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DIFFERENTIAL-GEOMETRY SCALING METHOD FOR ELECTROMAGNETIC FIELD AND ITS APPLICATIONS TO COAXIAL WAVEGUIDE JUNCTIONS

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ABSTRACT

A differential geometry scaling method, stemming from Baum's pioneering work, is thoroughly explored and developed for electromagnetic fields. This method creates a class of equivalent electromagnetic problems P each described by a complicated geometry and having a complicated medium from an electromagnetic problem P' described by a simple Cartesian geometry and having a simple medium. Application of this method to coaxial waveguide structures is examined with special emphasis. Various conditions and limitation of the method as imposed by special choices of geometry, medium, and field modes are obtained. Also two specific problems are solved in detail by applying this scaling method. In one a perfect matching section between a cylindrical and a conical coaxial waveguide is obtained by appropriately loading the section with inhomogeneous μ and ε, and all relevant electromagnetic quantities and geometrical boundaries are tabulated. In the other a perfect matching section between two cylindrical coaxial waveguides is found with the appropriately shaped matching section loaded by inhomogeneous ε, anisotropic conductivity $\underline{\underline{\sigma}}$, and constant μ. All results are tabulated and plotted. Also, we find the parallel-plate Cartesian scaled version of the fixed μ matching which may give matchings of other geometrical shapes by some variances in its $P' \to P$ scaling procedure. This aspect, and the non-orthogonal scaling which can make use of the Brewster angle transmission in a natural way are discussed for work of future interest.

Keywords: Electromagnetic pulses, differential geometry, waveguides

1. INTRODUCTION

A. Background and Motivation

It is well-known that in mechanics and fluid dynamics one can transform or scale one problem and its solution to create a whole class of equivalent problems and their solutions[1]. Different problems and their solution behaviors of one equivalent class may look very different, but among them there are properties they share. The essence of such a scaling is to get appropriate dimensionless parameters that are common to them all.

However, in electromagnetic (EM) theory the nature and application of such a similar scaling method[2], except for conformal mappings of static fields[3], has not been given extensive attention. Only a few articles have recently been devoted to it[4]. The purpose of the present work is to investigate and develop for EM theory the nature, the limitation, the usefulness, and the application of such a scaling or similarity transform by using a differential geometry approach.

B. Description and Outline

In this work we try to present the EM scaling method, from the most general theoretical formalism to the detailed solutions of some specific problems, in the simplest possible way that makes the reading extremely easy. We do not try to achieve the trivial task of being concise. On the contrary, we spell out most of the detail for such easy reading. We also tabulate the results for immediate engineering use.

In Chapter 2, we explore and develop the general differential geometry scaling method, which carries an EM problem P of complicated geometry into an equivalent problem P' of simple Cartesian or other simple geometry with its accompanying transformations for medium, geometry, source and field. The advantage of such a procedure is, hopefully, to make the complexities of the geometry and of the medium "cancel" each other in such a way that the resulting problem is simple and solvable. In Chapter 3, we study the time independent scaling for special cases. These include orthogonal coordinates and diagonal media, with coaxial systems especially emphasized.

Chapter 4 presents the inhomogeneous μ,ε loaded perfect matching between a cylindrical and a conical coaxial waveguide for TEM waves. Chapter 5 presents the inhomogeneous ε, constant μ, and anisotropic $\underline{\underline{\sigma}}$ loaded matching section between two cylindrical coaxial waveguides for TEM waves. Chapter 5-E contains conclusion, remarks, and discussion of works of possible further interest.

Furthermore, the reader interested in a fast grasp of applications may skip the general theory in Chapter 2 and part of Chapter 3 and start at Chapter 4 if he so desires. In doing so several "whys" referring to the previous general theory will arise, but despite these we have taken the effort to make such reading still easy and effortless.

Concerning notation, standard three dimensional vector analysis one is used. Also all results in the two aforementioned examples are in MKS units. In the general formalism, we have set the vacuum μ_o and ε_o equal to one so the μ and ε written are actually normalized with respect to μ_o and ε_o. This practice is just to keep the notation consistent with differential geometry and generalized EM theory, and an appendix is attached for the full recovery to MKS units.[7] Also part of this report includes the information which is contained in a paper to be published elsewhere.[5]

2. GENERAL DIFFERENTIAL-GEOMETRICAL EM SCALING METHOD (DGM)

A. Generalized Maxwell's Eqs.

The usual Maxwell's eqs. that describe the classical EM fields in an inertial frame[6] have been well tested and are fully accepted. In this case the observers who observe, measure, or "see" the fields are inertial observers. That is, they are attached to or fixed to an inertial coordinate frame $\{X^\mu\} \equiv \{X^\circ \equiv T, X^1, X^2, X^3\}$ such that each of them has his spatial location $(X^1, X^2, X^3) \equiv$ constants. The inertial frame $\{X^\mu\}$ can be described by a Cartesian geometry that has the differential length square [7]

$$(\Delta S)^2 = (\Delta T)^2 - (\Delta X^1)^2 - (\Delta X^2)^2 - (\Delta X^3)^2 \tag{2-1}$$

Let us consider a system of observers $\{0\}$ attached, in the above sense, to a coordinate frame $\{x^\mu\} \equiv \{x^\circ = t, x^1, x^2, x^3)$ which is not an inertial frame and cannot be described by a simple Cartesian geometry (2-1). Then to investigate the EM fields as "seen" by these observers $\{0\}$ we should use the postulated generalized relativistically covariant Maxwell's eqs. These eqs. have been so postulated because of their "naturalness" in a certain formalism - namely tensor calculus - and have been tested in special cases to a certain extent[8]. This relativistic classical EM theory is certainly correct for all known cases in special relativistic phenomena, and is probably correct to a high precision for general relativistic cases - within classical field theory. It is certainly the most popular and currently accepted one. We shall base our investigation on this generalized relativistic EM theory to get the most general EM scaling method which can be applied to moving media, time changing media, EM fields in gravity (general relativistic EM fields), accelerating media, and of course inhomogeneous and anisotropic media.

Now suppose we have a system of observers {0} attached to a general coordinate $\{x^\mu\} \equiv \{x^\circ = t, x^1, x^2, x^3\}$ where x° is the time coordinate and x^1, x^2, x^3 are the spatial coordinates. The geometry of this coordinate frame $\{x^\mu\}$ can be described by expressing its invariant length interval ds in terms of the metric coefficient functions $g_{\mu\nu}(x^\circ, x^1, x^2, x^3)$ as[9]

$$ds^2 = \sum_{\mu=0} \sum_{\nu=0} g_{\mu\nu} dx^\mu dx^\nu \equiv g_{\mu\nu} dx^\mu dx^\nu \qquad (2\text{-}2)$$

Here and in this work we have used the summation convention that repeated indices are summed over their whole ranges, except explicitly stated otherwise. Also, Greek letters μ, ν, etc. stand for 0, 1, 2, 3 and Roman letters i, j, etc. stand for the spatial 1, 2, 3. To these {0}, as the result of the relativistic EM theory, the Maxwell's eqs. become[10]

$$\left\{\begin{array}{ll} \underset{\sim}{\nabla} \cdot \left[\sqrt{\dfrac{-g}{g_{oo}}}\; \underset{\sim}{B} \cdot \underset{\approx}{e} - \sqrt{-g}\; \underset{\sim}{E} \cdot \underset{\sim}{c} \times \underset{\approx}{e}\right] = 0 & (2\text{-}3) \\ \underset{\sim}{\nabla} \cdot \left[\sqrt{-g}\; \underset{\approx}{e}^T \cdot \underset{\sim}{E} \times \underset{\approx}{e}\right] = -\dfrac{\partial}{\partial t}\left[\sqrt{\dfrac{-g}{g_{oo}}}\; \underset{\sim}{B} \cdot \underset{\approx}{e} - \sqrt{-g}\; \underset{\sim}{E} \cdot \underset{\sim}{c} \times \underset{\approx}{e}\right] & (2\text{-}4) \end{array}\right.$$

$$\begin{cases} \underset{\sim}{\nabla} \cdot \left[\sqrt{\dfrac{-g}{g_{oo}}}\, \underset{\sim}{D} \cdot \underset{\approx}{e} + \sqrt{-g}\, \underset{\sim}{H} \cdot \underset{\sim}{c} \times \underset{\approx}{e}\right] = \sqrt{-g}\left[\dfrac{q}{\sqrt{g_{oo}}} + \underset{\sim}{J} \cdot \underset{\sim}{c}\right] & (2\text{-}5) \\ \underset{\sim}{\nabla} \cdot \left[\sqrt{-g}\, \underset{\approx}{e}^{T} \cdot \underset{\sim}{H} \times \underset{\approx}{e}\right] = \dfrac{\partial}{\partial t}\left[\sqrt{\dfrac{-g}{g_{oo}}}\, \underset{\sim}{D} \cdot \underset{\approx}{e} + \underset{\sim}{H} \cdot \underset{\sim}{c} \times \underset{\approx}{e}\right] + \sqrt{-g}\, \underset{\sim}{J} \cdot \underset{\approx}{e} & (2\text{-}6) \end{cases}$$

The notations used here have the following meanings.[11] The field vectors $\underset{\sim}{E}$, $\underset{\sim}{B}$, $\underset{\sim}{D}$, and $\underset{\sim}{H}$ have their usual meanings of macroscopic electromagnetic fields as electric intensity, magnetic induction, electric displacement, and magnetic intensity respectively, with respect to $\{0\}$. The q and $\underset{\sim}{J}$ are the usual charge and current density relative to $\{0\}$. The vectors are decomposed or expressed on the <u>observer's local spatial unit vectors</u> $\underset{\sim}{e}_{(i)}$ which point in the pure spatial direction and are projected from the coordinate x^{i}-directions to be perpendicular to the proper time direction. The dyadic[12] $\underset{\approx}{e}$ is defined by its components

$$(\underset{\approx}{e})^{ij} \equiv e_{(i)}{}^{j} \qquad (2\text{-}7)$$

where $e_{(i)}{}^{j}$ is the projection of the <u>jth</u> covariant coordinate basis vector $\underset{\sim}{e}^{j}$ on the <u>ith</u> local spatial unit vector $\underset{\sim}{e}_{(i)}$. The $\underset{\approx}{e}^{T}$ is the transpose of the dyadic $\underset{\approx}{e}$, i.e., $(\underset{\approx}{e}^{T})^{ij} \equiv (\underset{\approx}{e})^{ji}$. The vector $\underset{\sim}{c}$ is defined by $(\underset{\sim}{c})^{i} \equiv e_{(i)}{}^{o}$ which is the <u>ith</u> component of the covariant time-coordinate basis vector on the $\underset{\sim}{e}_{(i)}$. Also the vector operators have their usual meaning and the quantity $g \equiv \det(g_{\mu\nu})$.

Notice that if $\{x^{\mu}\}$ is Cartesian with a diagonal matrix $g_{\mu\nu} \equiv (1,-1,-1,-1)$ as (2-1), then (2-3) to (2-6) immediately reduce to the familiar Maxwell eqs. Since in this case we have $\underset{\approx}{e} \equiv \underset{\approx}{U}$ (unit dyadic), $\underset{\sim}{c} \equiv 0$, $g = -1$, and

$g_{oo} = 1$, and thus for example the term on the left side of (2-4) becomes

$$\underset{\sim}{\nabla} \cdot (\underset{\approx}{U} \cdot \underset{\sim}{E} \times \underset{\approx}{U}) = \underset{\sim}{\nabla} \cdot (\underset{\sim}{E} \times \underset{\approx}{U}) \equiv \underset{\sim}{\nabla} \times \underset{\sim}{E} \cdot \underset{\approx}{U} - \underset{\sim}{E} \cdot \underset{\sim}{\nabla} \times \underset{\approx}{U} = \underset{\sim}{\nabla} \times \underset{\sim}{E} \qquad (2\text{-}8)$$

and (2-4) reduces to $\underset{\sim}{\nabla} \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}$. Other eqs., for this special case, reduce similarly to the usual familiar Maxwell eqs.

In this work, we are considering scaling for linear media only. Thus, we can assume the constitutive relations for these observers $\{0\}$ fixed in that $\{x^\mu\}$ as

$$\underset{\sim}{D} = \underset{\approx}{\varepsilon} \cdot \underset{\sim}{E} + \underset{\approx}{\alpha} \cdot \underset{\sim}{B} \qquad (2\text{-}9)$$

$$\underset{\sim}{H} = \underset{\approx}{\beta} \cdot \underset{\sim}{E} + \underset{\approx}{K} \cdot \underset{\sim}{B} \qquad (2\text{-}10)$$

$$\underset{\sim}{J} = \underset{\approx}{\sigma} \cdot \underset{\sim}{E} \qquad (2\text{-}11)$$

Here the dyadics $\underset{\approx}{\varepsilon}$, $\underset{\approx}{\alpha}$, $\underset{\approx}{\beta}$, $\underset{\approx}{K}$, $\underset{\approx}{\sigma}$ again have their usual meaning for a general linear media, and are local quantitative of the medium at positions of $\{0\}$.

Also, if there are perfectly conducting boundaries, they are given by $F(\underset{\sim}{x}) = 0$ on which $\underset{\sim}{E}$ has no tangential components. The above descriptions, i.e. the Maxwell's eqs. (2-3) to (2-6), constitutive relations (2-9) to (2-11), and conducting boundaries $F(\underset{\sim}{x}) = 0$, together with appropriate boundary conditions at far away define an EM problem which we call P.[13]

B. The General Scaling $P \rightarrow P'$

Now a scaling method can transform the problem P into an EM problem P' which is in a frame of simple Cartesian geometry and has correspondingly scaled medium properties, source strengths, and boundary conditions to be described in the following. To do so, we first define the scaled "mathematical" or "ficticious" EM fields $(\underset{\sim}{e},\underset{\sim}{b})$, $(\underset{\sim}{d},\underset{\sim}{h})$ by

$$\underset{\sim}{e} \equiv \frac{-1}{2} \sqrt{-g}\, \underset{\approx}{e}^{T} \dot{x} \, (\underset{\sim}{E} \times \underset{\approx}{e}) \tag{2-12}$$

$$\underset{\sim}{b} \equiv \sqrt{-g}\, \underset{\approx}{e}^{T} \cdot \left[\frac{\underset{\sim}{B}}{\sqrt{g_{oo}}} + \underset{\sim}{c} \times \underset{\sim}{E} \right] \tag{2-13}$$

$$\underset{\sim}{d} \equiv \sqrt{-g}\, \underset{\approx}{e}^{T} \cdot \left[\frac{\underset{\sim}{D}}{\sqrt{g_{oo}}} - \underset{\sim}{c} \times \underset{\sim}{H} \right] \tag{2-14}$$

$$\underset{\sim}{h} \equiv \frac{-1}{2} \sqrt{-g}\, \underset{\approx}{e}^{T} \dot{x} \, (\underset{\sim}{H} \times \underset{\approx}{e}) \tag{2-15}$$

in the coordinate frame $\{x^{\mu}\}$ which is <u>now taken</u> to be Cartesian with the simple metric geometry described by

$$dS^2 = dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{2-16}$$

Here and in some of the following, we define double operator ( ) for $\overset{\cdot}{\times}$, $\overset{\times}{\cdot}$, :, and $\overset{\times}{\times}$ between two dyadics as $\underset{\approx}{M}$ ( ) $\underset{\approx}{N}$ by the convention that the upper operator operates first on the inner indices and the lower operator operates next on the outer indices. For example, in component form we have

$$(\underset{\approx}{M} \overset{\cdot}{\times} \underset{\approx}{N})^{i} \equiv \eta^{ijk} M^{j\ell} N^{\ell k} \qquad (2\text{-}17)$$

$$(\underset{\approx}{M} \overset{\times}{\times} \underset{\approx}{N})^{ij} \equiv \eta^{i\ell m} \eta^{jkn} M^{k\ell} N^{mn} \qquad (2\text{-}18)$$

etc., where $\eta^{ijk} \equiv$ +1,-1, or 0 if ijk are even,odd permutations of 123, or otherwise.

To proceed with the scaling, we then define the scaled charge and current density source ρ and $\underset{\sim}{j}$ in this so-considered Cartesian frame $\{x^{\mu}\}$ by

$$\begin{cases} \rho \equiv \sqrt{-g}\left(\dfrac{q}{\sqrt{g_{oo}}} + \underset{\sim}{c} \cdot \underset{\sim}{J}\right) & (2\text{-}19) \\ \underset{\sim}{j} \equiv \sqrt{-g}\, \underset{\sim}{J} \cdot \underset{\approx}{e} & (2\text{-}20) \end{cases}$$

Then the Maxwell's eqs. (2-3) to (2-6) for these just defined "fictitious" fields assume respectively their familiar forms

$$\begin{cases} \underset{\sim}{\nabla} \cdot \underset{\sim}{b} = 0 & (2\text{-}21) \\ \underset{\sim}{\nabla} \times \underset{\sim}{e} = -\dfrac{\partial \underset{\sim}{b}}{\partial t} & (2\text{-}22) \end{cases}$$

$$\begin{cases} \underset{\sim}{\nabla} \cdot \underset{\sim}{d} = \rho & (2\text{-}23) \\ \underset{\sim}{\nabla} \times \underset{\sim}{h} = \underset{\sim}{j} + \dfrac{\partial}{\partial t} \underset{\sim}{d} & (2\text{-}24) \end{cases}$$

in the usual simplest Cartesian sense. That is, these "ficticious" fields and sources are taken to be in an ordinary Cartesian coordinate frame $\{x^\mu\}$ with metric geometry (2-16), and their vector components are expressed on the orthogonal Cartesian spatial unit vectors in that $\{x^\mu\}$ which is now considered to be Cartesian after the scaling.

To complete the scaling, the accompanying scaling of the medium's constitutive relations becomes

$$\underset{\sim}{d} = \underset{\approx}{\xi} \cdot \underset{\sim}{e} + \underset{\approx}{A} \cdot \underset{\sim}{b} \tag{2-25}$$

$$\underset{\sim}{h} = \underset{\approx}{B} \cdot \underset{\sim}{e} + \underset{\approx}{\lambda} \cdot \underset{\sim}{b} \tag{2-26}$$

$$\underset{\sim}{j} = \underset{\approx}{\Sigma} \cdot \underset{\sim}{e} \tag{2-27}$$

where the new ficticious constitutive parameters are expressed in terms of the original ones of (2-9) to (2-11) by

$$\underset{\approx}{\xi} \equiv \frac{1}{2\sqrt{g_{oo}}} \underset{\approx}{e}^T \cdot (\underset{\approx}{\Sigma} - \underset{\sim}{c} \times \underset{\approx}{\beta}) \cdot (\underset{\approx}{\hat{e}}^T \overset{x}{\times} \underset{\approx}{\hat{e}}) + \sqrt{g_{oo}} \, [\underset{\approx}{e}^T \cdot (\frac{\underset{\approx}{\alpha}}{\sqrt{g_{oo}}} - \underset{\sim}{c} \times \underset{\approx}{K})$$

$$\cdot \underset{\approx}{\hat{e}}] \times (\underset{\sim}{c} \cdot \underset{\approx}{\hat{e}}) \qquad (2\text{-}28)$$

$$\underset{\approx}{A} \equiv \sqrt{g_{oo}} \, \underset{\approx}{e}^T \cdot [\frac{\underset{\approx}{\alpha}}{\sqrt{g_{oo}}} - \underset{\sim}{c} \times \underset{\approx}{K}] \cdot \underset{\approx}{\hat{e}} \qquad (2\text{-}29)$$

$$\underset{\sim}{B} \equiv \frac{1}{2} (\underset{\approx}{e} \overset{x}{\times} \underset{\approx}{e}^T) \cdot [\frac{1}{2} \underset{\approx}{\beta} \cdot (\underset{\approx}{\hat{e}}^T \overset{x}{\times} \underset{\approx}{\hat{e}}) - (\underset{\approx}{K} \cdot \underset{\approx}{\hat{e}}) \times (\underset{\sim}{c} \cdot \underset{\approx}{\hat{e}})] \qquad (2\text{-}30)$$

$$\underset{\approx}{\lambda} \equiv \frac{\sqrt{g_{oo}}}{2} (\underset{\approx}{e} \overset{x}{\times} \underset{\approx}{e}^T) \cdot \underset{\approx}{K} \cdot \underset{\approx}{\hat{e}} \qquad (2\text{-}31)$$

$$\underset{\approx}{\Sigma} \equiv \frac{1}{2} \underset{\approx}{e}^T \cdot \underset{\approx}{\sigma} \cdot (\underset{\approx}{\hat{e}}^T \overset{x}{\times} \underset{\approx}{\hat{e}}) \qquad (2\text{-}32)$$

Here $\underset{\approx}{e}^T$ is the transpose of $\underset{\approx}{e}$, i.e., $(\underset{\approx}{e}^T)^{ij} \equiv (\underset{\approx}{e})^{ji}$. Also, $\underset{\approx}{\hat{e}}$ is the inverse of $\underset{\approx}{e}$ and always exists since $\det[(\underset{\approx}{e})^{ij}] \neq 0$.

Finally, the scaled boundary conditions are given through (2-12) to (2-15) and (2-27) to regulate the ficticious fields behaviors on the scaled shape of boundaries in the Cartesian frame $\{x^\mu\}$. The <u>mathematical description of such scaled boundaries</u> in the scaled and taken-to-be Cartesian frame $\{x^\mu\}$ is the <u>same</u> as its previous mathematical description in the original arbitrary coordinate frame $\{x^\mu\}$. For example conducting surfaces, if any, are still described by

$\underset{\sim}{F}(x) = 0$ in the scaled Cartesian frame $\{x^i\}$, on which the scaled $\underset{\approx}{e}$ satisfies

$$\underset{\sim}{N} \cdot (\underset{\approx}{\hat{e}} \times \underset{\sim}{e}) \cdot \underset{\approx}{\hat{e}}^T = 0 \qquad (2\text{-}33)$$

where $\underset{\sim}{N}$ is the normal of the conducting boundary surface $\underset{\sim}{F}(x) = 0$.

The above considerations, from eq. (2-12) and onward, describe how to obtain the new "fictitious" Cartesian problem P' from the original problem P and give the relations between these two. Apparently, the reduction from P to P' with greatly simplified geometry and differential eqs. is achieved at the expense of the much complicated medium properties. However, we must first realize that the scaled "fictitious" fields and "fictitious" problem P' are the equivalent of and are just as real as the original fields and problem P. Thus they can play a reverse role with respect to each other at our disposal. We can require the apparently complicated medium properties (2-25) to (2-32) to be simple enough so that we can solve or know the solution of the scaled Cartesian problem P'. Then through the inverse scaling $P' \to P$ we can obtain a whole class of problems P each with a known solution. Different problems P belonging to the same class just correspond to different choices of the metric of the scaling geometry $g_{\mu\nu}$. The advantage or the purpose of the scaling method in addition to being able to investigate the whole class of P's by investigating one of them, lies in the fact that one may choose the geometry and medium in such a way as to make their complications cancel each other so that the resulting problem is simple, solvable and possesses certain desirable features. This is precisely what the following work will demonstrate.

