Sensor and Simulation Notes

Note 439

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Optimization of Reflector IRA Aperture for Filling a Rectangle

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Abstract

In designing an impulse radiating antenna (IRA) (of either reflector or lens variety) various shapes of the antenna aperture are possible. In the case of a reflector IRA with four symmetrically positioned feed arms, the feed arms determine the feed impedance and the fields incident on the antenna aperture. By appropriate choice of the aperture boundary one can include those portions of the aperture with electric field oriented in a direction that adds to the aperture equivalent height, while deleting those portions that do not. As a canonical problem, using a thin-wire approximation for the feed arms, the equivalent height of a rectangular aperture is considered and compared to a circular aperture filling the same physical height.

Contents

1. **Introduction**

In [4] the positioning of the feed arms to illuminate an IRA aperture was extended from the configuration in [1], in which the four-arm configuration (nominally 200 Ω) with the planes of the arms separated by 90° (right angle), to more general angles (e.g., ϕ_0 of first feedarm plane moving from 45° to 60°). This increased the equivalent height h_{a_y} of the aperture proportional to $\sin(\phi_0)$, but also raised the feed impedance (for same feed-arm dimensions), giving competing factors to optimize.

In the present paper we consider other shapes than circular disks for the aperture. This will allow us to consider which portions contribute to h_{a_y} and which diminish it and, hence, can be profitably removed. Assuming that one has a confined space in which to fit the aperture (e.g., a maximum height) then one can design the aperture shape to maximize the far field (i.e., via h_{a_y}). We first have some qualitative considerations for the common case of four edge-on flat-plate feed arms. This is followed by the case of a four-thin-wire feed for which analytic formulae are obtained for a rectangular aperture, thereby extending the results of [2].

Summarizing, the impulsive part of the far field on boresight is

$$\vec{E}_f(\vec{r},t) = \frac{V_0}{r}\frac{1}{2\pi c f_g}\vec{h}_a\,\delta_a(t-\tfrac{r}{c})$$

δ_a ≡ approximate delta function → delta function as $r \to \infty$

$V_0 u(t)$ ≡ voltage on TEM feed at aperture plane

$t = 0$ ≡ time step-function plane wave arrives on antenna aperture S_a

r ≡ distance from aperture plane to observer

$$f_g \equiv \frac{Z_c}{Z_0} = \frac{\Delta u}{\Delta v} \equiv \text{geometrical impledance factor} \tag{1.1}$$

$$c = [\mu_0\varepsilon_0]^{-\frac{1}{2}} \equiv \text{speed of light} \simeq 2.997925 \times 10^8 \text{ m/s}$$

$$Z_0 = \left[\frac{\mu_0}{\varepsilon_0}\right]^{\frac{1}{2}} \simeq 376.73\ \Omega$$

≡ wave impedance of free space.

For more general exciting fast-rising waveforms $V(t)$, the impulsive part of the far field is

$$\vec{E}_f(\vec{r},t) = \frac{1}{2\pi r c f_g}\vec{h}_a\frac{dV(t-\frac{r}{c})}{dt} \tag{1.2}$$

A fundamental parameter is the aperture height, which in vector form is

$$\vec{h}_a = h_{a_x} \vec{1}_x + h_{a_y} \vec{1}_y \tag{1.3}$$

and which is expressed more conveniently in complex form as [2]

$$\begin{aligned} h_a &= h_{a_x} - jh_{a_y} = \frac{1}{\Delta v} \int_{S_a} \frac{dw(\zeta)}{d\xi} \, dx dy \\ &= \frac{j}{2\Delta v} \oint_{C_a} w(\zeta) d\xi^* = \frac{j}{\Delta v} \oint_{C_a} u(\zeta) d\xi^* = -\frac{1}{\Delta v} \oint_{C_a} v(\zeta) d\xi^* \end{aligned}$$

$$\begin{aligned} w(\xi) &= u(\xi) + jv(\xi) \equiv \text{complex potential function} \\ \xi &= x + jy \equiv \text{complex coordinate on aperture plane} \\ \Delta u &\equiv \text{change in the electric potential } u \text{ between conductors (taken positive)} \\ \Delta v &\equiv \text{change in the magnetic potential in going around the conductors} \\ &\quad \text{(if two, those with the same sign of currrent), also taken positive} \\ C_a &\equiv \text{aperture contour (boundary of } S_a\text{) with integral taken in the usual positive direction} \end{aligned} \tag{1.4}$$

Our convention here is such as to make h_{a_y} positive, corresponding to the positive E_y propagating from a paraboloidal reflector to S_a (after reflecting an incident negative E_y at the reflector) due to upper feed arms positive, and bottom ones negative. (For a lens IRA the sign is reversed.)

The reader can note tha the design of the aperture shape is separable (to some degree) from the feed-arm design. The spherical TEM wave incident on the reflector (or lens) has its spatial distribution determined by the feed arms (four in the typical case, two upper ones + , two lower ones –). Considering this as given, the problem is then to use the desirable portions of this incident field as it appears in the antenna aperture.

