System Design and Assessment Notes

Note 25

THEORY OF PERFORMANCE AND SURVIVABILITY OF LARGE C^3I NETWORKS UNDER NUCLEAR-STRESSED CONDITIONS*

JANUARY 1982

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* PRESENTED AT NATIONAL RADIO SCIENCE MEETING, BOULDER, COLORADO
13-15 JANUARY 1982

ABSTRACT

A theory for predicting the performance and survivability of large redundant path C^3I networks under nuclear-stressed and ECM conditions is developed. These communication systems (typified by the PACOM networks) possess nearly a thousand nodes (message centers, relay terminals) and propagation links ranging from VLF to SHF. The networks include numerous critical message centers (e.g., command posts), with the required performance between them generally being different. The performance of a network is evaluated in terms of a set of functions $\tilde{F}_\alpha(X,\vec{P},t)$ which for the α^{th} command post pair is defined as the probability that the character error rate (CER) is less than or equal to X, at time t following the onset of the threat. It is shown that $\tilde{F}_\alpha(X,\vec{P},t)$ is of the form:

$$\tilde{F}_\alpha(X,\vec{P},t) = \sum_{\ell} \tilde{G}_\ell(X,t)\hat{P}_\ell(t) + \sum_{m}\sum_{n} \tilde{G}_n(X,t)\,\tilde{G}_m(X,t)\hat{P}_n(t)\hat{P}_m(t) + \sum_{q}\sum_{r}\sum_{s} \cdots$$

where the $\hat{P}$'s are functions of the products of the individual probabilities-of-survival, P_i, for the nodes, and the $\tilde{G}$'s are functions of the link parameters. The $\hat{P}$ and $\tilde{G}$ functions depend upon the entire connectivity between the command post and hence incorporate a part of the redundancy in the system. Whereas the nodes are modelled as binary random variables, the links have a continuous distribution in CER due to degradation of the propagation medium caused by nuclear detonations. Numerical methods for computing $\tilde{F}_\alpha(X,\vec{P},t)$ are presented.

Using the aforementioned equation for $\tilde{F}_\alpha$ we find the minimum cost to achieve network survivability by first selecting the required CERs between command post pairs, X_α, and the time t_o at which minimum performance is achieved. We subsequently express the network cost function as: $C_{NT}=\Sigma C_i(P_i)$, where $C_i(P_i)$ is the cost required to harden node i to probability of survival P_i, and N is the number of nodes. Minimizing C_{NT} subject to the set of inequality constraints, $\tilde{F}_\alpha(X_\alpha,\vec{P},t_o) \geq T_a$ = constant = required time availability for each α gives the minimum cost for network survivability.

ACKNOWLEDGEMENT

The development of this effort involved discussions with several people. The author would like to thank Mr. Hank Root for consultations in the area of nodal probability-of-survival; Messrs. Charles Frye, Jim Hazelton, and Sheldon Price for discussions pertaining to the software utilization of the theory; and Dr. Harry Gelman, Lt. Commander Walter Rabke of DNA, and Messrs. Harry Hoogasian, Gordon Nelson, and Marshall Cross for providing useful insight into many of the problem areas. Thanks are also due to Chris Ashley of AFWL for reviewing the overall document and making several recommendations which were implemented in the report.

PREFACE

This paper addresses the theory underlying GTE's development of the Communications Network Assessment Simulator (CNAS) software which originated under the DNA-sponsored APACHE program. The purpose of APACHE was to evaluate the end-to-end (ETE) communications performance between selected command posts (end-point-pairs) of the PACOM networks under a postulated multi-burst high altitude nuclear scenario. The PACOM assessment was performed using a combination of site-related experimental EMP data and theoretical propagation models. It required the technical resources of a number of organizations with GTE and Boeing the principal contractors, and other organizations such as SRI, RDA, CSC, and BDM providing critical inputs to the data base in selected areas.

The basic information required for a network evaluation of the ETE performance of pre-, trans-, and post-attack scenarios is:

1. The determination of the time-dependent statistical distribution of the signal-to-noise ratio for the propagation paths (links).
2. The determination of the time behavior of the probability-of-survival, P_S, and recovery time, τ, for the functional elements (nodes) against the EMP pulse.
3. The development of a mathematical model for determining the ETE performance from the aforementioned information.
4. The development of a computer system for implementing the mathematical model using an appropriate data base for identification of the nodes and their respective values of P_S and τ.

This paper is only concerned with the development of item 3, the aforementioned mathematical model. No discussion of the data base or its method of development is addressed in this paper. It is also shown in this paper that it is theoretically possible to determine the minimum cost necessary to harden a network for a specified performance level, provided that one can determine the cost(s) required to harden the individual node(s) against EMP to respective variable probability-of-survival levels.

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PARTIAL LIST OF SYMBOLS

SYMBOL	DEFINITION
$P_i(t)$	Probability-of-Survival for node i
$C_i(P_i)$	Cost Function for node i
X	Character Error Rate (CER)
$\vec{P}$	Set of values of P_i
t	Time, measured from onset of threat
$\tilde{F}_\alpha(X,\vec{P},t)$	Performance Function, defined as the probability that the character error rate for the α^{th} end-point-pair is less than or equal to X at time t.
$\tilde{F}(X,\vec{P},t)$	General notation for performance function
P_{sur}	End-to-end probability-of-survival, deduced as the limit of the performance function when link deterioration can be neglected.
$\tilde{F}^*(X,t)$	Network Link Availability, defined as the limit of the performance function when all values of P_i are set equal to unity.
y	Signal-To-Noise Ratio (SNR) in dB
μ_y	Mean SNR
σ_s	Standard Deviation for signal
Z	Bit Error Rate (BER)
Z_i	BER for link i
X_i	CER for link i
$\tilde{f}_i(X_i)$	Probability density function (pdf) for CER on link i
$\tilde{f}_L(X)$	End-to-end link pdf for a single path
$\tilde{F}_L(X)$	End-to-end distribution function for a single path
T_a	Required time availability

SYMBOL	DEFINITION
X_o	Required CER
t_o	Required time at which minimum performance is to be achieved.
A_j	Event that node j is up
$E^{(i)}$	Event that path i occurs
$\tilde{F}(\vec{P})$	Performance Function for hard link network
$A_j^{(i)}$	Event that j^{th} node in the i^{th} path is up
$G^{(i)}$	Event that the CER along path i is less than or equal to X.
$W^{(i)}$	Event that the resultant CER due to link degradation along path i is less than or equal to X.
$X_j^{(i)}$	CER for link j along path i

1.0 Introduction

This investigation deals with the development of a theory and related analytical techniques for predicting the performance and survivability of large C^3I systems, such as the PACOM networks, under nuclear-stressed conditions. Characteristically, these networks include hundreds of nodes (e.g., message centers, relay terminals, transmitters, etc.) and propagation links covering the frequency spectrum from VLF to SHF. In addition, they possess numerous critical message centers (e.g., command posts) of varying degrees of operational importance, with the required performance between message centers generally being different.

It is desired to maintain the end-to-end performance (an entity which we shall precisely define) of these large networks in a hostile environment. Fig. 1-1 shows a pictorial representation of the situation envisioned. A communication systems may be susceptible to nuclear threats, electromagnetic countermeasures (jamming), commando strikes, etc. In this study we address only the nuclear threat.

As discussed in Section 2, a nuclear detonation may effect the system in two ways. The EMP generated by the weapon can cause permanent or temporary damage to the nodes (also called functional elements), while the nuclear-produced atmospheric ionization can degrade the performance of the radio links. For example, the 22 PACOM networks contain more than 600 functional elements (nodes) each of which has some degree of vulnerability to EMP.

Under GTE's portion of the APACHE program[1] an evaluation of the communications capability of the PACOM networks in a specified multi-burst nuclear laydown was rendered. In particular, the following considerations were addressed:

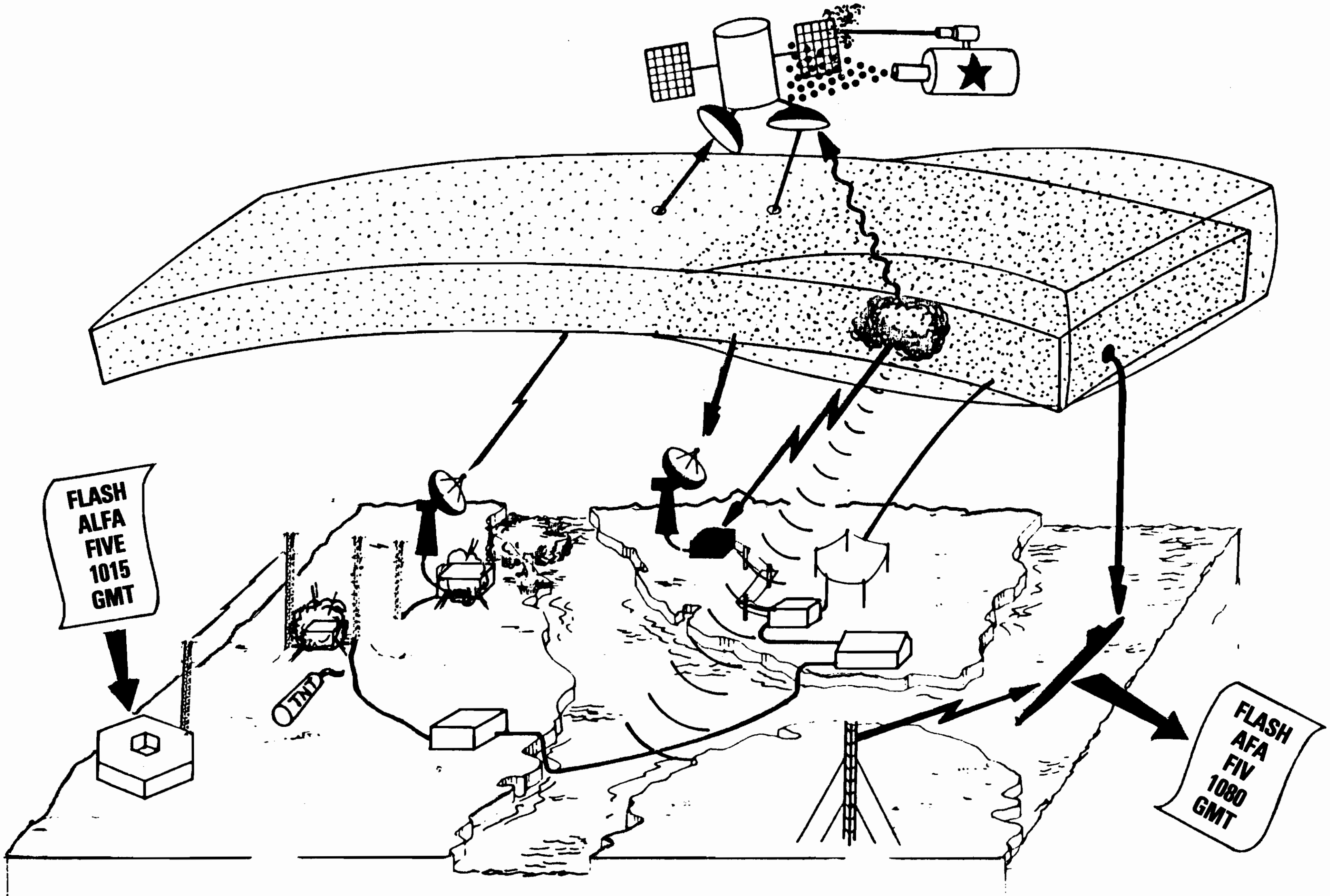

FIG. 1-1 END-TO-END PERFORMANCE AND THREATS

- The determination of the performance of selected end-point-pairs in the networks during the multi-burst scenario.
- The determination of the level of EMP hardness required at the individual nodes to achieve the desired performance between specified command posts throughout the network at the minimum cost.*

Although the theory and computer implementation for accomplishing the aforementioned tasks was initially developed with the PACOM networks in mind, the methodology can be, and has been, applied to other networks (e.g., NATO communication systems). Fig. 1-2 shows a generic network for which this analysis is applicable. In this diagram there are four command posts indicated by A, B, C, D. The circles are nodes, while the straight lines are links (propagation paths such as HF, cable, VLF, etc.). The object is to satisfactorily communicate between any or all of the command posts. Using end-point-pair A-B as an example we note that there are four possible paths connecting them. As observed, portions of these paths involve some common nodes and links. The theory developed shows how to rigorously compute the performance between any and all end-point-pairs for redundant path networks of the type just shown when the entire system is affected by multi-burst nuclear detonations. In addition, the mathematical relationship between performance and the minimum cost necessary to ensure a prescribed degree of required capability is derived.

* The basic building blocks for determining the minimum cost required to harden a system to some specified minimum performance level are the so-called cost functions $C_i(P_i)$. The entity $C_i(P_i)$ is the cost necessary to harden node i against EMP to probability-of-survival level P_i. In this analysis we do not use any specific form for the C_i functions. Reference 2 provides a discussion of minimum cost-to-harden techniques using specific cost functions for the functional elements of the PACOM networks.

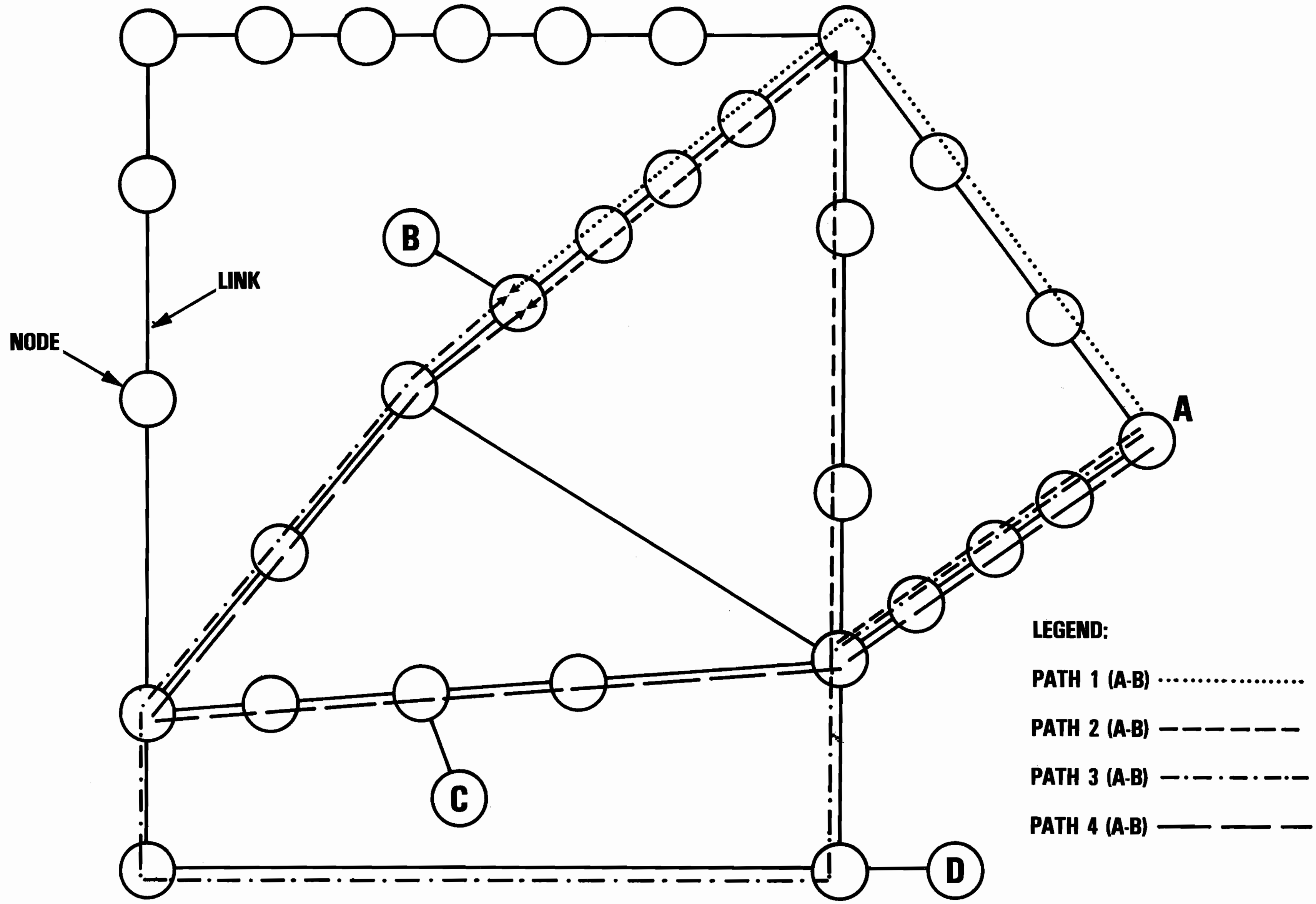

FIG. 1-2 GENERIC COMMUNICATION NETWORK WITH REDUNDANT PATHS

Up to this point we have loosely used the word "performance" to describe the capability of a communications network. In Section 2 we use a simple two-terminal radio network to present a meaningful physical interpretation and corresponding mathematical definition for an entity defined as the "performance function," $\tilde{F}(X,\vec{P},t)$. This function is the probability that the character error rate (CER) between a specified end-point-pair is less than or equal to X at time t, following the onset of the threat: $\vec{P}$ stands for the vector of all the probabilities-of-survival of the nodes involved in the paths connecting the end-point-pair: P_i is the probability-of-survival for node i.

The deduction of $\tilde{F}(X,\vec{P},t)$ requires the use of several fundamental assumptions. These will now be stated:

- The nodes, as used in this analysis, are taken to be individual functional elements. Thus, the behavior of circuits which pass through the same large facility or installation such as a building are analyzed in terms of their corresponding functional elements (nodes) which define the communications path.
- The susceptibility of a node, i, to the EMP threat is manifested in the probability-of-survival, P_i, for that node: P_i is the probability that node i will not go down. No assumption is made which limits the nature of P_i. On physical grounds one may assume that P_i will be time-dependent, reflecting the ability of the node to recover. A model for P_i which was used in the APACHE program is one in which restoration was assumed after a specified recovery time. However, this is not required in general.
- Statistical independence of the P_is for the nodes is assumed even though it is recognized that this assumption may not always be true. Information pertaining to the probability distribution for joint failure of two or more nodes is not available. When, and if such information becomes available the theory can be modified.
- Statistical independence of the behavior of the propagation paths (links) is assumed. Here, too, modification can be made if necessary.