C. The General Inverse Scaling $P' \rightarrow P$

Since in the application of the scaling method, we need the inverse scaling $P' \rightarrow P$ just as much as we need the scaling $P \rightarrow P'$, we thus list the inverse scaling for $P' \rightarrow P$ below. The fields are inversely scaled by

$$\underset{\sim}{E} = \frac{-1}{2\sqrt{-g}}\,\underset{\approx}{\hat{e}}\ \dot{\times}\ (\underset{\sim}{e} \times \underset{\approx}{\hat{e}}^{T}) \tag{2-34}$$

$$\underset{\sim}{B} = +\frac{\sqrt{g_{oo}}}{\sqrt{-g}}\,\underset{\approx}{\hat{e}} \cdot [\underset{\sim}{b} + \underset{\sim}{e} \times \underset{\approx}{\hat{e}}^{T} \cdot \underset{\sim}{c}] \tag{2-35}$$

$$\underset{\sim}{D} = \frac{\sqrt{g_{oo}}}{\sqrt{-g}}\,\underset{\approx}{\hat{e}} \cdot [\underset{\sim}{d} - \underset{\sim}{h} \times \underset{\approx}{\hat{e}}^{T} \cdot \underset{\sim}{c}] \tag{2-36}$$

$$\underset{\sim}{H} = \frac{-1}{2\sqrt{-g}}\,\underset{\approx}{\hat{e}}\ \dot{\times}\ (\underset{\sim}{h} \times \underset{\approx}{\hat{e}}^{T}) \tag{2-37}$$

and the sources are inversely scaled by

$$q = \frac{\sqrt{g_{oo}}}{\sqrt{-g}}\,(\rho - \underset{\sim}{c} \cdot \underset{\approx}{e} \cdot \underset{\sim}{j}) \tag{2-38}$$

$$\underset{\sim}{J} = \frac{1}{\sqrt{-g}}\,\underset{\approx}{\hat{e}} \cdot \underset{\sim}{j} \tag{2-39}$$

Also the inversely scaled medium's properties, i.e. the parameters $\underset{\approx}{\varepsilon}$, $\underset{\approx}{\alpha}$, $\underset{\approx}{\beta}$, $\underset{\approx}{K}$, and $\underset{\approx}{\sigma}$ of the problem P in eqs. (2-9) to (2-11), expressed in terms of the $\underset{\approx}{\xi}$, $\underset{\approx}{A}$, $\underset{\approx}{B}$, $\underset{\approx}{\lambda}$ and $\underset{\approx}{\sum}$ of P' in eqs. (2-25) to (2-27) are

$$\underset{\approx}{\varepsilon} = \sqrt{g_{oo}}\,\underset{\approx}{\hat{e}} \cdot [\tfrac{1}{2}(\underset{\approx}{\xi} + \underset{\sim}{c} \cdot \underset{\approx}{\hat{e}} \times \underset{\approx}{B}) \cdot (\underset{\approx}{e} \overset{\times}{\times} \underset{\approx}{e}^{T}) + (\underset{\approx}{A} + \underset{\sim}{c} \cdot \underset{\approx}{\hat{e}} \times \underset{\approx}{\lambda}) \cdot \underset{\approx}{e}^{T} \times \underset{\sim}{c}] \tag{2-40}$$

$$\underset{\approx}{\alpha} = \underset{\approx}{\hat{e}} \cdot (\underset{\approx}{A} + \underset{\sim}{c} \cdot \underset{\approx}{\hat{e}} \times \underset{\approx}{\lambda}) \tag{2-41}$$

$$\underset{\approx}{\beta} = \frac{1}{2} (\underset{\approx}{\hat{e}}^{T} \overset{\times}{\times} \underset{\approx}{\hat{e}}) \cdot [\underset{\approx}{\lambda} \cdot \underset{\approx}{e}^{T} \times \underset{\sim}{c} + \frac{1}{2}\underset{\approx}{B} \cdot (\underset{\approx}{e} \overset{\times}{\times} \underset{\approx}{e}^{T})] \tag{2-42}$$

$$\underset{\approx}{K} = \frac{1}{2\sqrt{g_{oo}}} (\underset{\approx}{\hat{e}}^{T} \overset{\times}{\times} \underset{\approx}{\hat{e}}) \cdot \underset{\approx}{\lambda} \cdot \underset{\approx}{e}^{T} \tag{2-43}$$

$$\underset{\approx}{\sigma} = \frac{1}{2}\underset{\approx}{\hat{e}} \cdot \underset{\approx}{\sum} \cdot (\underset{\approx}{e} \times \underset{\approx}{e}^{T}) \tag{2.44}$$

D. Remarks

From the above, we see that the scaling $P \to P'$ or the inverse scaling $P' \to P$ are actually equivalent. The problem P in any frame can be scaled into a problem in a Cartesian frame and vice versa. In manipulating the scaling processes, we should make sure the EM problems obtained have correct dimensions and represent true physical EM fields. For this reason the $g_{\mu\nu}$ metric coefficients should be made dimensionless.

Also, we can clearly see from (2-28) to (2-32) or from (2-40) to (2-44) that the nature of the medium after scaling depends on the scaling geometry as much as on the nature of the medium before scaling. This, plus realizability of the medium, poses various restrictions to the application of the differential geometry scaling method.

3 . SCALING IN SPECIAL CASES

For the $P \rightarrow P'$ scaling, the part of mixed constitutive parameters $\underset{\approx}{A}$ and $\underset{\approx}{B}$ in (2-25) and (2-26) that relate $\underset{\approx}{b}$ to $\underset{\approx}{d}$ and $\underset{\approx}{e}$ to $\underset{\approx}{h}$ are caused by two facts. As can be seen from (2-29) and (2-30), these mixings are caused partly by the medium's own constitutive mixings $\underset{\approx}{\alpha}$ and $\underset{\approx}{\beta}$ in (2-9) and (2-10), and partly by the non-time-orthogonality of the frame $\{x^\mu\}$ with $g_{oi} \neq 0$ which gives rise to $c^i \equiv e_{(i)}{}^{o}$. If we restrict ourselves to time orthogonal frames, i.e. only dealing with frames with $g_{oi} \equiv 0$ which exclude some particular non-stationary non-inertial frames such as accelerated frames, rotating frames, and frames of generally time changing gravitation, then $\underset{\sim}{c} \equiv 0$ and the scaling is simplified. Furthermore, we restrict ourselves for the present interest to media which have no electromagnetic mixture in their constitutive relations, i.e., $\underset{\approx}{\alpha} = \underset{\approx}{\beta} = 0$. Within these two restrictions, i.e.

$$\underset{\sim}{c} \equiv 0 \quad , \tag{3-1-a}$$

$$\underset{\approx}{\alpha} = \underset{\approx}{\beta} = 0 \tag{3-1-b}$$

we shall consider the following further restricted special cases.

A. Diagonal Geometry

If the original arbitrary coordinate frame $\{x^\mu\}$ has a diagonal metric, i.e.

$$g_{\mu\nu} \equiv 0 \quad \text{for} \quad \mu \neq \nu \tag{3-2-a}$$

such that

$$ds^2 = g_o^2(dt)^2 - [(g_1dx^1)^2 + (g_2dx^2)^2 + (g_3dx^3)^2] \equiv g_o^2dt^2 - d\sigma^2 \tag{3-2-b}$$

where $d\sigma^2$ is the three-dimensional invariant length square and

$$|g_{\mu\mu}|^{\frac{1}{2}} \equiv g_\mu \qquad \text{(no summation here)} \qquad (3\text{-}2\text{-}c)$$

then $\underset{\approx}{(e)}^{ij} \equiv \delta^{ij}/g_i$ (no summation). For this case, the $P \rightarrow P'$ scaling for the fields (2-12) to (2-15) reduces to

$$\begin{cases} \underset{\sim}{e} = g_o\, \underset{\approx}{\hat{e}} \cdot \underset{\sim}{E} & (3\text{-}3) \\ \underset{\sim}{b} = g_1 g_2 g_3\, \underset{\approx}{e} \cdot \underset{\sim}{B} & (3\text{-}4) \end{cases}$$

$$\begin{cases} \underset{\sim}{d} = g_1 g_2 g_3\, \underset{\approx}{e} \cdot \underset{\sim}{D} & (3\text{-}5) \\ \underset{\sim}{h} = g_o\, \underset{\approx}{\hat{e}} \cdot \underset{\sim}{H} & (3\text{-}6) \end{cases}$$

where the dyadic or matrix $\underset{\approx}{\hat{e}} \equiv \underset{\approx}{(e)}^{-1}$, and

$$\underset{\approx}{(e)}^{ij} \equiv \delta^{ij}/g_i \equiv \begin{pmatrix} \frac{1}{g_1} & & 0 \\ & \frac{1}{g_2} & \\ 0 & & \frac{1}{g_3} \end{pmatrix} \qquad (3\text{-}7\text{-}a)$$

$$\underset{\approx}{(\hat{e})}^{ij} \equiv g_i \delta^{ij} \equiv \begin{pmatrix} g_1 & & 0 \\ & g_2 & \\ 0 & & g_3 \end{pmatrix} \qquad (3\text{-}7\text{-}b)$$

Also, the sources are scaled, from (2-19) and (2-20), even more simply for the $P \to P'$

$$\rho = g_1 g_2 g_3 q \tag{3-8}$$

$$\underset{\sim}{j} = g_1 g_2 g_3 \underset{\approx}{e} \cdot \underset{\sim}{J} \tag{3-9}$$

But with this special restriction, on geometry the condition (3-2) and on media the condition (3-1-b), the most simplified scaling relations are the constitutive relations. These relations (2-28) to (2-32) for $P \to P'$ reduce to

$$\underset{\approx}{\xi} = \frac{g_1 g_2 g_3}{g_o} \underset{\approx}{e} \cdot \underset{\approx}{\varepsilon} \cdot \underset{\approx}{e} \tag{3-10}$$

$$\underset{\approx}{A} = \underset{\approx}{B} = 0 \tag{3-11}$$

$$\underset{\approx}{\eta} \equiv (\underset{\approx}{\lambda})^{-1} = \frac{g_1 g_2 g_3}{g_o} \underset{\approx}{e} \cdot \underset{\approx}{\mu} \cdot \underset{\approx}{e} \tag{3-12}$$

$$\underset{\approx}{\sum} = g_1 g_2 g_3 \underset{\approx}{e} \cdot \underset{\approx}{\sigma} \cdot \underset{\approx}{e} \tag{3-13}$$

such that for problem P' we have $\underset{\sim}{d} = \underset{\approx}{\xi} \cdot \underset{\sim}{e}$, $\underset{\sim}{b} = \underset{\approx}{\eta} \cdot \underset{\sim}{h}$ and $\underset{\sim}{j} = \underset{\approx}{\sum} \cdot \underset{\sim}{e}$. In the above $\underset{\approx}{\mu} \equiv (\underset{\approx}{K})^{-1}$ so that $\underset{\sim}{B} = \underset{\approx}{\mu} \cdot \underset{\sim}{H}$ for P in the present case. Also, the inverse scaling $P' \to P$ for this case is obtained simply by the inverse of the above matrix relations (3-3) to (3-13). We will not write them out.

Notice that for this special case, by using a coordinate frame of diagonal metric, a problem with a "non-mixed" constitutive medium still scales into a problem of the same nature. But the inhomogeneity and the time-dependence of the geometry can be transformed or scaled into such properties of the medium while the geometry is left simple and constant.

B. Diagonal Metric with $g_o \equiv 1$

This case corresponds to orthogonal curvilinear 3-dimensional coordinate frames in Euclidean space and leaves the time coordinate unchanged in the scaling. Since this is of particular interest to us, we now investigate it in further detail in the following.

B1. Diagonal Metric, $g_o \equiv 1$, and Diagonal Media

If the $\underline{\underline{\varepsilon}}$, $\underline{\underline{\mu}}$ and $\underline{\underline{\sigma}}$ for P are also diagonal, i.e. they have only diagonal elements in their matrices, then from (3-10) to (3-13) we immediately see that the scaled medium for P' is also diagonal. Thus requiring both media before and after the diagonal scaling to be diagonal imposes no further restriction on the geometry itself.

B2. Diagonal Metric, $g_o \equiv 1$, and Uniaxial Media

If we require both P and P' to have uniaxial media, i.e.

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon & & 0 \\ & \varepsilon & \\ 0 & & \varepsilon_3 \end{pmatrix} \qquad \underline{\underline{\xi}} = \begin{pmatrix} \xi & & 0 \\ & \xi & \\ 0 & & \xi_3 \end{pmatrix} \tag{3-14}$$

$$\underline{\underline{\mu}} = \begin{pmatrix} \mu & & 0 \\ & \mu & \\ 0 & & \mu_3 \end{pmatrix} \qquad \underline{\underline{\eta}} \begin{pmatrix} \eta & & 0 \\ & \eta & \\ 0 & & \eta_3 \end{pmatrix} \tag{3-15}$$

$$\underset{\approx}{\sigma} = \begin{pmatrix} \sigma & & 0 \\ & \sigma & \\ 0 & & \sigma_3 \end{pmatrix} \qquad \underset{\approx}{\Sigma} = \begin{pmatrix} \Sigma & & 0 \\ & \Sigma & \\ 0 & & \Sigma_3 \end{pmatrix} \tag{3-16}$$

then a certain restriction on our geometry is imposed, and as a result the above medium parameters are further related. From (3-10) and (3-14) we immediately have

$$g_1 = g_2 \tag{3-17}$$

and thus

$$\xi = g_3 \, \varepsilon \tag{3-18-a}$$

$$\xi_3 = \frac{(g_1)^2}{g_3} \, \varepsilon_3 \tag{3-18-b}$$

The same restriction $g_1 = g_2$ satisfies (3-15) and (3-16), and gives similar relations

$$\eta = g_3 \mu \tag{3-19-a}$$

$$\eta_3 = \frac{(g_1)^2}{g_3} \mu_3 \tag{3-19-b}$$

and

$$\begin{cases} \Sigma = g_3 \sigma & \text{(3-20-a)} \\ \Sigma_3 = \dfrac{(g_1)^2}{g_3} \sigma_3 & \text{(3-20-b)} \end{cases}$$

For this case, the restriction of uniaxiality on all $\underline{\underline{\sigma}}$, $\underline{\underline{\varepsilon}}$, and $\underline{\underline{\mu}}$ with respect to the same axis for both P and P' requires the scaling diagonal geometry be also "uniaxial" with $g_1 = g_2$. Notice that in this case, if we consider TEM wave propagation with respect to the x^3-axis, then the transverse "wave impedances"[14] satisfy

$$\sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\eta}{\xi}} \qquad \text{(3-21)}$$

and are unchanged during the scaling. This property will be used later in the next Chapter. Also μ_3 and ε_3 do not enter the TEM problem here.

Notice that a coordinate frame with metric satisfying (3-17) can be obtained from any orthogonal coordinate frame (v^1, v^2, v^3) with metric[15]

$$\begin{cases} ds^2 = (f_1\, dv^1)^2 + (f_2 dv^2) + (f_3 dv^3)^2 & \text{(3-22)} \\ f_1/f_2 \equiv \text{function of } v^1 \text{ only} \end{cases}$$

To do so, we simply transform (v^1, v^2, v^3) into (x^1, x^2, x^3) by

$$\begin{cases} x^1 \equiv \int (f_1/f_2)\, dv^1 + \text{const} \\ x^2 \equiv v^2 \\ x^3 \equiv F(v^3) \;\leftrightarrows\; v^3 \equiv F^{-1}(x^3) \end{cases} \qquad (3\text{-}23)$$

and the resulting metric for (x^1, x^2, x^3) is simply

$$ds^2 = (f_2)^2 \,[(dx^1)^2 + (dx^2)^2] + \frac{(f_3)^2 (dx^3)^2}{[F'(F^{-1}(x^3))]^2} \qquad (3\text{-}24)$$

A coordinate frame (x^1, x^2, x^3) so obtained from an orthogonal (v^1, v^2, v^3) of (3-22) is ready to be used in scaling involving transverse isotropy or uniaxiality.

If the uniaxiality axes for $\underset{\approx}{\varepsilon}$ $\underset{\approx}{\mu}$, and $\underset{\approx}{\sigma}$ are not the same, still further restrictions are imposed. For example, if $\underset{\approx}{\mu}$ is uniaxial relative to the x^2-direction such that

$$\underset{\approx}{\mu} = \begin{pmatrix} \mu & & 0 \\ & \mu_2 & \\ 0 & & \mu \end{pmatrix} \quad , \quad \mu \neq \mu_2 \qquad (3\text{-}25)$$

then, with $\underset{\approx}{\varepsilon}$ and $\underset{\approx}{\sigma}$ still required to satisfy (3-14) and (3-16), $\underset{\approx}{\eta}$ can be uniaxial with conditions in addition to (3-17)

$$\underset{\approx}{\eta} = g_1 \underset{\approx}{\mu} \qquad \text{if} \qquad g_3 = g_1 \tag{3-26}$$

and

$$\underset{\approx}{\eta} = g_1 \sqrt{\frac{\mu}{\mu_2}} \begin{pmatrix} \mu & & 0 \\ & \mu_2 & \\ 0 & & \mu_2 \end{pmatrix} \text{if} \quad g_3 = g_1 \sqrt{\frac{\mu}{\mu_2}} \tag{3-27}$$

B3. Diagonal Metric, $g_o \equiv 1$, and Isotropic Media

If we restrict the media of both P and P' to be isotropic, then the scaling is severely limited. Now such restrictions

$$\underset{\approx}{\varepsilon} = \varepsilon \underset{\approx}{U} \quad , \quad \underset{\approx}{\mu} = \mu \underset{\approx}{U} \quad , \quad \underset{\approx}{\sigma} = \sigma \underset{\approx}{U} \tag{3-28}$$

$$\underset{\approx}{\xi} = \xi \underset{\approx}{U} \quad , \quad \underset{\approx}{\eta} = \eta \underset{\approx}{U} \quad , \quad \underset{\approx}{\Sigma} = \Sigma \underset{\approx}{U} \tag{3-29}$$

immediately gives

$$g_1 = g_2 = g_3 \tag{3-30}$$

$$\frac{\xi}{\varepsilon} = \frac{\mu}{\eta} = \frac{\Sigma}{\sigma} = g_1 \tag{3-31}$$

from (3-10) to (3-13).

There are only two coordinate frames in the Euclidean space that can satisfy (3-30).[16] They are namely the Cartesian frame itself with $dS^2 = (dT)^2-(dX)^2-(dY)^2-(dZ)^2$, and the inverse Cartesian or inverse sphere frame with $dS^2 = dt^2 - a^4[(dx)^2 + (dy)^2 + (dz)^2)]/(x^3 + y^2 + z^2)$ which is obtained by inverting the Cartesian coordinate with respect to the sphere

$$X^2 + Y^2 + Z^2 = a^2 \quad \text{with} \quad \sqrt{x^2 + y^2 + z^2} \quad \sqrt{X^2 + Y^2 + Z^2} = a^2$$

C. Generalized Coaxial Geometry

For the geometry with $g_o \equiv 1$, a generalized coaxial coordinate frame $\{v^1, x^2 \equiv \phi, v^3\}$ has metric coefficients as given in the invariant length by the non-orthogonal

$$ds^2 = (f_1\, dv^1)^2 + \rho^2 d\phi^2 + (f_3 dv^3)^2 + 2f_{13} dv^1 dv^3 \qquad (3\text{-}32)$$

Here ϕ is the usual azimuthal angle for cylindrical coordinate and ρ is the usual cylindrical polar distance. Notice that the constant-ϕ half-planes are perpendicular to both the constant-v^1 and constant-v^3 surfaces which are not perpendicular to each other. Also, the f_1, f_3, f_{13}, and ρ can all be functions of v^1 and v^3, but not functions of ϕ. Such a generalized coaxial coordinate frame can be obtained by rotating any two dimensional coordinate frame possessing an axis of symmetry about the axis. Furthermore, if $f_{13} \equiv 0$ and f_1/f_3 is function of v^1, then this (v^1, v^2, v^3) can readily be used to obtain the (x^1, x^2, x^3) frame of (3-23) and (3-24).

D. Remarks and Trivial Examples

Viewing from the foregoing analyses, it might look as though the simpler the involved medium is, the more restrictive the scaling geometry must be and therefore the less useful the scaling method becomes in application. However, we should notice that if in a problem considered there are EM fields in certain directions only, then only constitutive relations in those directions enter in the analysis. So in such a situation a simple isotropic medium can be used as a substitute for the complicated media required by the full scaling method. Furthermore, a choice of anisotropic or directional conductors can help to select desired field components and suppress others such that the just-mentioned situation can be achieved. It is in the light of these that the scaling method becomes very powerful in both theoretical investigations and in applications.

Before we go on to solve special problems, let us illustrate the procedure and nature of the scaling method by some simple examples. Consider P' as a parallel plate waveguide with plates at $X = a$ and $X = b$. Let the region between the plates be filled with a uniform medium having constant dielectric constant ξ, magnetic permeability η, and conductivity $\sum \equiv 0$. Consider a TEM wave propagating in the Z-direction with field components

$$E^{(X)} = \sqrt{\frac{\eta}{\xi}}\ H^{(Y)} = e^{i\omega\sqrt{\xi\eta}\ Z} \tag{3-33}$$

This describes the Cartesian Problem P'.

Now a scaling or rather an inverse scaling can carry P' into a curvilinear problem P. If we choose the scaling geometry as

$$(X,\ Y,\ Z) \leftrightarrow (x^1,\ x^2,\ x^3) \leftrightarrow (c_1\theta,\ c_2\phi,\ r) \tag{3-34}$$

where (r,θ,ϕ) is the ordinary spherical coordinate and c_1, c_2 are constant lengths, then this choice corresponds to scale the parallel plate problem with TEM propagation in Z-direction into a conical problem with TEM propagation in the r-direction. Using (3-7) and the inverse of (3-10) to (3-13), we immediately obtain that the scaled medium must be inhomogeneous and diagonally anisotropic and must have constitutive parameters in the scaled $(c_1\theta, c_2\phi, r)$ coordinate as

$$\sigma \equiv 0 \tag{3-34-a}$$

$$\frac{\underline{\underline{\varepsilon}}}{\xi} = \frac{\underline{\underline{\mu}}}{\eta} = \begin{pmatrix} \frac{c_2}{c_1 \sin\theta} & & 0 \\ & \frac{c_1 \sin\theta}{c_2} & \\ 0 & & \frac{c_1 c_2}{r^2 \sin\theta} \end{pmatrix} \tag{3-34-b}$$

in the conical coaxial waveguide between conical boundaries $\theta = \frac{a}{c_1}$ and $\theta = \frac{b}{c_1}$. The TEM wave that propagates in the conical coaxial waveguide, obtained easily from (3-3) and (3-6), is

$$E^{(\theta)} = \frac{c_1 \sin\theta}{c_2} \sqrt{\frac{\eta}{\xi}}\, H^{(\phi)} = \frac{c_1}{r} e^{i\omega\sqrt{\xi\eta}\, r} \tag{3-35}$$

Notice that in this trivial example, we have some freedom to regulate the conical coaxial waveguide. Namely c_1 is at our disposal to control the angular span of the conical waveguide. The constant c_2 has to be chosen that

$$c_2 = \frac{\Delta Y}{2\pi} \tag{3-36}$$

where ΔY is the periodicity of the original problem in the Y-direction, but for this TEM case c_2 can be arbitrary and does not influence the scaling. Also the phase velocity during the scaling is unchanged here.