2. Qualitative Considerations Concerning Antenna-Aperture Shape

Consider the configuration in Fig. 2.1. This has the four flat-plate cones for the TEM feed as discused in [1]. Here we have the stereographic projection to give a two-dimensional structure as discussed in [3]. The complex ζ plane can also be written in polar form as

$$\zeta = \Psi e^{j\phi} = \Psi\cos(\phi) + j\Psi\sin(\phi)$$

$$\Psi \equiv \text{cylindrical radius} \; , \quad \phi \equiv \text{azimuth} \tag{2.1}$$

The feed structure has four symmetry planes if $\pi_0 = \pi/4$ (45°) located on $\phi = 0, \pi/4, \pi/2, 3\pi/2$ including the extension through the origin. By appropriate connections the feed plates can have the ones in quadrants 1 and 2 as $V_0/2$ with those in quadrants 3 and 4 as $-V_0/2$. This makes the $y = 0$ plane (with extension perpendicular to the y axis) a plane with respect to which the incident electric field is antisymmetric [5], allowing the $y = 0$ plane to be a conducting plane (metal sheet) if desired. With respect to the $x = 0$ plane the electric field is symmetric so that for $x = 0$ there is only an $E_y^{(inc)}$. Keeping at least one of these two planes as a symmetry plane for the antenna aperture S_a, then h_{a_y} is the only nonzereo component of the aperture equivalent height, and hence of the far electric field on boresight.

Note the circle of radius a defining possible choice of antenna apertuere. This is the case often considred in the past. In particular, this circle is significant as the radius for reciprocation symmetry [6]. By reciprocation symmetry we mean that a mapping from Ψ to a^2/Ψ for each ϕ leaves the geometry invariant. As indicated in Fig. 2.1, this is precisely the case for the feed arms (inner radius b, outer radius a^2/b). Not only is Z_c invariant under this transformation, but on the circle of radius a the incident electric field is tangential to the circle, i.e., has only a ϕ component.

With our constraint of reflection symmetry with respect to $x = 0$, then the incident electric field must be exactly *zero* at $\zeta = \pm ja$. If the electric field is zero there (and small near there), the presence of a reflector (or lens) near there adds very little to h_{a_y}. Then this potential part of the antenna aperture is not needed and the associated reflector (or lens) there can be eliminated.

First observation:

> Eliminating regions near $\zeta = \pm ja$ from the antenna aperture cuts top and bottom portions from the aperture. Equivalently for some $\pm y_{max}$ ($y_{max} < a$) this reduces the vertical aperture extent, or allows one to increase a (and hence h_{a_y}) for a given allowed y_{max} for our antenna aperture.

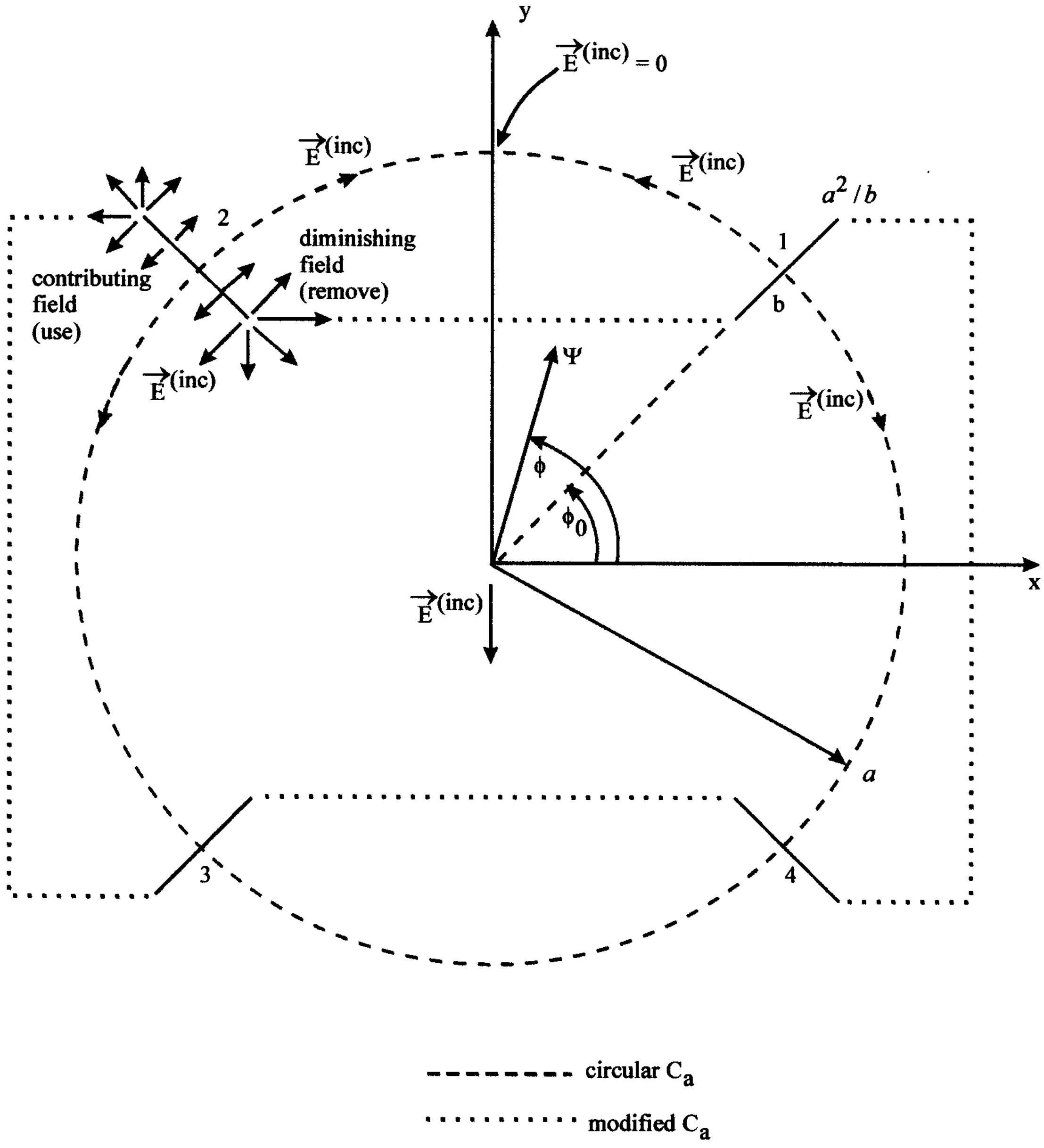

Fig. 2.1. Edge-On Four-Flat-Plate TEM Feed with Antenna Aperture.