Section 3 provides a derivation of $\tilde{F}(X,\vec{P},t)$ in the case of a serial combination of nodes and links. This particular result is useful not only in its own right, but also because it provides an important building block in the analysis of redundant-path networks. In addition, the serial case furnishes a good mathematical model for developing the concept of cost minimization: this is rendered in Section 4. In Section 5 we present the general method for determining $\tilde{F}(X,\vec{P},t)$ in large networks involving redundant paths, and discuss the associated cost-to-harden problem.

Based on certain mathematical properties of $\tilde{F}(X,\vec{P},t)$ we develop a rigorous definition of the probability-of-survival, P_{sur}, for an end-point-pair. It is shown in Section 6 that P_{sur} is equal to the performance function in the limit when link effects can be neglected. In addition, it is possible to define a function called the network link availability, $\tilde{F}^*(X,t)$ which is observed to be the limit of the performance function when all the nodes involved in the connectivity between the end-point-pair have probability-of-survival equal to unity.

In a large network the evaluation of $\tilde{F}(X,\vec{P},t)$ is seen to require numerical integration techniques. The essential problem is the computation of integrals involving several variables. It is well known that Monte-Carlo integration techniques are appropriate in such cases. For orientation purposes, we present a brief summary of the general concepts of this technique in Section 7 as it applies to our problem. It is beyond the intent of this paper to go into the refinements of the so-called "crude" Monte-Carlo numerical integration method. This will be presented in a subsequent publication. Concluding remarks are rendered in Section 8.

2.0 Physical Interpretation Of Performance in Nuclear-Stressed Environment: Definition Of Performance Function

The purpose of this section is to provide a physical basis for establishing $\tilde{F}(X,\vec{P},t)$ as a suitable measure of network performance in nuclear-stressed environments. This is accomplished by developing the natural evolution of $\tilde{F}(X,\vec{P},t)$ from the ambient to the nuclear case in a simple two-terminal communications network. The generalization to large C^3I networks is given in Section 5.

Figure (2-1a) shows a model of a two-terminal communication network, namely, one involving a single transmitter and receiver, with a single link (propagating medium) connecting them. Under ambient conditions the propagation medium may vary due to natural-occurring changes in the environment (e.g., rain, variations in the ionosphere, etc.), while the rms atmospheric noise is also variable. Because of the stochastic nature of the SNR it is necessary to deal with its probability distribution.

The probability distribution function, pdf, for the SNR in the ambient case is determined from a finite number, n, of experimental observations. If we denote

$$y = \text{SNR (in dB)}, \tag{2.1}$$

then the distribution function for y is estimated as[3]:

$$\hat{G}_n(y) = \frac{\text{Number of experiments in which measured SNR is less than or equal to } y}{\text{Total number of experiments} = n} \tag{2.2a}$$

Suppose that $\hat{G}_n(y)$ has been measured in a finite number, n, of experiments. What is actually desired is the true distribution function, which is defined as:

$$G(y) \stackrel{\Delta}{=} \lim_{n\to\infty} \hat{G}_n(y) \tag{2.2b}$$

From a practical point of view we cannot realistically determine G(y) since $n \ll \infty$. However, we can hypothesize that G(y), for example, is normal (i.e. Gaussian). In principle we could then use $\hat{G}_n(y)$ to conduct a statistical

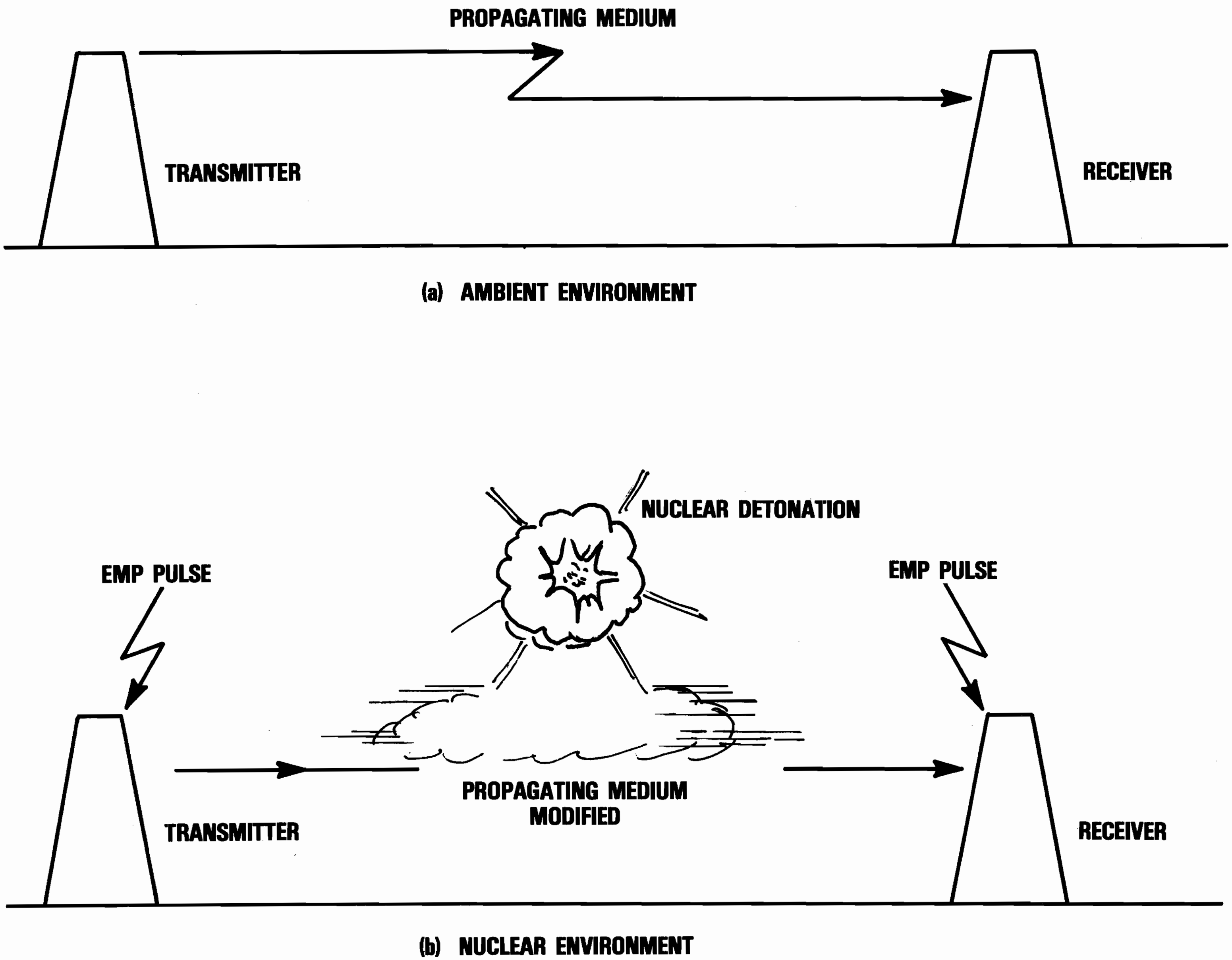

FIG. 2-1 TWO-TERMINAL COMMUNICATIONS NETWORK

test of this hypothesis. Of course there is some possibility that the hypothesis is false even if the test is passed. The probability that the hypothesis will pass a test even if it is false is called β. The β of a test of a normal hypothesis for $\hat{G}_n(y)$ against all "hazardous" alternative distributional hypotheses (e.g., the Cauchy distribution) is not known to this author at present.

We can nevertheless use the normal hypothesis as an example to show how probability and cost minimization calculations can be done if G(y) is known. In this case we have:

$$G[y] = \int_{-\infty}^{y} g(y')dy' \tag{2.3}$$

with g(y) being the normal pdf: g(y) is given by

$$g(y) = \frac{1}{\sqrt{2\pi\,\sigma^2}} \exp\left[-\frac{1}{2}\frac{(y-\mu_y)^2}{\sigma^2}\right], \tag{2.4}$$

where

μ_y = mean value of y

σ^2 = variance = $\sigma_n^2 + \sigma_s^2$ (assuming independence)

σ_n = standard deviation for rms noise

σ_s = standard deviation for signal level

The values of μ_y, σ_n, and σ_s are estimated by $\hat{\mu}_y$, $\hat{\sigma}_n$, and $\hat{\sigma}_s$ using a finite number of experiments. In this analysis we do not address the question regarding the confidence we have in the values of $\hat{\mu}_y$, $\hat{\sigma}_n$, and $\hat{\sigma}_s$.

For airborne platforms an additional uncertainty in the SNR arises from random variations in the antenna direction. In this case the variance is given by:

$$\sigma^2 = \sigma_n^2 + \sigma_s^2 + \sigma_a^2 \quad \text{(assuming independence)} \tag{2.5}$$

where σ_a is the standard deviation for the antenna fluctuation. Here again, an estimate, $\hat{\sigma}_a$ is used in lieu of the true value, σ_a.

Assuming a particular modulation scheme provides the relationship between the bit error rate, BER = Z, and SNR. We have the equation:

$$Z = \Gamma(y) \tag{2.6a}$$

and the inverse relationship

$$y = \Gamma^{-1}(Z) = \rho(Z) \quad , \tag{2.6b}$$

where Γ is the modulation function. Figure (2-2) shows examples of the relationship between bit error rates, and SNRs for selected modulation schemes. The pdf for the BER, denoted as $f_\ell(Z)$, is determined from the law of transformation[3] of pdfs:

$$f_\ell(Z)\, dZ = g(y)dy \quad , \tag{2.7}$$

which from Eq. (2.6) gives

$$f_\ell(Z) = g(y = \rho(Z))(\frac{d\rho}{dZ}) \tag{2.8}$$

The range of Z is from 0 to 0.5, while the probability that the BER is less than or equal to Z is given by:

$$F_\ell(Z) = \int_0^Z f_\ell(Z')dZ' \tag{2.9}$$

For the example of Fig. (2.1a) it is apparent that $F_\ell(Z)$ provides a usable measure of system performance. By virtue of the fundamental origin of P[y], and the relationship between Z and y, it follows that $F_\ell(Z)$ can be interpreted as the time availability. Physically, $F_\ell(Z)$ is the fraction of time that the BER will be less than or equal to Z.

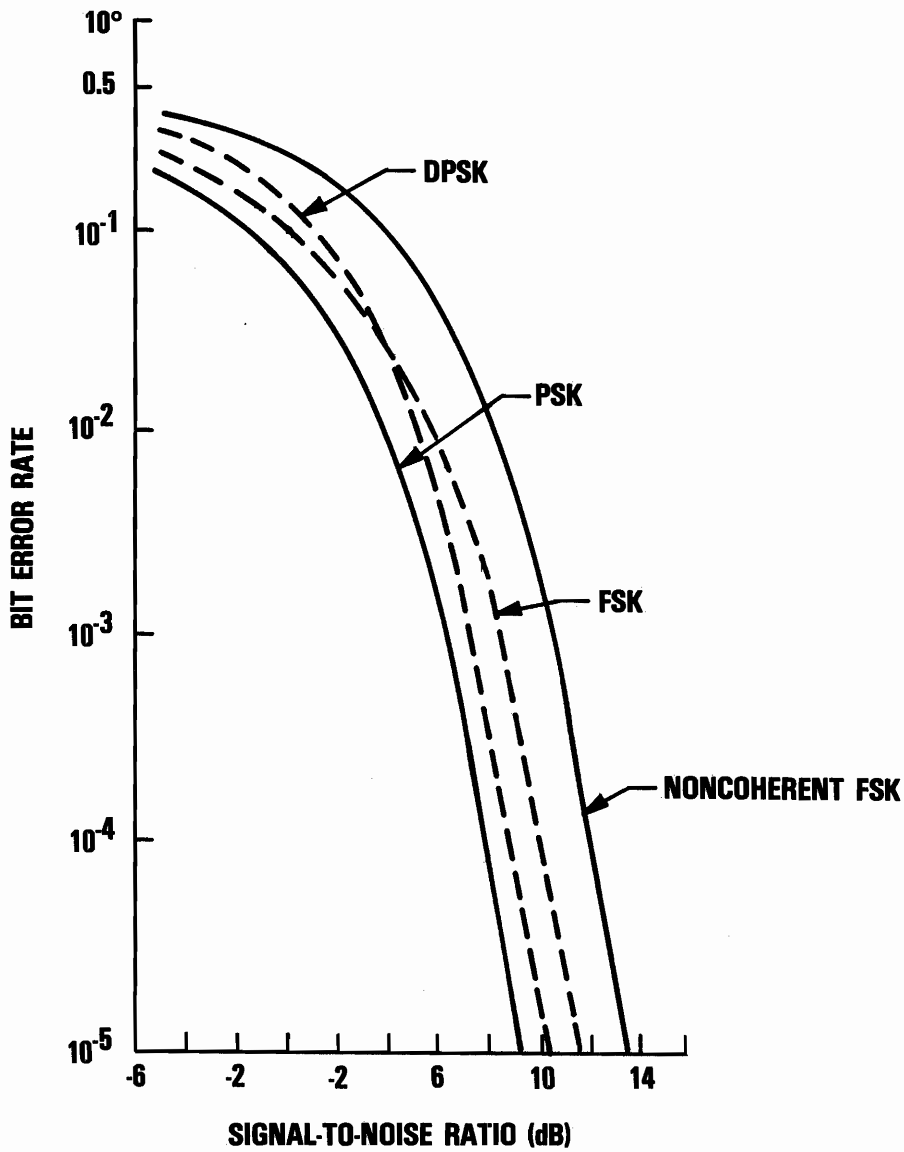

FIG. 2-2 RELATIONSHIP BETWEEN BIT ERROR RATE AND SIGNAL-TO-NOISE RATIO FOR SELECTED MODULATION SCHEMES

Within the foregoing context one can define and evaluate a quantity, Z*, which is the BER corresponding to a specified time availability H. The former is determined from the equation

$$H = F_\ell(Z^*) \tag{2.10}$$

In functional form Eq. (2.10) can be inverted to yield the relationship

$$Z^* = F_\ell^{-1}(H) = \psi(H) \tag{2.11}$$

One may carry the formalism one step further to the point of determining the pdf for the character error rate, CER. This quantity is perhaps more useful to the communications engineer than the BER. We can represent the functional relationship between CER and BER by the equation

$$\text{CER} \overset{\Delta}{=} X = G(Z) = \text{function of BER} \tag{2.12}$$

For example, if we assume 8-bit ASC II characters we have the result

$$X = 1 - (1 - Z)^8 = G(Z) \tag{2.13}$$

Although the upper limit of the BER is 0.5, the upper limit of X is near unity, as can be seen by substituting Z = 0.5 in Eq. (2.13). For Z = 0.5 Eq. (2.13) yields a value of $X_{max} = 0.996$. For all practical purposes we can approximate this upper limit as unity.

The pdf for the CER, denoted as $\tilde{f}_\ell(X)$ is deduced from f(Z) through the relationship:

$$\tilde{f}_\ell(X)dX = f_\ell(Z)dZ \tag{2.14}$$

Using Eq. (2.13) we have (for 8-bit ASCII characters):

$$Z = 1 - (1 - X)^{1/8} , \tag{2.15}$$

which then gives:

$$\tilde{f}_{\ell}(X) = \left(\frac{1}{8(1-X)^{7/8}}\right) f_{\ell}\left(Z = 1 - (1-x)^{1/8}\right) \tag{2.16}$$

The probability that the CER is less than or equal to X is given by:

$$\tilde{F}_{\ell}(X) = \int_0^X \tilde{f}_{\ell}(X')dx' \tag{2.17}$$

Let us now extend the previously developed ideas to the nuclear-stressed case. Consider initially the situation where only the propagating medium is affected by the detonation. Using nuclear weapons phenomenology computer programs such as WEPH VI in conjunction with propagation programs such as WEDCOM (VLF/LF), NUCOM (HF), WESCOM (UHF/SHF SATCOM), etc., one determines the time behavior of the median SNR for a particular link. The median SNR so determined is assumed to be deterministic.* As such, the pdf for the SNR is still given by Eq. (2.4) with $\mu_y(t)$ now determined from the solution of the nuclear propagation codes. Figure 2-3 shows the qualitative behavior of the median CERs for a broad spectrum of radio links. The median BER is determined from Eq. (2.6a), namely $Z_m = \Gamma(y_m)$, with the median CER then given by: $X_m = G(Z_m)$.

From a theoretical viewpoint, there is no change in the interpretation of $F_{\ell}(Z)$ or $\tilde{F}_{\ell}(X)$ vis-a-vis the ambient case. Thus, the predicted probability that a particular link will support a BER/CER less than certain specified amounts is still given by either Eq. (2.9) or Eq. (2.17) respectively.

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* Attempts[4,5] have been made to account for uncertainties in the prediction of the median-generated SNR. Investigations of this type are still in progress.

Using Fig. (2-1b) we now address the next level of complication, that being the case where the nuclear detonation affects both the link and the transmitter/receiver. Based on a combination of experimental data and theoretical analysis it is assumed that nodes such as a transmitter and receiver have a non-deterministic response to EMP. That is, one assigns a probability-of-survival for the nodes. For example, in the model of Fig. (2-1b) one may define P_T and P_R to be the probabilities-of-survival for the transmitter and receiver, respectively.

If we once again pose the question - "what is the probability that the end-to-end (transmitter-to-receiver) CER is less than X?," we must now account for the possibility that either or both the transmitter and receiver will be out of commission. The CER distribution function, $\tilde{F}(X,t)$ in this case can be written in the following form:

$$\tilde{F}(X,t) = (P_T(t)P_R(t))\tilde{F}_{\ell}(X,t) + U(1-X)(1-P_T(t)P_R(t)) \tag{2.18}$$

where U is the step function. The rationale for the above construct will now be explained.

