text{.... (3)}$$

Then the anode plane in the z-plane transforms into a circle in the w-plane of radius $r_B = 1/2p$ and centre (1/2p, o), ie it passes through the origin (Fig 3). The cathode transforms to a circle in the w-plane of radius

$$r_A = \frac{a}{(p+d)^2 - a^2}, \qquad \text{centre} \left[\frac{(p+d)}{(p+d)^2 - a^2}, o\right]$$

For the circles to be concentric, we have:

$$\frac{p+d}{(p+d)^2 - a^2} = \frac{1}{2p}$$

$$\text{ie} \quad p = \sqrt{d^2 - a^2} \qquad \text{.... (4)}$$

(2) $J_{(w)}$ IN THE w-PLANE

The cathode current density of a concentric cylindrical diode with cathode radius r_A, anode radius r_B and inter-electrode potential V_a is:

$$J_{(w)} = \frac{2.34 \times 10^{-6} V_a^{3/2}}{r_A r_B \beta^2 (r_B/r_A)} \text{ A/cm}^2 \qquad \text{.... (5)}$$

where $\beta^2 (r_B/r_A)$ is a tabulated function of the ratio r_B/r_A. (See, for example, ref. 3).

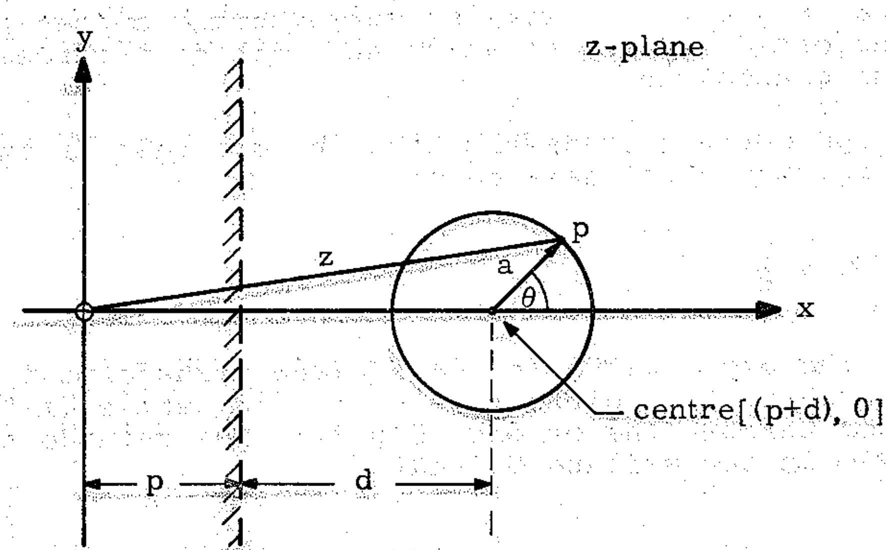


Figure 2

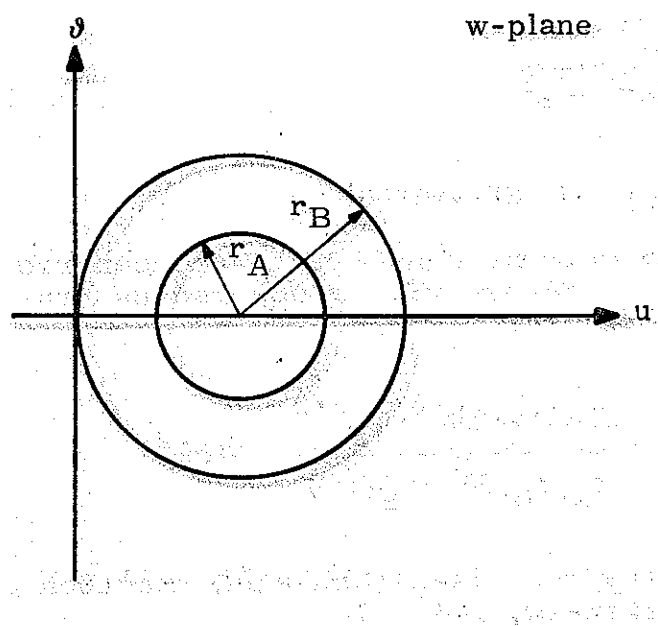


Figure 3

(3) INVERSION TO THE z-PLANE

For the inversion transformation (equation 3), the charge and current density scale ratio is:

$$\left|\frac{dw}{dz}\right|^2 = \left|\frac{1}{z}\right|^4 \qquad \text{.... (6)}$$

so that, from (2) we have, for z-plane current density:

$$J_{(z)} = J_{(w)} \left|\frac{dw}{dz}\right|^2 = J_{(w)} \left|\frac{1}{z}\right|^4$$

ie
$$J_{(z)} = \frac{2.34 \times 10^{-6} \, V_a^{3/2}}{r_A r_B \beta^2 (\frac{r_B}{r_A}) \cdot |z|^4} \qquad \text{.... (7)}$$

(4) TOTAL CURRENT

In order to derive a total current relation, we need to integrate (7) over a suitable contour in the z-plane. This is facilitated by expressing z in terms of the angle θ of Fig 2.

ie $|z|^2 = [(p+d) + a \cos \theta]^2 + [a \sin \theta]^2$.