Continuing, consider the region near one of the feed arms (as stereographically projected on the aperture plane), in this case near number 2 (on $\phi = 3\pi/2$) in the second quadrant of the ζ plane. The incident field above (and to the right) of this feed arm has an electric field with a component in the $+y$ direction which subtracts in the aperture integral from the dominant electric field pointed in the $-y$ direction as indicated in the center of the aperture ($\zeta = 0$).

Second observation:

> The region above arm 2 (near it) needs to be removed from the antenna aperture to increase the magnitude of h_{a_y}. This applies to symmetrical regions (with respect to the two symmetry planes) near the other three arms.

The electric field near the inner edge of arm 2 ($\Psi = b$) points perpendicular to this edge (line parallel to the z axis). Consider a line extending from this edge to the right (parallel to the x axis). Near the edge and above this line the electric field has a positive y component, reducing the magnitude of the aperture integral.

Third observation:

> Eliminate regions above $y = b\sin(\phi_0)$ near and "above" feed arms 1 and 2. Similarly remove those below $y = -b\sin(\phi_0)$ near and "below" the feed arms 3 and 4.

Comment:

> Straight line boundaries between the inner edges of arms 1 and 2, and between arms 2 and 4, as indicated in Fig. 2.1 may be a practical solution. Away from the edges one might restore some of the aperture in the directions of $\zeta = \pm ja$, but this may not help very much.

In a complementary way, the region below (and left of) feed arm 2 contributes to h_{a_y} because of the orientation of the y component of the incident field. This holds even outside the circle of radius a up to $\Psi = a^2/b$.

Fourth observation:

> Add regions below arm 2 (near it) to the antenna aperture to increase $|h_{a_y}|$. This again applies to the symmetrical regions near the other three arms.

The electric field near the outer edge of arm 2 ($\Psi = a^2 / b$) points perpendicular to this edge (line parallel to the z axis). Consider a line extending from this edge to the left. Near the edge and above this line the incident electric field has an unfavorable direction, while below it is favorable.

Fifth observation:

Include regions below $y = [a^2 / b]\sin(\phi_0)$ near and "below" feed arms 1 and 2. Similarly include those above $y = -[a^2 / b]\sin(\phi_0)$ near and "above" the feed arms 3 and 4.

Comment:

Straight line boundaries can be extended outward from the outer edges of the feed arms parallel to the x-axis to increase $| h_{a_y} |$. How far one may wish to extend the aperture outward is not so clear. Figure 2.1 then indicates possible vertical aperture boundaries outside the $\Psi = a$ circle as one possible choice, given some desire to constrain the aperture width.

So what we have left as an antenna aperture is indicated qualitatively in Fig. 2.1. The shape somewhat resembles a bow tie. The reciprocation circle of radius a can potentially be greater than y_{max}. There is still the matter of an optimal choice for ϕ_0 (not necessarily $\pi / 4$) while maintaining the two symmetry planes.

3. Four-Thin-Wire Feed to Rectangular Antenna Aperture

Now consider the rectangular antenna aperture with four thin wires as the TEM feed as illustrated in Fig. 3.1. The wire radius r_0 is assumed small compared to both x_0 and y_0 (coordinates of wire 1). Note that both $y = 0$ and $x = 0$ are taken as symmetry planes of both the feed and the aperture. With wires 1 and 2 positive and wires 2 and 3 negative, this assures that h_{a_y} (taken positive) is the only remaining component of the aperture equivalent height. Note that we have assumed $x_1 > x_0$ (or equal in a limiting sense) for these calculations.

3.1 Potential of right wire pair

For present purposes consider wires 1 and 4 as a wire pair. The potential from this first wire pair is distorted near the second wire pair (wires 2 and 3), but this is a small effect, occurring over distances of the order of r_0, which we can neglect. This potential then has the form [2, 4]

$$\begin{aligned} w(\zeta) &= \ell n\left(\frac{\frac{\zeta - x_0}{y_0} + j}{\frac{\zeta - x_0}{y_0} - j}\right) \\ u(\zeta) &= \frac{1}{2}\ell n\left(\frac{\left[\frac{x-x_0}{y_0}\right]^2 + \left[1 + \frac{y}{y_0}\right]^2}{\left[\frac{x-x_0}{y_0}\right]^2 + \left[1 - \frac{y}{y_0}\right]^2}\right) \\ \text{v}(\zeta) &= \arctan\left(\frac{2\frac{x-x_0}{y_0}}{\left[\frac{x-x_0}{y_0}\right]^2 + \left[\frac{y}{y_0}\right]^2 - 1}\right) \end{aligned} \tag{3.1}$$