The first term in Eq. (2.18), which also shows the explicit time dependence of P_T and P_R, is the only physically interesting one. The factor P_TP_R takes into account the obvious physical requirement that in order to receive any message at all, both transmitter and receiver must be operating. The interpretation of $\tilde{F}_{\ell}(X,t)$ is the same as before. However, the first term alone is not sufficient to define a proper probability distribution function because it does not satisfy the normalization condition, which requires that:

$$\int_0^1 \tilde{F}(X,t)dX = 1 \tag{2.19}$$

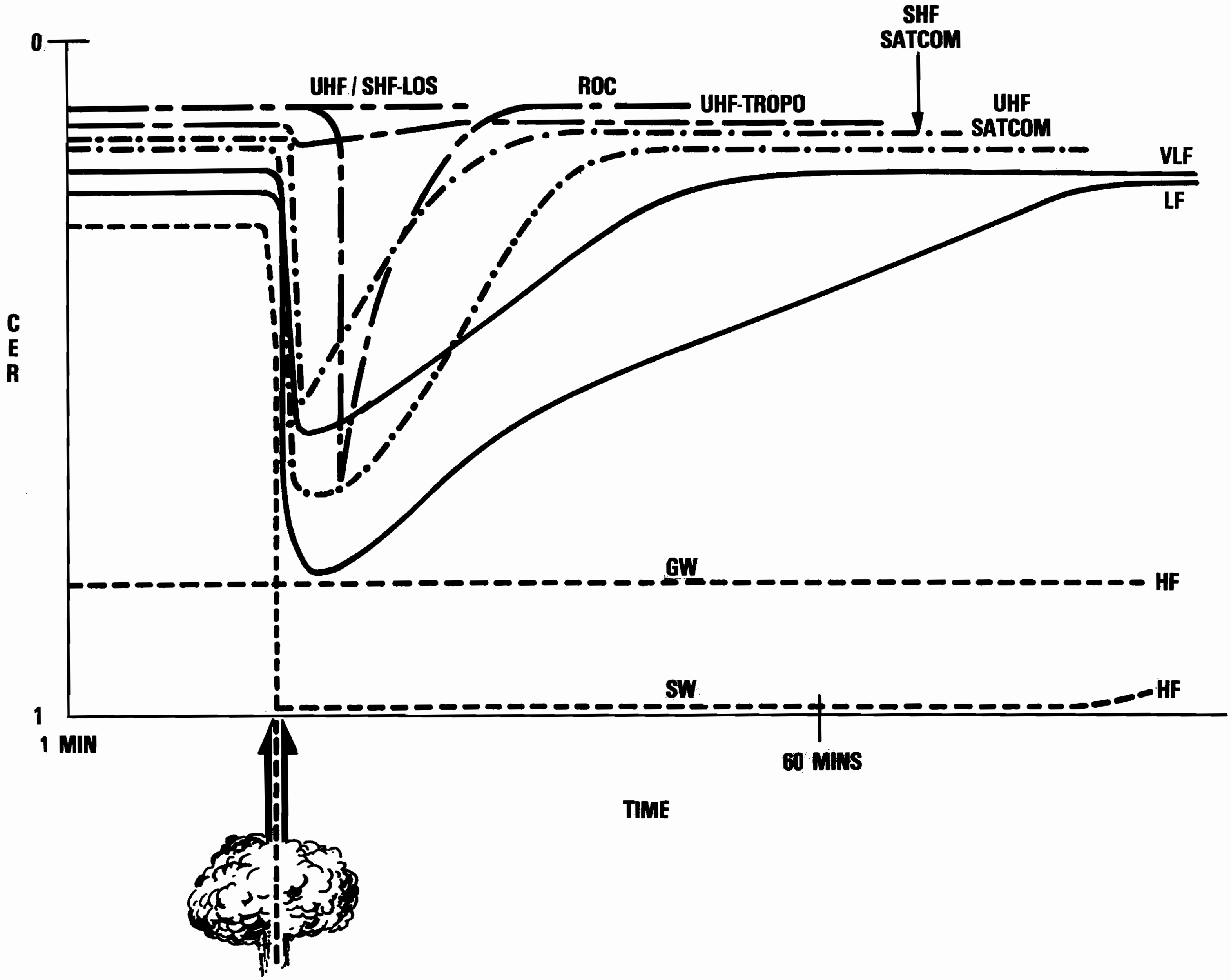


FIG. 2-3 QUALITATIVE BEHAVIOR OF MEDIAN CER FOR VARIOUS RADIO LINKS IN A NUCLEAR ENVIRONMENT

Since $\tilde{F}_\ell(X,t)$ is a properly normalized distribution function it follows that

$$\int_0^1 P_T P_R \tilde{F}_\ell(X,t)dx = P_T P_R \leq 1 \tag{2.20}$$

for nuclear-stressed conditions. In order to render $\tilde{F}(X,t)$ a proper distribution function, it is necessary to make a statement about the CER when the transmitter/receiver are down. We assume the failure of either or both of these nodes produces a CER of unity. This appears quite reasonable considering the discussion surrounding the upper limit of X as deduced from Eq. (2.13). Since no discernible signal is received when the transmitter/receiver malfunctions it follows that Z = 1/2 in this case. Inserting this value into Eq. (2.13) gives $X_{max} = 0.996 \simeq 1$. From a mathematical viewpoint this end-point singularity is taken into account by including a step function term at the value X = 1. Figure 2-4 shows a representative distribution function corresponding to Eq. (2.18).

Analogous to Eq. (2.18) the distribution function for the BER is given by:

$$F(Z,t) = P_T P_R \tilde{F}_\ell(Z,t) + U(\frac{1}{2} - Z)(1 - P_T P_R) \tag{2.21}$$

The interpretation of F(Z,t) parallels that for $\tilde{F}(X,t)$.

For values of X less than unity, which is always the case of physical interest, $\tilde{F}(X,t)$ is given by:

$$\tilde{F}(X,t) = P_T P_R \tilde{F}_\ell(X,t) \tag{2.22}$$

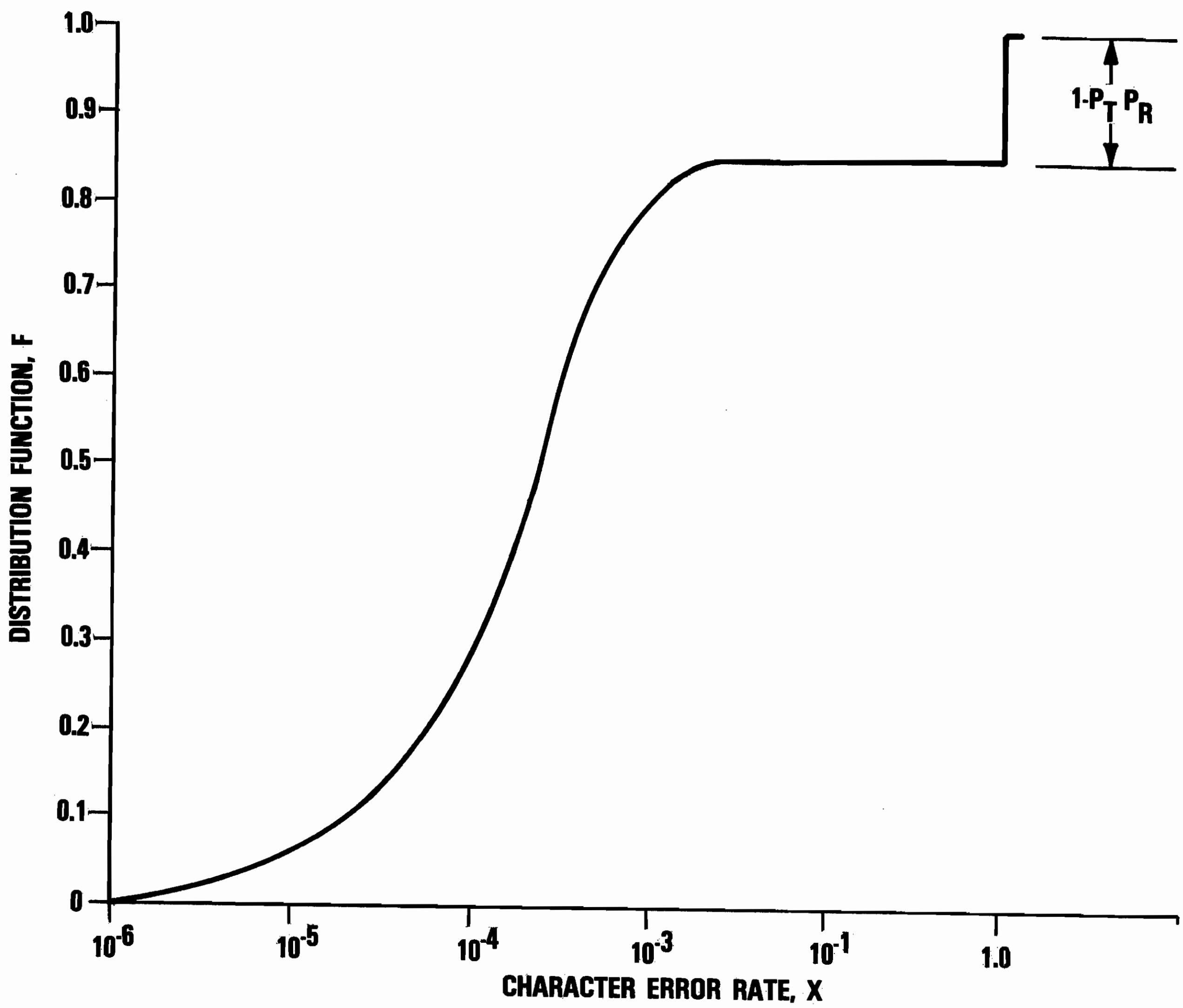


FIG. 2-4 GENERIC MODEL OF DISTRIBUTION FUNCTION VS. CHARACTER ERROR RATE IN NUCLEAR-STRESSED CASE

BY CONVENTION ONLY, $\tilde{F}(X,t)$ can be interpreted as time availability. In the absence of nodal considerations ($P_T = 1$, $P_R = 1$) one may more realistically interpret $\tilde{F}(X,t)$ as time availability if we assume that the median SNR is completely deterministic (cf. previous discussion). In this case one operates in the realm of predicted time availability vis-a-vis that which is based on experimental observations.

If $\tilde{F}(X,t)$ is the fraction of time that the CER is less than X, then

$$1 - \tilde{F}(X,t) = \text{fraction of time that the CER is greater than X} \tag{2.23}$$

For communication systems design it is frequently desirable to know the CER, X*, which is exceeded a certain percentage, r, of the time. This is determined from the equation

$$1 - \tilde{F}(X^*,t) = \frac{r}{100} = f_r \tag{2.24}$$

Figure 2-5 shows the time behavior of X* for three values of r for a particular link evaluated in GTE's APACHE study.

Inserting Eq. (2.22) into Eq. (2.24) gives

$$1 - P_T P_R \tilde{F}_\ell(X,t) = f_r \tag{2.25}$$

or equivalently

$$\tilde{F}_\ell(X^*,t) = \left(\frac{1 - f_r}{P_T P_R}\right) \tag{2.26}$$

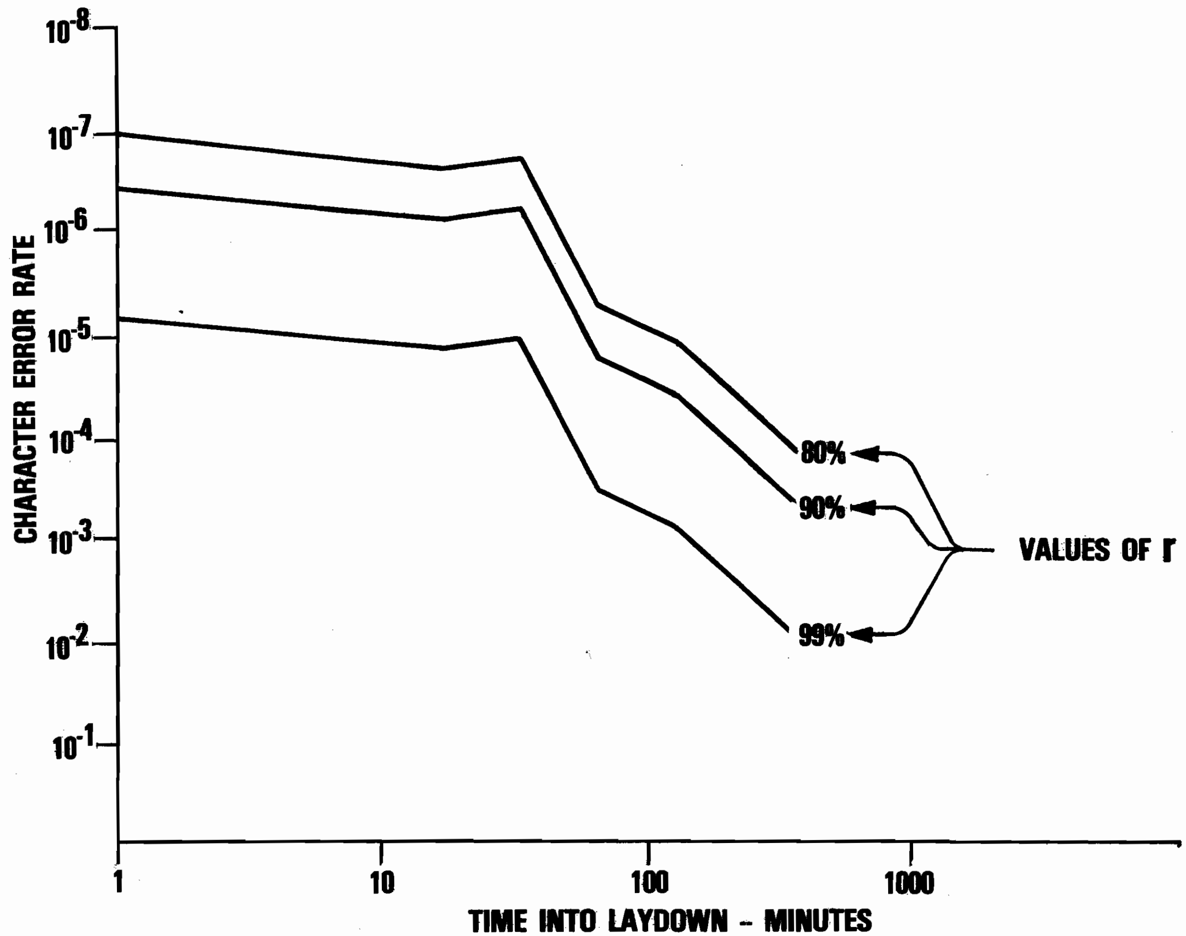


FIG. 2-5 CHARACTER ERROR RATE TIME HISTORY IN A NUCLEAR SCENARIO

Since the largest value of $\tilde{F}_\ell(X,t)$ is unity, Eq. (2.26) is observed to require

$$f_r \geq 1 - P_T P_R \tag{2.27}$$

Physically, Eq. (2.27) shows that for CERs less than unity there is a limit to the transmission capability of the system because of nodal failure.

3.0 Performance Of A Network Involving A Serial Combination Of Links And Nodes*

Figure (3-1a) shows a physical model of a communications system involving a serial combination of nodes and links, while Fig. (3-1b) shows its mathematical representation. This configuration is typical of the class of networks consisting of a series of "M" nodes and "M-1" links (cf. Fig. 3-2). The analyses of such networks are important not only in their own right but because they form the building blocks of larger, more complex networks such as the one shown in Fig. (1-1).

Let us consider a network consisting of M nodes and (M-1) links, and let each link have the capability of introducing a character error into the system. We define

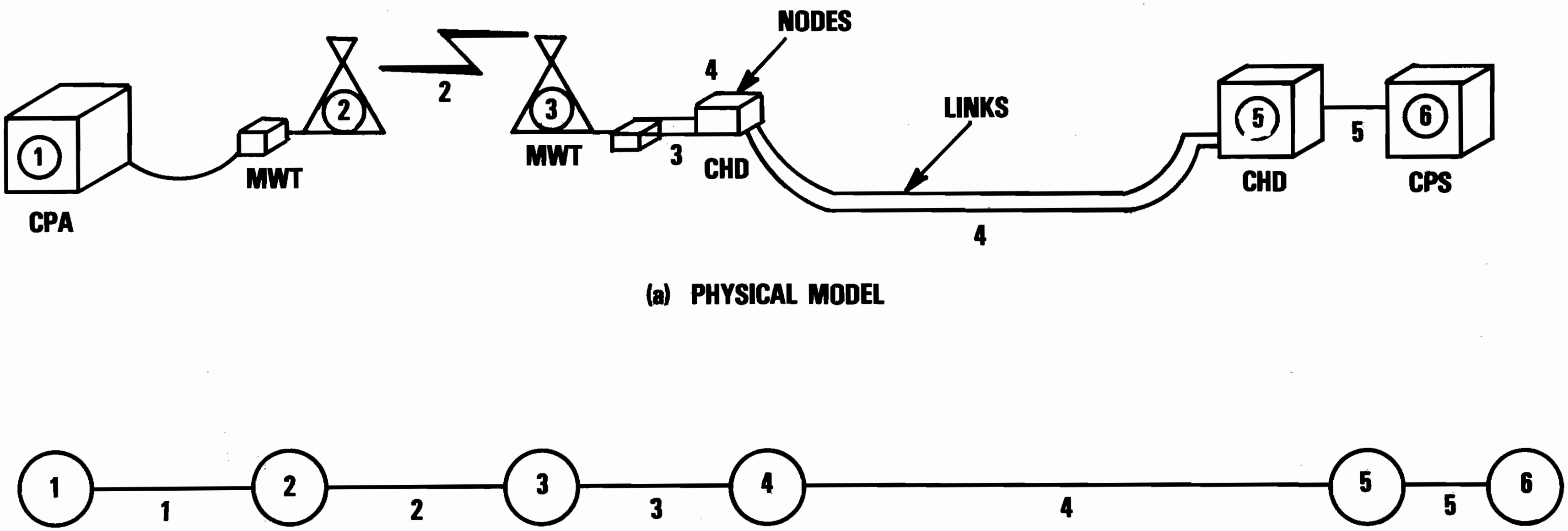
$$X_i = \text{CER for link } i \, (1 \leq i \leq M - 1) \qquad \text{(a)}$$

$$\tilde{f}_i(X_i)dX_i = \text{differential probability that the CER for link } i \text{ lies in the range } X_i \text{ to } X_i + dX_i \qquad \text{(b)} \qquad (3.1)$$

The foregoing definition for $\tilde{f}_i(X_i)$ is applicable for both the benign and nuclear-stressed cases, the only difference being in the choice of the model used for the median SNR (cf. previous discussion).