Just as another example, we can scale the above mentioned parallel-plate problem into a concentically bent parallel-plate waveguide by

$$(X, Y, Z) \leftrightarrow (x^1, x^2, x^3) \leftrightarrow (\rho, -z, c_3\phi) \tag{3-37}$$

where (ρ, ϕ, z) is the usual cylindrical coordinate system. In this case, the concentrically bent parallel-plate waveguide has boundary plates at $\rho = a$ and $\rho = b$, and the TEM wave is propagating along ϕ-direction with fields

$$E^{(\rho)} = -\sqrt{\frac{\eta}{\xi}}\, H^{(z)} = e^{i\omega\sqrt{\xi\eta}\, c_3\phi} \tag{3-38}$$

where the phase velocity along ϕ-direction is $\rho/(c_3\sqrt{\xi\eta})$ which is not constant on the ϕ = constant cross-section of the wave guide and differs from the phase velocity $1/\sqrt{\xi\eta}$ along the Z-direction before the scaling. Also the medium that fills the scaled waveguide between $\rho = a$ and $\rho = b$ has the following scaled constitutive parameters in the $(\rho, -z, c_3\phi)$ coordinate system

$$\frac{\underline{\underline{\varepsilon}}}{\xi} = \frac{\underline{\underline{\mu}}}{\eta} = \begin{pmatrix} \frac{c_3}{\rho} & & 0 \\ & \frac{c_3}{\rho} & \\ 0 & & \frac{\rho}{c_3} \end{pmatrix} \tag{3-39}$$

Notice that the arbitrary length constant c_3 provides a degree of freedom to regulate the angular $\Delta\phi$ bend of the concentric parallel plates from a given section length Δz of the original problem.

We conclude this Chapter by several remarks. First we see that there is a certain degree of freedom in the scaling which can be used to match boundary connections between different scalings. Second, there are conditions for the scaled fields to satisfy such as periodicity and boundary condition matches. Third, only the part of the scaled medium's properties which are relevant to the fields considered are needed; e.g., only the $\varepsilon^{(\theta)(\theta)}$, $\mu^{(\phi)(\phi)}$, and $\sigma^{(\theta)(\theta)}$ in (3-34) are needed for the TEM propagation in that example. All these details on freedoms and restrictions have to be properly taken care of in applications.

4. MATCHING SECTION WITH VARIABLE μ and ε BETWEEN CYLINDRICAL AND CONICAL COAXIAL WAVEGUIDES

A. The Problem

Consider a cylindrical coaxial waveguide described in the usual cylindrical coordinate system (ρ,ϕ,z) with inner conductor surface at ρ = B. and outer conductor surface at β = B. Also a homogeneous simple medium with constant μ,ε, and $\sigma \equiv 0$ fills the coaxial region in this waveguide and a TEM wave is propagating along the z-direction in it (see fig. 1)

$$E^{(\rho)} = \sqrt{\frac{\mu}{\varepsilon}}\; H^{(\phi)} = \frac{1}{\rho}\, e^{i\omega\sqrt{\mu\varepsilon}\, z-i\omega t} \tag{4-1}$$

Now this cylindrical coaxial waveguide is to be connected to a conical waveguide filled with the same simple medium in such a way that a TEM wave in the cylindrical guide propagates into the conical guide without any reflection and distortion. The problem is whether such a transition section exists and how one will go about finding it.

B. The Application of Scaling in the Cylindrical Part

Denote the cylindrical part as region I, the transitional part to be found as region II, and the conical part region III. To find the matching section region II, we first realize that it will likely be a coaxial structure since both I and III are coaxial. Further, since only TEM waves propagate in I and III, so probably the simplest matching structure in II carries also TEM wave only. Since for such a TEM wave the longitudinal medium properties along the axial direction play no role, only transverse medium parameters are of importance. Moreover, whether the TEM wave is reflected or not depends on the matching of the transverse wave impedances. Therefore we are naturally led to consider a transverse isotropic scaling of III-B2 which preserves the transverse wave impedance. With all these in mind, we can try to investigate in this way the probable simplest scaling for the desired matching section. Even if one such section is obtained, it may only be a convenient one and is not at all necessarily the unique one.

Now the problem of finding the desired matching section of region II is really tantamount to finding the common Cartesian problem P' that is common to all regions I, II, and III. For such a P', different scalings in different regions should then scale the P' into the different configurations of our problem P as required, namely, a cylindrical coaxial waveguide filled with uniform simple medium in I, an appropriately loaded perfect matching section in II, and the final conical coaxial waveguide filled with the same uniform simple medium as in region I. A TEM wave propagates in all regions and should be connected smoothly without reflection, for both P and P'.

Now the P in the region I is given. Hence fixed in region I is the scaled P' which can be obtained easily. First use the (ρ,ϕ,z) as the (v^1,v^2,v^3) to get the (x^1,x^2,x^3), as was done in (3-32) to (3-24); we get for region I

$$\begin{cases} x^1 \equiv c_1 \ln \dfrac{\rho}{\rho_o} \\ x^2 \equiv c_1 \phi \\ x^3 \equiv z \end{cases} \qquad (4\text{-}2)$$

and the metric geometry for the coordinate system (x^1,x^2,x^3)

$$d\ell^2 = \left(\frac{\rho_o e^{\frac{x^1}{c_1}}}{c_1}\right)^2 \quad [(dx^1)^2 + (dx^2)^2] + (dx^3)^2 \tag{4-3}$$

Then in region I, the given cylindrical problem P in this coordinate frame $(x^1,x^2,x^3) \equiv (c_1\ \ell n(\rho/\rho_o),\ c_1\phi, z)$ has, from (4-1), fields

$$E^{(1)} = \frac{e^{i\omega\sqrt{\mu\varepsilon}x^3}}{\rho_o \exp(\frac{x^1}{c_1})} \tag{4-4a}$$

$$H^{(2)} = \frac{\sqrt{\frac{\varepsilon}{\mu}}\ e^{i\omega\sqrt{\mu\varepsilon}x^3}}{\rho_o\ \exp(\frac{x^1}{c_1})} \tag{4-4b}$$

and has conducting boundaries at $x^1 = c_1 \ell n(A/\rho_o)$ and $x^1 = c_1 \ell n(B/\rho_o)$. Also, it of course has the same constant μ,ε simple medium. Up to here we have only rewritten P in the region I. Now we scale the P described above into a parallel-plate P' by

$$(x^1,x^2,x^3) \leftrightarrow (X,Y,Z) \tag{4-5}$$

i.e., we take these (x^1,x^2,x^3) to be a Cartesian coordinate frame after the scaling. Then the P' is a parallel-plate waveguide with the plates located at

$$X = c_1 \ell n(A/\rho_o)$$
$$X = c_1\ \ell n(B/\rho_o) \tag{4-6}$$

and with a TEM wave in this parallel-plate waveguide given by

$$e^{(1)} \equiv e^{(X)} = \frac{e^{i\omega\sqrt{\mu\varepsilon}\, x^3}}{c_1} \tag{4-7a}$$

$$h^{(2)} = h^{(Y)} = \frac{\sqrt{\frac{\varepsilon}{\mu}}\, e^{i\omega\sqrt{\mu\varepsilon}\, x^3}}{c_1} \tag{4-7b}$$

This is obtained by using the inverse of (3-3) and (3-6). The medium scaled to fill this parallel-plate waveguide is neither homogeneous nor isotropic, but has constitutive parameters in this Cartesian-taken frame $(x^1,x^2,x^3) \leftrightarrow (X,Y,Z)$

$$\underset{\approx}{\Sigma} = 0 \tag{4-8a}$$

$$\frac{\underset{\approx}{\xi}}{\varepsilon} = \frac{\underset{\approx}{\eta}}{\mu} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \frac{\rho_o^2}{c_1^2}\, e^{\frac{2x^1}{c_1}} \end{pmatrix} \tag{4-8b}$$

This P' should be the problem common to all the regions I, II, and III if the original perfect matching problem has a solution. The constant ρ_o, c should be determined later by matching conditions.

Now, for P' the TEM wave (4-7) propagating in the parallel-plate waveguide with plates (4-6) and medium (4-8) certainly satisfies Maxwell's equations and the relevant boundary conditions, and propagates in the x^3 direction of the (x^1,x^2,x^3) Cartesian-taken frame without reflection and without distortion. The task next is to inversely scale this P' by different ways for region II and III into our original problem.

C. The Scaling and Design of the Matching Section

In region II, from the original problem we see that we need a rotational frame that carries its constant x^3-surfaces from plane surface to spherical surfaces. Looking at the table, we see that a toroidal coordinate system (η,ϕ,θ) does that very simply.[17] So in region II we choose this toroidal coordinate frame (η,ϕ,θ) as the (v^1,v^2,v^3) and obtain, by using (3-23) and (3-24), for region II (see Fig. 1)

$$\begin{cases} x^1 = a \ln(\text{th}\,\frac{\eta}{2}) + c_2 \\ x^2 = a\,\phi \\ x^3 = a\,F(\theta) \end{cases} \qquad (4\text{-}9)$$

with metric coefficients

$$d\ell^2 = \frac{\sinh^2}{(\cosh\eta + \cos\theta)^2}\,[(dx^1)^2 + (dx^2)^2] + \frac{(dx^3)^2}{[F'(\theta)(\cosh\eta+\cos\theta)]^2} \qquad (4\text{-}10)$$

As a footnote, we remind ourselves that the toroidal coordinate frame (η,ϕ,θ) has the metric length

$$d\ell^2 = \frac{a^2}{(\cosh\eta + \cos\theta)^2}\,[d\eta^2 + \sinh^2\eta\, d\phi^2 + d\theta^2] \quad , \qquad (4\text{-}11)$$

$$0 \leqslant \eta < \infty$$
$$0 \leqslant \phi \leqslant 2\pi$$
$$-\pi \leqslant \theta \leqslant \pi$$

If we identify the z-axis of our cylindrical coordinate frame in region I as the $\eta = 0$ straight line, then the toroidal (η,ϕ,θ) has a rotational symmetry about the z-axis and has constant coordinate surfaces given by

$$\eta = \text{constant:} \quad z^2 + (\rho - a \coth \eta)^2 = a^2 \operatorname{csch}^2 \eta \qquad (4\text{-}12a)$$

which is a toroidal surface obtained by rotating the circle of radius $a \operatorname{csch} \eta$ and centered at a distance $a \coth \eta$ from the z-axis and on the azimuth $\theta = \pi$ plane. Also we have

$$\phi = \text{const:} \quad \text{half plane intersecting z-axis} \qquad (4\text{-}12b)$$

and

$$\theta = \text{const:} \quad \rho^2 + (z + a \cot \theta)^2 = \frac{a^2}{\sin^2\theta} \qquad (4\text{-}12c)$$

which are spheres centered at $z = a \cot \theta$ on the z-axis and of radius $a/|\sin\theta|$. Notice that the constant a regulates the (η,ϕ,θ) coordinate frame by changing the radius of the circle to which the toroidal surface converge as $\eta \to \infty$.

Thus in region II, the inverse scaling

$$(X,Y,Z) \leftrightarrow (x^1,x^2,x^3) \leftrightarrow (a\,\ell n\,(\text{th}\,\tfrac{\eta}{2}) + c_2,\ a\phi,\ aF(\theta)) \qquad (4\text{-}13)$$

will give the shape of the matching section and the whole description of desired problem P in that region. Before we write out the fields and medium properties in region II for P, we first have to make sure the (x^1, x^2, x^3) of region I and II as given by (4-2) and (4-9) respectively join smoothly. This smooth joint will then ensure matching of the boundary conditions for the scaled fields. Thus at the boundary surface $z = 0$ or $\theta = 0$ between regions I and II, we require

$$x^1(\mathrm{I}) \equiv x^1(\mathrm{II}) \leftrightarrow c_1 \ell n \frac{\rho}{\rho_o} \equiv a\ \ell n\ \mathrm{th}\ \frac{\eta}{2} + c_2 \Rightarrow c_2 = a\ \ell n \frac{a}{\rho} \tag{4-14a}$$

$$x^2(\mathrm{I}) \equiv x^2(\mathrm{II}) \Leftrightarrow c_1 \phi = a\phi \Leftrightarrow c_1 = a \tag{4-14b}$$

$$x^3(\mathrm{I}) \equiv x^3(\mathrm{II}) \Leftrightarrow F(\theta = 0) \equiv 0 \tag{4-14c}$$

where the final $\Rightarrow$ in (4-14a) is obtained by the help of (4-14b). Also the scaled medium in region II for our matching section has the transverse dielectric constant

$$\varepsilon(\mathrm{II,\ transverse}) = \varepsilon(\cosh \eta + \cos \theta) F'(\theta) \tag{4-15}$$

which, if for realizability purpose is required to be greater than or equal to ε, implies

$$F'(\theta) \geq \frac{1}{(\cosh \eta + \cos \theta)} \tag{4-16}$$

This condition (4-16) can be satisfied by choosing

$$F'(\theta) = \frac{1}{1+\cos\theta} \geq \frac{1}{\cosh\eta + \cos\theta} \Rightarrow F(\theta) = \tan\frac{\theta}{2} \qquad (4\text{-}17)$$

From the above, we obtain the inversely scaled matching section in region II. It is a toroidal coaxial waveguide with boundaries

at $\eta = 2\tanh^{-1}A/a$, $\eta = 2\tanh^{-1}B/a$ (4-18)

obtained by using (4-6), (4-9) and (4-13). The purely TEM fields in the matching section, by means of the inverse of (3-3) and (3-6), are

$$E^{(\eta)} \equiv E^{(1)} = \frac{(\cosh\eta + \cos\theta)e^{i\omega\sqrt{\mu\varepsilon}\ a\tan\frac{\theta}{2}}}{a\sinh\eta} \qquad (4\text{-}19)$$

$$H^{(\phi)} \equiv H^{(2)} = \sqrt{\frac{\varepsilon}{\mu}}\ \frac{(\cosh\eta + \cos\theta)}{a\sinh\eta} e^{i\omega\sqrt{\mu\varepsilon}\ a\tan\frac{\theta}{2}} \qquad (4\text{-}20)$$

in the matching region II. And the medium in the matching section has constitutive parameters

$$\underline{\underline{\sigma}} = 0 \qquad (4\text{-}21a)$$

$$\frac{\underline{\underline{\varepsilon}}}{\varepsilon} = \frac{\underline{\underline{\mu}}}{\mu} = \begin{pmatrix} \frac{\cosh\eta + \cos\theta}{1+\cos\theta} & & 0 \\ & \frac{\cosh\eta + \cos\theta}{1+\cos\theta} & \\ 0 & & \frac{(\cosh\eta + \cos\theta)(1+\cos\theta)}{(\cosh\eta + 1)^2} \end{pmatrix}$$

(4-21b)

where the dyadic components are expressed in the (η,ϕ,θ) coordinate frame or the (x^1,x^2,x^3) of (4-9). These are obtained by using the inverse of (3-10) to (3-13) for the present (x^1,x^2,x^3) of (4-9) and scaling relation (4-13).

The purely TEM propagation from the cylindrical region I to the toroidal region II loaded as (4-21) undergoes no distortion and no reflection. They are just the differently scaled versions of the basic parallel-plate problem P' and are smoothly joined such that the tangential TEM fields $\underset{\sim}{E}$ and $\underset{\sim}{H}$ match. Next to be found is how does this toroidal transition section connect to the conical region III by another inverse scaling.

D. From the Matching Section to the Conical Part

Here we want to inversely scale our "basic common" parallel-plate problem P' into the conical coaxial waveguide in a smooth way. First since the spherical coordinate frame (Θ,ϕ,r) satisfies the requirement (3-22), we can take it as the (v^1,v^2,v^3) to obtain the "transverse isotropic" (x^1,x^2,x^3) by (3-23) and get for region III

$$\begin{cases} x^1 = c_3 \ \ell n \tan \frac{x}{2} + c_4 \\ x^2 = c_3 \phi \\ x^3 = a\, G(r) \end{cases} \tag{4-22}$$

with metric

$$d\ell^2 = \frac{r^2 \sin^2\Theta}{(c_3)^2} \left[(dx^1)^2 + (dx^2)^2 \right] + \frac{(dx^3)^2}{a^2 [G'(r)]^2} \tag{4-23}$$

Again the smooth match of (x^1, x^2, x^3) at the spherical boundary $\theta = \theta_1$ or $r = \frac{a}{|\sin\theta_1|}$ between regions II and III requires

$$\left\{\begin{array}{ll} x^1(\mathrm{II}) \equiv x^1(\mathrm{III}) \Longleftrightarrow a\ln(\mathrm{th}\,\frac{\eta}{2}) + c_2 = c_3\,\ln(\tan\frac{\Theta}{2}) + c_4 & \text{(4-24a)} \\ x^2(\mathrm{II}) \equiv x^2(\mathrm{III}) \Leftrightarrow c_3\phi = a\phi \Leftrightarrow c_3 = a & \text{(4-24b)} \\ x^3(\mathrm{II}) \equiv x^3(\mathrm{III}) \Leftrightarrow aG(a/|\sin\theta_1|) = a\,\tan\frac{\theta_1}{2} & \text{(4-24c)} \end{array}\right.$$

from which the first two conditions and (4-14a) imply

$$c_4 = a\,\ln\left(\frac{a}{\rho_o\,\tan\frac{\theta_1}{2}}\right) \qquad \text{(4-25)}$$

Thus for region III, we use the scaling

$$(X,Y,Z) \leftrightarrow (x^1,x^2,x^3) \leftrightarrow \left(a\,\ln\left(\frac{\tan\frac{\Theta}{2}}{\tan\frac{\theta_1}{2}}\right) + a\,\ln\frac{a}{\rho_o},\ a\phi,\ aG(r)\right) \qquad \text{(4-26)}$$

Notice that the G(r) is not restricted by our scaling itself. But the scaled medium in the conical region III has the following transverse dielectric constant, obtained by using (3-18a) from the common P',

$$\varepsilon_{(\mathrm{III,\ transverse})} = \varepsilon\,aG'(r) \qquad \text{(4-27)}$$

Now since also in our original problem we require the conical section to be filled with the same uniform simple medium as the cylindrical part in region I, then we need

$$\varepsilon\ aG'(r) = \varepsilon \tag{4-28}$$

Thus the $G(r)$, implied by (4-28) and (4-24a), is

$$G(r) = \frac{r}{a} - \cot\theta_1 \tag{4-29}$$

With the scaling geometry completed, the scaling (4-26) itself then gives the propagating fields in region III

$$E^{(1)} \equiv E^{(\Theta)} = \frac{1}{r\sin\Theta}\ e^{i\omega\sqrt{\mu\varepsilon}(r-a\ \cot\theta_1)} \tag{4-30}$$

$$H^{(2)} \equiv H^{(\phi)} = \sqrt{\frac{\varepsilon}{\mu}}\ \frac{e^{i\omega\sqrt{\mu\varepsilon}(r-a\ \cot\theta_1)}}{r\sin\Theta} \tag{4-31}$$

by using again the inverse of (3-3) and (3-6) for the present scaling (4-26). The resulting conical wave guide then has boundaries, described in the spherical coordinates, at

$$\Theta = 2\ \tan^{-1}\left(\frac{A}{a}\ \tan\frac{\theta_1}{2}\right) \tag{4-32a}$$

$$\Theta = 2\ \tan^{-1}\left(\frac{B}{a}\ \tan\frac{\theta_1}{2}\right) \tag{4-32b}$$

The scaled medium to be filled in this conical coaxial region III expressed in the spherical (θ,ϕ,r) or the (x^1,x^2,x^3) of (4-26) is simply

$$\underset{\approx}{\sigma} = 0 \qquad (4\text{-}33a)$$

$$\frac{\underset{\approx}{\varepsilon}}{\varepsilon} = \frac{\underset{\approx}{\mu}}{\mu} = \begin{pmatrix} 1 & & O \\ & 1 & \\ O & & \dfrac{a^2(1+\cos\theta_1)}{r^2\,(1+\cos\theta)^2(1-\cos\theta_1)} \end{pmatrix} \qquad (4\text{-}33b)$$

E. Conclusion of the μ,ε Loaded Matching Section

We have achieved an appropriately loaded perfect matching section between the cylindrical and the conical coaxial guides by using the scaling method. Before summarizing the whole result of this chapter IV in a convenient table, we here make several remarks again. First, the underlying basic problem P' common to all regions I, II, and III is simply a parallel plate wave guide with TEM propagation. Second, the scaling geometry in different regions are connected smoothly and ensure the continuity of the tangential fields which are the only fields. Third, the no-reflection at junction surfaces is very clear because of the no-reflection for P' along its propagation and the smoothness of different scaling geometry at their junctions. Another way of looking into this property is that the scaling chosen possesses the transverse isotropy and preserves the transverse wave impedances. Such constancy of transverse impedances, in addition to the smoothness of the joining scaling geometries, clearly guarantees no reflection and no distortion for the TEM mode considered. Fourth, to realize the problem we need only an isotropic medium in II with its isotropy given by $\varepsilon^{\eta\eta}$ and $\mu^{\theta\theta}$ as in (4-21b), since only a coaxial TEM wave exists. Fifth, the solution so found is by no means unique. It perhaps is the simplest one. Because any orthogonal rotational coordinate frame (v^1,v^2,v^3) that can carry its constant coordinate surfaces from a plate to spherical surfaces curved away relative to that plane can be used in II to join the cylindrical I at the left side and the conical III at the right side for the $P' \to P$ scaling. Of course, each such choice needs its different accompanying inhomogeneous loading medium in II. Sixth, the free parameters in the solution we obtained are the cylindrical radii A and B, the toroidal pole distance 2a, and the constant θ_1 on the spherical boundary surface dividing region II and III. These parameters are bounded to the extent

$$0 < A < B < a \tag{4-34}$$

$$0 < \theta_1 < \pi \tag{4-35}$$

Seventh, we emphasize that the toroidal (η,ϕ,θ) is only one of the many admissible (v^1,v^2,v^3).[15]

Here is the table 1 summarizing all the results. We remind ourselves again that for the toroidal coordinate (η,θ,ϕ) the η = const. toroidal surfaces are described by (4-12a) and the θ = const. spherical surfaces are described by (4-12c). (See Fig. 1).