On the right conductors we have

$$u_+ = -u_- = \operatorname{arcsinh}\left(\frac{y_0}{r_0}\right) \rightarrow \ell n\left(2\frac{y_0}{r_0}\right) \quad \text{as} \quad \frac{r_0}{y_0} \rightarrow 0 \tag{3.2}$$

Circling one of these wires we find

$$\Delta \text{v} = 2\pi \tag{3.3}$$

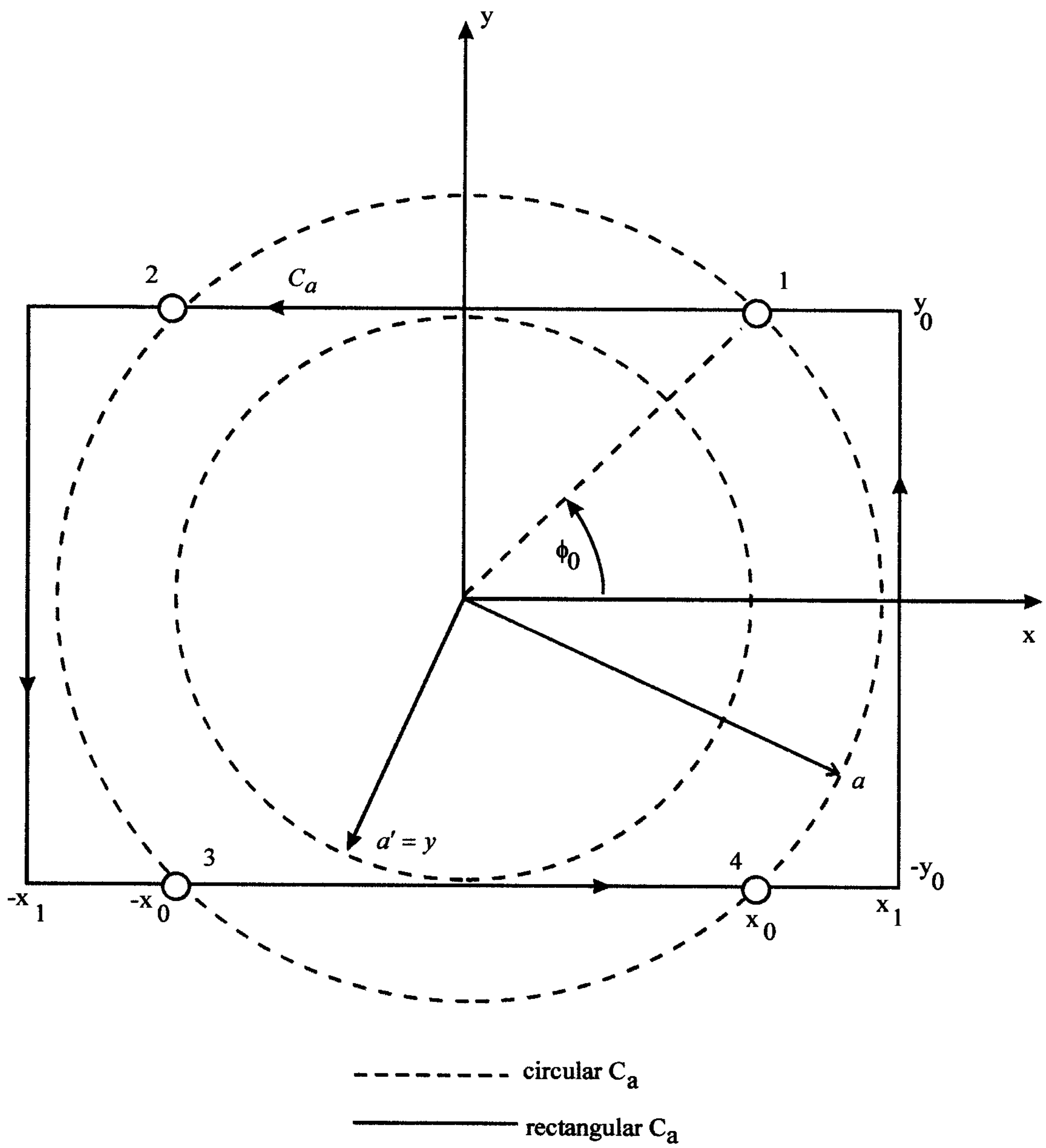

Fig. 3.1 Four Thin Wires Feeding Rectangular Antenna Aperture.

3.2 General formulae

To calculate h_{a_y} we can integrate u from the right wire pair, double this (by symmetry) to account for the left wire pair, and divide by 4π accounting for both wire pairs. Alternately, one can divide one integral by 2π giving