Initially, let us assume that the nodes are operating perfectly; and proceed to compute the CER at the terminating node (e.g., node 6 in Fig. 3-1). If X_i is the probability of a character error for link i then $(1 - X_i)$ is the probability of end-to-end error-free transmission, with X being the terminating CER. We assume that the terminal end-to-end error-free probability is the product of the individual ones. Thus, for a path consisting of (M-1) links in series we have:

* An earlier version of this model was formulated in Reference 6.



(a) PHYSICAL MODEL

(b) EQUIVALENT MATHEMATICAL REPRESENTATION

FIG. 3-1 COMMUNICATION SYSTEM INVOLVING SERIAL COMBINATION OF NODES AND LINKS

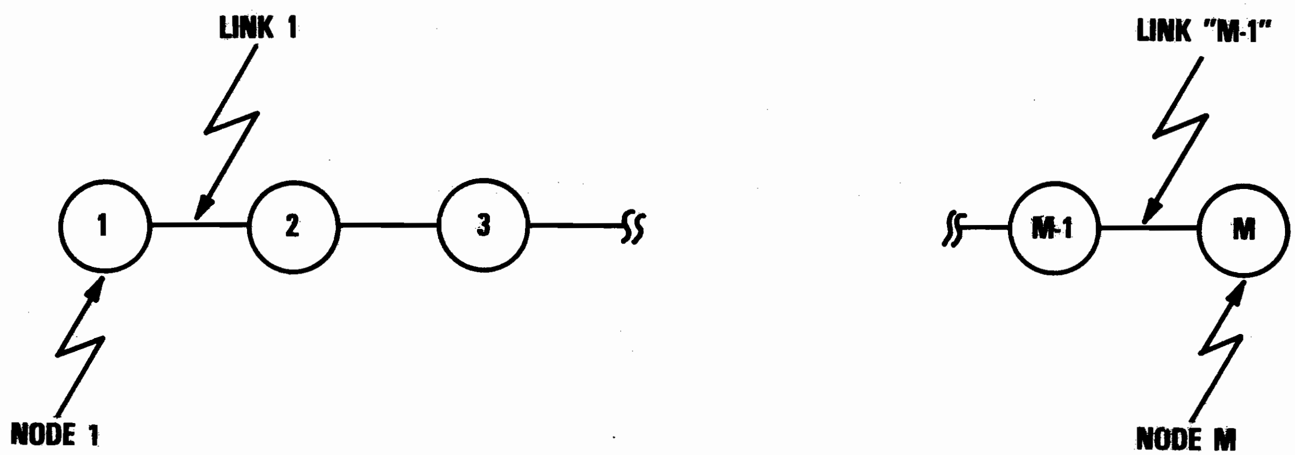


FIG. 3-2 SERIAL COMBINATION OF "M" NODES AND "M-1" LINKS

$$1 - X = \prod_{i=1}^{M-1} (1 - X_i) \qquad (3.2)$$

Equation (3.2) is based on the neglect of character error correction. In this respect the model is conservative as it will predict a lower bound on end-to-end network performance. A similar equation can be written for the end-to-end BER, namely,

$$1 - Z = \prod_{i=1}^{M-1} (1 - Z_i) \qquad (3.3)$$

which is applicable[7,8] in the range of small Z_i. The reason that Eq. (3.2) is not necessarily restricted to small values of X_i, whereas Eq. (3.3) requires small values of Z_i can be traced to the intrinsic relationship between BER and CER, as for example given by Eq. (2.13). That is, the probability for a random correction of a character is much less than that for a bit.

For the remainder of this section we shall be concerned with the determination of $\tilde{f}_L(X)$, the pdf for the end-to-end CER. To the extent that Eq. (3.3) is applicable, the methodology can be applied to compute $f_L(Z)$, the end-to-end pdf for BER.

Since each X_i is a random variable so is the end-to-end value, X. The pdf for X is determined as follows. We first take the natural logarithm of both sides of Eq. (3.2). There results:

$$\ell n(1 - X) = \sum_{i=1}^{M-1} \ell n(1 - X_i) \qquad (3.4)$$

If we now define:

$$Q_i = \ell n(1 - X_i) \tag{3.5}$$

then Eq. (3.4) becomes:

$$Q = \sum_{i=1}^{M-1} Q_i \tag{3.6}$$

The pdf for Q_i is determined from the transformation law:

$$\hat{f}_i(Q_i)dQ_i = \tilde{f}_i(X_i)dX_i \tag{3.7}$$

Using Eq. (3.5) we have:

$$X_i = 1 - e^{-Q_i} \qquad \text{(a)}$$

$$\frac{dQ_i}{dX_i} = \left(\frac{1}{1 - X_i}\right) = e^{Q_i} \qquad \text{(b)} \tag{3.8}$$

Substituting Eq. (3.8) into Eq. (3.7) gives:

$$\hat{f}_i(Q_i) = \tilde{f}_i(X_i = 1 - e^{-Q_i})e^{-Q_i} \tag{3.9}$$

The computation of the pdf for Q, $\hat{f}(Q)$, follows from straightforward application of the laws of probability(3). Using a two-link path gives

$$Q = Q_1 + Q_2 \tag{3.10}$$

The pdf for Q in this case is given by the convolution integral[3]

$$\hat{f}(Q) = \int_0^Q \hat{f}_1(Q - Q')\hat{f}_2(Q')dQ \qquad (3.11)$$

The foregoing integral can be evaluated in terms of the pdfs for the CERS $\tilde{f}_1$, $\tilde{f}_2$ using the following technique. From the transformation law for pdfs we can write:

$$\hat{f}(Q)dQ = \tilde{f}_L(X)dX \qquad (a)$$

$$\hat{f}_2(Q')dQ' = \tilde{f}_2(X')dX' \qquad (b) \qquad (3.12)$$

where

$$Q' = -\ln(1 - X') \qquad (a)$$

$$Q = -\ln(1 - X) \qquad (b) \qquad (3.13)$$

Using Eq. (3.13b) then gives:

$$\hat{f}(Q) = (1 - X)\tilde{f}_L(X) \quad , \qquad (3.14)$$

where $\tilde{f}_L(X)$ is yet to be evaluated.

Finally, we introduce the variable

$$Q^* = Q - Q' \qquad (3.15)$$

and consider the pdf defined by the relationship

$$\tilde{f}_1(\theta)d\theta = \hat{f}_1(Q^*)dQ^* \qquad (3.16)$$

where

$$Q^* = -\ell n(1 - \theta) \tag{3.17}$$

$\tilde{f}_1(\theta)$ is the pdf for the CER of link 1 expressed as a function of the variable, θ.

There results

$$\hat{f}_1(Q^*) = (1 - \theta)\tilde{f}_1(\theta) \tag{3.18}$$

Since

$$\theta = 1 - e^{-Q^*} \tag{3.19}$$

we obtain

$$\theta = 1 - e^{-(Q - Q')} \quad \text{(a)}$$

$$= 1 - e^{-Q}e^{Q'} \quad \text{(b)} \tag{3.20}$$

Making the substitutions

$$e^{-Q} = 1 - X \quad \text{(a)}$$

$$e^{Q'} = \left(\frac{1}{1 - X'}\right) \quad \text{(b)} \tag{3.21}$$

then yields

$$\theta = 1 - \frac{(1 - X)}{(1 - X')} = \frac{X - X'}{(1 - X')} \tag{3.22}$$

and

$$(1 - \theta) = \frac{(1 - X)}{(1 - X')} \tag{3.23}$$

Substituting Eqs. (3.14) and (3.18) into Eq. (3.11) then gives:

$$(1 - X)\tilde{f}_L(X) = \int_0^X \frac{(1 - X)}{(1 - X')} \tilde{f}_1\left(\frac{X - X'}{1 - X'}\right)\tilde{f}_2(X')dX' \tag{3.24}$$

Since (1 - X) can be brought outside the integral sign we obtain the final result

$$\tilde{f}_L(X) = \int_0^X \left(\frac{1}{1 - X'}\right) \tilde{f}_1\left(\frac{X - X'}{1 - X'}\right) \tilde{f}_2(X')dX' \quad , \tag{3.25}$$

which is valid in the range $X < 1$. In functional form we can write

$$\tilde{f}_L(X) = \tilde{f}_1 \hat{\otimes} \tilde{f}_2 \tag{3.26}$$

where the convolution sign, $\hat{\otimes}$, stands for the irregular convolution integration given by Eq. (3.25).

If we now return to the general expression of Eq. (3.6) it readily follows that for (M-1) links in series we have:

$$\tilde{f}_L(X) = \tilde{f}_1 \hat{\otimes} \tilde{f}_2' \hat{\otimes} \ldots . \tilde{f}_{M-1} \tag{3.27}$$

In the ambient environment the CERs are always much less than unity (except in extremely rare cases).

which permits X' to be neglected in comparison to unity in the convolution integral. Thus, for example, Eq. (3.25) becomes

$$\tilde{f}_L(X) = \int_0^X \tilde{f}_1(X - X')\tilde{f}_2(X')dX' \quad , \tag{3.28}$$

which is recognized as the standard convolution integral. Equation (3.28) could have been deduced by noting that in the small X_i range we have:

$$Q_i = -\ell n(1 - X_i) = X_i \tag{3.29}$$

For a nuclear-stressed environment the CERs may indeed become large (cf. Fig. 2-3) so that the irregular convolution integral as given by Eq. (3.25) may be more appropriate.

The probability that the end-to-end CER is less than X is the path distribution function

$$\tilde{F}_L(X) = \int_0^X \tilde{f}_L(X')dX' \quad , \tag{3.30}$$

where $\tilde{f}_L(X')$ is given by Eq. (3.27). Since each of the $\tilde{f}_i(X_i)$s is normalized it is easy to show:

$$\tilde{F}_L(X = 1) = \int_0^1 \tilde{f}_L(X')dX' = 1 \tag{3.31}$$

It is also clear that if a single node in a series of nodes* is down then the communication path is destroyed. Following the same reasoning as for the simple transmitter/receiver model of Section 2, the end-to-end pdf for the network shown in Fig. 3-2 is given by (cf. Eq. 2.18):

---

* The reader should recall that a node as used in this report is an individual functional element.

$$\tilde{F}(X,\vec{P},t) = \left(\prod_{i=1}^{M} P_i(t)\right)\tilde{F}_L(X,t) + U(1 - X)\left(1 - \prod_{i=1}^{M} P_i(t)\right) \qquad (3.32)$$

$$= P^*\tilde{F}_L(X,t) + U(1 - X)(1 - P^*) \qquad (3.33)$$

where

$$P^* = \prod_{i-1}^{M} P_i(t), \qquad (3.34)$$

and $\vec{P}$ stands for the set of all P_is. For subsequent usage we have taken the liberty of explicitly showing the dependence of the performance function, $\tilde{F}$, on $\vec{P}$.

Analogous to the discussion surrounding Eqs. (2.21) and (2.22) we note that for the physically-interesting regime, $X < 1$, the first term in Eq. (3.33) is the only useful part. Thus, for all practical purposes $\tilde{F}(X,\vec{P},t)$ is given by:

$$\tilde{F}(X,\vec{P},t) = P^*\tilde{F}_L(X,t) \qquad (3.35)$$

Examination of Eq. (3.35) leads to the following conclusion:

A SERIES OF NODES AND LINKS CAN BE REPRESENTED BY A SINGLE EQUIVALENT NODE WHOSE PROBABILITY OF SURVIVAL IS THE PRODUCT OF THE INDIVIDUAL PROBABILITIES OF SURVIVAL; AND A SINGLE LINK WHOSE PDF IS DETERMINED BY A SERIES OF CONVOLUTION INTEGRALS INVOLVING THE PDFs FOR THE INDIVIDUAL LINKS.

In order to use the foregoing result in a network involving branch points, it is necessary to develop an appropriate geometric interpretation of the equivalence statement. For example, even in the serial network case we must still retain the end-points if the model is to look realistic. This is accomplished by reducing Fig. 3-2 to the equivalent network of Fig. 3-3. For cosmetic purposes, an arrangement is used in which the interior nodes - 2,3,...,M-1 are combined to yield an equivalent interior node whose probability-of-survival is given by

$$P^{**} = \prod_{i=2}^{M-1} P_i \qquad (3.36)$$

The interior node so deduced is called a "Virtual Node."

Since an interior node must have an input and output link, it is convenient to artificially construct one of the links on either side of the node to be a "perfect link," while the other is the "Equivalent Link."* This is shown in Fig. 3-3. Using this rule the end-to-end performance of the equivalent network of Fig. 3-3 is the same as that of Fig. 3-2.

The use of virtual nodes and equivalent links is found to provide considerable simplification in the analysis of the network by reducing the number of variables (cf. Section 5). Figure 3-4 shows the reduction of a network to a simpler form using virtual nodes and equivalent links. It is to be noticed in this network simplification that the branch points (labeled b1-b8) remain unaffected.

It is also instructive at this point to examine the mathematical properties of $\tilde{F}(X,\vec{P},t)$ in the regime where link degradation effects can be neglected. We define such cases as "hard-link" models. Although the forthcoming discussion of the hard-link case is cast within the framework of the serial node and link model, the concept is of general applicability since many actual networks (particularly those involving cables) exhibit such behavior.

*An alternate way of treating the links would be to retain the first link and group the remaining M-2 links around the equivalent node. Although this would avoid the use of a physically unrealizable perfect link, one would still be left the computational requirement of convolving link #1 with the remaining ones. It is mathematically convenient to use the perfect link technique.

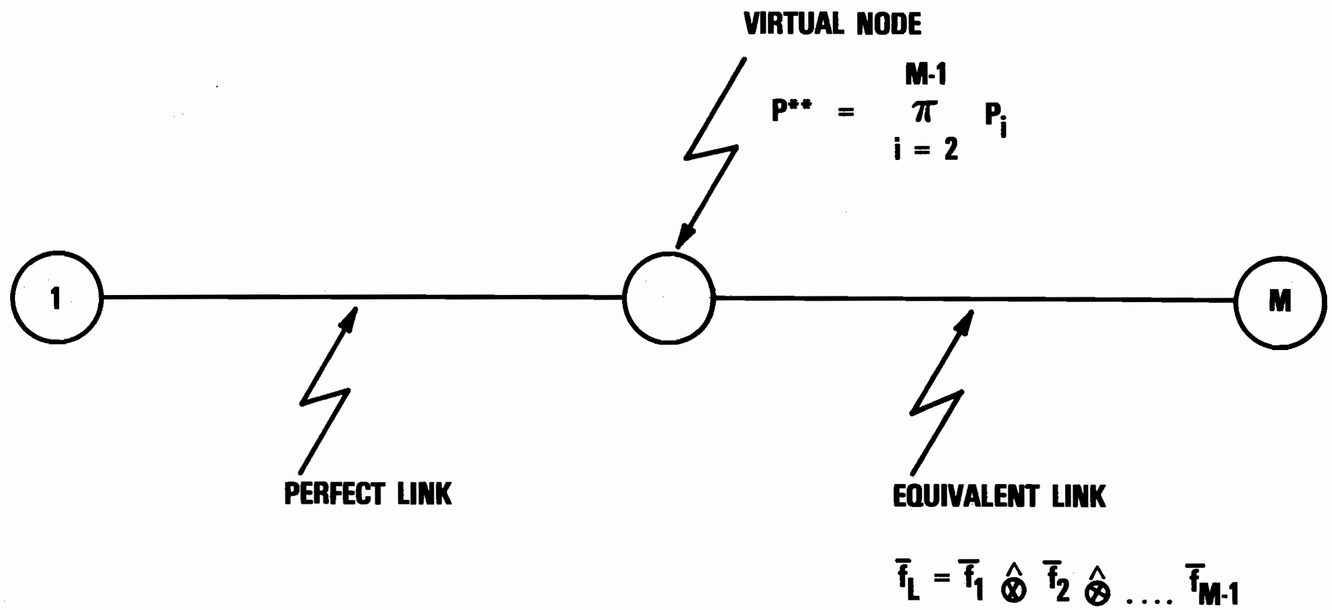


FIG. 3-3 VIRTUAL NODE REPRESENTATION OF SERIAL COMBINATION

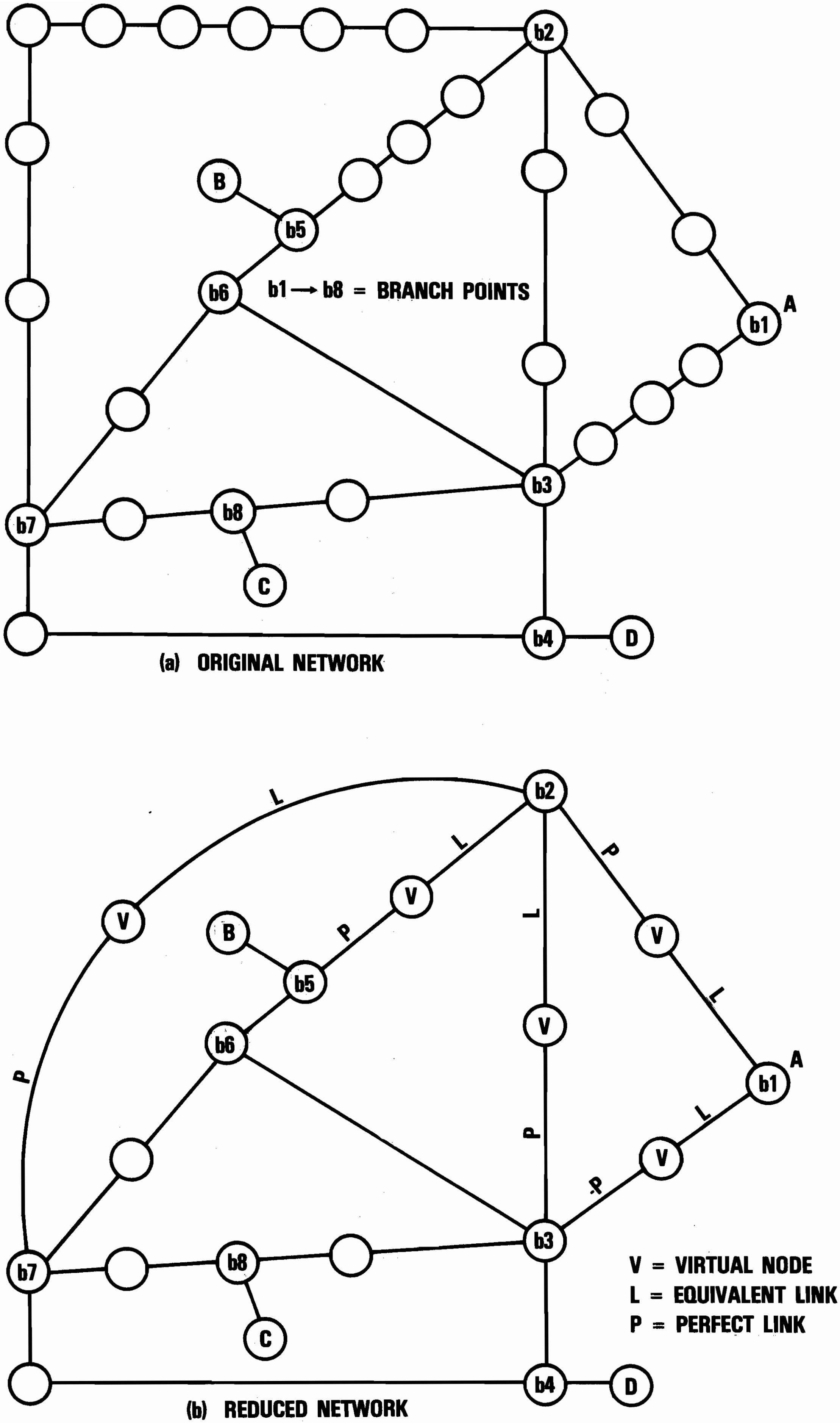


FIG. 3-4 REDUCTION OF NETWORK THROUGH USE OF VIRTUAL NODES AND EQUIVALENT LINKS

The statement that link degradation effects are negligible can be approached with varying degrees of rigor. On one extreme we could require that each pdf for a link be of the form