TABLE 1

Regions / Quantities	I $z=0 \rightarrow$	$\leftarrow \theta=0$ II $\theta=\theta_1 \rightarrow$ $0<\theta_1<\pi$	$\leftarrow r = \frac{a}{\sin\theta_1}$ III
x^1	$a\,\ln(\rho/\rho_o)$	$a\,\ln(\tanh\frac{\eta}{2}) + a\,\ln(\frac{a}{\rho_o})$	$a\,\ln\left(\frac{\tan\frac{\theta}{2}}{\tan\frac{\theta_1}{2}}\right) + a\,\ln(\frac{a}{\rho_o})$
x^2	$a\phi$	$a\phi$	$a\phi$
x^3	z	$(\equiv aF(\theta)) = a\tan\frac{\theta}{2}$ and $F(0) = 0$	$(\equiv aG(r)) = r - a\cot\theta_1$ and $G(a/\sin\theta_1) = \tan\frac{\theta_1}{2}$
$g_1 = g_2$	$(\frac{\rho}{a})$	$\frac{\sinh\eta}{(\cosh\eta + \cos\theta)}$	$\frac{r\sin\theta}{a}$
g_3	1	$(\equiv \frac{1}{(\cosh\eta + \cos\theta)F'(\theta)})$ $= \frac{1+\cos\theta}{\cosh\eta + \cos\theta}$	$(\equiv \frac{1}{aG'(r)})$ 1
Boundaries $x^1 = a\,\ln\frac{A}{\rho_o}$ to $x^1 = a\,\ln\frac{B}{\rho_o}$	$\rho = A$ to $\rho = B$ $0 < A < B < a$	$\eta = 2\tanh^{-1}\frac{A}{a}$ to $\eta = 2\tanh^{-1}\frac{B}{a}$	$\Theta = 2\tan^{-1}(\frac{A}{a}\tan\frac{\theta_1}{2})$ to $\Theta = 2\tan^{-1}(\frac{B}{a}\tan\frac{\theta_1}{2})$

Quantities/Regions	I	II	III
Media $\frac{\varepsilon^{(i)}{}_{(j)}}{\varepsilon} = \frac{\mu^{(i)}{}_{(j)}}{\mu} = \frac{\sigma^{(i)}{}_{(j)}}{\sigma}$ $\left(\varepsilon^{(1)}{}_{(1)} = \varepsilon^{(2)}{}_{(2)} = \frac{\xi}{g_3}\right.$ and $\left.\varepsilon^{(3)}{}_{(3)} = \xi_3 \frac{g_3}{g_1^2}\right)$	δ^{ij}	$\begin{pmatrix} \frac{(\cosh\eta + \cos\theta)}{(1+\cos\theta)}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & 0 \\ 0 & \frac{(\cosh\eta + \cos\theta)(1+\cos\theta)}{(1+\cosh\eta)^2} \end{pmatrix}$	$\begin{pmatrix} \begin{matrix}1 & \\ & 1\end{matrix} & 0 \\ 0 & \frac{a^2(1+\cos\theta_1)}{r^2(1-\cos\theta_1)(1+\cos\theta)^2} \end{pmatrix}$
	(now $\sigma^{(i)}{}_{(j)} \equiv 0$ for all regions, since $\sigma = 0$)		
Fields			
$\underset{\sim}{E}\, e^{i\omega t}$	$\underset{\sim}{e}_{(\rho)} \frac{e^{i\omega\sqrt{\mu\varepsilon}z}}{\rho}$	$\underset{\sim}{e}_{(\eta)} \frac{(\cosh\eta + \cos\theta)}{a \sinh\eta} \cdot e^{i\omega\sqrt{\mu\varepsilon}\, a \tan\frac{\theta}{2}}$	$\underset{\sim}{e}_{(\Theta)} \frac{1}{r \sin\Theta} \cdot e^{i\omega\sqrt{\mu\varepsilon}(r - a\cot\theta_1)}$
$\underset{\sim}{H}\, e^{i\omega t}$	$\underset{\sim}{e}_{(\phi)} \sqrt{\frac{\varepsilon}{\mu}} \cdot \frac{e^{i\omega\sqrt{\mu\varepsilon}z}}{\rho}$	$\underset{\sim}{e}_{(\phi)} \sqrt{\frac{\varepsilon}{\mu}} \frac{(\cosh\eta + \cos\theta)}{a \sinh\eta} \cdot e^{i\omega\sqrt{\mu\varepsilon}\, a\tan\frac{\theta}{2}}$	$\underset{\sim}{e}_{(\phi)} \sqrt{\frac{\varepsilon}{\mu}} \cdot \frac{e^{i\omega\sqrt{\mu\varepsilon}(r-a\cot\theta_1)}}{r \sin\Theta}$

Notice again that in the above table $F(\theta)$, $G(r)$ in general are dimensionless arbitrary smooth functions that satisfy $F(0) = 0$, $G(a/\sin\theta_1) = \tan\frac{\theta_1}{2}$, and the choices as shown are the results of requiring the $\varepsilon_{(\text{transverse})} \geq \varepsilon$ in II and the $\varepsilon_{(\text{transverse})} \equiv \varepsilon$ in III. Also arrows indicated at the top of the table denote boundary surfaces that divide the regions.

5. $\underset{=}{\sigma}$,ε Loaded Matching Section Between Two Cylindrical Coaxial Waveguides

The matching section found in the previous Chapter requires a loading material with tapered μ and ε. Such a tapering of inhomogeneous μ is difficult to realize. In this chapter we shall consider a matching section loaded with fixed μ, but with inhomogeneous ε and anisotropic $\underset{=}{\sigma}$.

A. The Problem

Consider, in a cylindrical coordinate frame (ρ,ϕ,z), two cylindrical coaxial waveguides with different sizes. The first one has inner and outer cylindrical conducting radii $\rho = A$ and $\rho = B \equiv \lambda_t A$, and the second one has its respective radii $\rho = A'$ and $\rho = B' \equiv \lambda_t' A'$. Here the transverse outer-to-inner ratios $\lambda_t > 1$ and $\lambda_t' > 1$. Also we assume $A < A'$ so the second one has larger size. Let both waveguides be filled with the same simple uniform medium of constant ε, constant μ, and $\sigma \equiv o$. (See Fig. 3).

Now the problem is to find a perfect matching section of the conical coaxial shape between these two cylindrical coaxial waveguides such that a TEM wave incident from the left side in the first smaller one can propagate into a TEM wave in the second larger one without reflection and without distortion. We allow ourselves to have variable $\varepsilon(\underset{\sim}{x})$ and anisotropic conductivity $\underset{=}{\sigma}$ in the matching section, but we require the matching section to have the same fixed μ as in the cylindrical regions.

The task is to see if such a matching is possible, to find it if it exists, and to look into ways of generalizing it to matchings of other kinds by the differential geometry method (DGM). As before, we denote as region I the left side smaller cylindrical part, as region II the matching part to be found, and as region III the right side larger cylindrical part. Also the TEM wave in the cylindrical I is

$$E^{(\rho)} \equiv E^{(1)} = \frac{e^{i\omega\sqrt{\mu\varepsilon}z}}{\rho} \tag{5-1}$$

$$H^{(\phi)} \equiv H^{(2)} = \sqrt{\frac{\varepsilon}{\mu}}\,\frac{e^{i\omega\sqrt{\mu\varepsilon}z}}{\rho} \tag{5-2}$$

B. Approach With Impedance Concept

Before launching into full detail of EM field consideration, we examine the problem by a rough impedance concept by looking at Fig. 2. For a perfect matching, we need the impedance to be matched all the way for regions I, II, and III. This can be achieved by inserting many coaxial conducting layers in all the regions with appropriately shaped boundary interfaces, with their spacings d and the thickness δ of each such sheath satisfying

$$\delta << d << \lambda \tag{5-3}$$

where λ is the wave length of the TEM wave, and with the conductivity σ of each sheath satisfying

$$\frac{1}{\sqrt{\omega\mu\sigma}} << \delta \qquad \text{and} \qquad \sqrt{\frac{\omega\varepsilon}{\sigma}}\,L << d \tag{5-4}$$

where L, is the longitudinal dimension of the matching section.

The condition (5-3) "ensures" no reflection and (5-4) "ensures" only the TEM field exists. Moreover, for a TEM wave to propagate from I, through II, and into III without distortion, a plane wave front in I should go into a plane wave front in III and the traveling time should be the same for waves following paths of different radii. These two requirements are intuitively necessary for our matching to exist. We shall see if they will indeed give such a matching in the following.

B1. Conditions for Shape and Medium of the Matching Section

Now, referring to Fig. 2, we see that if we require equal traveling time along MM'M" and its infinitesimally changed version OO'O", we obtain

$$\sqrt{\mu\varepsilon(\theta)}[r_2(\theta) - r_1(\theta)] + \sqrt{\mu\varepsilon}\Delta_1 = [r_2(\theta+d\theta) - r_1(\theta+d\theta)]\sqrt{\mu\varepsilon(\theta+d\theta)} + \sqrt{\mu\varepsilon}\Delta_2 \tag{5-5}$$

where

$$\Delta_1 \equiv r_1(\theta)\cos\theta - r_1(\theta+d\theta)\cos(\theta+d\theta) = [r_1(\theta)\sin\theta - r_1'(\theta)\cos\theta]d\theta \tag{5-6a}$$

$$\Delta_2 \equiv r_2(\theta)\cos\theta - r_2(\theta+d\theta)\cos(\theta+d\theta) = [r_2(\theta)\sin\theta - r_2'(\theta)\cos\theta]d\theta \tag{5-6b}$$

Here $r = r_1(\theta)$ and $r = r_2(\theta)$ describe the boundary intersurfaces Γ_1 connecting I to II and Γ_2 connecting II to III respectively. These eqs. (5-5) and (5-6), as a result of equal traveling time requirement, relate the boundary intersurfaces $\Gamma_1 : r_1(\theta)$ and $\Gamma_2 : r_2(\theta)$ and the inhomogeneous loading

$\varepsilon(\theta)$ in the conical II by

$$[r_2'(\theta) - r_1'(\theta)](\cos\theta - \sqrt{\frac{\varepsilon(\theta)}{\varepsilon}}) - [r_2(\theta) - r_1(\theta)][\sin\theta + \frac{(\varepsilon(\theta)/\varepsilon)'}{2\sqrt{\varepsilon(\theta)/\varepsilon}}] = 0 \tag{5-7}$$

Now consider the matching of impedances for each of the thin coaxial layers. First notice that the impedance for the TEM wave of a cylindrical coaxial waveguide is (see Fig. 2)

$$Z_{cyl.} \equiv \frac{V}{I} = \frac{\sqrt{\frac{\mu}{\varepsilon}}\,\ln(\rho_{outer}/\rho_{inner})}{2\pi} \tag{5-8}$$

and of a conical coaxial waveguide is

$$Z_{con.} \equiv \frac{V}{I} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{2\pi}\,\ln\left|\frac{\tan(\frac{\theta_{outer}}{2})}{\tan(\frac{\theta_{inner}}{2})}\right| \tag{5-9}$$

This infinitesimal impedance matching on Γ_1 then becomes

$$\sqrt{\frac{\mu}{\varepsilon}}\,\ln\frac{\rho+d\rho}{\rho} = \sqrt{\frac{\mu}{\varepsilon(\theta)}}\,\ln\left(\frac{\tan(\frac{\theta+d\theta}{2})}{\tan(\frac{\theta}{2})}\right) \tag{5-10}$$

which gives the differential eq.

$$\frac{r_1'(\theta)}{r_1(\theta)} + \frac{1}{\sin\theta}\left(\cos\theta - \frac{1}{\sqrt{\varepsilon(\theta)/\varepsilon}}\right) = 0 \qquad (5\text{-}11)$$

by expanding and using $\rho \equiv r_1(\theta)\sin\theta$ in (5-10). Then (5-11) can be readily integrated to relate $r_1(\theta)$ and $\varepsilon(\theta)$ on Γ_1 by

$$r_1(\theta) = \text{const.}\ \exp\left[\int \frac{d\theta}{\sin\theta}\left(\frac{1}{\sqrt{\varepsilon(\theta)/\varepsilon}} - \cos\theta\right)\right] \qquad (5\text{-}12)$$

Here we immediately see that the $r_2(\theta)$ of Γ_2 must behave similarly and can differ from $r_1(\theta)$ only by a multiplicative amplification constant λ_a

$$r_2(\theta) = \lambda_a r_1(\theta) \quad , \quad \lambda_a > 1 \qquad (5\text{-}13)$$

where λ_a is greater than one because III is larger than I.

Therefore, with both impedance matchings and equal traveling time requirements, we have from (5-13) and (5-7) the differential equation

$$r'(\theta)\left(\cos\theta - \sqrt{\varepsilon(\theta)/\varepsilon}\right) - r(\theta)\left(\sin\theta + \frac{(\varepsilon(\theta)/\varepsilon)'}{2\sqrt{\varepsilon(\theta)/\varepsilon}}\right) = 0 \qquad (5\text{-}14)$$

and from (5-13) and (5-11) the differential equation

$$\frac{r'(\theta)}{r(\theta)} + \frac{1}{\sin\theta}\left(\cos\theta - \frac{1}{\sqrt{\varepsilon(\theta)/\varepsilon}}\right) = 0 \qquad (5\text{-}15)$$

Both the $r_1(\theta)$ of Γ_1 and the $r_2(\theta)$ of Γ_2, which are related by an amplification constant λ_a as in (5-13), should satisfy these two equations.

We here make several remarks before we proceed to solve (5-14) and (5-15). First, we choose $\varepsilon(\theta)$ being a function of θ only in II because the TEM propagation is independent of ϕ and any r-dependence as longitudinal inhomogeneity will give rise to unwanted reflection. Second, the relation (5-13) for the intersurfaces Γ_1 and Γ_2 implies that the transverse dimension ratios λ_t for I and λ_t' for III are the same

$$\lambda_t' = \lambda_t \tag{5-15}$$

since

$$\lambda_t \equiv \frac{B}{A} \equiv \frac{r_1(\theta_B)\sin\theta_B}{r_1(\theta_A)\sin\theta_A} \equiv \frac{\lambda_a r_1(\theta_B)\sin\theta_B}{\lambda_a r_1(\theta_A)\sin\theta_A} = \frac{r_2(\theta_B)\sin\ _B}{r_1(\theta_B)\sin\ _A} \equiv \frac{B'}{A'} \equiv \lambda_t' \tag{5-16}$$

This constancy of the transverse dimension ratios indeed checks as it should because the impedance of the matched cylindrical coaxial lines should be equal no matter what happens in the transition matching section. Third, that the relation (5-13) itself should hold is intuitively clear <u>if</u> we require the voltage on each layered conductor sheath be constant throughout regions I, II, and III, and if we already require the $\lambda_t = \lambda_t'$ from a direct impedance matching concept for I and III. Because then the layered sheaths in I and III must have a similar geometrical outlay and differ only by a scale length. That each of such sheaths should be linked by a sheath of constant θ means the two intersurfaces Γ_1 and Γ_2 must be similar and differ by the same scale length. Fourth, we must point out that the cylindrical coaxial conducting sheaths in I and III are only conceptual and need not be realized. Because the TEM wave in them does not see such sheaths. Fifth, the conical coaxial conductor sheaths in II are needed to ensure TEM wave there. When (5-3) and (5-4) are satisfied, such property in region II can formally be treated as though it has an anisotropic conductivity

$$\underset{\approx}{\sigma} = \begin{pmatrix} 0 & \infty & \bigcirc \\ \bigcirc & & \infty \end{pmatrix} \qquad (5\text{-}17)$$

in the spherical coordinate frame (θ,ϕ,r) in region II.

B2. Shape of the Matching Section

Now we will solve (5-14) and (5-15) to determine the loading $\varepsilon(\theta)$ and the boundary shapes Γ_1 and Γ_2 for the conical region II. From (5-14), we have

$$\frac{r'(\theta)}{r(\theta)} + \frac{(\sqrt{\varepsilon(\theta)/\varepsilon})' + \sin\theta}{\sqrt{\varepsilon(\theta)/\varepsilon} - \cos\theta} = 0 \qquad (5\text{-}18)$$

which can be integrated to give

$$\sqrt{\varepsilon(\theta)} = \sqrt{\varepsilon}\,\frac{c_1 + r(\theta)\cos\theta}{r(\theta)} \qquad (5\text{-}19)$$

where c_1 is a constant with dimension of length. Inserting (5-19) into

(5-15), we get

$$\frac{r'(\theta)}{r(\theta)}\sin\theta + (\cos\theta - \frac{r(\theta)}{c_1 + r(\theta)\cos\theta}) = 0 \qquad (5\text{-}20)$$

which is a non-linear first order ordinary differential eq. for $r(\theta)$. To solve (5-20), we try first to rewrite it as

$$\frac{d}{d\theta}(r(\theta)\sin\theta) = \frac{r^2(\theta)}{(c_1 + r(\theta)\cos\theta)} \qquad (5\text{-}21)$$

which leads us further trying to separate variables for $r(\theta)\cos\theta$ by using

$$\frac{d}{d\theta}(r(\theta)\sin\theta) \equiv \frac{d}{d\theta}(r(\theta)\cos\theta\tan\theta) = \tan\theta\frac{d}{d\theta}(r(\theta)\cos\theta) + \frac{r(\theta)\cos}{\cos^2\theta} \qquad (5\text{-}22)$$

Now substituting (5-22) into (5-21) gives

$$\left(\frac{1}{r(\theta)\cos\theta} + \frac{1}{c_1}\right)\frac{d}{d\theta}\, r(\theta)\cos\theta = \frac{-1}{\sin\theta\cos\theta} \qquad (5\text{-}23)$$

which can immediately be integrated to give

$$r(\theta)\sin\theta = c_2 e^{\frac{-r(\theta)\cos\theta}{c_1}} \qquad (5\text{-}24)$$

that gives the boundary intersurface function $r(\theta)$ implicitly by a transcendental algebraic eq. Here c_2 is another constant length to be determined. Notice that on either boundaries Γ_1 or Γ_2, (5-24) can be rewritten in terms of the cylindrical coordinates

$$\rho(z) = c_2 e^{\frac{-z}{c_1}} \tag{5-25}$$

which states that the Γ_1 dividing I and II and the Γ_2 dividing II and III are nothing more complicated than two exponentially-shaped boundary intersurfaces.

Now for Γ_1, which can be described by either $r = r_1(\theta)$ or $\rho = \rho_1(z)$, we have (see Fig. 3)

$$A \equiv \rho_1(z_A) = c_2 e^{\frac{-z_A}{c_1}} \equiv c_2 e^{\frac{-a\cot\theta_A}{c_1}} \tag{5-26a}$$

$$\lambda_t A \equiv B = \rho(z_B) = c_2 e^{\frac{-z_B}{c_1}} \equiv c_2 e^{\frac{-\lambda_\ell z_A}{c_1}} \tag{5-26b}$$

Here z_A and z_B are the abscissas, along the coaxial axis from the conical origin, of the points where the innermost and outermost conical surfaces meet the $\rho = A$ and $\rho = \lambda_t A$ cylindrical waveguide surfaces respectively. (See Fig. 3). Using the inner cylindrical radius A of I, the transverse dimension ratio λ_t of I, the inner conical angle θ_a of II, and the longitudinal abscissa ratio for Γ_1

$$\lambda_\ell \equiv z_B/z_A \tag{5-27}$$

As parameters, we have from (5-26)

$$c_1 = \frac{(1-\lambda_\ell)A\cot\theta_A}{\ell n\lambda_t} \tag{5-28a}$$

$$c_2 = A\, e^{\frac{\ell n\lambda_t}{1-\lambda_\ell}} \tag{5-28b}$$

Thus, with the c_1 and c_2 so determined, the intersurface Γ_1 between regions I and II can be described either by $r = r_1(\theta)$ with $r_1(\theta)$ determined by (5-24), or described by $\rho = \rho_{1(z)}$ using (5-25) and (5-28).

For the Γ_2 that divides regions II and III, we have already found its relation to Γ_1 by (5-13), i.e. $r = r_2(\theta) \equiv \lambda_a r_1(\theta)$. Thus the description of Γ_2 in terms of the cylindrical coordinates in III can be obtained easily

$$\rho = \rho_2(z) = r_2(\theta)\sin\theta = \lambda_a r_1(\theta)\sin\theta = \lambda_a c_2 e^{\frac{-\lambda_a r_1(\theta)\cos\theta}{\lambda_a c_1}} = \lambda_a c_2 e^{\frac{-z}{\lambda_a c_1}} \tag{5-29}$$

which differs from the $\rho_1(z)$ only by replacing c_1 and c_2 by $\lambda_a c_1$ and $\lambda_a c_2$ respectively.

Notice that the $r_1(\theta)$ or $\rho_1(z)$ of Γ_1 and the $r_2(\theta)$ or $\rho_2(z)$ of Γ_2 relate all corresponding lengths in I and III by the amplification factor λ_a of (5-13). Now in terms of the independent free parameters $A \equiv$ inner radius of I, $\lambda_t \equiv$ transverse dimension ratio of I, $\theta_A \equiv$ inner half-conical angle of II, $\lambda_\ell \equiv$ longitudinal abscissa ratio of Γ_1, and $\lambda_a \equiv$ amplification factor of III relative to I, we refer to Fig. 3 and summarize here the results on geometrical shapes for all regions I, II, and III.

I: $B \equiv$ outer cylindrical radius of I $= \lambda_t A,\quad 1 < \lambda_t$ (5-30a)

Γ_1: $r = r_1(\theta)$ where $r_1(\theta)\sin\theta = c_2 e^{\frac{-r_1(\theta)\cos\theta}{c_1}}$, $0<\theta_A<\theta<\theta_B<\frac{\pi}{2}$; or

$$\rho = \rho_1(z) = c_2 e^{\frac{-z}{c_1}}, \quad 0<z_B<z<z_A \tag{5-30b}$$

$$z_A = A\cot\theta_A \quad \text{and} \quad z_B = \lambda_\ell z_A = \lambda_\ell A\cot\theta_A$$

II: $\theta_B \equiv$ outer half-conical angle of II $= \tan^{-1}\left(\frac{B}{z_B}\right) = \tan^{-1}\left(\frac{\lambda_t}{\lambda_\ell}\tan\theta_A\right)$ (5-30c)

Γ_2: $r = r_2(\theta) = \lambda_a r_1(\theta)$; or $\rho = \rho_2(z) = \lambda_a c_2 e^{\frac{-z}{\lambda_a c_1}}$, $\lambda_a z_B<z<\lambda_a z_A$ (5-30d)

III: $A' \equiv$ inner cylindrical radius of III $= \lambda_a A$, $\lambda_a>1$

$B' \equiv$ outer cylindrical radius of III $= \lambda_t\lambda_a A$ (5-30e)

We remark that the $\lambda_\ell<1$ of (5-30b) will be explained in the following Chapter 5-B3.

B3. Medium of the Matching Section

From (5-19), the inhomogeneous dielectric constant $\varepsilon(\theta)$ in II is

$$\varepsilon(\theta) = \varepsilon \left[\frac{c_1+r_1(\theta)\cos\theta}{r_1(\theta)}\right]^2 \qquad (5\text{-}31)$$

where c_1 is given by (5-28a). Notice that if we insert $r_2(\theta)$ and its $\lambda_a c_1$ instead of c_1 for Γ_2 into (5-19), we get the same $\varepsilon(\theta)$ as a function of the conical angle θ in region II.

To study the behavior of $\varepsilon(\theta)$, of course we can plot it numerically. But we can get some insight by examining it analytically without any difficulty. Now on Γ_1 and from (5-30b) we have

$$\lambda_t = \frac{B}{A} = \frac{\rho_1(z_B)}{\rho_1(z_A)} = e^{\frac{-(z_B-z_A)}{c_1}} > 1 \qquad (5\text{-}32)$$

and thus

$$\frac{z_A-z_B}{c_1} > 0 \qquad (5\text{-}33)$$

Also from (5-31) we have

$$c_1 = r_1(\theta)\left[\sqrt{\frac{\varepsilon(\theta)}{\varepsilon}} - \cos\theta\right] \qquad (5\text{-}35)$$

Now in order to have an easy realization of the required $\varepsilon(\theta)$ in II, we require

$$\varepsilon(\theta) \geq \varepsilon \text{ for } 0<\theta_A \leq \theta_B < \frac{\pi}{2} \text{ in II} \tag{5-35}$$

because ε may often be just the ε_o of the vacuum. Thus with the requirement (5-35); we have from (5-34) and (5-33) the relations

$$c_1 > 0 \tag{5-36}$$

$$z_A > z_B \quad \text{or } \lambda_\ell \equiv \frac{z_B}{z_A} < 1 \tag{5-37}$$

which explains the inequality in (5-30b). This guarantees that the Γ_1 and Γ_2 are both of expontially decreasing shape for ρ as a function of z on these intersurfaces.

