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Antenna-Aperture Synthesis for Hyperband SAR Antennas

Carl E. Baum

Air Force Research Laboratory
Directed Energy Directorate

Abstract

This paper introduces an aperture synthesis procedure for producing a desired pulse shape, including the desired frequency spectrum of the pulse. This is accomplished by controlling the time-of-arrival of fields on the aperture plane, thereby synthesizing a delay as a function of radius for the arrival of a stop-function TEM-like wave on the aperture plane. Amplitude taper as a function of radius can also be included. The procedure is illustrated with gate and sawtooth waveforms radiated on boresight.

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1. Introduction

While an impulse-radiating antenna (IRA) is appropriate for a narrow beam (focused at infinity) [7], for synthetic aperture radar (SAR) one may wish to have a broader beam. This allows as the antenna is moved for the target of interest to remain in the important part of the antenna pattern for a significant angular spread to take advantage of the SAR processing. However, for target identification purposes, one would still like to have hyperband operation (band ratio greater than a decade) to encompass the complex natural frequencies of the target (aspect independent) that are a fundamental part of the singularity-expansion-method (SEM) representation of the target scattering [8]

A previous paper [4] addresses this problem by illuminating the aperture with a spherical TEM wave. There it is observed that with a circular aperture centered on the direction to the observer (the z axis) and the spherical-wave center also on this axis, the radiated field on boresight has notches in the frequency spectrum. In terms of step-function temporal excitation this is due to a second step associated with the truncation of the aperture at a particular radius, a. In the previous paper one method for eliminating the second step and replacing it with a constant-times-time decay to zero was investigated. This involved a resistive sheet with a special profile (function of radius, Ψ) on the aperture.

The present paper delves further into this problem. In particular we consider altering the spherical wave illuminating the aperture by increasing the delay near the aperture edge. As we shall see, this also can be designed to eliminate the second step.

2. Far Fields from Spherical TEM Wave on Circular Antenna Aperture

Summarizing from [4] let us consider a spherical TEM wave with coordinates as in Fig. 2.1 as an ideal step-function wave of the form

$$\vec{E}^{(s)}(\vec{r},t_r) = -\frac{V_0}{r\Delta\Phi^{(s)}}\nabla_s\Phi^{(s)}(\theta,\phi)\, u(t_r)$$

$$t_r = t - \frac{r}{c} \equiv \text{retarded time}$$

$$T_r \equiv ct_r \equiv \text{retarded time in length units}$$

$$\nabla_s = \vec{1}_\theta \frac{\partial}{\partial\theta} + \vec{1}_\theta \csc(\theta)\frac{\partial}{\partial\phi} \tag{2.1}$$

$$c = [\mu_0\varepsilon_0]^{-\frac{1}{2}} \equiv \text{speed of light}$$

$$r \equiv |\vec{r}|$$

For convenience we have cylindrical (Ψ, ϕ, z) and spherical (r, θ, ϕ) coordinates related to Cartesian (x, y, z) coordinates as

$$\begin{aligned} x &= \Psi\cos(\phi)\ , \quad y = \sin(\phi) \\ \Psi &= r\sin(\theta)\ , \quad z = r\cos(\theta) \\ \vec{1}_\Psi &= \cos(\phi)\,\vec{1}_x + \sin(\phi)\,\vec{1}_y \\ \vec{1}_\phi &= -\sin(\phi)\,\vec{1}_x + \cos(\phi)\,\vec{1}_y \end{aligned} \tag{2.2}$$

The potential function $\Phi^{(s)}$ gives a change in voltage $\Delta\Phi^{(s)}$ between appropriate conductors, but these need not concern us here. The potential function takes the form for spherical TEM waves as

$$\begin{aligned} \Phi^{(s)}(\theta,\phi) &= \sum_{m=0}^{\infty}\left[2\tan\left(\frac{\theta}{2}\right)\right]^m \left[a_m\cos(m\phi) + b_m\sin(m\phi)\right] \\ &= \sum_{m=0}^{\infty}\left[\theta + O\left(\theta^3\right)\right]^m \left[a_m\cos(m\phi) + b_m\sin(m\phi)\right] \\ &\qquad \text{as } \theta\to 0 \end{aligned} \tag{2.3}$$