$$\tilde{f}_i(X) = \delta(X) = \text{delta function at } X \qquad (3.37)$$

Using Eq. (3.38) in Eq. (3.28) then gives

$$\tilde{f}_L(X) = \delta(X) \qquad (3.38)$$

and

$$\tilde{F}_L(X) = U(X) \qquad (3.39)$$

Thus, for any value of $X > 0$ we have $\tilde{F}_L(X) = 1$ and

$$\tilde{F}(X,\vec{P},t) = P* \qquad (3.40)$$

In practice it is not necessary to use the severe approximation of Eq. (3.37) in order to construct a hard-link approximation. Instead, we simply assume that there exists some value $X = X_u$, where X_u satisfies the condition

$$0 < X_u < 1, \qquad (3.41)$$

such that

$$\tilde{F}_L(X_u,t) = 1 - \varepsilon : \varepsilon \to 0 \quad (\varepsilon > 0) \qquad (3.42)$$

Indeed, the upper limit of X_u can be as close to (but always less than) unity so long as Eq. (3.42) applies.

4.0 Concept Of Cost Minimization: Application For Single Path

The purpose of this section is to introduce the concept of cost minimization and show its intrinsic relationship to the measure of performance. For orientation purposes, this is explicitly demonstrated for the relatively simple model of the serial chain of nodes and links (a single path). Indeed, for this case, the analytic technique of Lagrange Multipliers[9] can be used. As indicated in Section 5, the methodology for determining the minimum cost-to-harden a network involving multiple end-point-pairs requires the numerical techniques of non-linear optimization[10,11].

Let us initially consider the communication model of Fig. 2-1. As shown in Eq. (2.22) the performance of this network is given by:

$$\tilde{F}(X,t) = P_T(t)P_R(t)\tilde{F}_\ell(X,t), \qquad (4.1)$$

where it is assumed that the probabilities P_T and P_R are time dependent. An example of such behavior is shown in Fig. 4-1. Initially we have:

$$P_T = 1 \qquad (a)$$

$$P_R = 1 \qquad (b) \qquad (4.2)$$

which means that prior to the detonations, everything is assumed to be working perfectly. For illustrative purposes, the transmitter/receiver are assumed to be far enough so that each is affected by a different nuclear weapon. Thus, for example, P_T drops from unity to 0.8 at time t_1, the detonation time for the first weapon, while P_R drops from 1 to 0.5 at nuclear detonation time t_2^*. If both nodes are influenced by the same event, $t_1 \doteq t_2$. A concept introduced into the survivability picture is that nodes

*Based on experimental observations, it is noted (cf. Ref 12) that some nodes can recover within a certain time interval, τ. The length of τ depends upon the particular node and the type of degradation caused by EMP[12]. In the event that the node never recovers, we set $\tau = \infty$.

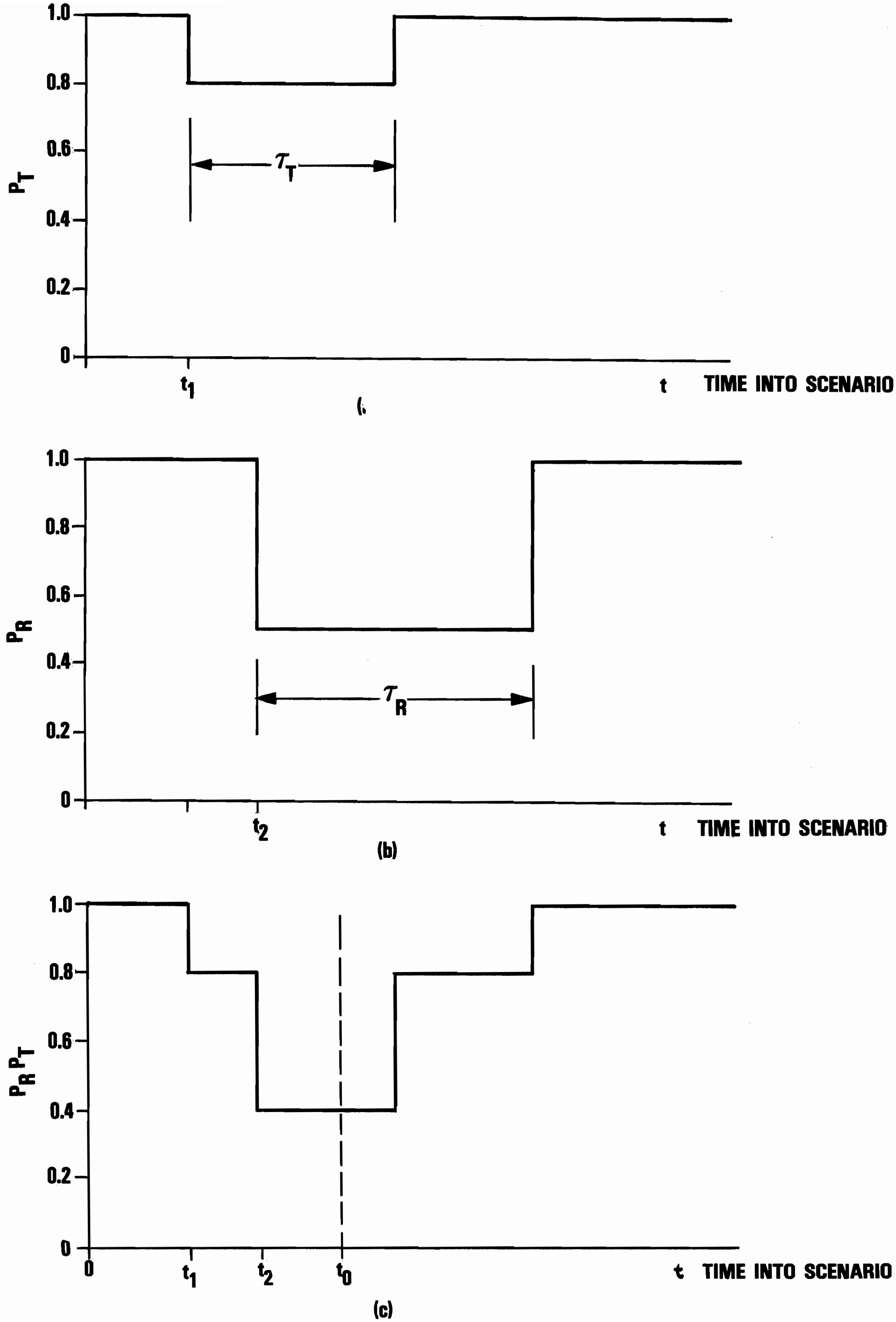


FIG. 4-1 EXAMPLE OF BEHAVIOR OF P_T AND P_R

can recover; τ_T, τ_R are the recovery times for the transmitter and receiver respectively. Combining the results from Figs. 4-1a, 4-2b gives the result shown in Fig. (4-1c).

Suppose, from an operational viewpoint we require that at time t_o following the onset of detonations, a certain time availability, T_a, with a maximum allowable CER given by X_o (we naturally assume $X_o < 1$). Using Eq. (4.1) then yields

$$\tilde{F}(X_o,t_o) = P_T(t_o)P_R(t_o)\tilde{F}_\ell(X_o,t_o) \geq T_a, \quad (4.3)$$

or equivalently

$$P_T(t_o)P_R(t_o) \geq \frac{T_a}{\tilde{F}_\ell(X_o,t_o)} \quad (4.4)$$

If the value $(T_a/\tilde{F}_\ell(X_o,t_o))$ is the larger than P_TP_R it means that the performance requirements cannot be met for the nodal-behavior shown in Fig. 4-1. On the other hand it is deemed possible - given enough financial resources - to increase the value $P_T(t_o)P_R(t_o)$ so that the equality condition is met. Of course, should the value $(T_a/\tilde{F}_\ell(X_o,t_o))$ be greater than unity, no solution to Eq. (4.4) is possible.

Defining

$$\Gamma_o = \frac{T_a}{\tilde{F}_\ell(X_o,t_o)} , \quad (4.5)$$

we seek to find the minimum cost which will satisfy the condition

$$P_TP_R = \Gamma_o , \quad (4.6)$$

with the equality sign being appropriate[9]. We assume that the transmitter and receiver are each characterized by cost functions $C_T(P_T)$, $C_R(P_R)$, which are the costs required to harden the respective facilities against EMP to probability-of-survival levels P_T, P_R respectively.*

The total hardening cost is given by:

$$C_N = C_T(P_T) + C_R(P_R) \tag{4.7}$$

For the simple two-node case under consideration it is easy to determine the minimum value of C_N, and the corresponding values of P_T and P_R at the minimum which will satisfy Eq. (4.6). Substituting for P_T from Eq. (4.6) in Eq. (4.7) gives:

$$C_N = C_T\left(\frac{\Gamma_o}{P_R}\right) + C_R(P_R) \tag{4.8}$$

Solving the equation

$$\left(\frac{dC_N}{dP_R}\right) = 0 \tag{4.9}$$

---

* The development of the cost functions is a rather complex process involving detailed knowledge of the equipment in the facility, and the hardening know-how. SRI has performed such an evaluation for DNA under the APACHE program,[13] although refinement of their results (toward lower costs) seems appropriate. For the purposes of this analysis it suffices to say that one can ascribe a cost function to any node in the system. Since it obviously costs more money to harden a node to a higher probability-of-survival level, the Cs are monotonically increasing functions of the Ps; that is;

$$\frac{\partial C}{\partial P} \geq 0$$

for all nodes.

yields the value of P_R at the minimum cost, $P_{R,min}$. The corresponding value of P_T is given by:

$$P_{T,min} = \frac{\Gamma_o}{P_{R,min}} \quad , \tag{4.10}$$

while the minimum cost is given by:

$$C_{N,min} = C_T(P_{T,min}) + C_R(P_{R,min}) \tag{4.11}$$

It is also possible to analytically determine the minimum cost for hardening a serial chain of nodes and links, consistent with maintaining a certain level of performance. The problem is of interest for orientation purposes - as it demonstrates the transition between the two-node case and that for many variables; and also shows how to develop an equivalent cost function for the virtual node. This latter concept is useful in analyzing the general network problem.

Consider the network of Fig. 3-2, whose end-to-end pdf for the CER is given by Eq. (3.32). It is desired to find the total minimum cost to harden this network subject to the constraint:

$$\left(\prod_{j=1}^{M} P_j\right)\tilde{F}_L(X_o) \geq T_a \quad , \tag{4.12}$$

where the implicit dependence on time has been suppressed. The total cost is given by:

$$C_N = \sum_{j=1}^{M} C_j(P_j) \quad , \tag{4.13}$$

with the C_js being monotonically increasing functions of the Ps. In this case, the equality sign of Eq. (4.12) applies. Taking the logarithm of this equation gives:

$$\sum_{j=1}^{M} \ln P_j = \ln \beta \tag{4.14}$$

where

$$\beta = \frac{T_a}{\tilde{F}_L(X_o)} \tag{4.15}$$

Following the method of Lagrange Multipliers we introduce the function

$$\Phi(P_1, P_2, \ldots. P_M, \beta) = \sum_{j=1}^{M} \ln P_j - \ln \beta = 0 \tag{4.16}$$

The auxiliary function which now is to be minimized is[9]:

$$\tilde{C} = C_N - \lambda \Phi \tag{4.17}$$

where

$$\lambda = \text{Lagrange Multiplier} \tag{4.18}$$

If $\tilde{C}$ is to be a minimum then we require that the first variation of $\tilde{C}$ with respect to all the P_j's vanishes. We have:

$$\delta \tilde{C} = \sum_{j=1}^{M} \frac{\partial \tilde{C}}{\partial P_j} \delta P_j = 0 \tag{4.19}$$

Using Eq. (4.17) yields:

$$\sum_{j=1}^{M} \left(\frac{\partial C_N}{\partial P_j} - \lambda \frac{\partial \Phi}{\partial P_j}\right) \delta P_j = 0 \qquad (4.20)$$

where

$$\frac{\partial C_N}{\partial P_j} = \frac{\partial C_j}{\partial P_j} \qquad (4.21)$$

$$\frac{\partial \Phi}{\partial P_j} = \frac{1}{P_j} \qquad (4.22)$$

Inserting Eqs. (4.21) and (4.22) into Eq. (4.20) gives:

$$\sum_{j=1}^{M} \left(P_j \frac{\partial C_j}{\partial P_j} - \lambda\right) \delta P = 0 \qquad (4.23)$$

The solution for Eq. (4.23) requires that the coefficient for each P_j vanish. We thus have:

$$\underline{j = 1,2,\ldots M} \; : \quad P_j \frac{\partial C_j}{\partial P_j} = \lambda \qquad (4.24)$$

which represent M equations. Combining these M equations with the constraint equation

$$\Phi(P_1,P_2,\ldots P_M,\beta) = \sum_{j=1}^{M} \ell n P - \ell n \, \beta = 0 \qquad (4.25)$$

gives (M + 1) equations in the (M + 1) variables P_1, P_2, P_M, λ. The procedure for solving for P_j is straightforward. For example, $C_j(P_j)$ is a function only of P_j. Thus, each solution of Eq. (4.24) is of the form:

$$P_j = \psi_j(\lambda) \qquad (4.26)$$

Alternatively,

$$\ell n P_j = \ell n \ \psi_j(\lambda) \tag{4.27}$$

Inserting Eq. (4.27) into Eq. (4.25) gives:

$$\sum_{j=1}^{M} \ell n \ \psi_j(\lambda) = \ell n \ \beta \tag{4.28}$$

Since the ψ_j's are known, Eq. (4.28) is a solution for λ in terms of β. Let us denote the solution of Eq. (4.28) by the relationship:

$$\lambda = \lambda^*(\beta) \tag{4.29}$$

Equation (4.29) gives the value of the Lagrange Multiplier at the minimum total cost. The corresponding values of P_j at this minimum are determined from Eq. (4.26). We have:

$$P_j^* = \psi_j(\lambda^*) \tag{4.30}$$

and the minimum cost for hardening the system subject to the constraint of H_o and X_o is given by:

$$C_{N,min} = \text{minimum cost} = \sum_{j=1}^{M} C_j(P_j^*) \tag{4.31}$$

An academic example of this technique applied to simple analytic form for $C_j(P_j)$ is rendered in Appendix A.

In retrospect what has actually been accomplished is to find the minimum cost for hardening a chain of nodes whose P_js satisfy the condition

$$\prod_{j=1}^{M} P = \beta \ , \tag{4.32}$$

where β is an arbitrary number between 0 and 1. Indeed, by examining Eqs. (4.29) - (4.31) we note that we have been able to express the minimum cost $C_{N,min}$ as a function of the variable β. That is,

$$C_{N,min} = C_{N,min}(\beta) \tag{4.33}$$

Physically, β is the resulting probability-of-survival for the serial chain of M nodes, and can assume any value between 0 and 1. It therefore follows that the virtual node of Fig. 3-3 can be described by an equivalent cost function, $C_{min}(P^{**})$, which is the minimum cost required to harden the chain of nodes i = 2 to i = M-1 to net probability-of-survival, P**. An example of this is presented in Reference 2.

5.0 Performance And Cost-To-Harden Large Networks Involving Redundant Paths

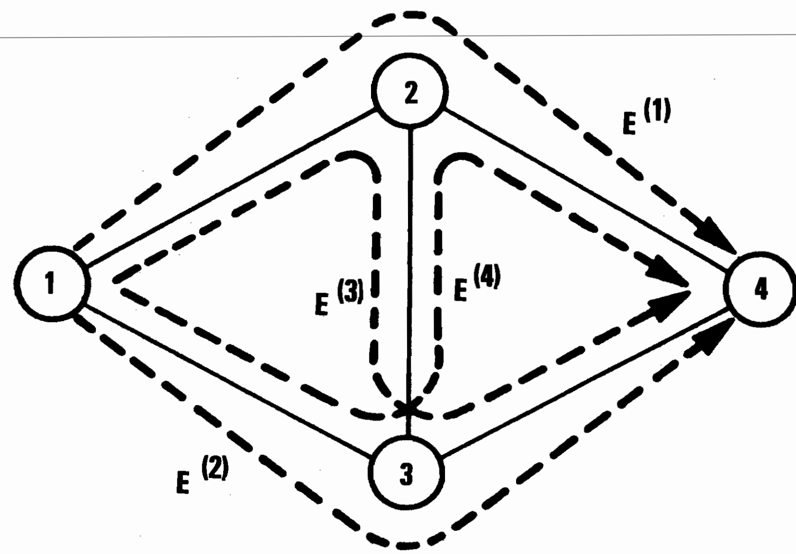
In the previous sections we established the conceptual basis for relating the performance and cost-to-harden in a single-path network; that is, one which did not involve redundant paths. We now show how these ideas can be extended to the general network case. Here again, we choose to proceed from the simple to the more complex. With this in mind we first consider the so-called "hard-link" model in Section 5.1. This case is one in which we neglect degradation of the propagation links. Situations like this may occur, for example, in networks where the nodes are connected by hardened cable systems, or any other links which are not affected by nuclear detonations. Subsequently, in Section 5.2 we determine $\tilde{F}(X,\vec{P},t)$ for networks involving both nodal susceptability and links whose performance can be degraded by nuclear detonations.