To see more of the θ-dependence of $\varepsilon(\theta)$, let us rewrite (5-31) on Γ_1 and express $\varepsilon(\theta)$ as a function of z

$$\varepsilon(\theta(z)) = \varepsilon \cdot \frac{(c_1 + z)^2}{z^2 + \rho_1^2(z)} \quad , \quad 0<\lambda_\ell z_A \leq z \leq z_A \tag{5-38}$$

or

$$\varepsilon(\theta(v)) = \frac{(v + \tau)^2}{v^2 + \tan^2\theta_A \, e^{\frac{2(1-v)}{\tau}}} \cdot \varepsilon \quad , \quad \lambda_\ell \leq v \equiv \frac{z}{z_A} \leq 1 \tag{5-39}$$

by using the $\rho_1(z)$ for Γ_1 explicitly. Here v is just the z_A-normalized coordinate of points on Γ_1 and

$$\tau \equiv \frac{1-\lambda_\ell}{\ell n\lambda_t} > 0 \tag{5-40}$$

Taking $\varepsilon(\theta(v))$ as a function of v, the requirement $\varepsilon(\theta) \geq \varepsilon$ for Θ becomes

$$\varepsilon(\theta(v)) \geq \varepsilon \text{ for } 0<\lambda_\ell \leq v \leq 1 \tag{5-41}$$

which is equivalent to

$$\tan^2\theta_A \leq \tau \quad (\tau + 2v)e^{\frac{2(v-1)}{\tau}}, \quad \lambda_\ell \leq v \leq 1 \tag{5-42}$$

The right hand side of (5-42) is a monotomically increasing function of v, therefore the Θ_A should be chosen to satisfy

$$\tan^2\Theta_A \leq \tau \quad (\tau + 2\lambda_\ell)e^{\frac{2(\lambda_\ell-1)}{\tau}} \tag{5-43}$$

or in terms of λ_t and λ_ℓ for θ_B

$$\tan^2\Theta_B \leq \frac{(1-\lambda_\ell)[1-\lambda_\ell + 2\lambda_\ell \ell n\lambda_t]}{(\lambda_\ell \ell n\lambda_t)^2} \tag{5-44}$$

So the requirement of $\varepsilon(\Theta) \geq \varepsilon$ in II imposes restrictions on the independent parameters λ_ℓ, λ_t, and θ_A by (5-37) and (5-44).

For simplicity of the results, let us choose Θ to satisfy the equality of (5-44), i.e.

$$\tan^2\Theta_B = \frac{(1-\lambda_\ell)[(1-\lambda_\ell) + 2\lambda_\ell \ell n\lambda_t]}{(\lambda_\ell \ell n\lambda_t)^2} \tag{5-55}$$

Then the $\varepsilon(\Theta(v))$ satisfies

$$\varepsilon(\Theta(\lambda_\ell)) = \varepsilon \tag{5-56a}$$

$$\varepsilon(\Theta(1)) = \varepsilon \cdot \frac{\lambda_t^{\,2}\,(1+\tau)^2}{\lambda_t^{\,2} + \tau(\tau + 2\lambda_\ell)} > \varepsilon \tag{5-56b}$$

Examining $\varepsilon(\Theta(v))$ for $\Theta_A < \Theta < \Theta_B$ or for $\lambda_\ell < v < 1$, we see that

$$\frac{d\varepsilon(\Theta(v))}{dv} \sim (+)\ [(2\tau + v)(\tau + 2\lambda_\ell)(\lambda_t)^{\frac{2(\lambda_\ell - v)}{1-\lambda_\ell}} - \tau v]$$

$$\geq (+)\ [2\tau^2 + 4\lambda_\ell\tau + \tau v + 4\lambda_\ell v] > 0 \tag{5-57}$$

Therefore $\varepsilon(\Theta(v))$ is a monotomically increasing function of v for $\lambda_\ell \leq v \leq 1$. In terms of θ, these results (5-56) and (5-57) are equivalent to

$$\varepsilon(\theta_B) = \varepsilon \qquad (5\text{-}58a)$$

$$\varepsilon(\theta<\theta_B)>\varepsilon \qquad (5\text{-}58b)$$

and,

$\varepsilon(\theta)$ $\equiv$ monotonically decreasing function of θ for $\theta_{max}\leq\theta\leq\theta_B$ (5-59)

where θ_{max} is the place the maximum of $\varepsilon(\theta)$ occurs and is described in Fig. 5. This is clearly so because on Γ_1, as given by (5-30b), z is a strictly decreasing function of ρ and therefore a strictly decreasing function of θ for $\theta_{max}\leq\theta\leq\theta_B$. Looking at Fig. 2, this should be obvious because the outer geometrical length OO'O" is longer than the inner geometrical length MM'M". Therefore for OM and O"M" to have the same constant phase front the wave in II at $\theta + d\theta$ should travel faster than the wave at θ, and this precisely requires $\varepsilon(\theta + d\theta) < \varepsilon(\theta)$ for $\theta_{max}\leq\theta\leq\theta_B$.

B4. Fields and Their Matchings on Γ_1

Up to here we have only treated the matching geometry and the matching medium using a rough impedance approach. Now we want to see if the matching so obtained indeed matches a reflectionless and distortionless TEM wave from I to III.

In the cylindrical region I, the medium has constant simple parameters μ and ε, and the TEM fields are given by (5-1) and (5-2). In the conical region II, the medium has a constant μ, an $\varepsilon(\theta)$ implicitly given by (5-31) and (5-30b), and an anisotropic conductivity

(5-17), which in reality can be replaced by conical coaxial conducting sheaths satisfying (5-3) and (5-4), in the spherical coordinate system (Θ,ϕ,r). In such a II, we can easily verify that a TEM wave

$$E^{(\Theta)} = \frac{e^{i\omega\sqrt{\mu\varepsilon(\Theta)}\,r}}{r\sin\Theta} f(\Theta) \tag{5-60a}$$

$$H^{(\phi)} = \sqrt{\frac{\varepsilon(\theta)}{\mu}}\ \frac{e^{i\omega\sqrt{\mu\varepsilon(\Theta)}\,r}}{r\sin\Theta}\ f(\Theta) \tag{5-60b}$$

can exist and satisfy Maxwell eqs. provided an induced current density

$$J^{(r)} = \frac{1}{r^2\sin\Theta}\frac{\partial}{\partial\Theta}\left[\sqrt{\frac{\varepsilon(\Theta)}{\mu}}\ e^{i\omega\sqrt{\mu\varepsilon(\Theta)}\,r}\ f(\Theta)\right] \tag{5-61}$$

exist also. But the anisotropic conductivity (5-17) in II does have a $\sigma^{(r)(r)} = \infty$ in the radial direction and can suppress the electric field to be transverse by providing such a current. So the fields (5-60) and current density (5-61) are legitimate in region II. Here the $f(\Theta)$ is an arbitrary smooth function of Θ to be used for matching conditions on Γ_1 and Γ_2.

To match the field on Γ_1 between I and II, we need those components of (5-1), (5-2) and (5-60) tangential to Γ_1 to be equal on the intersurface Γ_1. Referring to Fig. 4, we see that the tangent to Γ_1 makes an angle Ψ with the axial z-axis and

$$\tan\Psi = \frac{c_2}{c_1} e^{\frac{-r_1(\Theta)\cos\Theta}{c_1}} \tag{5-62}$$

by using (5-30b). Thus we have on Γ_1

$$E_I^{(\rho)} \cos(\frac{\pi}{2} - \psi) = E_{II}^{(\Theta)} \cos(\frac{\pi}{2} - \psi - \Theta) \Rightarrow E_I^{(\rho)} = E_{II}^{(\Theta)} (\cos\Theta + \sin\Theta \cot \Psi) \tag{5-63a}$$

$$H_I^{(\phi)} = H_{II}^{(\phi)} \qquad \Longrightarrow \qquad E_I^{(\rho)} = \sqrt{\frac{\varepsilon(\Theta)}{\varepsilon}} E_{II}^{(\Theta)} \tag{5-63b}$$

where the subscripts I and II just emphasize the regions the fields are in. From (5-62) and (5-63), we see that the (5-63a) and (5-63b) can be consistent if and only if

$$\sqrt{\frac{\varepsilon(\Theta)}{\varepsilon}} = (\cos\Theta + \frac{c_1 \sin\Theta}{c_2 e^{-r_1(\Theta)\cos\Theta/c_1}}) \tag{5-64}$$

on Γ_1. But the $\varepsilon(\Theta)$ we found in (5-31) satisfies this relation (5-64) precisely. So with our appropriately shaped Γ_1 and inhomogeneous $\varepsilon(\theta)$, the matching of the tangential components of the TEM field on Γ_1 can be achieved by one condition (5-63a) or equivalently (5-63b). This one condition then determines the arbitrary function $f(\theta)$ and gives

$$f(\theta) = \frac{e^{-i\omega\sqrt{\mu\varepsilon}\, c_1}}{\cos\theta + (c_1/r_1(\theta))} \tag{5-65}$$

To summarize, the matched fields in I on II are

$$\text{I:} \quad \begin{cases} E_I^{(\rho)} = \dfrac{e^{i\omega\sqrt{\mu\varepsilon}\, z}}{\rho} \\ H_I^{(\phi)} = \sqrt{\dfrac{\varepsilon}{\mu}}\, \dfrac{e^{i\omega\sqrt{\mu\varepsilon}\, z}}{\rho} \end{cases} \tag{5-66a}$$

$$\text{II:} \quad \begin{cases} E_{II}^{(\theta)} = \dfrac{e^{i\omega\sqrt{\mu\varepsilon}\left[\frac{c_1+r_1(\theta)\cos\theta}{r_1(\theta)} r - c_1\right]}}{\left(\cos\theta + \frac{c_1}{r_1(\theta)}\right) r \sin\theta} \\ H_{II}^{(\phi)} = \sqrt{\dfrac{\varepsilon}{\mu}}\, \dfrac{e^{i\omega\sqrt{\mu\varepsilon}\left[\frac{c_1+r_1(\theta)\cos\theta}{r_1(\theta)} r - c_1\right]}}{r \sin\theta} \end{cases} \tag{5-66b}$$

B5. Fields and Their Matchings on Γ_2

Since the Γ_2 that divides regions II and III is of similar shape to Γ_1, the fields just obtained from Γ_1 matchings have no difficulty to be matched on Γ_2.

First, the tangent angle ψ_2 on Γ_2 is the same as the one on Γ_1, namely ψ. This is easily seen by using the $\rho_2(z)$ in (5-30d):

$$\tan\psi_2 = -\frac{d\rho_2(z)}{dz} = \frac{c_2}{c_1} e^{\frac{-z}{\lambda_a c_1}} = \frac{c_2}{c_1} e^{\frac{-r_1(\theta)\cos\theta}{c_1}} \tag{5-67}$$

Therefore, the TEM fields in III that match tangentially on Γ_2 the fields in II are

$$E_{III}^{(\rho)} = \frac{e^{i\omega\sqrt{\mu\varepsilon}\,[z+(\lambda_a-1)c_1]}}{\rho} \tag{5-68a}$$

TABLE 2

Regions / Description	Region I	Region II	Region III
Coordinate System	cylindrical (ρ,ϕ,z)	spherical (θ,ϕ,r)	cylindrical (ρ,ϕ,z)
Waveguide	$\rho = A$ to $\rho = \lambda_t A$ $(\lambda_t > 1)$	$\theta = \theta_A$ to $\theta = \tan^{+}(\frac{\lambda_t}{\lambda_\ell} \tan\theta_A)$ $(0 < \theta_A \leq \theta \leq \theta_B < \frac{\pi}{2},\ \lambda_\ell < 1)$	$\rho = \lambda_a A$ to $\rho = \lambda_t \lambda_a A$ $(\lambda_a > 1)$
Intersurface dividing regions described by each coordinate system	Γ_1 dividing I and II: $\rho = \rho_1(z) \equiv c_2 e^{-z/c_1}$ for $\lambda_\ell z_A \leq z \leq z_A$ where $z_A \equiv A \cot\theta_A$	Γ_1: $r = r_1(\theta)$ Γ_2: $r = r_2(\theta) \equiv \lambda_a r_1(\theta)$ where $r_1(\theta)$ is determined by $r_1(\theta)\sin\theta = c_2 e^{(-r_1(\theta)\cos\theta)/c_1}$ for $0 < \theta_A \leq \theta \leq \theta_B$	Γ_2 dividing II and III: $\rho = \rho_2(z) \equiv \lambda_a c_2 e^{-z/\lambda_a c_1}$ for $\lambda_\ell \lambda_a z_A \leq z \leq \lambda_a z_a$
Constitutive parameters for medium	Constant μ and ε, and $\sigma \equiv 0$	Same constant μ, but $\varepsilon(\theta) = \varepsilon[\cos\theta + \frac{c_1}{r_1(\theta)}]^2 \geq \varepsilon$ such that $\varepsilon(\theta_A) > \varepsilon$, $\varepsilon(\theta_B) = \varepsilon$, and $\varepsilon(\theta)$ monotonically decreasing for θ in $[\theta_A, \theta_B]$, and $\underset{\sim}{\sigma} = \begin{pmatrix} 0 & & O \\ O & \infty & \\ & & \infty \end{pmatrix}$	Constant μ and ε, and $\sigma \equiv 0$

TABLE 2 (continued)

Regions / Descriptions		Region I	Region II	Region III
Fields		$E^{(\rho)} = \frac{e^{i\omega\sqrt{\mu\varepsilon}\, z}}{\rho}$	$E^{(\theta)} = \frac{e^{i\omega\sqrt{\mu\varepsilon}[(c_1+r_1(\theta)\cos\theta)r/r_1(\theta) - c_1]}}{(\cos\theta + \frac{c_1}{r_1(\theta)})r\sin\theta}$	$E^{(\rho)} = \frac{e^{i\omega\sqrt{\mu\varepsilon}[z+(\lambda_a-1)c_1)}}{\rho}$
		$H^{(\phi)} = \sqrt{\frac{\varepsilon}{\mu}}\, \frac{e^{i\omega\sqrt{\mu\varepsilon}\, z}}{\rho}$	$H^{(\phi)} = \sqrt{\frac{\varepsilon}{\mu}}\, \frac{e^{\omega\sqrt{\mu\varepsilon}[(c_1+r_1(\theta)\cos\theta)r/r_1(\theta) - c_1]}}{r\sin\theta}$	$H^{(\phi)} = \sqrt{\frac{\varepsilon}{\mu}}\, \frac{e^{i\omega\sqrt{\mu\varepsilon}[z+(\lambda_a-1)c_1]}}{\rho}$
Current		0	$J^{(r)} = \sqrt{\frac{\varepsilon}{\mu}}\, \frac{1}{r^2\sin\theta}\, e^{-i\omega\sqrt{\mu\varepsilon}\, c_1}\, \frac{\partial}{\partial\theta}\, [e^{i\omega\sqrt{\mu\varepsilon(\theta)}\, r}]$	0
Constants		$c_1 \equiv \frac{(1-\lambda_\ell)A\cot\theta_A}{\ell n\lambda_t}$	$c_2 \equiv A\, e^{\frac{\ell n\lambda_t}{(1-\lambda_\ell)}}$	(See caption of Fig. 5 on p. 82)

$$H_{III}^{(\phi)} = \frac{\varepsilon}{\mu} \frac{e^{i\omega\sqrt{\mu\varepsilon}\,[z+(\lambda_a-1)c_1]}}{\rho} \qquad (5\text{-}68b)$$

B6. Summary of the Matching Problem between Two Cylinders

We summarize the solution of the fixed-μ perfect matching problem in Table 2.

We remind ourselves that in this table we use A, $\lambda_t(>1)$, θ_B (from (5-55), which in general should satisfy (5-54)), λ_ℓ (<1 as implied by $\lambda_t > 1$ and $\varepsilon(\theta) \geq \varepsilon$), and $\lambda_a(>1)$ as the free parameters of our problem. Also the conductivity $\underset{\approx}{\sigma}_{II}$ in II is only needed for its rr radial component and in reality can be replaced by conical coaxial perfectly conducting sheaths or radial conducting wires satisfying (5-3) and (5-4).

C. Approach with Fields Concept

In the previous Chapter 5-B, we see that the necessary requirements on circuit impedance and ray travelling time indeed result in the field matching. This may seem to be a lucky coincidence. But all these can be understood and expected more easily and obviously from an approach with fields concept for the original matching problem.

First, let the problem P still be described as in Chapter 5-A. Let the anisotropic conductivity in II be described by (5-3), (5-4) and (5-17). Then the reflectionlessness in II obviously dictates the $\varepsilon(\theta)$ being a function of θ only in II. Second, requiring the field in III to be distortionless TEM gives

$$E_{III}^{(\rho)} = \frac{e^{i\omega\sqrt{\mu\varepsilon}(z + K)}}{\rho} \qquad (5\text{-}69a)$$

$$H_{III}^{(\rho)} = \sqrt{\frac{\varepsilon}{\mu}}\frac{e^{i\omega\sqrt{\mu\varepsilon}(z + K)}}{\rho} \tag{5-69b}$$

where K is a phase constant. Third, the use of the $\underset{\approx}{\sigma}$ in II for surpressing mode to be purely TEM as the simplest matching possibility makes the wave form in II as (5-60).

Now, consider the intersurface Γ_1 between I and II. Let this intersurface be described either by $r = r_1(\theta)$ or $\rho = \rho_1(z_1)$ where the z_1 emphasizes the description for Γ_1. From the tangential field matching on Γ_1, clearly we must require (5-63) which immediately gives

$$\sqrt{\frac{\varepsilon(\theta)}{\varepsilon}} = (\cos\theta + \sin\theta\cot\psi(\theta)) \tag{5-70}$$

where $\psi(\theta)$ is the angle Γ_1 makes with the -z axis. Substituting the explicit expressions of the fields in I and II and (5-63) on Γ_1, we obtain

$$f(\theta) = \frac{e^{-i\omega\sqrt{\mu\varepsilon}\, r_1(\theta)\sin\theta\cot\psi(\theta)}}{\cos\theta + \sin\theta\cot\psi(\theta)} \tag{5-71}$$

Doing the same thing on Γ_2 which is described by $r = r_2(\theta)$ or $\rho = \rho_2(z_2)$, we get

$$\sqrt{\frac{\varepsilon(\theta)}{\varepsilon}} = (\cos\theta + \sin\theta\cot\psi_2(\theta)) \tag{5-72}$$

which, with the (5-70), implies

$$\psi_2(\theta) = \psi(\theta) \tag{5-73}$$

where $\psi_2(\theta)$ is the angle Γ_2 makes with the -z axis. Similar to (5-71) we also get

$$f(\theta) = \frac{e^{-i\omega\sqrt{\mu\varepsilon}[r_2(\theta)\sin\theta \cot\psi(\theta) + K]}}{\cos\theta + \sin\theta \cot\psi(\theta)} \tag{5-74}$$

in which (5-73) has been used.

To determine the $\psi(\theta)$, $r_1(\theta)$, $r_2(\theta)$, $\rho_1(z_1)$ and $\rho_2(z_2)$, we compare (5-71) with (5-74) and get

$$[r_2(\theta)-r_1(\theta)]\sin\theta \cot\psi(\theta) + K = 0 \tag{5-75}$$

But on Γ_1 we have $\rho_1(z_1) = r_1(\theta)\sin\theta$ and $\tan\psi = -d\rho_1(z_1)/dz_1$, and on Γ_2 we have $\rho_2(z_2) = r_2(\theta) \sin\theta$ and $\tan\psi_2 = -d\rho_2(z_2)/dz_2$, therefore (5-75) becomes

$$\left[\frac{\rho_2(z_2)}{\rho_1(z_1)} - 1\right] dz_1 = \frac{K d\rho_1(z_1)}{\rho_1(z_1)} \tag{5-76}$$

Since the right hand side of (5-76) is a function of z_1 only, so must be the left hand side. Thus $\rho_1(z_2)/\rho_1(z_1)$ can be a function of z_1 only. Because of (5-73), we can conclude similarly $\rho_2(z_2)/\rho_1(z_1)$ can be a function of z_2 only. Thus

$$\frac{\rho_2(z_2)}{\rho_1(z_1)} \equiv \lambda_a \tag{5-77}$$

where λ_a is a constant. Combining (5-76) and (5-77), we have for Γ_1

$$\Gamma_1 : \quad \rho_1(z_1) = c_2 e^{\frac{-z_1}{c_1}} \quad \text{or} \quad r_1(\theta)\sin\theta = c_2 e^{\frac{-r_1(\theta)\cos\theta}{c_1}} \tag{5-78}$$

where the constant $c_2 \equiv K/(\lambda_a - 1)$.

Now, from (5-78), the constants c_1 and c_2 can be determined just as in (5-26) and are given by (5-28). Then (5-77) yields again (5-30d) as the description $\rho_2(z_2)$ or $r_2(\theta)$ for the intersurface Γ_2. Also from (5-78) we have

$$\tan\psi(\theta) = -\frac{d\,\rho_1(z_1)}{d\,z_1} = \frac{c_2}{c_1}\, e^{\frac{-z_1}{c_1}}\, e^{\frac{-r_1(\theta)\sin\theta}{c_1}} \tag{5-79}$$

which with (5-70) implies

$$\varepsilon(\theta) = \varepsilon\Big(\cos\theta + \frac{c_1\sin\theta}{c_2 e^{\frac{-r_1(\theta)\sin\theta}{c_1}}}\Big)^2 = \varepsilon\Big(\cos\theta + \frac{c_1}{r_1(\theta)}\Big)^2 \tag{5-80}$$

precisely as obtained in the previous (5-31). Finally, the complete field expressions in regions II and III are obtained by inserting $K = (\lambda_a - 1)c_2$ and $f(\theta)$ from (5-74) into their respective expressions (5-60) and (5-69). If, furthermore, we require that in region II $\varepsilon(\theta) \geq \varepsilon$ and $\varepsilon(\theta_B) = \varepsilon$, then we obtain results all identical as listed in table 2.

In conclusion, we see that a field approach to our original matching problem directly can yield all the results in a logical and clear way. Besides, along the way it also makes clear why such a matching is possible in reducing the two tangential matchings of $\underset{\sim}{E}$ and $\underset{\sim}{H}$ to one condition by an appropriate choice of $f(\theta)$. Also, plots of the boundary surfaces Γ_1 and Γ_2 and the $\varepsilon(\theta)$ are shown in Fig. 5.

D. Scaling into the Parallel Plate Waveguide $P \to P'$

We have already solved the special matching problem P concerning two cylindrical TEM waveguides with fixed - μ loaded matching in two ways.

Now we want to see whether the P can be solved by using a DGM scaling technique via which a general procedure to obtain matchings of other geometrical shapes may reveal. To be specific, we want to scale P into its simplest parallel plate version P'. If such a P' can be obtained satisfactorily, then it may be possible to attain various different matchings by the $P' \to P$ scaling each with a different but appropriate geometry of particular choice.

D1. $P \to P'$ in Region I

Now for P the region I is described in Chapter 5-A. To get the scaled region I' for the scaled P', we choose (4-2) again:

$$(\rho,\phi,z) \leftrightarrow (\alpha\ \ell n \frac{\rho}{\rho_o}\ ,\ \alpha\phi, z) \leftrightarrow (x^1,x^2,x^3) \qquad (5\text{-}81)$$

because such a choice preserves the transverse wave impedance and gives constant transverse $\xi = \varepsilon$ and $\eta = \mu$ for the present $P \to P'$.