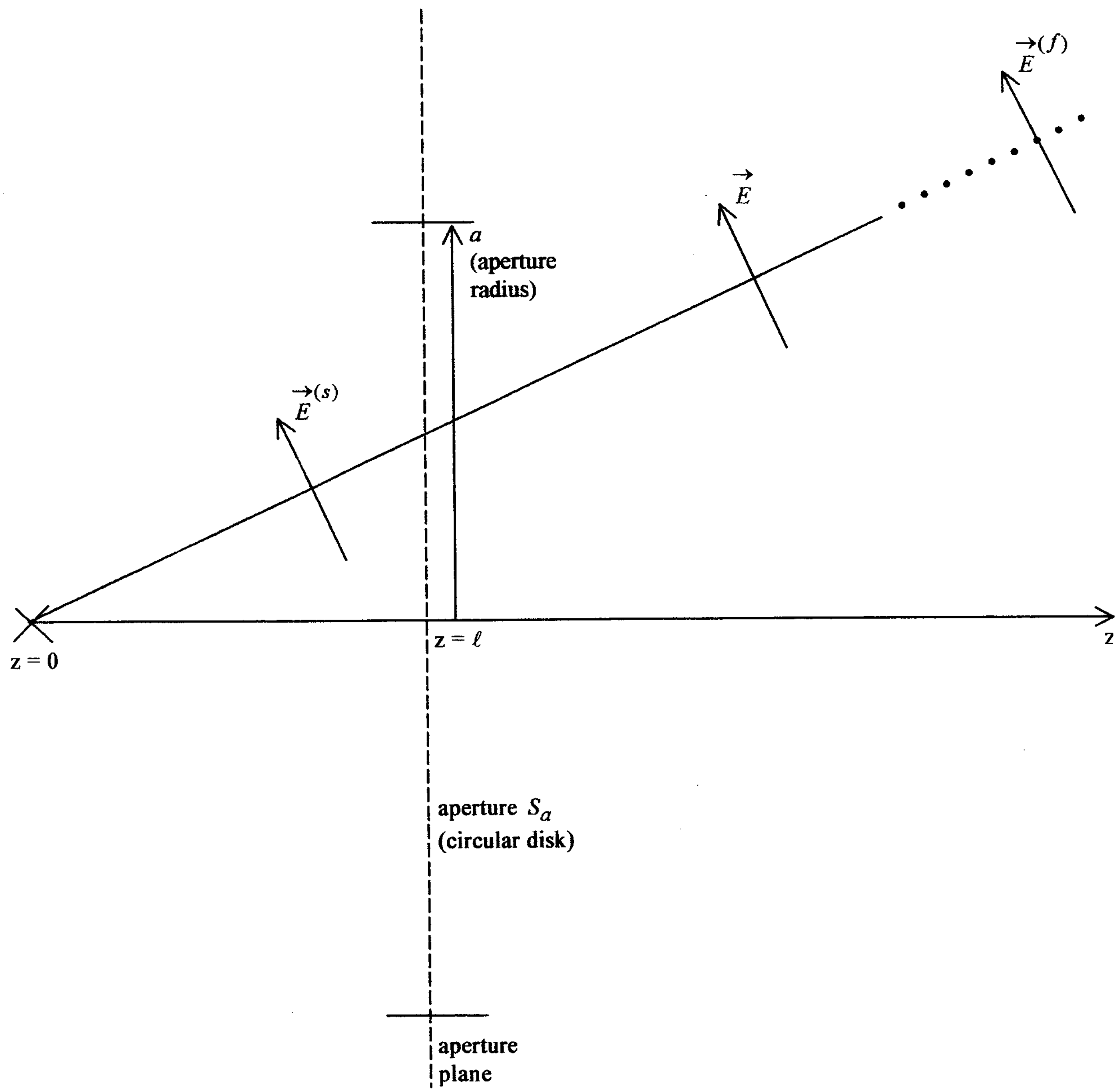

Fig. 2.1 TEM Step-Function Wave Incident on Antenna Aperture

On the aperture plane (z = ℓ) this becomes

$$\Phi^{(a)}(\Psi,\phi) = \sum_{m=0}^{\infty}\left[2\frac{\left[\ell^2+\Psi^2\right]^{1/2}-\ell}{\Psi}\right]^m \left[a_m\cos(m\phi) + b_m\sin(m\phi)\right]$$

$$= \sum_{m=0}^{\infty}\left[\frac{\Psi}{\ell} + O\left(\left[\frac{\Psi}{\ell}\right]^3\right)\right]^m \left[a_m\cos(m\phi) + b_m\sin(m\phi)\right] \tag{2.4}$$

$$\text{as } \frac{\Psi}{\ell}\to 0$$

The electric field is represented via

$$\nabla_s\Phi^{(s)}(\theta,\phi) = \frac{\partial\Phi^{(s)}(\theta,\phi)}{\partial\theta}\vec{1}_\theta + \csc(\theta)\frac{\partial\Phi^{(s)}(\theta,\phi)}{\partial\phi}\vec{1}_\phi$$

$$= \sum_{m=1}^{\infty} m\left[\left[2\tan\left(\frac{\theta}{2}\right)\right]^{m-1}\sec^2\left(\frac{\theta}{2}\right)\left[a_m\cos(m\phi) + b_m\sin(m\phi)\right]\vec{1}_\phi \right. \tag{2.5}$$

$$\left. + \csc(\theta)\left[2\tan\left(\frac{\theta}{2}\right)\right]^m\left[-a_m\sin(m\phi) + b_m\cos(m\phi)\right]\vec{1}_\phi\right]$$