Putting $p + d = x_c$, the origin-to-cylinder centre distance:

$$|z|^2 = (x_c^2 + a^2) + 2ax_c \cos \theta \ . \qquad \text{.... (8)}$$

The total current per unit length contained between angles θ_1 and θ_2 on the cathode is thus:

$$\left[\frac{I}{l}\right]_{\theta_1}^{\theta_2} = \frac{2.34 \times 10^{-6} \, V_a^{3/2} \cdot a}{r_A r_B \ \beta^2 (\frac{r_B}{r_A})} \int_{\theta_1}^{\theta_2} \frac{d\theta}{[a_1 + b_1 \cos \theta]^2} \qquad \text{... (9)}$$

where $a_1 = x_c^2 + a^2$

$b_1 = 2 \, ax_c$.

Now
$$\int_{\theta_1}^{\theta_2} \frac{dd}{[a_1 + b_1 \cos\theta]^2} = \left\{ \frac{b_1 \sin\theta}{(b_1^2 - a_1^2)(a_1 + b_1 \cos\theta)} \right.$$

$$\left. + \frac{a_1}{a_1^2 - b_1^2} \left[\frac{2}{\sqrt{a_1^2 - b_1^2}} \tan^{-1} \frac{(a_1 - b_1 \tan\frac{\theta}{2}}{\sqrt{a_1^2 - b_1^2}} \right] \right\}_{\theta_1}^{\theta_2} \qquad \ldots. \ (10)$$

For the current from the whole of the cathode circumference, we have

$$\left[\frac{I}{I}\right]_T = 2 \int_o^{\pi} J_{(z)} \ a \cdot d\theta$$

and, from (10):

$$\int_o^{\pi} \frac{d\theta}{|z|^4} = \frac{\pi a_1}{(a_1^2 - b_1^2)^{3/2}}$$

the integral vanishing at the lower limit.

As $(a_1^2 - b_1^2) = (x_c^2 - a^2)^2$, we have finally:

$$\left[\frac{I}{I}\right]_T = \frac{14.66 \times 10^{-6} \ V_a^{3/2} \cdot a}{r_A r_B \ \beta^2 (\frac{r_B}{r_A})} \left[\frac{x_c^2 + a^2}{(x_c^2 - a^2)^3} \right] \qquad \ldots. \ (11)$$

where, in terms of the geometry in the z-plane (a,d) :

$$
\left.
\begin{aligned}
x_c &= d + \sqrt{d^2 - a^2} \\
r_A &= \frac{a}{2(d^2 - a^2) + 2d\sqrt{d^2 - a^2}} \\
r_B &= \frac{1}{2\sqrt{d^2 - a^2}}
\end{aligned}
\right) \quad \ldots\ldots (12)
$$

4 PRACTICAL FORM OF THE RESULTS

In order to put equation (11) into a form more readily applicable to experiment, a series of calculations of effective perveance K have been performed, where

$$
\left.
\begin{aligned}
K &= \frac{\left[\frac{I}{l}\right]_T}{10^{-6}\, V_a^{3/2}} \qquad \frac{A/cm}{(volts)^{3/2}} \\
&= \frac{\left[\frac{I}{l}\right]_T}{V_{MV}^{3/2}} \qquad \frac{KA/cm}{MV^{3/2}}
\end{aligned}
\right) \quad \ldots\ldots (13)
$$