The results for the I' of P' are similar to those of Chapter 4. Namely, we have a parallel plate waveguide with plates at

$$x^1 = x^1_L = \alpha \ \ell n \frac{A}{\rho_o}$$

$$x^1 = x^1_U \equiv \alpha \ \ell n \frac{\lambda_t A}{o} \qquad (5-82)$$

with a medium given by

$$\frac{\xi^{ij}}{\varepsilon} = \frac{\eta^{ij}}{\mu} = \begin{pmatrix} 1 & & \bigcirc \\ & 1 & \\ \bigcirc & & \left(\frac{\rho_o e^{\frac{x^1}{\alpha}}}{\alpha}\right)^2 \end{pmatrix} \qquad (5-83)$$

in the Cartesian (x^1, x^2, x^3) coordinate frame, with a right side boundary intersurface $\Gamma^{I'}_1$ given by a linear function

$$\Gamma^{I'}_1 : \quad x^3 = c_1 \left[\ell n \frac{c_2}{\rho_o} - \frac{x^1}{\alpha}\right] \qquad (5-84)$$

and with a scaled TEM wave

$$e^{(1)}_I = \frac{1}{\alpha} e^{i\omega\sqrt{\mu\varepsilon}\ x^3} \qquad (5-85a)$$

$$h^{(2)}_I = \sqrt{\frac{\varepsilon}{\mu}}\, e^{i\omega\sqrt{\mu\varepsilon}\ x^3} \qquad (5-85b)$$

D2. $P \to P'$ in Region II

Now for the $P \to P'$ in II, we choose for II the scaling

$$(\theta,\phi,r) \leftrightarrow (u(\theta),\alpha\phi,\frac{r-k}{\beta}) \leftrightarrow (x^1,x^2,x^3) \tag{5-86}$$

where the ϕ-matching results the choice $\alpha\phi$, the orthogonality results $u(\theta)$, and the fixed μ results the choice of x^3 as linear function of r. The metric geometry of the scaling is then

$$d\sigma^2 = (\beta x^3+k)^2\left[\frac{dx^1}{u'(\theta(x^1))}\right]^2 + (\beta x^3+k)^2(\sin\theta(x^1))^2\frac{(dx^2)^2}{\alpha^2}$$

$$+\beta^2(dx^3)^2 \tag{5-87}$$

The Γ_1, described by $r = r_1(\theta)$ in II, gives the $\Gamma_1^{II'}$ as the intersurface between I' and II' by

$$\Gamma_1^{II'} : \; r_1(\theta(x^1)) = \beta x^3 + k \tag{5-88}$$

in terms of the (x^1,x^3) in II', where $\theta = \theta(x^1)$ is the inverse function of $x^1 = u(\theta)$. Since (5-84) and (5-88) should describe the same boundary intersurface Γ_1', therefore

$$r_1(\theta(x^1)) = \beta c_1[\ln\frac{c_2}{\rho_o} - \frac{x^1}{\alpha}] + k \tag{5-89}$$

where α,β,k are constants of the scaling. But the $r_1(\theta)$ is known implicitly through (5-30b), thus the $x^1 = u(\theta)$ or the $\theta = \theta(x^1)$

is determined implicitly by

$$\{\beta c_1[\ell n \frac{c_2}{\rho_o} - \frac{x^1}{\alpha}] + k\} \sin\theta = c_2 e^{\frac{-\{\beta c_1[\ell n \frac{c_2}{\rho_o} - \frac{x^1}{\alpha}] + k\}\cos\theta}{c_1}} \tag{5-90}$$

Also the Γ_2', scaled from Γ_2, as intersurface between II' and III' becomes

$$\Gamma_2^{II'}: x^3 = \lambda_a c_1[\ell n \frac{c_2}{\rho_o} - \frac{x^1}{\alpha}] + \frac{k}{\beta}(\lambda_a - 1) \tag{5-91}$$

if it is viewed in II'. Notice that the two exponential intersurfaces Γ_1 and Γ_2 for P becomes simply two straight line plane intersurfaces Γ_1' and Γ_2' for the scaled parallel plate problem P'.

D3. P → P' for Region III

Because the geometrical similarity of I and III, "naturally" we try for the scaling of III

$$(\rho,\phi,z) \leftrightarrow (\alpha\ell n \frac{\rho}{\lambda_a\rho_o}, \alpha\phi, \frac{z-p}{s}) \leftrightarrow (x^1,x^2,x^3) \tag{5-92}$$

where the $\alpha\phi$ and the $\lambda_a\rho_o$ instead of ρ_o are obvious choices. Now in III' the $\Gamma_2^{III'}$, scaled from the Γ_2: $\rho = \rho_2(z)$ of the (5-30d), becomes

$$\Gamma_2^{III'}: \frac{sx^3 + p}{\lambda_a c_1} + \frac{x'}{\alpha} = \ell n \frac{c_2}{\rho_o} \tag{5-93}$$

Since (5-93) should be identical to (5-91), as they describe different expressions of the same interface Γ_2', we have

$$\begin{cases} s = 1 \\ p = (\lambda_a - 1)\,\dfrac{k}{\beta} \end{cases} \qquad (5\text{-}94)$$

D4. Determine the Scaling Constants

We have $\rho_o, \alpha, k, \beta, s,$ and p as constants of scaling. Already (5-94) reduces s and p in terms of k and β. To determine the other constants in terms of parameters in P before the scaling, we compare the length along θ_A and θ_B of II in P· to the lengths along x_L^1 and x_U^1 of II' in P'. We get

$$(\lambda_a - 1)\,\frac{r_1(\theta_A)}{\beta} = (\lambda_a - 1)\,\frac{k}{\beta} + (\lambda_a - 1)c_1 \ln \frac{c_2}{A} \qquad (5\text{-}95a)$$

$$(\lambda_a - 1)\,\frac{r_1(\theta_B)}{\beta} = (\lambda_a - 1)\,\frac{k}{\beta} + (\lambda_a - 1)c_1 \ln \frac{c_2}{\lambda_t A} \qquad (5\text{-}95b)$$

which immediately gives

$$k = \frac{r_1(\theta_B)\ln\frac{c_2}{A} - r_1(\theta_A)\ln\left(\frac{c_2}{\lambda_t A}\right)}{\ln\lambda_t} = A \cdot \frac{\sqrt{\lambda_\ell^{\,2} + \lambda_t^{\,2}\tan^2\theta_A}}{(1 - \lambda_\ell)\tan\theta_A} \qquad (5\text{-}96a)$$

$$\beta = \frac{r_1(\theta_A) - r_1(\theta_B)}{\ln\lambda_t} = \frac{\sec\theta_A - \sqrt{\lambda_\ell^{\,2} + \lambda_t^{\,2}\tan^2\theta_A}}{(1 - \lambda_\ell)} \qquad (5\text{-}96b)$$

As to ρ_o and α, they are the degrees of parameter freedom left in the $P \to P'$ scaling.

D5. Summary of the Parallel Plate Problem P'

We have found a parallel plate Cartesian scaled version P' of P, the original matching problem and its solution. The regions I', II', and III' become a combined and extended parallel plate region. The scaled media in I' and III' are simple and are the same. The scaled medium in II' is complicated with inhomogeneous ε_{II}', inhomogeneous μ_{II}', and longitudinal perfect conductivity $\underset{\approx}{\sigma}_{II'}$.

The scaling geometry itself is given explicitly by (5-8) for I', and by (5-92) for III'. The scaling for II' is given implicitly by (5-86) and (5-90). Also all but two scaling constants of P' are found in terms of parameters of P. The two scaling constants ρ_o and α left undertermined represent the degree of parameter freedom in the $P \to P'$.

E. Concluding Remarks

In this report we have investigated and developed the differential geometry scaling method for EM theory and examined its applications. The essence is, at least two of the three constitutive parameters $\underset{\approx}{\mu}$, $\underset{\approx}{\varepsilon}$, and $\underset{\approx}{\sigma}$ have to be allowed to change according to the scaling need. With such accompanying change of scaled media, the scaling method is very powerful in creating an equivalent class of problems and their solutions from a given problem of known solution. Through this we can obtain solutions to many interesting problems from the knowledge of a simple problem.

In the applications we solved two problems. One is a μ and ε loaded matching section between a cylindrical and a conical coaxial TEM waveguide. The other is a fixed-μ matching section between two cylindrical coaxial TEM waveguides of different dimensions.

In the fixed-μ matching, it seems possible that similar matching problems of different geometry can be obtained by some variance on the inverse scaling of the basic problem P' that we have found in Chapter 5D. This will be of interest for further investigations.

Finally, if we relieve the restriction of being in orthogonal scaling, more problems can be treated by the DGM. For example, non-normal incidence and Brewster angle transmission can be used for the reflectionlessness requirement. Also, E-mode or H-mode in the waveguide can be treated easily. These and many other aspect of application of DGM will also be of interest for further investigation.

Acknowledgement

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References

1. See, e.g., L. D. Landau and E. M. Lifshitz, Fluid Dynamics, Pergamon Press (1959), Secs. 19, 53, 118, 119; also, Mechanics, Pergamon Press (1960), Sec. 10; for an earlier example of such transforms, see E. J. Routh, "Some Applications of Conjugate Functions", Proc. London Math. Soc. 12, 73-89 (1881).

2. Some different examples of the electromagnetic scaling can be found in the following. For conformal mapped waveguide see, e.g., F. E. Borgnis and C. H. Papas, "Electromagnetic Waveguides and Resonators" in Handbuch der Physik, Springer-Verlag, Berlin (1958) 16, 358; F. J. Tischer, "Conformal Mapping in Waveguide Considerations" (Correspondence) Proc. IEEE 51, 1050 (July 1963), and "Conformal Mapping Applied to Three-Dimensional Wave Problems", (Correspondence) Proc. IEEE 53, 168 (Feb. 1965); also J. A. Stratton, Electromagnetic Theory, McGraw-Hill (1941), 217; also P. Krasnooshkin, "Acoustic and Electromagnetic Waveguides of Complicated Shape", J. Phys. USSR 10, 434 (1946); for an approach using invariance groups in differential forms, see B. K. Harrison and F. B. Estabrook, "Geometric Approach to Invariance Groups and Solution of Partial Differential Systems", J. Math. Phys. 12, 653 (April 1971); for a frequency scaling of reflection see, J. H. Davis and J. R. Cogdell, "Reflection Efficiency Evaluation by Frequency Scaling", IEEE Trans. AP-19, 58 (Jan. 1971); for a scaling for reducing constantly moving uniform simple media, see R. J. Pogorzelski, "A Technique for Solution of Maxwell's Equations in a Moving Dielectric Medium", IEEE Trans. AP-19, (Communication) 455 (May 1971).

3. See any textbook on basic electromagnetic theory, e.g., see D. T. Paris and F. K. Hurd, Basic Electromagnetic Theory, McGraw-Hill (1969), Chapters 3 and 4.

4. For most recent treatments on scaling, see C. E. Baum, "A Lens Technique for Transitioning Waves between Conical and Cylindrical Transmission Lines", EMP Sensor and Simulation Notes 32, (Jan. 1967), and "A Scaling Technique for the Design of Idealized Electromagnetic Lenses", EMP

Sensor and Simulation Notes 64 (1968), and also Ph.D. Thesis, Caltech Antenna Lab Report 47 (1968), California Institute of Technology. For an example of tapering the dielectric to suit propagation, see P. L. Uslenghi, "On the Generalized Luneberg Lenses", IEEE Trans. AP-17 (Communication), 644 (Sept. 1969).

5. T. C. Mo, C. H. Papas, and C. E. Baum, "General Scaling Method for Electromagnetic Fields with Application to a Matching Problem", J. Mathematical Phys, April 1973.

6. See any text on relativistic electrodynamics, e.g., V. Fock, The Theory of Space, Time and Gravitation, Pergamon Press (1964), Sec. 24.

7. Notice that the signature (+−−−) is used, therefore, for special relativity flat spacetime $dS^2 = dt^2 - dx^2 - dy^2 - dz^2$. Also geometrized MKS unit with $\mu_o = \varepsilon_o = 1$ for vacuum is used for simple formalisms. For retrieval to MKS see, e.g., attached table in T. C. Mo, "Electromagnetic Wave Propagation in a Uniformly Accelerated Simple Medium", Radio Science 6, 673 (June 1971). For any applications in special relativistic EM theory, only insert appropriate powers of c (3×10^8 meters/sec) in the final answer to fix dimensions right.

8. See, e.g., R. H. Dicke, The Theoretical Significance of Experimental Relativity, Gordon & Breach, 1964.

9. See any textbook on relativity, e.g., J. L. Synge, General Relativity, Interscience (1960), Sec. 3.

10. T. C. Mo, "Theory of Electrodynamics in Media in Noninertial Frames and Applications", J. Mathematical Phys. 11, 2589 (1970), Sec. 4. Also for 3-vectors, $D^{(\ell)}$ $-D_{(\ell)}$, etc.

11. All notations here are identical to the usual three-dimensional vector analysis ones.

12. See Ref. 10.

13. The problem p can be any EM problem. It consists of fields, media, and boundary geometry. Also for both P and P' to be really physical, care must be taken into account to make x^i have dimensions of length such that $g_{\mu\nu}$ are dimensionless pure numbers.

14. Wave impedance is always defined as E/H . Here it is defined using the transverse E_t and H_t .

15. See also Ref. 4, C. E. Baum, Sensor and Simulation Note 64.

16. See L. P. Eisenhart, A Treatise on the Differential Geometry of Curves and Surfaces, Dover, 1960, p. 449, see also Ref. 15.

17. P. Moon and D. E. Spencer, Field Theory Handbook, Springer-Verlag, Berlin, 1961, p. 112, see also Ref. 15.

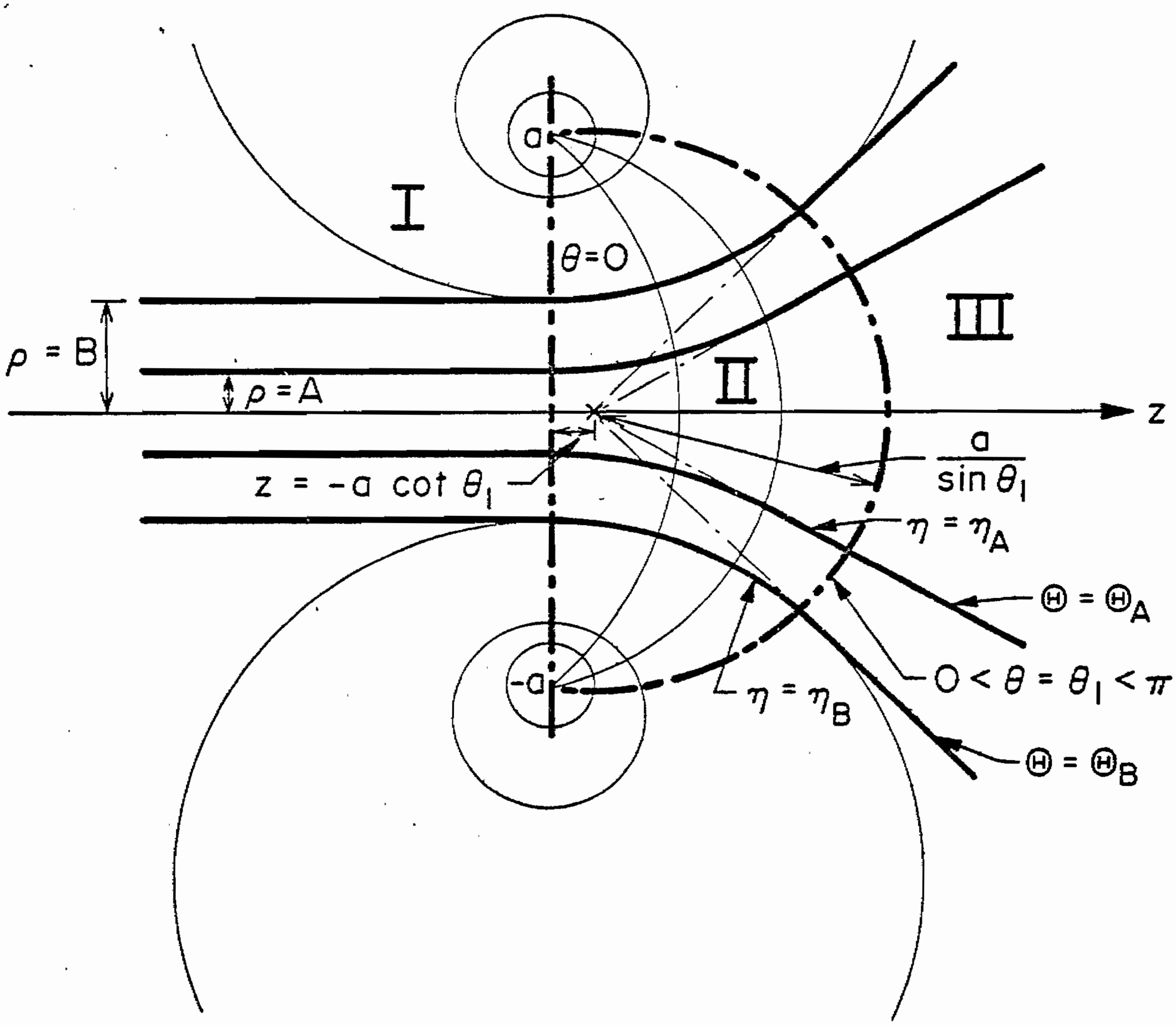


Fig. 1; Geometry of Variable μ and ε Matching Section for Chapter 4

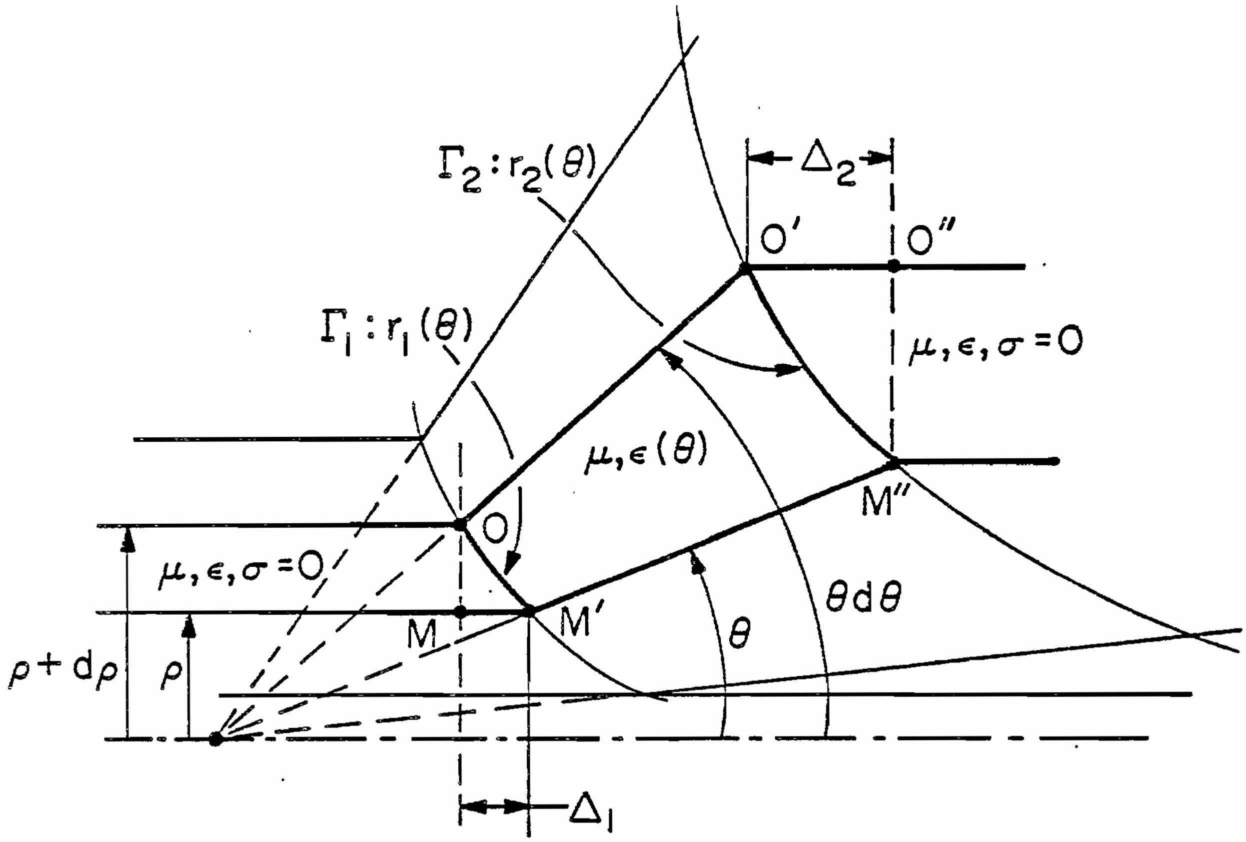


Fig. 2; Geometry of Infinitesimal Impedance and Travelling Time Considerations for Chapter 5-B

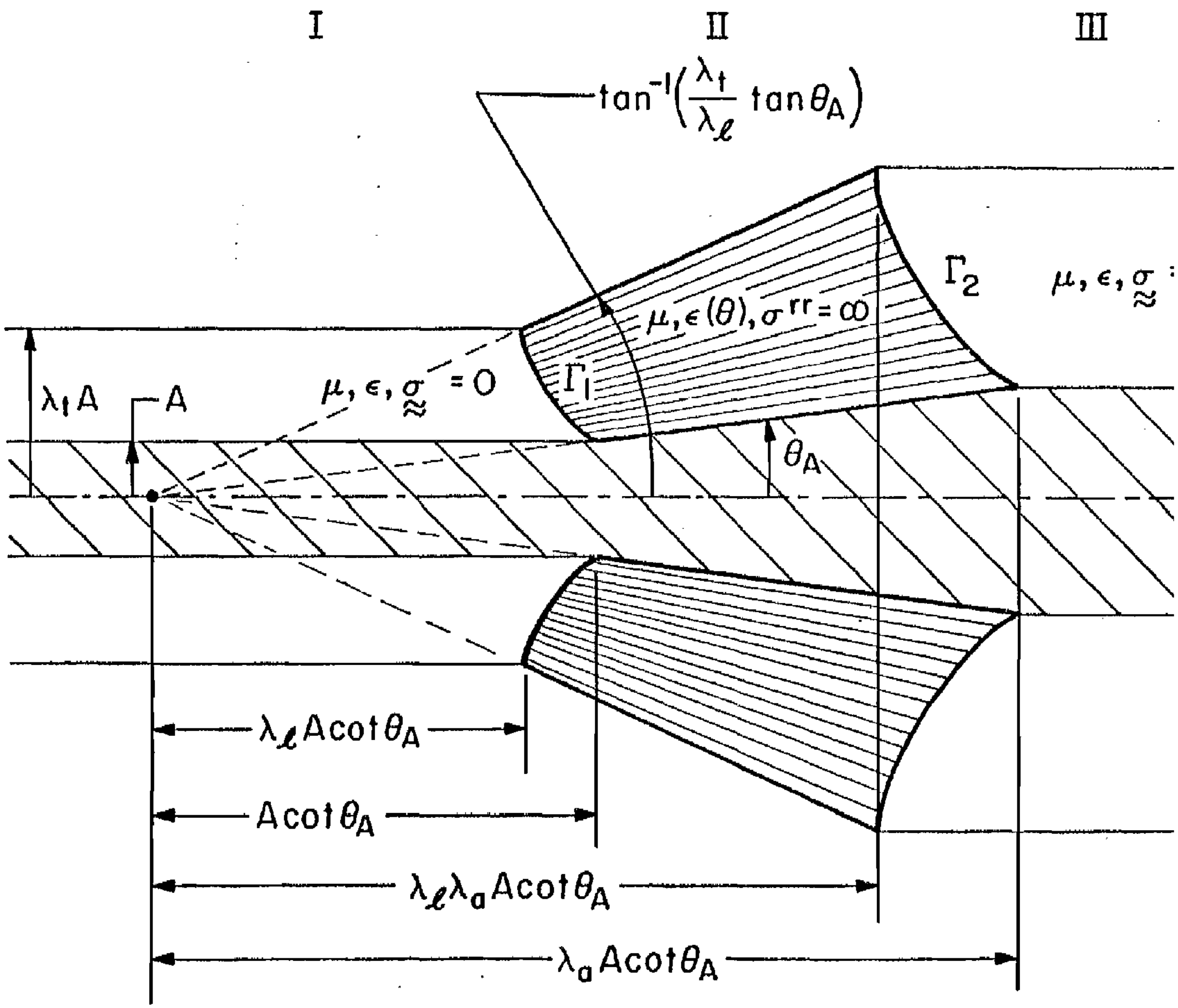


Fig. 3; Geometry of Fixed μ, Variable ε, and Anisotropic Conductivity Matching Section for Chapter 5