5.1 Hard-Link Networks

In Section 3 we introduced the concept of a hard-link network; namely one for which link degradation effects are negligible. For this type of network we showed that for a serial chain of nodes, the performance function, as given by Eq. (3.40), is only a function of the P_is. We now consider the deduction of $\tilde{F}$ for all networks in which the P_is alone determine the communications capability of the system. The determination of the performance function and cost-to-harden for this class of networks is not only important in its own right, but in addition provides insight into the theoretical development of the general case considered in Section 5.2.

5.1.1 Theoretical Formulation Of Performance Function

For orientation purposes, let us initially consider the bridge circuit model shown in Fig. 5-1 in which we seek to determine the probability that the CER between nodes 1 and 4 is less than X. That is, we desire to calculate $\tilde{F}(\vec{P},X,t)$ for all paths connecting nodes 1 and 4. Since by hypothesis



$E^{(i)}$ = event that path i occurs

A_j = event that node j is up

$P[A_j]$ = P_j probability-of-survival for node

$F_{14} = P\left[\bigcup_{i=1}^{4} E^{(i)}\right] = P\left[\bigcup_{i=1}^{3} E^{(i)}\right]$

$E^{(1)} = A_1A_2A_4$

$E^{(2)} = A_1A_3A_4$

$E^{(3)} = A_1A_2A_3A_4$

$E^{(4)} = A_1A_3A_2A_4 = E^{(3)}$

FIG. 5-1 REDUNDANT PATH NETWORK WITH HARD-LINKS FOR BRIDGE CIRCUIT the links are assumed hard, we only consider the effects of node survivability. Thus, $\tilde{F}(\vec{P},X,t)$ reduces to $\tilde{F}(\vec{P})$, where the time dependence is implicit.

In contrast to the intuitive development of Section 3, the deduction of $\tilde{F}(\vec{P})$ in this case requires the use of the formal aspects of probability theory.* Thus, for example, we introduce the event A_j defined by the condition:

$$A_j = \text{event that node } j \text{ is up} \tag{5.1}$$

The probability of survival for the j^{th} node is given by:

$$P_j = P\left[A_j\right] = \text{probability of event } A_j \text{ occurring} \tag{5.2}$$

As observed from Fig. 5-1, there are four possible paths connecting nodes 1 and 4. These are indicated as follows:

Path	Sequence
1	1-2-4
2	1-3-4
3	1-2-3-4
4	1-3-2-4

Now let $E^{(i)}(i = 1,2,3,4)$ be the event that path i occurs, that is communication is achieved along path i. From Fig. 5-1 we readily deduce:

$$E^{(1)} = A_1A_2A_4 \qquad \text{(a)}$$

$$E^{(2)} = A_1A_3A_4 \qquad \text{(b)}$$

$$E^{(3)} = E^{(4)} = A_1A_3A_2A_4 \qquad \text{(c)} \tag{5.3}$$

* This can be found in any standard text in probability theory, for example, reference 3.

For this discussion we assume statistical independence between the nodes (this assumption is easy to modify).

If there were only one path connecting nodes 1 and 4 (e.g., path 1) then $\tilde{F}(\vec{P})$ would be given by:

$$\tilde{F}(\vec{P}) = P\left[E^{(1)}\right] = P\left[A_1 A_2 A_4\right] = P\left[A_1\right] P\left[A_2\right] P\left[A_4\right] \quad \text{(a)}$$
$$= P_1 P_2 P_4 \quad \text{(b)} \qquad (5.4)$$

As to be expected, the foregoing result is the one which would have been deduced using the method of Section 3.

However, for the redundant path case of Fig. 5-1 $\tilde{F}(\vec{P})$ is given the equation:

$$\tilde{F}(\vec{P}) = P\left[\bigcup_{i=1}^{4} E^{(i)}\right] = P\left[E^{(1)} \cup E^{(2)} \cup E^{(3)} \cup E^{(4)}\right] \qquad (5.5)$$

where U stands for the union of events. Let us now apply the foregoing result to the bridge circuit of Fig. 5-1. As a first step we use the relationship

$$E^{(3)} = E^{(4)} \qquad (5.6)$$

which simplifies the calculation.

We thus obtain

$$\tilde{F}(\vec{P}) = P\left[\bigcup_{i=1}^{4} E^{(i)}\right] = P\left[\bigcup_{i=1}^{3} E^{(i)}\right] \qquad (5.7)$$

The general formula for the probability of the union of N events is given by

$$\tilde{F} = P\left[\bigcup_{i=1}^{N} E^{(i)}\right] = \sum_{i=1}^{N} P\left[E^{(i)}\right] - \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} P\left[E^{(i)}E^{(j)}\right]$$

$$+ \sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}\sum_{k=j+1}^{N} P\left[E^{(i)}E^{(j)}E^{(k)}\right] - \dots$$

$$+ \dots\dots (-1)^{N+1}\ P\left[E^{(1)}E^{(2)} \dots\dots E^{(N)}\right] \tag{5.8}$$

Applying Eq. (5.8) for N = 3 gives

$$\tilde{F} = P\left[E^{(1)}\right] + P\left[E^{(2)}\right] + P\left[E^{(3)}\right] - P\left[E^{(1)}E^{(3)}\right] - P\left[E^{(1)}E^{(2)}\right] - P\left[E^{(2)}E^{(3)}\right]$$

$$+ P\left[E^{(1)}E^{(2)}E^{(3)}\right] \tag{5.9}$$

From the additional relationships:

$$E^{(3)} = E^{(1)}\ E^{(2)},\quad E^{(1)}\ E^{(3)} = E^{(3)},\quad E^{(2)}\ E^{(3)} = E^{(3)}, \tag{5.10}$$

it is easy to show that Eq. (5.9) reduces to:

$$\tilde{F} = P\left[E^{(1)}\right] + P\left[E^{(2)}\right] - P\left[E^{(1)}\ E^{(2)}\right] \tag{5.11}$$

Using Eq. (5.3) gives

$$P\left[E^{(1)}\right] = P\left[A_1A_2A_4\right] = P\left[A_1\right]\ P\left[A_2\right]\ P\left[A_4\right] = P_1P_2P_4 \qquad \text{(a)}$$

$$P\left[E^{(2)}\right] = P\left[A_1A_3A_4\right] = P\left[A_1\right]\ P\left[A_3\right]\ P\left[A_4\right] = P_1P_3P_4 \qquad \text{(b)}$$

$$P\left[E^{(1)}E^{(2)}\right] = P\left[A_1A_2A_4A_1A_3A_4\right] = P\left[A_1A_2A_3A_4\right]$$

$$= P\left[A_1\right]\ P\left[A_2\right]\ P\left[A_3\right]\ P\left[A_4\right] = P_1P_2P_3P_4 \qquad \text{(c)} \tag{5.12}$$

In deducing Eq. (5.12) we have used the result

$$A_\kappa A_\kappa = A_\kappa \tag{5.13}$$

for all κ.

Inserting Eq. (5.12) into (5.10) yields

$$\tilde{F} = P_1 P_4 \left[P_2 + P_3 - P_2 P_3 \right] \tag{5.14}$$

The foregoing calculation has shown by example the method which is used to compute the performance of hard-link networks. The deduction of $\tilde{F}(\vec{P})$ between any two endpoints for an arbitrary network is accomplished using the following procedure: First, we identify all the paths connecting the end-point-pair, and thereby construct the events

$$E^{(i)} = A_1^{(i)} A_2^{(i)} \ldots\ldots A_{N_i}^{(i)} = \text{event that path i exists} \tag{5.15}$$

In the foregoing equation N_i is the number of nodes in path i, and $A_j^{(i)}$ is the event that the j^{th} node in the i^{th} path is up. We then compute the function

$$\tilde{F}(\vec{P}) = P\left[\bigcup_{i=1}^{M} E^{(i)} \right], \tag{5.16}$$

where M is the number of paths connecting the end-point-pair. Finally, we use the expansion of Eq. (5.8) and carry-out the compution of the individual terms in accordance with the usual rules of probability theory. The result of the expansion will be a sum of terms involving products of the individual P_is, similar to that given by Eq. (5.14). It should also be noted that none of the terms will ever involve P_i to any power other than unity.

To illustrate the point let us consider the calculation of say

$$P\left[E^{(i)}E^{(k)}\right] , \tag{5.17}$$

terms of which occur in the expansion of Eq. (5.16). Using Eq. (5.15) we have:

$$P\left[E^{(i)}E^{(k)}\right] = P\left[A_1^{(i)}A_2^{(i)} \ldots A_{N_i}^{(i)}A_1^{(k)} \ldots . A_{N_k}^{(k)}\right] \tag{5.18}$$

In view of the derivation of Eq. (5.12c) we note that some of the events indicated in Eq. (5.18) are the same even though they are labeled differently. This results from the fact that the same node(s) may be common to two different paths. However, in view of the result of Eq. (5.13) we note that it is counted only once in the calculation of $P\left[E^{(i)}E^{(k)}\right]$. We thus obtain the following result:*

$$P\left[E^{(i)}E^{(k)}\right] = \prod_{\ell} P_{\ell} \quad , : \quad \ell \in \text{set of all nodes contained in paths } i \text{ and } k, \tag{5.19}$$

The computation of all higher-order terms such as $P\left[E^{(i)}E^{(k)}E^{(j)}\right]$ follows the identical procedure with the resultant product of P_{ℓ} consisting of all nodes contained in paths i,k and j.

5.1.2 Cost-To-Harden With Multiple End Points

For a single end-point-pair, such as the bridge circuit of Fig. 5-1, the minimum cost-to-harden, subject to the constraint that $\tilde{F}(\vec{P})$ is greater than or equal to some value T_a, is found using the techniques of Section 4. We simply minimize the function

$$C_N = \sum_{i=1}^{N} C_i(P_i) \tag{5.20}$$

* The derivation of Eq. (5.19) is based on the assumption of statistical independence (cf. Section 1).

subject to the constraint

$$\tilde{F}(\vec{P}) \geq T_a \tag{5.21}$$

Under normal circumstances the equality sign of Eq. (5.21) will prevail[8]. The summation in Eq. (5.20) extends over all the nodes involved in the connectivity between the end-point-pair.

Now consider the network shown in Fig. 5-2 where it is required that communication between end-point-pairs AD and AG be simultaneously satisfied, and with different performance levels. Using the techniques of the previous section, we readily deduce

$$\tilde{F}_{AG} = P_A P_E P_F P_G + P_A P_B P_F P_G - P_A P_B P_E P_F P_G \geq T_{AG} \quad \text{(a)}$$

$$\tilde{F}_{AD} = P_A P_B P_C P_D \geq T_{AD}, \quad \text{(b)} \tag{5.22}$$

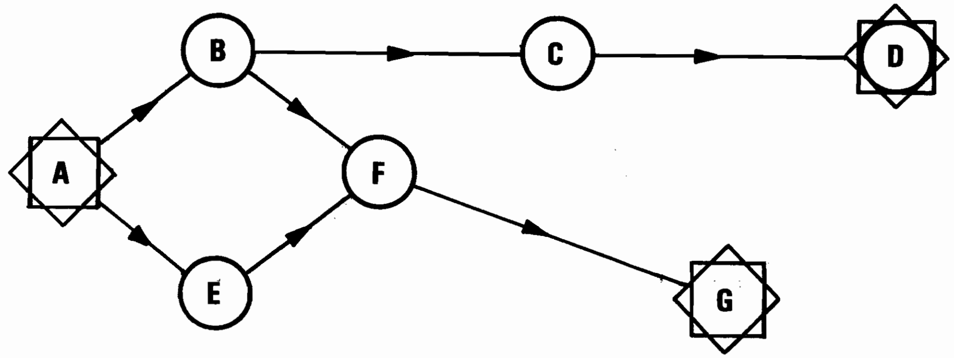
where T_{AG} and T_{AD} are the required performance levels for end-point-pairs AG and AD respectively.

The minimum cost is found by minimizing the function

$$C_N = C_A(P_A) + C_B(P_B) + C_C(P_C) + C_D(P_D) + C_E(P_E) + C_F(P_F) + C_G(C_G) \tag{5.23}$$

subject to the two individual constraints of Eq. (5.22).

In general, the result of the minimization will be such that only one of the two equations of Eq. (5.22) will satisfy the equality condition. To the author's knowledge, the only methods for solving Eq. (5.22) are numerical, and fall in the category of "Nonlinear Optimization Techniques." The method of solution selected for actual networks is the one developed by Dr. R. Holmes[11] of Lincoln Laboratory based on the Box Algorithm[10]. The reader is referred to the references for more detail.



<u>NETWORK CONSTRAINTS</u>:

TIME AVAILABILITY, $\tilde{F}_{AG}$, FOR PATHS A $\longrightarrow$ G MUST BE $\geq T_{AG}$

TIME AVAILABILITY, $\tilde{F}_{AD}$, FOR PATH A $\longrightarrow$ D MUST BE $\geq T_{AD}$

<u>MINIMIZE</u>:

$$C_T = C_A(P_A) + C_B(P_B) + C_C(P_C) + C_D(P_D) + C_E(P_E) + C_F(P_F) + C_G(P_G)$$

SUBJECT TO THE FOLLOWING TWO CONSTRAINTS:

$$\tilde{F}_{AG} = P_A P_E P_F P_G + P_A P_B P_F P_G - P_A P_B P_E P_F P_G \geq T_{AG}$$

$$\tilde{F}_{AD} = P_A P_B P_C P_D \geq T_{AD}$$

FIG. 5-2 COST-TO-HARDEN FOR A MULTIPLY-CONSTRAINED NETWORK (HARD-LINK CASE)

Using the result developed for the model of Fig. 5-2 we can write a general statement concerning the mathematical form of the nonlinear optimization problem, which will observed to be applicable not only to the hard-link case of this subsection but also to the general network problem of Section 5.2. Let R = number of end-point-pairs in a network (for orientation purposes the reader may find it convenient to refer to the network model of Fig. 1-2). Using the techniques of this subsection we evaluate the performance functions, $\tilde{F}_\alpha(\vec{P})$, where α = particular end-point-pair ($\alpha = 1,2,\ldots.R$) and $\vec{P}$ stands for the set of all P_is. The $\tilde{F}_\alpha$s would be of the form given by Eq. (5.22). We now define T_α to be the required time availability between end-point-pair α. From these definitions the general statement for the nonlinear optimization problem is:

$$\text{Minimize } C_N = \sum_{i=1}^{N} C_i(P_i) \tag{5.24}$$

subject to the R constraints:

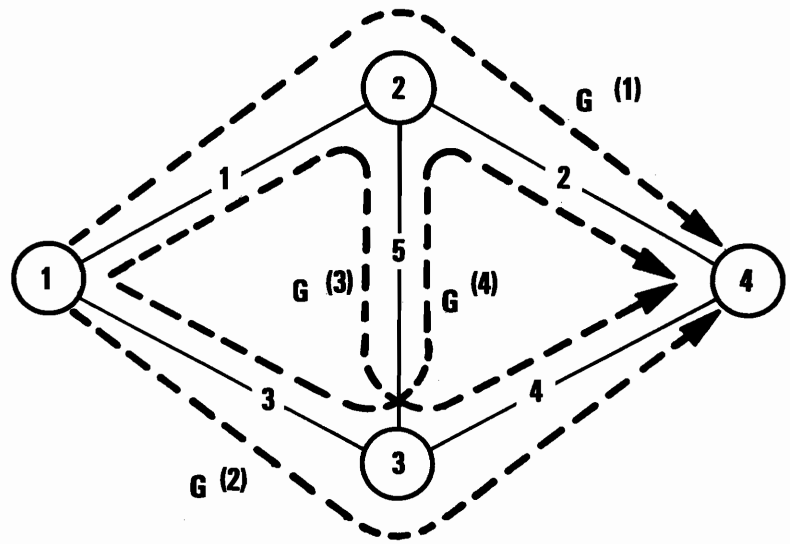
$$\tilde{F}_\alpha(\vec{P}) \geqslant T_\alpha \qquad (\alpha = 1,2,\ldots R) \tag{5.25}$$

5.2 General Network Case

The mathematical development for determining $\tilde{F}(X,\vec{P},t)$ for the general network case is facilitated using a specific model. In order to highlight the differences between the general ("soft-link") case and the hard-link case, let us once again use the bridge circuit model of Fig. 5-1, however, this time assuming that the links have CERs which are not equal to zero. This model is shown on Fig. 5-2.