For use on the aperture plane we have the tangential electric field via

$$\frac{1}{r_a}\vec{1}_z \bullet \nabla_s\Phi^{(a)}(\theta,\phi)$$

$$= \sum_{m=1}^{\infty} m\left[\left[\ell^2+\Psi^2\right]^{-1/2}\left[2\frac{\left[\ell^2+\Psi^2\right]^{1/2}=\ell}{\Psi}\right]^{m-1}\frac{2\ell}{\left[\ell^2+\Psi^2\right]^{1/2}+\ell}\left[a_m\cos(m\phi) + b_m\sin(m\phi)\right]\vec{1}_\Psi \right.$$

$$\left. + \frac{1}{\Psi}\left[2\frac{\left[\ell^2+\Psi^2\right]^{1/2}-\ell}{\Psi}\right]^m\left[-a_m\sin(m\phi) + b_m\cos(m\phi)\right]\vec{1}_\phi\right]$$

$$r_a = \left[\ell^2+\Psi^2\right]^{-1/2} = r \text{ evaluated on aperture plane} \tag{2.6}$$

Further details can be found in [4]. For later use we can note that on the z axis ($\theta = 0$) the terms for $m > 1$ are zero and the above reduces to

$$\begin{aligned}\nabla_s \Phi^{(a)}(0,\phi) &= \left[a_1 \cos(\phi) + b_1 \sin(\phi)\right] \vec{1}_\Psi + \left[-a_1 \sin(\phi) + b_1 \cos(\phi)\right] \vec{1}_\phi \\ &= a_1 \vec{1}_x + b_1 \vec{1}_y \\ &(\phi \text{ being irrelevant for } \theta = 0)\end{aligned} \tag{2.7}$$