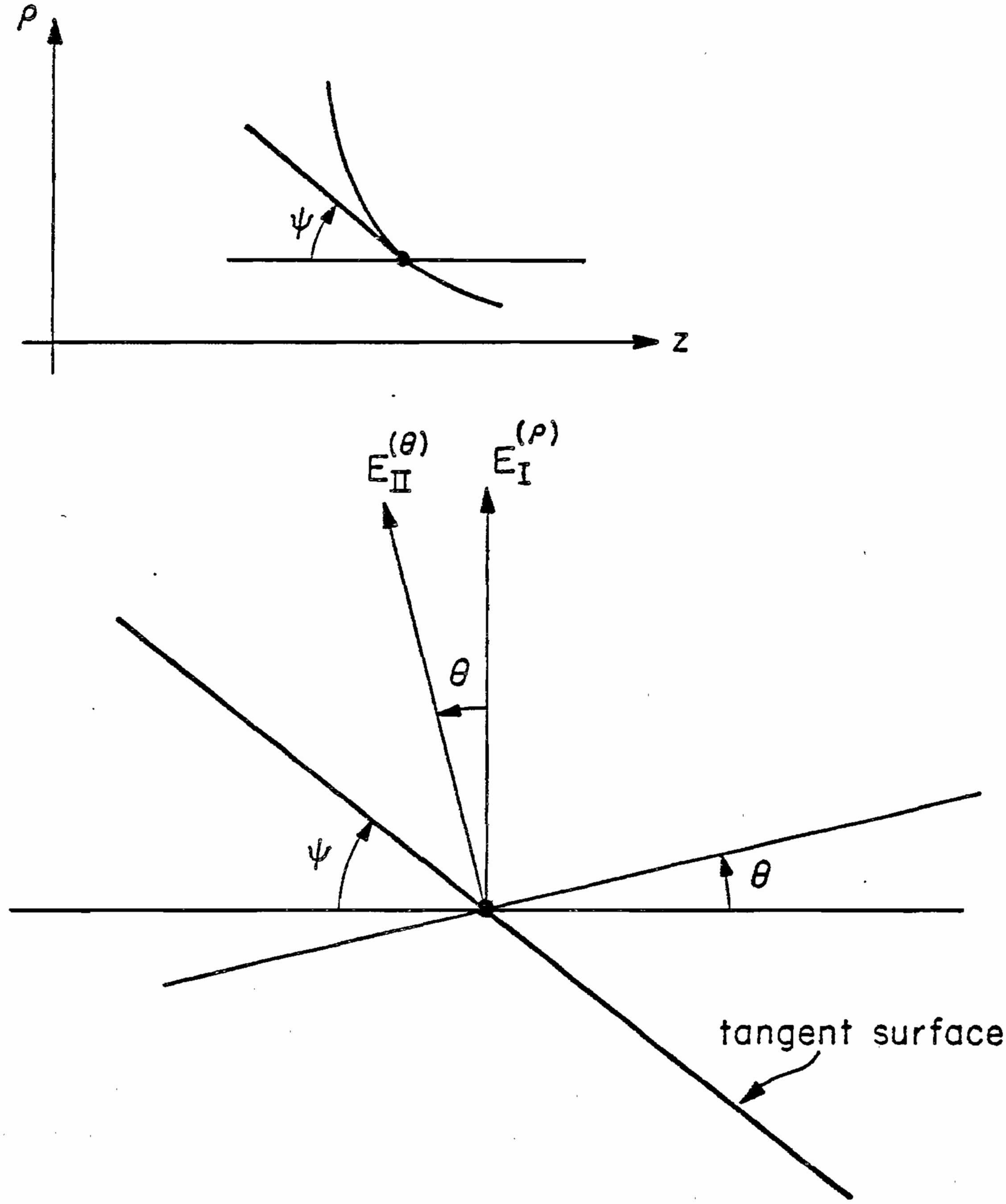


Fig. 4; Tangential Field Matchings on Γ_1 Between I and II

Fig. 5 Plots of $r_1(\theta)$, $r_2(\theta)$ of Γ_1, Γ_2, and $\varepsilon(\theta)$ as Function of θ

The equations for $r_1(\theta)$ of Γ_1 and $r_2(\theta)$ of Γ_2 as given in Table 2 on p. 62, can be simplified by normalizing them relative to their value at the outer conical angle θ_B .

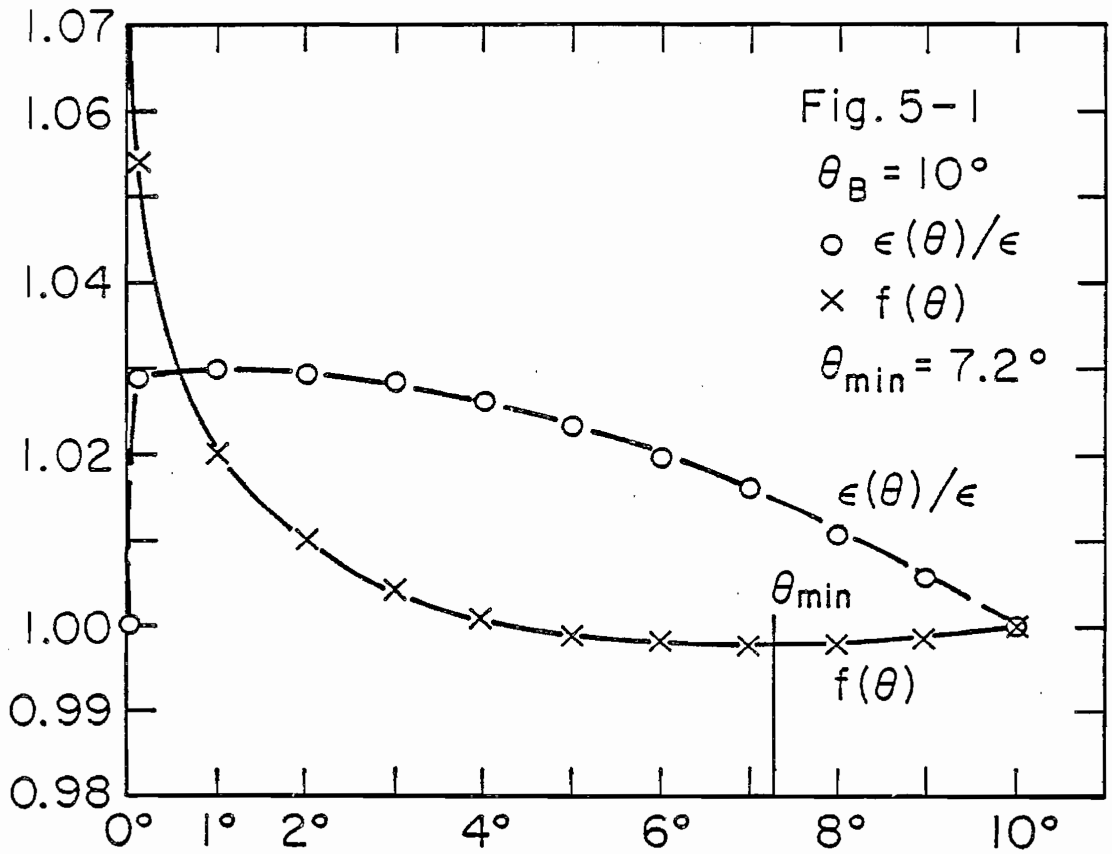
$$\frac{r_1(\theta)}{r_1(\theta_B)} = \frac{r_2(\theta)}{r_2(\theta_B)} \equiv f(\theta)$$

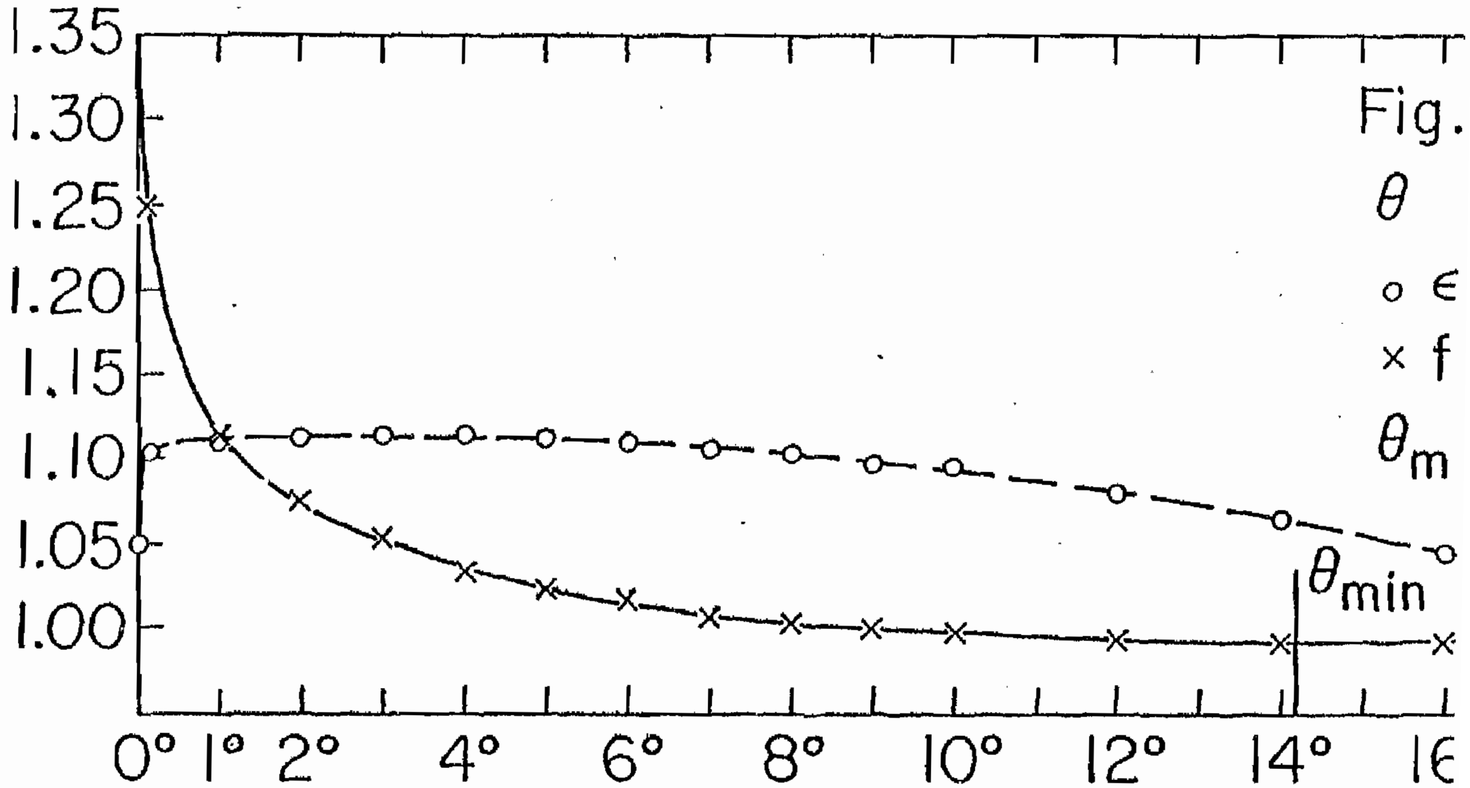
which satisfies

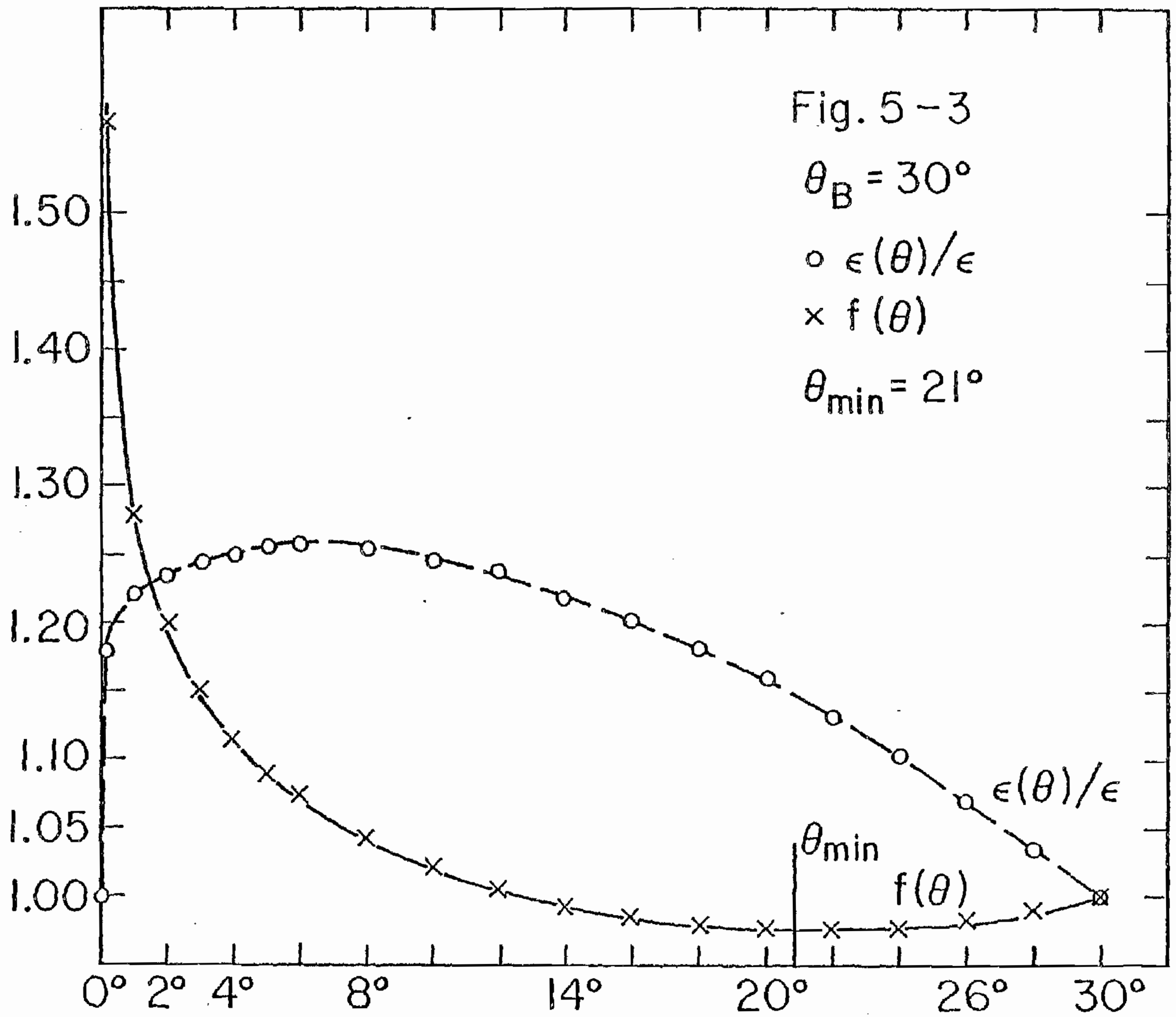
$$f(\theta)\sin\theta = \sin\theta_B \, e^{\frac{\cos\theta_B - f(\theta)\cos\theta}{1-\cos\theta_B}}$$

Also, correspondingly the expression for $\varepsilon(\theta)$ is $\frac{\varepsilon(\theta)}{\varepsilon}$ = $[\cos\theta + \frac{(1-\cos\theta_B)}{f(\theta)}]^2$

Notice that there is only one parameter θ_B for the curves $f(\theta)$ and $\varepsilon(\theta)/\varepsilon$, and each of the following plots for $f(\theta)$ and $\varepsilon(\theta)/\varepsilon$ is for a different parameter value θ_B. Furthermore, we have $f(\theta) \to \infty$ as $\theta \to o$ and $f(\theta)\sin\theta \to o$ as $\theta \to o$. Also, since $f'(\theta) \to -\infty$ as $\theta \to o$ and $f'(\theta_B) > o$, therefore the minimum of $f(\theta)$ must occur at $\theta_{min} < \theta_B$ and this θ_{min} satisfies $f(\theta)\sin\theta\tan\theta = 1 - \cos\theta_B$. Finally, since $(\sqrt{\varepsilon(\theta)/\varepsilon})' > o$ at $\theta = o$ and $< o$ at $\theta = \theta_B$, therefore the maximum for $\varepsilon(\theta)$ must occur at $\theta_{max} < \theta_B$ and this θ_{max} satisfies $f^2(\theta)\sin\theta + (1-\cos\theta_B)f'(\theta) = 0$.


1.07
1.06
1.04
1.02
1.00
0.99
0.98
0° 1° 2° 4° 6° 8° 10°
Fig. 5-1
$\theta_B = 10°$
o $\epsilon(\theta)/\epsilon$
× $f(\theta)$
$\theta_{min} = 7.2°$
$\epsilon(\theta)/\epsilon$
θ_{min}
$f(\theta)$


1.35
1.30
1.25
1.20
1.15
1.10
1.05
1.00
0° 1° 2° 4° 6° 8° 10° 12° 14° 16
Fig.
θ
o ∈
× f
θm
θmin



θB = 30°
○ ε(θ)/ε
× f(θ)
θmin = 21°
1.50
1.40
1.30
1.20
1.10
1.05
1.00
0° 2° 4° 8° 14° 20° 26° 30°
ε(θ)/ε
θmin
f(θ)

Fig. 5-3

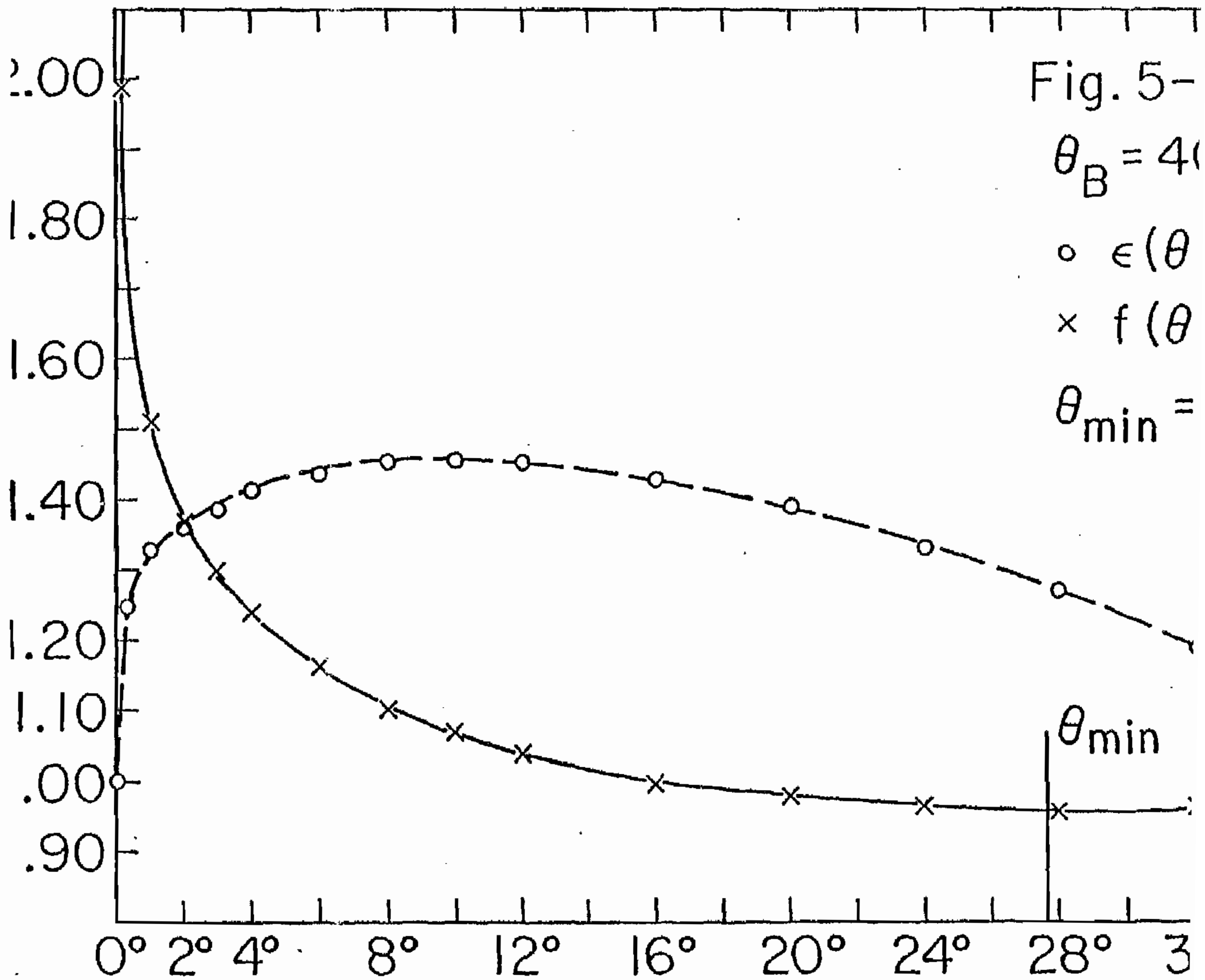


Fig. 5-
θ_B = 4
∘ ϵ(θ
× f (θ
θ_{min} =
θ_{min}
2.00
1.80
1.60
1.40
1.20
1.10
.00
.90
0° 2° 4° 8° 12° 16° 20° 24° 28° 3