As in the hard-link case we first deduce the number of paths connecting nodes 1 and 4 which are the end-point-pair of interest. Once again, we assume there are four possible paths. We now let

$$G^{(i)} = \text{event that the CER along path } i \text{ is less than or equal to } X. \tag{5.26}$$



$$G^{(i)} = E^{(i)}W^{(i)}$$

$E^{(i)}$ = EVENT THAT NODES ALONG PATH i ARE UP

$W^{(i)}$ = EVENT THAT RESULTANT CER ALONG PATH i IS LESS THAN X

$$\tilde{F}(X,\vec{P},t) = P\left[\bigcup_{i=1}^{4} G^{(i)}\right]$$

FIG. 5-3 GENERAL NETWORK CASE USING BRIDGE CIRCUIT AS EXAMPLE

We now assume that the CER received at node 4 is the minimum of all possible CERs entering that node. The event that at least one of the CERs is less than or equal to X is given by:

$$G = \bigcup_{i=1}^{4} G^{(i)} \tag{5.27}$$

The probability that at least one of the CERs is less than X is the performance function $\widetilde{F}(X,\vec{P},t)$, which is given by:

$$\widetilde{F}(X,\vec{P},t) = P\Big[G\Big] = P\Big[\bigcup_{i=1}^{4} G^{(i)}\Big] \tag{5.28}$$

Equation (5.28) is evaluated using Eq.(5.8) with $E^{(i)}$ replaced by $G^{(i)}$. However, in contrast to the hard-link case, the evaluation of the individual terms is much more complicated. This can be seen by first noting that each event $G^{(i)}$ is itself a product of two separate subspaces, links and nodes. We write

$$G^{(i)} = E^{(i)}W^{(i)}, \tag{5.29}$$

where

$E^{(i)}$ = event that the nodes in path i are up (a)

$W^{(i)}$ = event that the resultant CER due to link degradation along path i is less than or equal to X (b) (5.30)

$E^{(i)}$ is the same event defined by Eq. (5.15), and is therefore given by

$$E^{(i)} = A_1^{(i)}A_2^{(i)} \;\ldots.\; A_{N_i}^{(i)} \quad , \tag{5.31}$$

where, as in Section 5.1, $A_j^{(i)}$ is the event that the j^{th} node in path i is up. Using the results from Section 3, $W^{(i)}$ is the event given by the condition

$$Q^{(i)} = \sum_{j=1}^{N_i-1} Q_j^{(i)} \leqslant Q \tag{5.32}$$

where

$$Q = -\ln(1-X) \qquad (a)$$

$$Q_j^{(i)} = -\ln(1-X_j^{(i)}) \qquad (b)$$

$$X_j^{(i)} = \text{CER for link } j \text{ along path } i \qquad (c)$$

$$N_i-1 = \text{number of links in path } i \qquad (d) \qquad (5.33)$$

Assuming statistical independence for the links and nodes yields the following result for $P\left[G^{(i)}\right]$, the probability of event $G^{(i)}$ occurring.

$$P\left[G^{(i)}\right] = P\left[E^{(i)}\right]P\left[W^{(i)}\right] \qquad (5.34)$$

Using Eq. (5.31) yields

$$P\left[E^{(i)}\right] = P_1^{(i)}P_2^{(i)} \; \ldots\ldots \; P_{N_i}^{(i)} \qquad (5.35)$$

On the other hand, $P\left[W^{(i)}\right]$is just the link distribution function developed in Section 3. We have:

$$P\left[W^{(i)}\right] = \tilde{F}_L^{(i)}(X) = \int_0^X \tilde{f}_L^{(i)}(X^1)dX^1 \qquad (5.36)$$

where $\tilde{f}_L^{(i)}(X)$ is the pdf defined by Eq. (3.27) evaluated along path i: there results

$$\tilde{f}_L^{(i)} = \tilde{f}_1^{(i)} \,\hat{\otimes}\, \tilde{f}_2^{(i)} \,\hat{\otimes}\, \ldots\ldots \tilde{f}_{N_i-1}^{(i)} \quad , \qquad (5.37)$$

In the foregoing equation $\tilde{f}_j^{(i)}$ is the link pdf for the j^{th} link in path i.

Combining Eqs. (5.35) and (5.36) yields

$$P\left[G^{(i)}\right] = \left(\prod_{j=1}^{N_i-1} P_j^{(i)}\right) \tilde{F}_L^{(i)} \tag{5.38}$$

As to be expected, the foregoing equation is the same result given for the single path case developed in Section 3, as given by Eq. (3.35).

Now let us consider the next level of difficulty, which is the calculation of the probability of the $G^{(i)}G^{(k)}$, terms of which occur in the expansion of Eq. (5.28). Using Eq. (5.29) we have:

$$\begin{aligned} P\left[G^{(i)}G^{(k)}\right] &= P\left[E^{(i)}W^{(i)}E^{(k)}W^{(k)}\right] && \text{(a)} \\ &= P\left[E^{(i)}E^{(k)}\right]P\left[W^{(i)}W^{(k)}\right] && \text{(b)} \end{aligned} \tag{5.39}$$

The method for the calculation of $P\left[E^{(i)}E^{(k)}\right]$ are identical to the procedure for the hard-link case discussed in Section 5.1.1, as given by Eqs. (5.18) and (5.19). That is,

$$P\left[E^{(i)}E^{(k)}\right] \triangleq P_\ell^{(i,k)} = \prod_\ell P_\ell \quad : \ell\epsilon \text{ paths } i \text{ and } k \quad , \tag{5.40}$$

where $P_\ell^{(i,k)}$ stands for all the P_ℓs included in paths i and k.

The computation of $P\left[W^{(i)}W^{(k)}\right]$ proceeds by first interpreting the joint event $W^{(i)}W^{(k)}$, once again we assume statistical independence for this discussion. $W^{(i)}W^{(k)}$ is the event defined by the simultaneous inequalities:

$$\sum_{j=1}^{N_i-1} Q_j^{(i)} \leqslant Q \qquad \text{(a)}$$

$$\sum_{m=1}^{N_k-1} Q_m^{(k)} \leq Q \quad , \qquad \text{(b)} \qquad (5.41)$$

where the foregoing quantities have the meaning of Eq. (5.32). Similar to the case of common nodes, there are situations where there exist common links, as for example, link 5 in Fig. 5-2.

The probability $P\left[W^{(i)}W^{(k)}\right]$ can either be computed in "Q-space" or in "X-space." For brevity, let us carry-out the calculation in the physically-interesting regime where $X \ll 1$. Then from the approximation

$$Q = -\ell n(1-X) \simeq X \qquad (5.42)$$

we have in lieu of Eq. 5.41 the following inequalities:

$$\sum_{j=1}^{N_i-1} X_j^{(i)} \leq X \qquad \text{(a)}$$

$$\sum_{m=1}^{N_k-1} X_m^{(k)} \leq X \qquad \text{(b)} \qquad (5.43)$$

From the definition of probability we have[3]:

$P\left[W^{(i)}W^{(k)}\right] \stackrel{\Delta}{=} \tilde{F}_L^{(i,k)}$ = integration over all variables $(X_j^{(i)}, X_m^{(k)})$ included in both paths, but counted <u>only once</u>. The domain of integration is defined by Eq. (5.43). (5.44)

The mathematical interpretation of Eq. (5.44) is:

$$\tilde{F}_L^{(i,k)} = \int_{\Omega^{(i,k)}} (\tilde{f}_\alpha f_\beta \ldots . \tilde{f}_\gamma) dX_\alpha dX_\beta \ldots .. dX_\gamma \qquad (5.45)$$

where

$$dX_\alpha dX_\beta \ldots dX_\gamma = \text{CER space defined by all links common to paths i and k, and counted only once} \tag{5.46}$$

$$\tilde{f}_\alpha \tilde{f}_\beta \ldots\ldots \tilde{f}_\gamma = \text{product space of all link pdfs included in both paths} \tag{5.47}$$

$$\Omega^{(i,k)} = \text{domain of integration defined by the equation:} \tag{5.48}$$

$$\sum_{\mu=1}^{N_\sigma - 1} X_\mu^{(\sigma)} < X \quad : \sigma = i,k$$

Equation (5.48) is a short-hand way of expressing the two equations of Eq. (5.43). In Appendix B we work out a simple example which illustrates the calculational procedure.

When more than two paths are involved, as for example, in the computation of $P\left[W^{(i)}W^{(k)}W^{(j)}\right]$, the result is again an integral of the type given by Eq. (5.45), namely

$$\tilde{F}^{(i,k,j)} = \int_{\Omega^{(i,j,k)}} (\tilde{f}_{\alpha'} \tilde{f}_{\beta'} \ldots\ldots \tilde{f}_{\gamma'}) dX_{\alpha'} dX_{\beta'} \ldots dX_{\gamma'} \quad , \tag{5.49}$$

where the variables in this case are the CERs included in all three paths. The domain of integration in this case is defined by the three inequalities:

$$\sum_{\mu=1}^{N_\sigma - 1} X_\mu^{(\sigma)} \leqslant X \quad : \sigma = i,k,j \tag{5.50}$$

Using the definition just developed, the performance function $\tilde{F}(X,\vec{P},t)$ between any end-point-pair connected by N paths is given by:

$$\tilde{F}(X,\vec{P},t) = P\left[\bigcup_{i=1}^{N} G^{(i)}\right] = \sum_{i=1}^{N} P\left[G^{(i)}\right] - \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} P\left[G^{(i)}G^{(j)}\right] \quad (5.51)$$

$$+ \sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}\sum_{k=j+1}^{N} P\left[G^{(i)}G^{(j)}G^{(k)}\right] +\ldots (-1)^{N+1}P\left[G^{(1)}G^{(2)}\ldots G^{(N)}\right]$$

From the product space decomposition of the Gs, we reduce Eq. (5.51) to the form:

$$\tilde{F}(X,\vec{P},t) = P\left[\bigcup_{i=1}^{N} G^{(i)}\right] = \sum_{i=1}^{N}\left(\prod_{\ell} P_{\ell}^{(i)}\right) F_{L}^{(i)}(X,t)$$

$$- \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\prod_{\ell} P_{\ell}^{(i,j)}\right) F_{L}^{(i,j)}(X,t)$$

$$+ \sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}\sum_{k=j+1}^{N}\left(\prod_{\ell} P_{\ell}^{(i,j,k)}\right)\left(F_{L}^{(i,j,k)}(X,t)\right)$$

$$+ \ldots. (-1)^{N+1}\prod_{\ell} P_{\ell}^{(i,j,\ldots.N)}F_{L}^{(i,j,\ldots.N)}(X,t) \quad (5.52)$$

When link degradation effects can be neglected, all the $\tilde{F}_L$s in Eq. (5.52) are unity, and the expansion for $\tilde{F}(X,\vec{P},t)$ becomes that for the hard-link approximation.

Equation (5.52) is the general expression for the performance function between any end-point-pair. By requiring a certain performance level for the CER, X_0, at time t_0 one can determine the minimum cost required to achieve acceptable system performance using the results of Section 5.1.2. That is, the mathematical form of Eq. (5.52) is that of Eq. (5.25). The generalization of the cost minimization for R end-point-pairs in the case of soft-links is the same as for the hard-link case.

6.0 Mathematical Properties Of $\tilde{F}(X,\vec{P},t)$

The purpose of this section is to discuss certain important mathematical properties of the performance function $\tilde{F}(X,\vec{P},t)$, in the general network case. Of particular interest is the establishment of the rigorous connection between $\tilde{F}(X,\vec{P},t)$ and the so-called probability-of-survival, P_{sur}. In addition, it is possible to define a new physical entity called the "network time availability," $\tilde{F}_{NET}$, which is defined to be the value of $\tilde{F}(X,\vec{P},t)$ when all the values of P_i are set equal to unity. Thus, $\tilde{F}_{NET}$ accounts solely for the effects of the links.

For expository purposes, let us re-examine the serial chain model of Section 3 in a new light. The performance function for this case is given by

$$\tilde{F}(X,\vec{P},t) = \left(\prod_{i=1}^{M} P_i(t)\right) \tilde{F}_L(X,t) + U(1-X)\left(1 - \prod_{i=1}^{M} P_i(t)\right), \qquad (6.1)$$

where the variables have their previously-defined meaning. The reader may recall from Section 3 that the second term in Eq. (6.1) was included solely for the purpose of ensuring that $\tilde{F}(X,\vec{P},t)$ was properly normalized. Indeed, the physically significant part of Eq. (6.1) is the first term.

In the regime $X < 1$ we have

$$\tilde{F}(X,\vec{P},t) = \left(\prod_{i=1}^{M} P_i(t)\right) \tilde{F}_L(X,t) \qquad (6.2)$$

Equation (5.52), which is likewise valid in the regime $X < 1$, is the counterpart to Eq. (6.2) for a general network. Moreover, from a theoretical viewpoint (cf. Sections 2 and 3) one initially derives the useful part of $\tilde{F}(X,\vec{P},t)$ and then tacks on the step function for normalization purposes.

Using Eq. (6.2) let us assume that there exists some value of X, say $X_\lambda < 1$ such that

$$\tilde{F}_L(X_\lambda,t) \simeq 1 \qquad (6.3)$$

Then Eq. (6.2) becomes:

$$\tilde{F}(X_\lambda,\vec{P},t) = \prod_{i=1}^{M} P_i(t) \tag{6.4}$$

The value X_λ defines the regime where link effects are no longer of interest, and as expected, in this case the performance function depends only on the probability-of-survival for the nodes. Physically, this case encompasses the range where network performance is determined by the nodes. It is appropriate to define the performance function for this case as the probability-of-survival for the network. We thus have:

$$P_{sur} \stackrel{\Delta}{=} \tilde{F}(X_\lambda,\vec{P},t) \tag{6.5}$$

= limit of network performance when link effects can be neglected

Using Eq. (6.4) then yields

$$P_{sur} = \prod_{i=1}^{M} P_i(t) \tag{6.6}$$

Inserting Eq. (6.6) into Eq. (6.1) then provides the following alternate way of expressing $\tilde{F}$ in terms of the probability-of-survival for the network:

$$\tilde{F}(X,\vec{P},t) = P_{sur}\,\tilde{F}_L(X,t) + U(1-X)(1-P_{sur}) \tag{6.7}$$

If we now assume that all the nodes are up, $P_i = 1$, $P_{sur} = 1$, and the last term of Eq. (6.7) vanishes, there results:

$$\tilde{F}(X,\vec{P}=1,t) = \tilde{F}_L(X,t) \tag{6.8}$$

As to be expected, Eq. (6.8) shows that in the case where all the nodes are up, the performance function reduces to the link distribution function.

With the foregoing considerations in mind, let us consider the interpretation of the performance function in the general network case. The starting point for this evaluation is Eq. (5.52). Once again, we hypothesize the existence of some value, of CER, X_λ, such that

$$\tilde{F}_L^{(i,j\ldots)}(X_\lambda,\vec{P},t) \approx 1 \tag{6.9}$$

for all i,j,...etc. Thus, all the link-dependent terms in Eq. (5.52) become unity, and the expression for $\tilde{F}(X_\lambda,\vec{P},t)$ reduces to that of Eq. (5.8). This equation is analogous to Eq. (6.6), and is likewise interpreted as the probability-of-survival for the network. We thus have, in the general case,

$$P_{sur} \stackrel{\Delta}{=} \tilde{F}(X_\lambda,\vec{P},t), \tag{6.10}$$

where $\tilde{F}$ in this case (vis-a-vis the serial chain) is given by Eq. (5.8). Thus, P_{sur} is a function of the P_is for the nodes in the system; that is;

$$P_{sur} = P_{sur}\ (\vec{P}) \tag{6.11}$$

Since $\tilde{F}(X,\vec{P},t)$ is a monotonically increasing function of X it follows that

$$P_{sur} = \tilde{F}(X_\lambda,\vec{P},t) \geq \tilde{F}(X,\vec{P},t) \tag{6.12}$$

Thus, even at the highest CER of interest $\tilde{F}(X,\vec{P},t)$ is less than unity which indicates that an additional term must be affixed to Eq. (5.52) to ensure normalization of the distribution function. As in the serial link case, we construct the normalized distribution function as follows:

$$\tilde{F}(X,\vec{P},t) = \left[\text{Eq.}(5.58)\right] + U(1-X)(1-P_{sur}) \tag{6.13}$$

In the range $X < 1$ it is clearly the first term of Eq. (6.13) which is of interest, also analogous to the serial chain model.

It is also instructive to examine the behavior of Eq. (6.13) in the case where all the P_is are unity. In this situation $P_{sur} = 1$ and Eq. (6.13) reduces to:

$$\tilde{F}(X,\vec{P}=1,t) \triangleq \tilde{F}_L^*(X,t) = \sum_{i=1}^{N} \tilde{F}_L^{(i)}(X,t) - \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \tilde{F}_L^{(i,j)}(X,t) \tag{6.14}$$

$$+ \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} \sum_{k=j+1}^{N} \tilde{F}_L^{(i,j,k)}(X,t) + \ldots .(-1)^{N+1}\tilde{F}_L^{(i,j\ldots N)}(X,t)$$

The function $\tilde{F}_L^*(X,t)$ can be interpreted as the "network link availability." Physically, it accounts for the ability of the links to support a CER when all the nodes are up. In a complex system such as the general network, it may be of interest to evaluate that part of the performance which attributable to link effects only. We can thus introduce the "conditional network link availability," $\tilde{F}_{LC}^*(X,t)$, which is defined as

$$\tilde{F}_{LC}^*(X,t) = \frac{\tilde{F}^*(X,\vec{P},t)}{P_{sur}} \tag{6.15}$$

Since

$$\int_0^1 \tilde{F}^*(X,\vec{P},t)dX = P_{sur} , \tag{6.16}$$

we note that $\tilde{F}_{LC}^*(X,t)$ is a properly normalized distribution function, satisfying the condition

$$\int_0^1 \tilde{F}_{LC}^*(X,t)dX = 1 \tag{6.17}$$

In conclusion, there is an inequality relationship which establishes a bound on $\tilde{F}(X,\vec{P},t)$. Because the performance must increase with a corresponding increase in the probability-of-survival for any node, we have the additional relationships:

$$\tilde{F}(X,P_i,\ldots P_{i-1},P_i=1,P_{i+1},\ldots t) \geqslant \tilde{F}(X,P_i,\ldots P_{i-1},P_i,P_{i+1}\ldots t)=\tilde{F}(X,\vec{P},t) \quad \text{(a)}$$

$$\tilde{F}(X,P=1,t) = \tilde{F}_L^*(X,t) \geqslant \tilde{F}(X,\vec{P},t) \quad \text{(b)}$$

(6.18)

Combining Eqs. (6.12) and (6.18b) we deduce the following inequality,

$$P_{sur}\ \tilde{F}_L^*(X,t) \geqslant \left(\tilde{F}(X,\vec{P},t)\right)^2 \qquad (6.19)$$

or equivalently

$$\sqrt{P_{sur}\tilde{F}_L^*} \geqslant \tilde{F} \qquad (6.20)$$

Eq. (6.20) permits an upper limit to the performance to be determined when simpler methods can be used to calculate the individual entities, P_{sur} and $\tilde{F}_L^*$.