$$
\begin{aligned}
h_{a_y} &= -\frac{1}{\Delta v} \oint_{C_a} u(\zeta) d\zeta^* \\
&= \frac{1}{4\pi} \int_{-x_1}^{x_0-r_0} \ell n \left(\frac{\left[\frac{x-x_0}{y_0}\right]^2 + 4}{\left[\frac{x-x_0}{y_0}\right]^2} \right) dx + \frac{1}{4\pi} \int_{x_0+r_0}^{x_1} \ell n \left(\frac{\left[\frac{x-x_0}{y_0}\right]^2 + 4}{\left[\frac{x-x_0}{y_0}\right]^2} \right) dx \quad \text{(top of } C_a\text{)} \\
&\quad - \frac{1}{4\pi} \int_{-x_1}^{x_0-r_0} \ell n \left(\frac{\left[\frac{x-x_0}{y_0}\right]^2}{\left[\frac{x-x_0}{y_0}\right]^2 + 4} \right) dx - \frac{1}{4\pi} \int_{x_0+r_0}^{x_1} \ell n \left(\frac{\left[\frac{x-x_0}{y_0}\right]^2}{\left[\frac{x-x_0}{y_0}\right]^2 + 4} \right) dx \quad \text{(bottom of } C_a\text{)} \\
&\quad + O(u_+ r_0) \text{ as } \frac{r_0}{y_0} \to 0 \quad \text{(negligible contribution integrating near wires)} \\
&= \frac{1}{2\pi} \int_{-x_1}^{x_0-r_0} \ell n \left(\frac{\left[\frac{x-x_0}{y_0}\right]^2 + 4}{\left[\frac{x-x_0}{y_0}\right]^2} \right) dx + \frac{1}{2\pi} \int_{x_0+r_0}^{x_1} \ell n \left(\frac{\left[\frac{x-x_0}{y_0}\right]^2 + 4}{\left[\frac{x-x_0}{y_0}\right]^2} \right) dx \\
&\quad + O(u_+ r_0) \text{ as } \frac{r_0}{y_0} \to 0
\end{aligned}
\tag{3.4}
$$

Substitute

$$
\xi = \mp \frac{2y_0}{x-x_0}, \quad dx = \pm \frac{2y_0}{\xi^2} d\xi \tag{3.5}
$$

with the upper sign for the first integral and the lower sign for the second integral. Similarly define

$$
\xi_0 \equiv \frac{2y_0}{r_0}, \quad \xi_\ell = \frac{2y_0}{x_1+x_0}, \quad \xi_r = \frac{2y_0}{x_1-x_0} \tag{3.6}
$$

This then gives

$$h_{a_y} = \frac{y_0}{\pi}\int_{\xi_\ell}^{\xi_0} \xi^{-2}\ell n(1+\xi^2)d\xi + \frac{y_0}{\pi}\int_{\xi_r}^{\xi_0} \xi^{-2}\ell n(1+\xi^2)d\xi \tag{3.7}$$

$$+ \; O(u_{+}r_0) \quad \text{as} \quad \frac{r_0}{y_0} \to 0$$

Following the procedure in [2 (Appendix C.5)] this is integrated by parts giving

$$h_{a_y} = \frac{y_0}{\pi}\left[-\xi^{-1}\ell n(1+\xi^2)\Big|_{\xi_\ell}^{\xi_0} + 2\int_{\xi_\ell}^{\xi_0}\left[1+\xi^2\right]^{-1} d\xi \; - \xi^{-1}\ell n(1+\xi^2)\Big|_{\xi_r}^{\xi_0} + 2\int_{\xi_r}^{\xi_0}\left[1+\xi^2\right]^{-1} d\xi \right.$$

$$\left. + \; O\left(u_{+}\frac{r_0}{y_0}\right)\right] \quad \text{as} \quad \frac{r_0}{y_0} \to 0$$

$$= \frac{y_0}{\pi}\left[-2\xi_0^{-1}\ell n(1+\xi_0^2) \; + \; \xi_r^{-1}\ell n(1+\xi_\ell^2) \; + \; \xi_r^{-1}\ell n(1+\xi_r^2) \right. \tag{3.8}$$

$$+ 4\arctan(\xi_0) - 2\arctan(\xi_\ell) - 2\arctan(\xi_r)$$

$$\left. + O\left(\xi_0^{-1}\ell n(\xi_0)\right)\right] \quad \text{as} \quad \xi_0 \to \infty$$

Taking the limit as $\xi_0 \to \infty$ $(r_0 \to 0)$ gives the simpler form

$$\frac{h_{a_y}}{y_0} = \frac{1}{\pi}\left[\xi_\ell^{-1}\ell n\left(1+\xi_\ell^2\right) + \xi_r^{-1}\ell n\left(1+\xi_r^2\right) + 2\pi - 2\arctan(\xi_\ell) - 2\arctan(\xi_r) \right] \tag{3.9}$$

The error in h_{a_y} for small r_0 / y_0 is rather small and can be neglected. However, for f_g (appearing in (1.1)) the value of r_0 / y_0 is important. This factor has been treated for various ϕ_0 in [4].

3.3 Canonical comparison to circular aperture

In [4] we have found for four thin wires that for a circular aperture of radius a', we have

$$h_{a_y} = \sin(\phi_0)a' \equiv h_0 \tag{3.10}$$

If we fit a circular aperture in a maximum aperture half height of y_0, then y_0 replaces a' in the above formula giving

$$h_0 = \sin(\phi_0) y_0 \tag{3.11}$$

In (3.9) then we have a relative efficiency of

$$\frac{h_{a_y}}{h_0} = \csc(\phi_0)\frac{h_{a_y}}{y_0} \tag{3.12}$$

which we can use for comparison.

3.4 Infinitely wide aperture

Keeping x_0 fixed let $x_1 \to \infty$ giving

$$\xi_\ell \to 0 \quad , \quad \xi_r \to 0 \quad , \phi_0 = \arctan\left(\frac{y_0}{x_0}\right) \tag{3.13}$$

The aperture equivalent height is then

$$\begin{aligned} \frac{h_{a_y}}{y_0} &= 2 \\ \frac{h_{a_y}}{h_0} &= 2\csc(\phi_0) \end{aligned} \tag{3.14}$$

This result is like two single-wire-pair circular apertures of radius y_0 driven in parallel. The factor $\csc(\phi_0)$ blows up for small ϕ_0, but this needs to be balanced with the lowering f_g of the feed. For practical reasons the aperture width needs to be limited anyway.