The far field radiated from the aperture is [1, 2]

$$\vec{E}^{(f)}(\vec{r}, t_r) = \frac{1}{2\pi c r}\left[\left[\vec{1}_z \cdot \vec{1}_r\right]\overleftrightarrow{1} - \vec{1}_z \vec{1}_r\right] \cdot \frac{\partial}{\partial t} \int_{S_a} \vec{E}_t\left(\vec{r}_a, t - \frac{\vec{1}_r \cdot \vec{r}_a}{c}\right) dS$$

$$\begin{aligned}&\vec{r}_a = \Psi \vec{1}_\Psi + \ell \vec{1}_z \equiv \vec{r} \text{ evaluated on aperture plane} \\ &r_a \equiv |\vec{r}_a| \\ &\vec{E}_t \equiv \text{tangential electric field on aperture plane}\end{aligned} \tag{2.8}$$

Note that $\vec{r} = \vec{0}$ is not taken on the aperture plane for convenience. However, the r^{-1} dependence for the far field is still asymptotically correct.

Considering a circular aperture (unloaded) of radius a we can define

$$\theta_a \equiv \arctan\left(\frac{a}{\ell}\right) \tag{2.9}$$

For $0 \le \theta < \theta_a$ the initial far field is exactly the same as the formula in (2.1) (by causality). Consider the first ray (straight line) from $\vec{r} = \vec{0}$ passing through aperture, as compared to a second ray from $\vec{r} = \vec{0}$ to the aperture edge at $\vec{r} = \vec{r}_a = a\vec{1}_\Psi + \ell \vec{1}_z$. The second ray arrives in the far field at a retarded time later than the first ray by

$$\begin{aligned}t_1 &= \frac{1}{c}\left[\ell^2 + a^2\right]^{1/2} + \frac{r}{c}\left|\vec{1}_r - \frac{a}{r}\vec{1}_\Psi - \frac{\ell}{r}\vec{1}_z\right| \\ &= \frac{1}{c}\left[\ell^2 + a^2\right]^{1/2}\left[1 - \cos(\theta_a - \theta)\right] \text{ as } r \to \infty\end{aligned} \tag{2.10}$$

For retarded times $< t_1$ the spherical TEM wave gives the *exact* far-field result. After t_1 the aperture integral as in (2.7) gives a more complicated result, including portions with zero electric field. After a second retarded time given by

$$t_2 = \frac{1}{c}\left[\ell^2 + a^2\right]^{1/2} + \frac{r}{c}\left|\vec{1}_r + \frac{a}{r}\vec{1}_\Psi - \frac{\ell}{r}\vec{1}_z\right|$$
$$= \frac{1}{c}\left[\ell^2 + a^2\right]^{1/2}\left[1-\cos\left(\theta_a + \theta\right)\right] \text{ as } r \to \infty \tag{2.11}$$

the far field is zero from (2.8) (the time derivative of a constant vector given by the integral over the entire aperture of radius a). Between t_1 and t_2 the radiated waveform decays from that given by (2.1) to zero.

For the special case of $\theta = 0$ we have $t_1 = t_2$ implying a step function giving

$$\vec{E}^{(f)}(\vec{r}, t_r) = -\frac{V_0}{r\Delta\Phi^{(s)}}\nabla_s\Phi^{(s)}(0,\phi)\left[u(t_r) - u(t_r - t_1)\right]$$
$$= \frac{\ell}{r}\vec{E}_t\left(\vec{0},\, t - \frac{\ell}{c}\right)\left[u(t_r) - u(t_r - t_1)\right] \tag{2.12}$$

As discussed in [4] this waveform has problems due to notches in the frequency spectrum corresponding to periods which are positive integer submultiples of $2t_1$. In that case a resistively tapered aperture was considered to smooth the decay of the waveform, eliminating the second step. Here we take a different approach to removing the spectral notches from the boresight waveform.

Of course, the aperture need not be circular. Even with a circular aperture there may be protrusions into the circular area by conductors such as feed arms. As discussed in [3] one can use the concept of a circular aperture (discussed above) as a model pertaining to a smaller circular aperture, which does not have any such protrusions.