7.0 Computational Technique

Computational complexities arise from the evaluation of the link-dependent integrals as given for example by Eqs. (5.45) and (5.49). For the case of M paths connecting an end-point-pair we have:

$$\tilde{F}_L^{(1,2,\ldots.M)} = \int_{\Omega_M} \left(\tilde{f}_\alpha(X_\alpha) \ldots\ldots \tilde{f}_\gamma(X_\gamma)\right) dX_\alpha \ldots\ldots dX_\gamma, \tag{7.1}$$

where

M	=	number of paths	(a)	
Ω_M	=	domain of integration defined by inequalities: $\sum_{\mu=1}^{N_\sigma-1} X_\mu^{(\sigma)} \leqslant X : \sigma = 1,2,\ldots.M$	(b)	
N_σ	=	number of nodes in path σ; hence $N_\sigma-1$ = number of links in path σ	(c)	
$X_\mu^{(\sigma)}$	=	μ^{th} link in path σ	(d)	
$X_\alpha,\ldots,X_\gamma$	=	the set of distinct CERs involved in the entire set of M paths connecting end-point-pair	(e)	(7.2)

Without going into extensive explanations, it suffices to say that the multi-dimensional integral of Eq. (7.1) is too costly to perform using explicit numerical integration techniques when a large number of links are involved. The Monte Carlo technique[14-16] provides a method of estimating the numerical value of the integral based on a statistical sampling procedure. In this section* we shall present a discussion of the conceptual basis of this technique as it applies to the computation of Eq. (7.1).

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* The method described in this section is the so-called "crude Monte Carlo." In a subsequent publication we shall present a discussion of more efficient Monte Carlo techniques based on the mathematical properties of the pdfs.

Since the CERs for each link are confined to the range

$$0 \leqslant X_i \leqslant 1 \tag{7.3}$$

we can write in lieu of Eq. (7.1),

$$\tilde{F}_L(1,2,\ldots M) = \int_0^1 \int_0^1 \cdots\cdots \int_0^1 \tilde{f}(\vec{X}^{(s)})\, d\vec{X}^{(s)} \tag{7.4}$$

where

$$\tilde{f}(\vec{X}^{(s)}) = \left(\tilde{f}_\alpha(X_\alpha)\ldots\ldots\tilde{f}_\gamma(X_\gamma)\right)\psi(X_\alpha\ldots\ldots X_\beta) \qquad \text{(a)}$$

$$d\vec{X}^{(s)} = dX_\alpha\ldots\ldots dX_\gamma\,, \qquad \text{(b)} \tag{7.5}$$

and ψ is a function defined by the condition

$$\psi(X_\alpha\ldots X_\beta) = 1, \quad \text{if Eq. (7.2b) is satisfied:} \qquad \text{(a)}$$

$$= 0 \text{ otherwise} \qquad \text{(b)} \tag{7.6}$$

Since $\tilde{f}(\vec{X}^{(s)})$ is square-integrable the integration of Eq. (7.4) can be performed using the crude Monte Carlo method discussed in reference 14. In this method we let $\vec{X}^{(s)}$ be a multi-dimensional random variable with each component $X_i^{(s)}$ belonging to the set defined by Eq. (7.2e), and defined in the range given by Eq. (7.3). Essentially, we identify $\vec{X}^{(s)}$ as a random variable with each component being uniformly distributed between 0 and 1. That is, we define a pdf for $\vec{X}^{(s)}$ defined by the equation:

$$P_0(\vec{X}^{(s)}) = 1 \text{ for } 0 \leqslant X_i \leqslant 1 \tag{7.7}$$

The mathematical expectation, M_e, of $\tilde{f}(\vec{X}^{(s)})$ is then given by:

$$M_e\left[\tilde{f}(\vec{X}^{(s)})\right] = \int_0^1 \cdots \int_0^1 \tilde{f}(\vec{X}^{(s)})\, P_0(\vec{X}^{(s)})\, d\vec{X}^{(s)} \,, \qquad \text{(a)}$$

which using Eq. (7.7) gives:

$$= \int_0^1 \cdots \int_0^1 \tilde{f}(\vec{X}^{(s)})\, d\vec{X}^{(s)} \qquad \text{(b)}$$

$$= \tilde{F}_L^{(1,2,\ldots M)} \qquad (7.8)$$

Now suppose we have N_0 trials, with

$$X_{i,K}^{(s)} = \text{value of } X_i^{(s)} \text{ in the } k^{th} \text{ trial;} \qquad \text{(a)}$$

$$K = 1,2,,\ldots\ldots N_0$$

$$\vec{X}_K^{(s)} = \text{set of all } X_{i,K}^{(s)} \qquad \text{(b)} \qquad (7.9)$$

Assuming a uniform distribution for each $X_i^{(s)}$ in accordance with Eq. (7.7) then gives the following value of $\tilde{f}$ for the K^{th} trial

$$\tilde{f}_K = \tilde{f}(\vec{X}_K^{(s)}) \qquad (7.10)$$

The average value of $\tilde{f}(\vec{X}^{(s)})$ for N_0 trials is:

$$\left\langle \tilde{f}_{N_0}(\vec{X}^{(s)}) \right\rangle = \left(\frac{1}{N_0}\right) \sum_{K=1}^{N_0} \tilde{f}(\vec{X}_K^{(s)}) \triangleq \tilde{F}_{L,N_0}^{(1,2,\ldots M)} \qquad (7.11)$$

The accuracy of the estimate for $\tilde{F}_L$ improves as the number of terms, N_0, increases. For a fixed level of confidence in the estimate for $\tilde{F}_L$ the variance of $(\tilde{F}_L - \tilde{F}_{L,N_0})$ behaves as $N_0^{-\frac{1}{2}}$ and is independent of the dimension space. Kahn[15] and Davis and Rabinowitz[16] discuss variance-reducing techniques which ultimately require fewer samples for a fixed level of confidence.

8.0 Conclusion

In this report we present a theoretical method for determining the performance and cost-to-harden large C^3I networks involving redundant paths under nuclear-stressed conditions. With the exception of the link calculations the analytical models developed in this investigation have been implemented on a large computer system, called CNAS[17]. This computer system has been used successfully to evaluate the performance and cost-to-harden the 22 PACOM networks under the APACHE program; and in addition has been incorporated at SHAPE for subsequent application to NATO C^3I systems.

It is shown that the performance of a network can be evaluated in terms of a set of functions, $\tilde{F}_\alpha(X,\vec{P},t)$, which for the α^{th} command post pair is defined as the probability that the character error rate (CER) is less than or equal to X at time t following the onset of the threat. The parameter $\vec{P} = \{P_1, P_2, .., P_i, \;\; ,N_\alpha\}$ where P_i is the probability-of-survival for node i and N_α is the number of nodes involved in the connectivity between the α^{th} end-point-pair. $\tilde{F}_\alpha(X,\vec{P},t)$ is called the "performance function" and is rigorously deduced from the principles of probability theory based on the modeling of nodes as binary random variables (they are either up or down) while all the links may have a continuous distribution in CER due to degradation of the propagation medium caused by nuclear detonations. Methods for computing $\tilde{F}_\alpha(X,\vec{P},t)$ are presented.

When link degradation effects can be neglected the performance function depends only on $\vec{P}$; i.e., $\tilde{F}_\alpha = \tilde{F}_\alpha(\vec{P})$. In this special case, defined as a hard-link network, it is convenient to call $\tilde{F}_\alpha(\vec{P})$ the end-to-end probability-of-survival, P_{sur}. On the other hand, when the nodes are assumed to survive, as perhaps would be the case in mobile systems involving a limited number of nodes, the limitation on performance is controlled by link degradation effects. In this case one can set $\vec{P} = 1$ in the expression for $\tilde{F}_\alpha(X,\vec{P},t)$ and thereby deduce a quantity called the "network link availability" $\tilde{F}^*_\alpha(X,t) = \tilde{F}_\alpha(X,\vec{P}=1,t)$.

Using the equations for $\tilde{F}_\alpha$ we find the minimum cost to achieve network survivability by first selecting the required CERs between command post pairs, X_α, and the time t_0 at which minimum performance is to be achieved. We subsequently express the network cost function as: $C_{NT}=\Sigma C_i(P_i)$, where $C_i(P_i)$ is the cost required to harden node i to probability of survival P_i, and N is the number of nodes. Minimizing C_{NT} subject to the set of inequality constraints, $\tilde{F}_\alpha(X_\alpha,\vec{P},t) \geq T_\alpha$ = constant = required time availability for each α gives the minimum cost for network survivability.

9.0 References

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APPENDIX A: EXAMPLE OF METHOD OF LAGRANGE MULTIPLIERS

For purely illustrative purposes let us consider a simple analytical form for C_j. Assume that the cost for hardening node j is given by (using the notation of Section 4)

$$C_j = A_j + B_j P_j \qquad \text{(A.1)}$$

where A_j and B_j are constants. We then have:

$$\frac{\partial C_j}{\partial P_j} = B_j \qquad \text{(A.2)}$$

$$P_j \frac{\partial C_j}{\partial P_j} = P_j B_j = \lambda \qquad \text{(A.3)}$$

which gives,

$$P_j = \lambda / B_j = \Psi_j(\lambda), \qquad \text{(A.4)}$$

or equivalently,

$$\ell n\ P_j = \ell n\ \lambda - \ell n\ B_j = \ell n\ \Psi_j \qquad \text{(A.5)}$$

Substituting the foregoing result into the equation:

$$\sum_{j=1}^{M} \ell n\ \psi_j(\lambda) = \ell n \beta \qquad \text{(A.6)}$$

gives:

$$\sum_{j=1}^{M} \ell n\ \Psi_j = M\ \ell n\ \lambda - \sum_{j=1}^{M} \ell n\ B_j = n\beta \qquad \text{(A.7)}$$

The solution of Eq. (A.7) for λ is given by:

$$\lambda^* = \sqrt[M]{\beta \prod_{j=1}^{M} B_j} \qquad (A.8)$$

The value of the probability of survival for node j at the minimum overall cost is determined from the equation

$$P_j^* = \psi_j(\lambda^*) = \frac{\lambda^*}{B_j} \qquad (A.9)$$

Using Eq. (A.9), the cost of hardening node j at the minimum value is

$$C_j^* = A_j + B_j P_j^* = A_j + \lambda^* \quad , \qquad (A.10)$$

while the overall minimum cost is:

$$C^* = \sum_{j=1}^{M} C_j^* = \sum_{j=1}^{M} A_j + M\lambda^* \qquad (A.11)$$

APPENDIX B: CALCULATION OF PERFORMANCE FUNCTION FOR A SIMPLE NETWORK INVOLVING A REDUDANT LINK

The purpose of this appendix is to demonstrate via a simple example the method for computing the performance function in the soft-link case. Fig. B-1 depicts a model in which there are two paths by which communication can be achieved between nodes #1 and #3. Using the notation Section 5 we have the following expression for the performance function

$$\tilde{F}(X,\vec{P},t) = P\left[G\right] = P\left[G^{(1)} \cup G^{(2)}\right] \quad (a)$$

$$= P\left[G^{(1)}\right] + P\left[G^{(2)}\right] - P\left[G^{(1)}G^{(2)}\right] \quad (b)$$

$$= P\left[E^{(1)}\right] P\left[W^{(1)}\right] + P\left[E^{(2)}\right] P\left[W^{(2)}\right]$$

$$- P\left[E^{(1)}E^{(2)}\right] P\left[W^{(1)}W^{(2)}\right], \quad (c) \qquad (B.1)$$

Using Eq. (5.15) we obtain the following results for $E^{(1)}$ and $E^{(2)}$.

$$E^{(1)} = A_1^{(1)}A_2^{(1)}A_3^{(1)}A_4^{(1)} \quad (a)$$

$$E^{(2)} = A_1^{(2)}A_2^{(2)}A_3^{(2)}A_4^{(2)} , \quad (b) \qquad (B.2)$$

where

$$A_1^{(1)} = A_1;\ A_2^{(1)} = A_4;\ A_3^{(1)} = A_2;\ A_4^{(1)} = A_3 \quad (a)$$

$$A_1^{(2)} = A_1;\ A_2^{(2)} = A_5;\ A_3^{(2)} = A_2;\ A_4^{(2)} = A_3 \quad (b) \qquad (B.3)$$

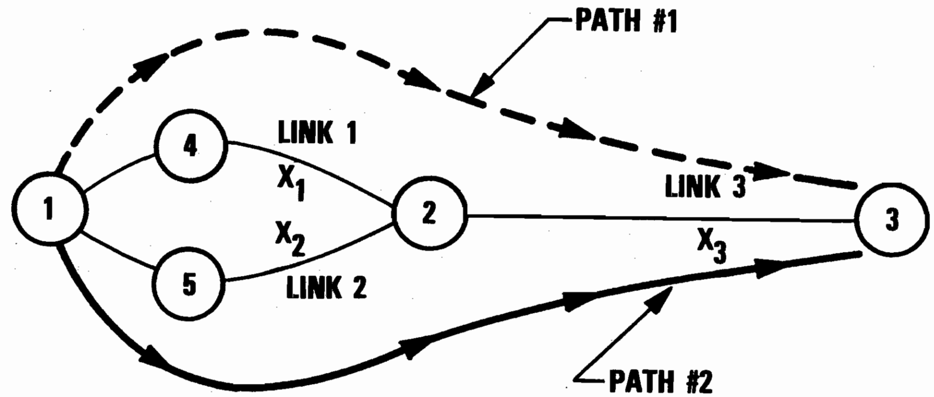

FIG. B-1 SOFT LINK MODEL

In the foregoing equation A_i (i=1,2,...5) is the event that node i is up.

The event $E^{(1)}E^{(2)}$ is given by:

$$E^{(1)}E^{(2)} = \left(A_1A_4A_2A_3\right)\left(A_1A_5A_2A_3\right) = A_1A_2A_3A_4A_5 \qquad \text{(B.4)}$$

Assuming statistical independence gives:

$$P\left[E^{(1)}\right] = P_1P_4P_2P_3 \qquad \text{(a)}$$

$$P\left[E^{(2)}\right] = P_1P_5P_2P_3 \qquad \text{(b)}$$

$$P\left[E^{(1)}E^{(2)}\right] = P_1P_2P_3P_4P_5 \qquad \text{(c)} \qquad \text{(B.5)}$$

The results for $P\left[W^{(1)}\right]$, $P\left[W^{(2)}\right]$, and $P\left[W^{(1)}W^{(2)}\right]$ in the realm where $X_i << 1$ are determined in accordance with the method of Section 5.2. We have:

$$P\left[W^{(1)}\right] = \tilde{F}_L^{(1)} = \int_0^X \tilde{f}_L^{(1)}(X')dX' \qquad \text{(B.6)}$$

where

$$\tilde{f}_L^{(1)}(X') = \int_0^{X^1} \tilde{f}_3(X'')\tilde{f}_1(X'-X'')dX'', \qquad \text{(B.7)}$$

and $\tilde{f}_1, \tilde{f}_3$ are the pdfs for the links. Equation (B.7) is recognized as the convolution integral. Equation (B.6) can be simplified in the following way: We let

$$\tilde{f}_1(\lambda) = \frac{d}{d\lambda}\left(\tilde{F}_1(\lambda)\right), \qquad \text{(B.8)}$$

where $\tilde{F}_1(\lambda)$ is the CER distribution function for link 3. Substituting Eq. (B.8) into Eq. (B.7) gives:

$$\tilde{f}_L^{(1)}(X') = \int_0^{X'} \tilde{f}_3(X'') \frac{\partial \tilde{F}_1(X'-X'')}{\partial X'} dX'' \qquad (B.9)$$

Using the general result for any function $\psi(X',X'')$; namely

$$\frac{d}{dX'} \int_0^{X'} \psi(X',X'')dX'' = \int_0^{X} \frac{\partial \psi}{\partial X'} dX'' + \psi(X',X''=X') \quad , \qquad (B.10)$$

combined with the condition

$$\tilde{F}_1(0) = 0 \qquad (B.11)$$

gives:

$$\tilde{f}_L^{(1)}(X') = \frac{d}{dX'} \int_0^{X'} \tilde{f}_3(X'')\tilde{F}_1(X'-X'')dX'' \qquad (B.12)$$

Substituting Eq. (B.12) into Eq. (B.6) gives:

$$\tilde{F}_L^{(1)} = \int_0^{X} \tilde{f}_3(X'')\tilde{F}_1(X-X'')dX'' \qquad (B.13)$$

In a similar way we can readily show:

$$P[W^{(2)}] = \tilde{F}_L^{(2)} = \int_0^{X} \tilde{f}_3(X'')\tilde{F}_2(X-X'')dX'' \qquad (B.14)$$

Using Eqs. (5.44)-(5.46) gives:

$$P[W^{(1)}W^{(2)}] = \tilde{F}_L^{(1,2)} = \iiint_{\Omega^{(1,2)}} \tilde{f}_1(X_1)\tilde{f}_2(X_2)\tilde{f}_3(X_3)dX_1dX_2dX_3 \qquad (B.15)$$

where $\Omega^{(1,2)}$ is the domain of integration determined by the inequality conditions:

$$X_1 + X_3 < X \qquad \text{(a)}$$

$$X_2 + X_3 < X \qquad \text{(b)} \qquad \text{(B.16)}$$

For this case it is possible to integrate over X_1 and X_2 independently up to the limits $X-X_3$, thus yielding

$$\tilde{F}_L^{(1,2)} = \int_0^X \tilde{f}_3(X_3)\tilde{F}_1(X-X_3)\tilde{F}_2(X-X_3) \qquad \text{(B.17)}$$

where $\tilde{F}_1, \tilde{F}_2$ are the CER distribution functions for links 1 and 2.

Combining the individual calculations gives the following final result:

$$\tilde{F}(X,\vec{P},t) = P_1P_2P_3 \int_0^X \tilde{f}_3(X'')\Big\{P_4\tilde{F}_1(X-X'') +$$

$$P_5\tilde{F}_2(X-X'')-P_4P_5\tilde{F}_1(X-X'')\tilde{F}_2(X-X'')\Big\}dX'' \qquad \text{(B.18)}$$

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