3.5 Rectangular aperture with wires at corners

3.5.1 General case

A rectangle aperture with wires at the corners has

$$x_1 = x_0 \quad , \quad \xi_r \to \infty \quad , \quad \xi_\ell = \frac{y_0}{x_0} \quad , \quad \phi_0 = \arctan(\xi_\ell) \tag{3.15}$$

Then (3.9) becomes

$$\frac{h_{a_y}}{y_0} = \frac{1}{\pi}\left[\xi_\ell^{-1}\ell n\left(1+\xi_\ell^2\right) + \pi - 2\arctan(\xi_\ell)\right] \tag{3.16}$$

3.5.2 Infinitely wide aperture

As a special case, for comparison to the previous subsection, now let the aperture become infinitely wide so that

$$\xi_\ell \to 0 \tag{3.17}$$

Then (3.16) gives

$$\frac{h_{a_y}}{y_0} = 1 \tag{3.18}$$

This is precisely half the result in (3.14). Considering the discussion in Section 2, the deficiency here is the lack of aperture illumination for $|x| > x_0$. The field "outside" the wire positions contributes as much as that for $|x| < x_0$. Looking near the wires where the field is strong, wires positioned at the aperture corners each have one quadrant of field with favorable orientation within the aperture, while for $x_1 > x_0$ there are two quadrants for each wire with favorable field orientation in the aperture. So $x_1 > x_0$ seems to be important.

3.5.3 Narrow aperture

For completeness we can also see what happens for a narrow aperture with wires at the corners. In this case we have

$$\xi_\ell = \frac{y_0}{x_0} \to \infty$$

$$\frac{h_{a_y}}{y_0} = \frac{1}{\pi}\left[2\frac{\ell n(\xi_\ell)}{\xi_\ell}\left[1+O\left(\xi_\ell^{-2}\right)\right] + \pi - 2\left[\frac{\pi}{2} - \xi_\ell^{-1} + O\left(\xi_\ell^{-3}\right)\right]\right]$$

$$= \frac{2}{\pi}\,\frac{\ell n(\xi_\ell)+1}{\xi_\ell}\left[1+O\left(\xi_\ell^{-2}\right)\right] \quad \text{as } \xi_\ell \to \infty$$

$$\frac{h_{a_y}}{x_0} = \frac{2}{\pi}\left[\ell n(\xi_\ell)+1\right]\left[1+O\left(\xi_\ell^{-2}\right)\right] \text{ as } \xi_\ell \to \infty \tag{3.19}$$

As one would expect h_{a_y} is proportional to the width, but there is a logarithmic correction due to the field near the wires.

3.5.4 Square aperture: $\phi_0 = \pi/4$

As a special case, consider a square aperture with

$$\begin{aligned} x_1 &= x_0 = y_0, \quad \phi_0 = \frac{\pi}{4}, \quad \sin(\phi_0) = \frac{1}{\sqrt{2}} \\ \xi_r &\to \infty, \quad \xi_\ell = 1 \end{aligned} \tag{3.20}$$

Then (3.9) gives

$$\begin{aligned} \frac{h_{a_y}}{y_0} &= \frac{\ell n(2)}{\pi} + \frac{1}{2} \simeq 0.7206 \\ \frac{h_{a_y}}{y_0} &= \sqrt{2}\left[\frac{\ell n(2)}{\pi} + \frac{1}{2}\right] \simeq 1.019 \end{aligned} \tag{3.21}$$

Thus we see only a slight improvement over a circular aperture of the same half height and the same ϕ_0. Again we may wish to use more of the field near the wires.

3.6 Aperture half width x_1 a little larger than x_0

3.6.1 General case

In order to better appreciate the contribution of the field near the wires, let us define

$$\Delta x \equiv x_1 - x_0 \tag{3.22}$$

and consider the case of small positive $[\Delta x]/y_0$. For this purpose we have

$$\xi_\ell^{(0)} = \frac{y_0}{x_0}$$

$$\xi_\ell = \frac{2y_0}{2x_0+\Delta x} = \xi_\ell^{(0)}\left[1+\frac{1}{2}\frac{\Delta x}{y_0}\right]^{-1} = \xi_\ell^{(0)}\left[1-\frac{1}{2}\frac{\Delta x}{y_0}+\mathrm{O}\left(\left[\frac{\Delta x}{y_0}\right]^2\right)\right] \text{ as } \frac{\Delta x}{y_0}\to 0 \quad (3.23)$$

$$\xi_r = 2\frac{y_0}{\Delta x}$$

Then we write (from (3.16))

$$\frac{h_{a_y}}{y_0} = \frac{h_{a_y}^{(0)}}{y_0} + \frac{\Delta h_{a_y}}{y_0}$$

$$\frac{h_{a_y}^{(0)}}{y_0} = \frac{1}{\pi}\left[\xi_\ell^{(0)^{-1}}\ell n\left(1+\xi_\ell^{(0)^2}\right) + \pi - 2\arctan\left(\xi_\ell^{(0)}\right)\right] \quad (3.24)$$