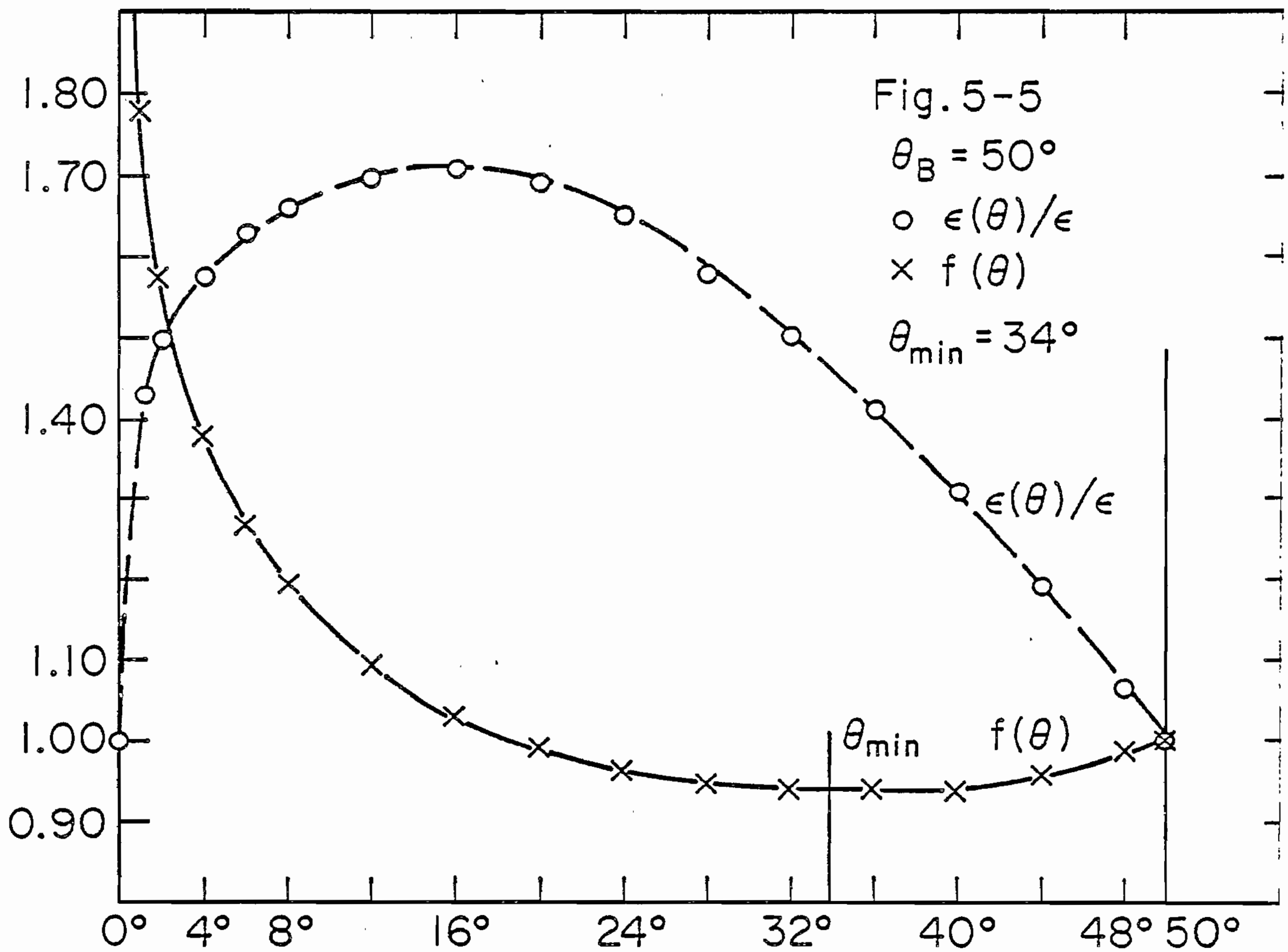


Fig. 5-5
$\theta_B = 50°$
○ $\epsilon(\theta)/\epsilon$
× $f(\theta)$
$\theta_{min} = 34°$
1.80
1.70
1.40
1.10
1.00
0.90
$\epsilon(\theta)/\epsilon$
θ_{min}
$f(\theta)$
0°
4°
8°
16°
24°
32°
40°
48°
50°

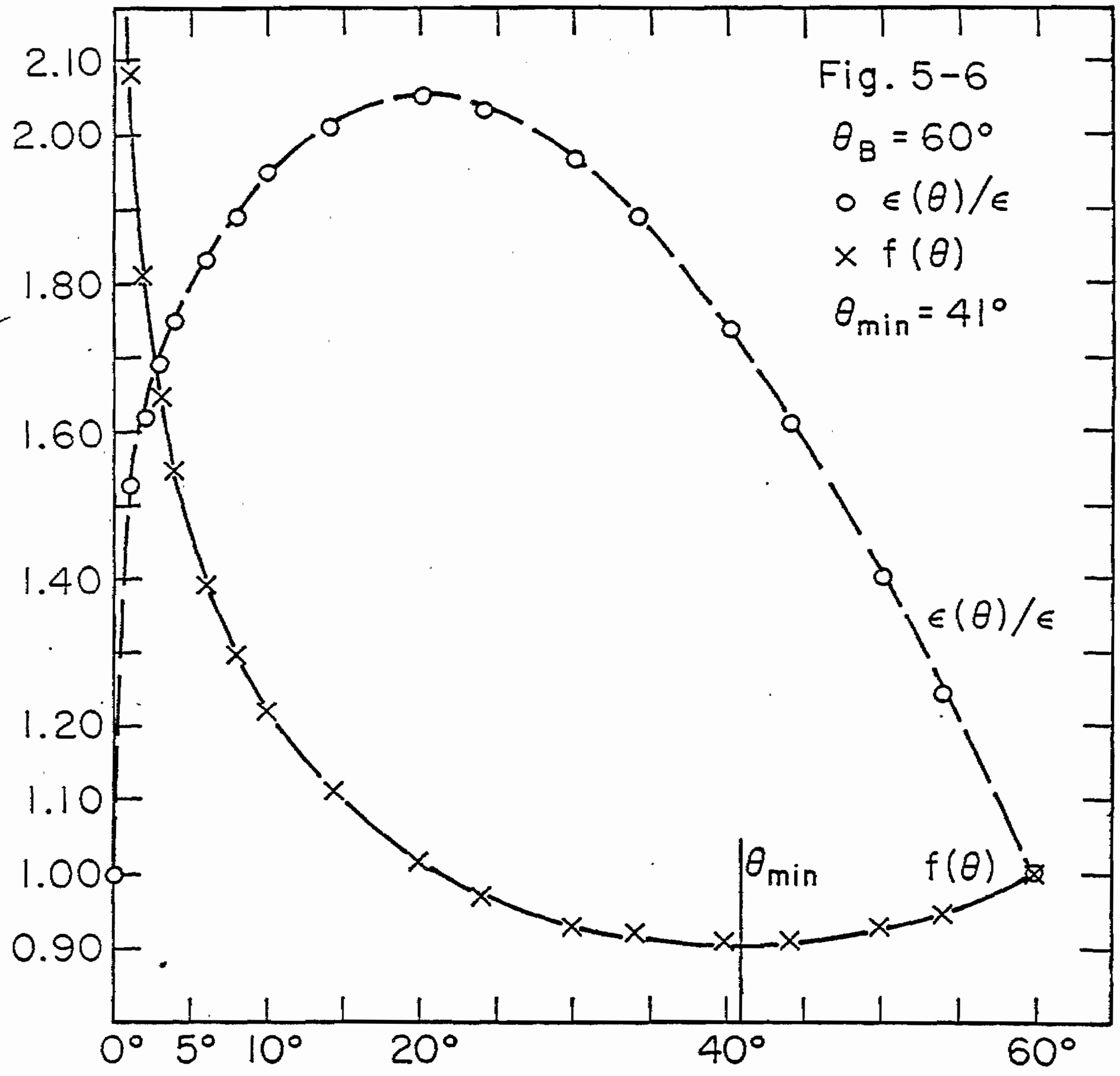


Fig. 5-6
θ_B = 60°
o $\epsilon(\theta)/\epsilon$
× f(θ)
θ_{min} = 41°
2.10
2.00
1.80
1.60
1.40
1.20
1.10
1.00
0.90
$\epsilon(\theta)/\epsilon$
θ_{min}
f(θ)
0° 5° 10° 20° 40° 60°

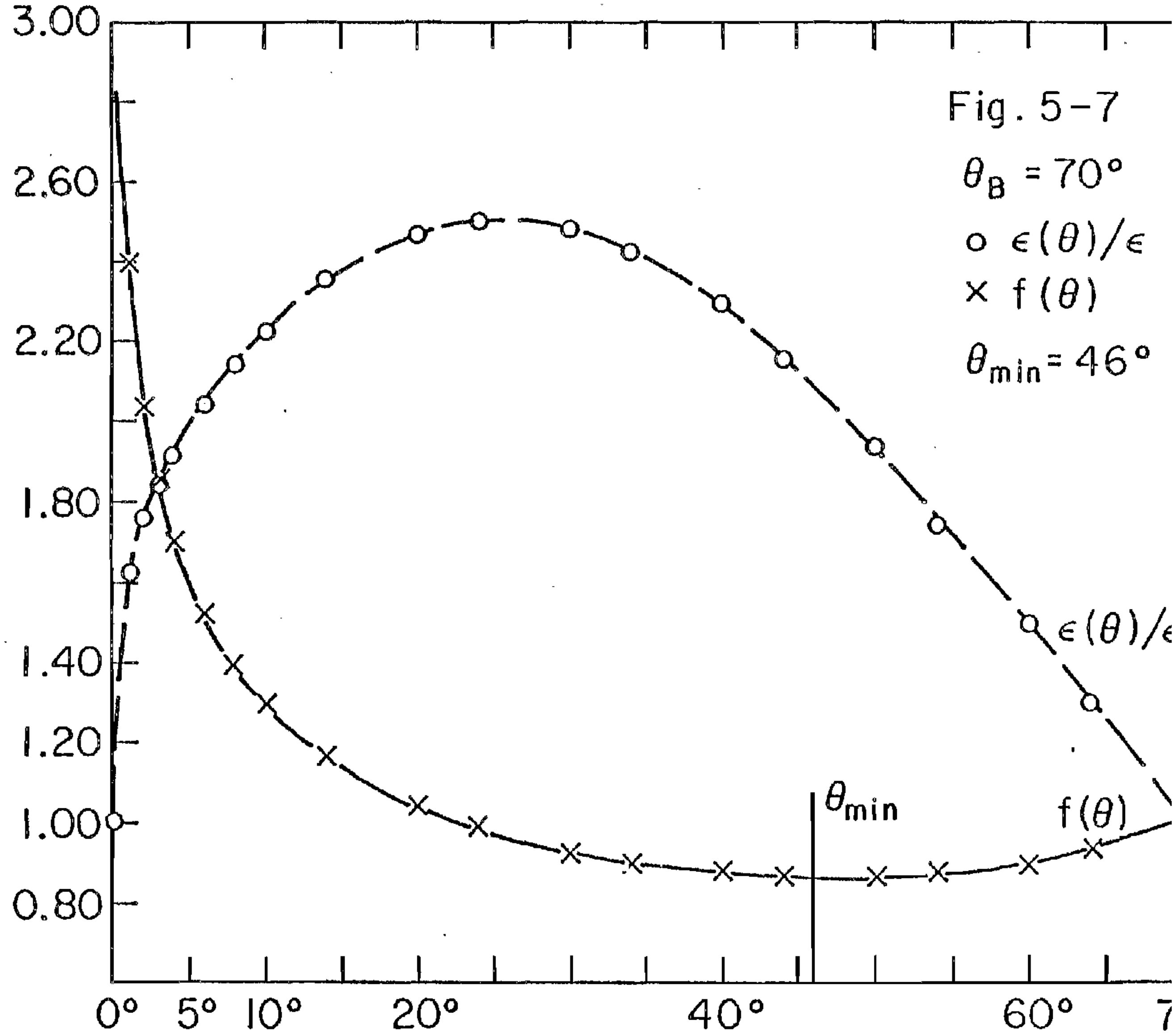
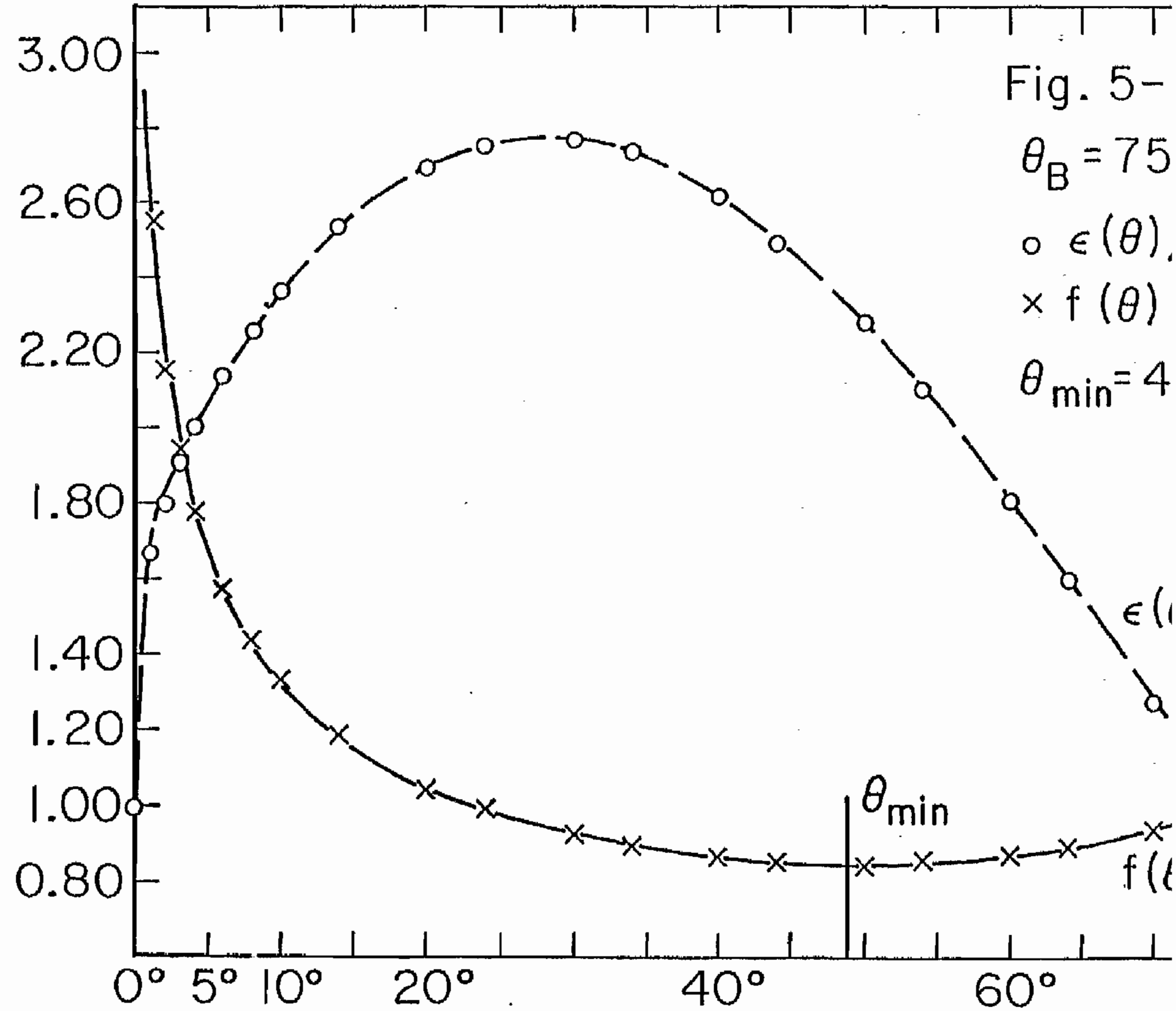


Fig. 5-7
$\theta_B = 70°$
o $\epsilon(\theta)/\epsilon$
× $f(\theta)$
$\theta_{min} = 46°$
$\epsilon(\theta)/\epsilon$
θ_{min}
$f(\theta)$
3.00
2.60
2.20
1.80
1.40
1.20
1.00
0.80
0°
5°
10°
20°
40°
60°



3.00
2.60
2.20
1.80
1.40
1.20
1.00
0.80
0° 5° 10° 20° 40° 60°
Fig. 5-
$\theta_B = 75$
○ $\epsilon(\theta)$,
× $f(\theta)$
$\theta_{min} = 4$
θ_{min}

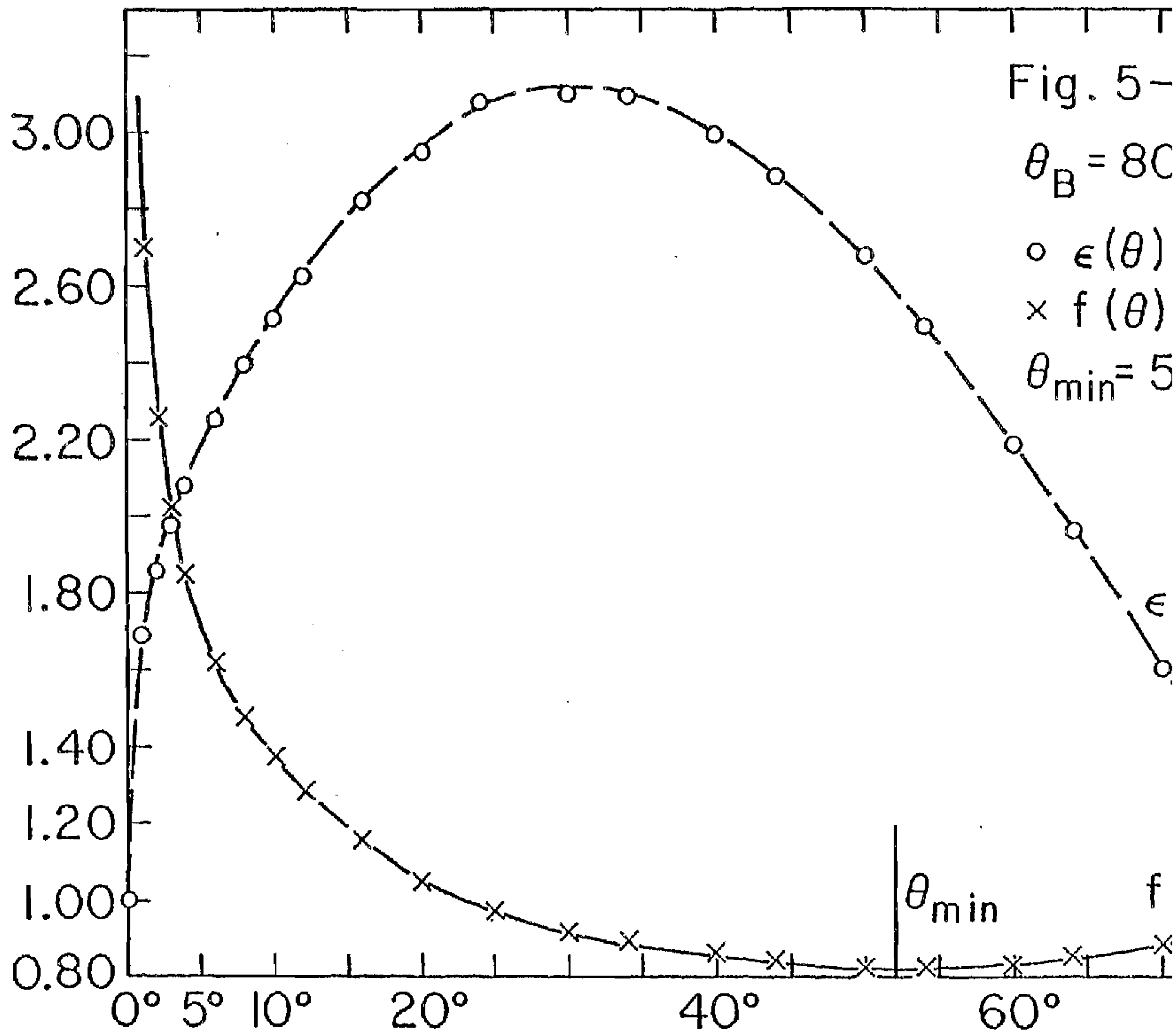
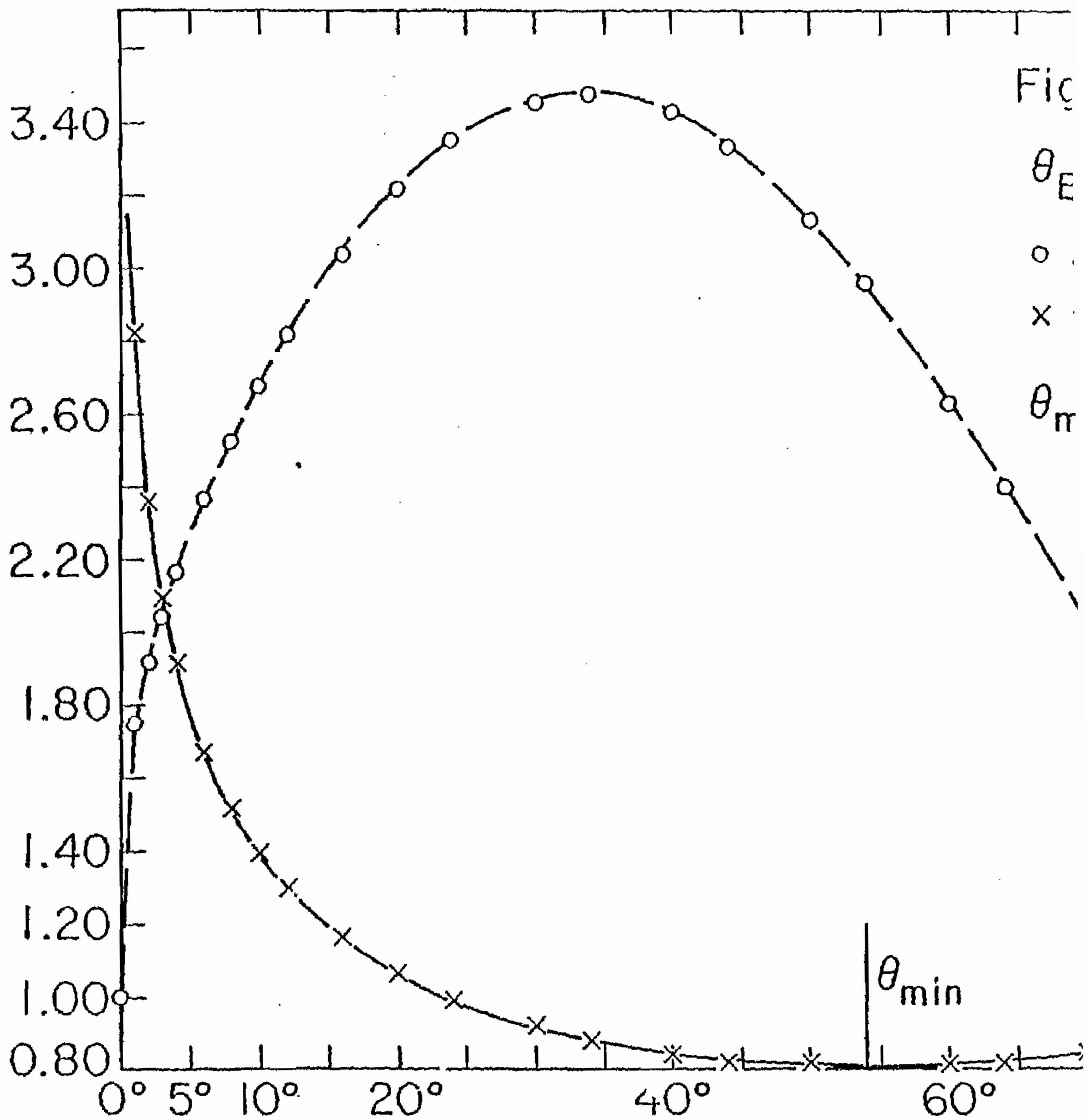


Fig. 5-
θ_B = 8C
○ $\epsilon(\theta)$
× f (θ)
θ_{min}= 5
3.00
2.60
2.20
1.80
1.40
1.20
1.00
0.80
0° 5° 10° 20° 40° 60°
ϵ
θ_{min}
f



3.40
3.00
2.60
2.20
1.80
1.40
1.20
1.00
0.80
0° 5° 10° 20° 40° 60°
θ_{min}
Fi
θ_E
o
x
θ_m