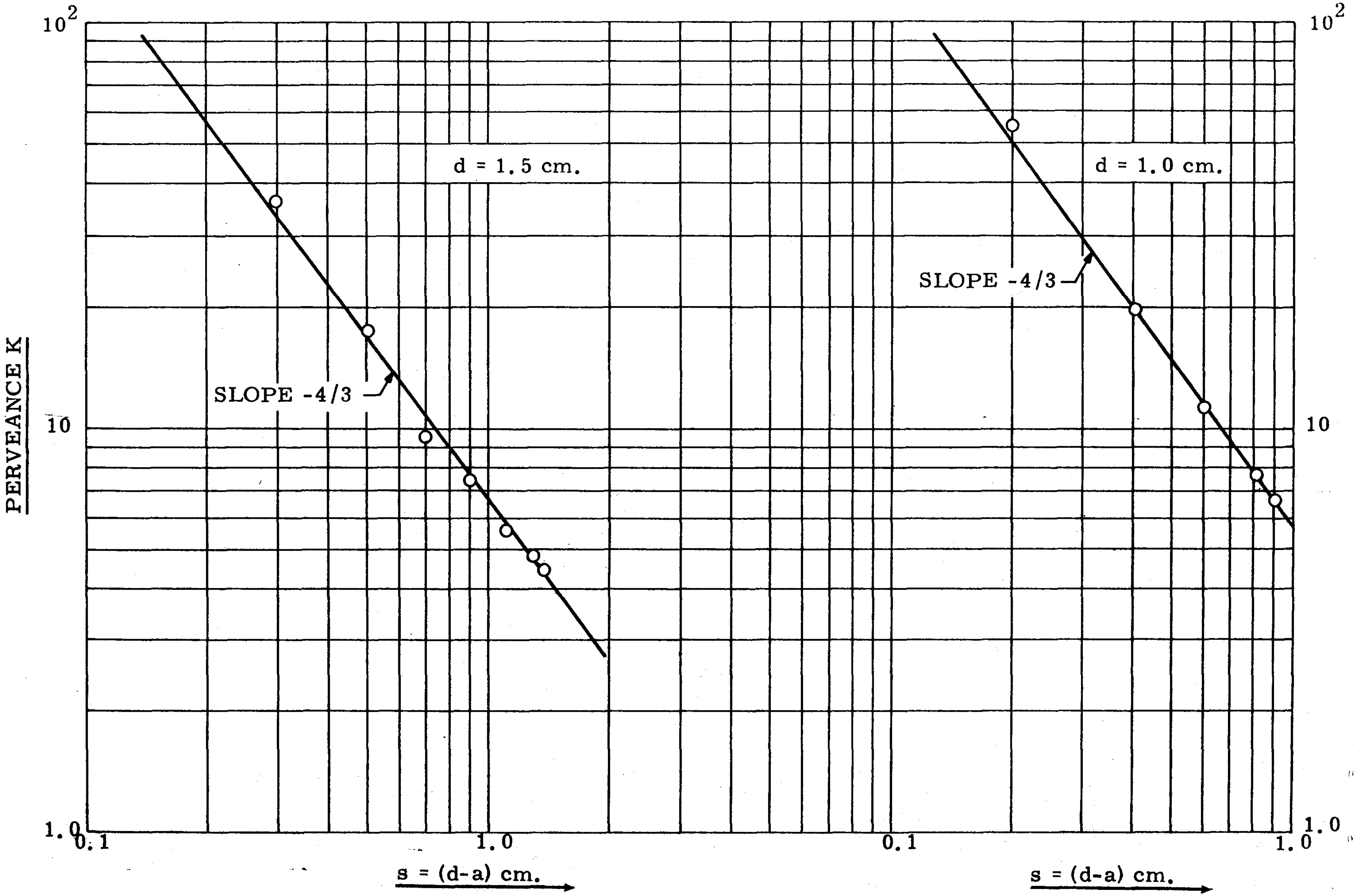
In these calculations, K was computed as a function of the effective diode spacing (d - a) for a series of values of cathode centre to anode plane distance d , using tabulated values of $\beta^2(r_B/r_A)$. Two typical results are shown in Figure 4. A good fit is provided in each case by a straight line (log-log paper) of slope -4/3; ie the dependence of K on d and a can be approximated by a relation of the form:

$$
K = \frac{\alpha\ (d)}{(d - a)^{4/3}}
$$

where $\alpha(d)$ is a function of d alone.

Now it is readily shown that as $a \to o$, $\beta^2(r_B/r_A) \to 1$ and $K \tilde{\to} 7/d$, so that we would expect $\alpha(d)$ to be of the form given by:

FIG. 4: PERVEANCE v. s.



$$\frac{7}{d} \sim \frac{\alpha(d)}{d^{4/3}} \quad \text{ie} \quad \alpha(d) \sim 7\, d^{1/3}.$$

In fact, a slightly better fit to the numerical results is:

$$\alpha(d) = 5.8\, d^{1/3}$$

so that, finally, the perveance can be written:

$$K\,(a,d) = \frac{5.8\, d^{1/3}}{(d-a)^{4/3}} \qquad \ldots.\ (14)$$

From the definition of perveance (13), the total current per unit length of diode is thus:

$$\boxed{\left[\frac{I}{l}\right]_T = 5.8\, V_{MV}^{3/2} \cdot \frac{d^{1/3}}{(d-a)^{4/3}} \quad \text{KA/cm}} \qquad \ldots.\ (15)$$

and the diode impedance Z_T for one cm length, given by

$$Z_T = \frac{V_{KV}}{I_{KA}} = \frac{10^3}{KV_{MV}^{1/2}}\ \Omega$$

per cm length, becomes:

$$\boxed{Z_{T1}\ell = 172\, V_{MV}^{-1/2}\, \frac{(d-a)^{4/3}}{d^{1/3}} \quad \Omega - \text{cm}} \qquad \ldots.\ (16)$$

REFERENCES

1 K J Binns and P J Lawrenson: "Analysis and Computation of Electric and Magnetic Field Problems", p. 138. (Pergamon Press, 1963).

2 E Weber: "Electromagnetic Fields", Vol. I, p. 304 (Wiley, 1960).

3 Beck: "Thermionic Values", p. 148 (Cambridge, 1953).