To find the increase we expand the terms in (3.9) as

$$\xi_\ell^{-1}\ell n\left(1+\xi_\ell^2\right) = \xi_\ell^{(0)^{-1}}\left[1+\frac{1}{2}\frac{\Delta x}{y_0}\right]\left[\ell n\left(1+\xi_\ell^{(0)^2}\right) + \ell n\left(1+\frac{\xi_\ell^2-\xi_\ell^{(0)^2}}{1+\xi_\ell^{(0)^2}}\right)\right]$$

$$= \xi_\ell^{(0)^{-1}}\ell n\left(1+\xi_\ell^{(0)^2}\right) + \frac{\Delta x}{y_0}\ \frac{1}{2}\xi_\ell^{(0)^{-1}}\ell n\left(1+\xi_\ell^{(0)^2}\right)$$

$$+ \xi_\ell^{(0)^{-1}}\ell n\left(1-\frac{\Delta x}{y_0}\xi_\ell^{(0)^2}\left[1+\xi_\ell^{(0)^2}\right]^{-1} + \mathrm{O}\left(\left[\frac{\Delta x}{y_0}\right]^2\right)\right)$$

$$= \xi_\ell^{(0)^{-1}}\ell n\left(1+\xi_\ell^{(0)^2}\right) + \frac{\Delta x}{y_0}\left[\frac{1}{2}\xi_\ell^{(0)^{-1}}\ell n\left(1+\xi_\ell^{(0)^2}\right) - \xi_\ell^{(0)}\left[1+\xi_\ell^{(0)^2}\right]^{-1}\right]$$

$$+ \mathrm{O}\left(\left[\frac{\Delta x}{y_0}\right]^2\right) \text{ as } \frac{\Delta x}{y_0}\to 0$$

$$\xi_r^{-1}\ell n\left(1+\xi_r^2\right) = \frac{1}{2}\frac{\Delta x}{y_0}\ell n\left(1+\left[2\frac{y_0}{\Delta x}\right]^2\right)$$

$$= \frac{1}{2}\frac{\Delta x}{y_0}\left[2\ell n\left(\frac{2y_0}{\Delta x}\right) + \ell n\left(1+\left[\frac{\Delta x}{4y_0}\right]^2\right)\right] \quad (3.25)$$

$$= \frac{\Delta x}{y_0}\left[\ell n\left(\frac{2y_0}{\Delta x}\right) + \mathrm{O}\left(\left[\frac{\Delta x}{4y_0}\right]^2\right)\right] \text{ as } \frac{\Delta x}{4y_0}\to 0$$

$$-2\arctan(\xi_\ell) = -2\arctan\left(\xi_\ell^{(0)} - \frac{\Delta x}{y_0}\frac{\xi_\ell^{(0)}}{2} + O\left(\left[\frac{\Delta x}{y_0}\right]^2\right)\right)$$

$$= -2\arctan\left(\xi_\ell^{(0)}\right) + \frac{\Delta x}{y_0}\xi_\ell^{(0)}\left[1+\xi_\ell^{(0)^2}\right]^{-1}$$

$$+ O\left(\left[\frac{\Delta x}{y_0}\right]^2\right) \text{ as } \frac{\Delta x}{y_0} \to 0 \tag{3.25}$$

$$\pi - 2\arctan(\xi_r) = \operatorname{arccot}(\xi_r) = \xi_r^{-1} + O\left(\xi_r^{-3}\right)$$

$$= \frac{1}{2}\frac{\Delta x}{y_0} + O\left(\left[\frac{\Delta x}{y_0}\right]^2\right) \text{ as } \frac{\Delta x}{y_0} \to 0$$

Collecting terms we have

$$\frac{\Delta h_{a_y}}{y_0} = \frac{1}{\pi}\frac{\Delta x}{y_0}\left[\ell n\left(\frac{2y_0}{\Delta x}\right) + \frac{1}{2}\xi_\ell^{(0)^1}\ell n\left(1+\xi_\ell^{(0)^2}\right) + \frac{1}{2}\right]$$

$$+ O\left(\left[\frac{\Delta x}{y_0}\right]\right) \text{ as } \frac{\Delta x}{y_0} \to 0 \tag{3.26}$$

As we can see, we have not only $[\Delta x]/y_0$ but a logarithmic term which increases the contribution for small Δx.

3.6.2 Aperture a little wider than a square

Section 3.5.4 considers the square aperture with $\phi_0 = \pi/4$. Keeping the same ϕ_0 for the feed arms, let us widen the aperture a little to see what may be gained. From (3.26) we have

$$\xi_\ell^{(0)} = 1$$

$$\frac{\Delta h_{a_y}}{y_0} = \frac{1}{\pi}\frac{\Delta x}{y_0}\left[\ell n\left(\frac{2y_0}{\Delta x}\right) + \frac{1}{2}[\ell n(2)+1] + O\left(\frac{\Delta x}{y_0}\right)\right] \text{ as } \frac{\Delta x}{y_0} \to 0 \tag{3.27}$$

$$\simeq \frac{1}{\pi}\frac{\Delta x}{y_0}\left[\ell n\left(\frac{2y_0}{\Delta x}\right) + 0.847\right]$$

As an example, let us consider

$$\frac{\Delta x}{y_0} = 0.1 \quad \text{(10\% width increase)}$$

$$\frac{\Delta h_{a_y}}{y_0} \simeq 0.112 \tag{3.28}$$

or

$$\frac{\Delta x}{y_0} = 0.05 \quad \text{(5\% width increase)}$$

$$\frac{\Delta h_{a_y}}{y_0} = 0.0722 \tag{3.29}$$

Comparing to 0.7206 for h_{a_y} / y_0 in (3.21) we see that a 5% width increase gets about a 10% increase in the aperture equivalent height, and of course somewhat more for a 10% width increase.

3.7 Optimization of x_0 within a rectangular aperture

3.7.1 General results

Considering x_1 / y_0 fixed one can ask what value of x_0 / y_0 maximizes h_{a_y} / y_0. For this purpose we can rewrite (3.9) in integral form as in (3.7) as

$$\frac{h_{a_y}}{y_0} = \frac{1}{\pi} \int_{\xi_\ell}^{\infty} \xi^{-2} \ell n\left(1+\xi^2\right) d\xi + \frac{1}{\pi} \int_{\xi_r}^{\infty} \xi^{-2} \ell n\left(1+\xi^2\right) d\xi \tag{3.30}$$

Then we note

$$\begin{aligned} \xi_\ell &= 2\left[\frac{x_1}{y_0} + \frac{x_0}{y_0}\right]^{-1}, \quad \frac{d\xi_\ell}{dx_0} = -\frac{2}{y_0}\left[\frac{x_1}{y_0} + \frac{x_0}{y_0}\right]^{-2} = -\frac{\xi_\ell^2}{2y_0} \\ \xi_r &= 2\left[\frac{x_1}{y_0} - \frac{x_0}{y_0}\right]^{-1}, \quad \frac{d\xi_r}{dx_0} = \frac{2}{y_0}\left[\frac{x_1}{y_0} + \frac{x_0}{y_0}\right]^{-2} = \frac{\xi_r^2}{2y_0} \end{aligned} \tag{3.31}$$

Setting the derivative of (3.30) with respect to x_0 to zero gives

$$0 = \frac{1}{\pi y_0^2}\ell n\left(1+\xi_\ell^2\right) - \frac{1}{\pi y_0^2}\ell n\left(1+\xi_r^2\right)^2 \tag{3.32}$$

which is solved by inspection to give

$$\frac{x_0}{y_0} = 0 \ , \quad \xi_\ell = \xi_r = 2\frac{y_0}{x_1} \equiv \xi_1 \ , \quad \phi_0 = \frac{\pi}{2} \tag{3.33}$$

as the only acceptable solution. Substituting this back in (3.9) gives us

$$\frac{h_{a_y}}{y_0} = \frac{2}{\pi}\left[\xi_1^{-1}\ell n\left(1+\xi_1^2\right) + \pi - 2\arctan\left(\xi_1\right)\right] \tag{3.34}$$

as the maximum value obtained for this parameter. This agrees with the results in [2 (Appendix C.5)] for a single wire-pair feed.

As discussed in [4] there is a disadvantage of having $\phi_0 \to 0$ $\left(x_0 \to 0\right)$ in a four-arm feed system. While it does increase h_{a_y} (in a circular aperture in that case) the wire radius r_0 increases considerably if one wishes to maintain f_g independent of ϕ_0 (say for a 200 Ω four-wire feed). Alternately, if one maintains r_0 as ϕ_0 is increased toward $\pi/2$, then f_g increases, this appearing in the denominator in the expression for the far field in (1.1). So while some increase in ϕ_0 beyond $\pi/4$ may be warranted, this should not be pushed too far.

3.7.2 Square aperture

Considering the special case of a square aperture with $\phi_0 = \pi/2$ we have

$$\begin{aligned} \xi_1 &= 2 \\ \frac{h_{a_y}}{y_0} &= \frac{2}{\pi}\left[\frac{1}{2}\ell n(5) + \pi - 2\arctan(2)\right] \\ &\simeq 1.10 \end{aligned} \tag{3.35}$$

This is a 10% increase over the circular aperture with radius y_0, effectively a two-wire feed in each case.

3.8 Square aperture with $\phi_0 = \pi/3$

In the spirit of [4] let us move the arm-plane angle ϕ_0 from $\pi/2$ (45°) up to $\pi/3$ (60°). However, let the aperture now be square instead of circular. Then we have

$$\frac{x_1}{y_0} = 1\,, \quad \frac{x_0}{y_0} = \cos(\pi/3) = \frac{1}{2}$$

$$\xi_\ell = \frac{4}{3}\,, \quad \xi_r = 4$$

$$\frac{h_{a_y}}{y_0} = \frac{1}{\pi}\left[\frac{3}{4}\ell n\left(\frac{25}{9}\right) + \frac{1}{4}\ell n(17) + 2\pi - 2\arctan\left(\frac{4}{3}\right) - 2\arctan(4)\right] \simeq 1.0350 \tag{3.36}$$

This is compared to a circular aperture of radius y_0 with $\phi_0 = \pi/3$, which from (3.11) gives

$$\frac{h_0}{y_0} = \sin(\phi_0) = \frac{\sqrt{3}}{2} \simeq 0.8660$$
$$\frac{h_{a_y}}{h_0} \simeq 1.1950 \tag{3.37}$$

which may be a useful increase. Note that f_g is the same for the two cases in this comparison

4. Concluding Remarks

So now we have some information concerning optimiaing IRA aperture design within constraints, specifically within a specified rectangle with a four-wire feed. Modest increases in aperture equivalent height are possible by appropriate positioning of the feed conductors within the rectangle, as compared to a circular aperture which fits in the same rectangle.

The practical case of edge-on flat-plate-cone feed arms is also of interest. Section 2 has considered this case with qualitative results based on the orientation of the fields near the feed arms. More detailed calculations of this case would give more precise estimates of the improvement attainable.

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