3. Rotationally Symmetric Temporal Taper of Aperture Field

Now let the aperture field be related to a modified form of (2.1) and (2.8). Specifically, the form of $\vec{E}_t$ remains the same except for the dependence on $t - \vec{1}_r \cdot \vec{r}_a / c$, which is changed to some other time-of-arrival taper on the aperture. For convenience this taper will be rotationally symmetric about the z axis. Specializing to boresight we write

$$\vec{E}^{(f)}(z\,\vec{1}_z, t_r) = \frac{1}{2\pi c r}\frac{\partial}{\partial t}\int_{S_a} \vec{E}_t\left(\vec{r}_a\,,\; t - \frac{\tau(\Psi_a)}{c}\right) dS$$
$$t_r = t - \frac{z}{c} \quad , \quad T_r = ct - z \tag{3.1}$$

Using cylindrical (Ψ , ϕ , ℓ) coordinates on the aperture, we have

$$\int_{S_a} \vec{E}_t\left(\vec{r}_a\,,\; t - \frac{\tau(r_a)}{c}\right) dS = \int_0^{2\pi}\int_0^{a} \vec{E}_t\left(\Psi\,,\phi\,;\; t - \frac{\tau(\Psi)}{c}\right)\Psi\, d\Psi\, d\phi \tag{3.2}$$

where the aperture radius is limited to a. Noting that τ is not a function of ϕ let us evaluate (from (2.1) and (2.6))

$$\int_0^{2\pi} \vec{E}_t(\Psi,\phi;t-\tau(\Psi))\, d\phi = \frac{V_0}{\Delta\Phi^{(s)}}\left[\ell^2 + \Psi^2\right]^{-1/2} u\left(t-\tau(\Psi)\right)\int_0^{2\pi} \overleftrightarrow{1}_z \cdot \nabla_s \Phi^{(a)}(\theta,\phi)\, d\phi$$

$$\left[\ell^2 + \Psi^2\right]^{-1/2}\int_0^{2\pi} \overleftrightarrow{1}_z \cdot \nabla_s \Phi^{(a)}(\theta,\phi)\, d\phi$$

$$= \sum_{m=1}^{\infty} m\left[\left[\ell^2+\Psi^2\right]^{-1/2}\left[2\frac{\left[\ell^2+\Psi^2\right]^{-1/2} - \ell}{\Psi}\right]^{m-1}\frac{2\ell}{\left[\ell^2+\Psi^2\right]^{-1/2}+\ell}\int_0^{2\pi}\left[a_m\cos(m\phi)+b_m\sin(m\phi)\right]\vec{1}_\Psi\, d\phi\right.$$
$$\left. + \frac{1}{\Psi}\left[2\frac{\left[\ell^2+\Psi^{-2}\right]^{1/2} - \ell}{\Psi}\right]^{m}\int_0^{2\pi}\left[-a_m\sin(m\phi)+b_m\cos(m\phi)\right]\vec{1}_\phi\, d\phi\right] \tag{3.3}$$
$$\vec{1}_\Psi = \cos(\phi)\,\vec{1}_x + \sin(\phi)\,\vec{1}_y \quad , \quad \vec{1}_\phi = -\sin(\phi)\,\vec{1}_x + \cos(\phi)\,\vec{1}_y$$

Note that for $m \geq 2$ all the integrals are zero (orthogonality). From [4 (Appendix B)] we have

$$\int_0^{2\pi} \left[a_m \cos(m\phi) + b_m \sin(m\phi)\right] \vec{1}_\Psi \, d\phi = \begin{cases} \pi\left[a_1 \vec{1}_x + b_1 \vec{1}_y\right] & \text{for } m = 1 \\ \vec{0} & \text{for } m > 1 \end{cases}$$

$$\int_0^{2\pi} \left[-a_m \sin(m\phi) + b_m \cos(m\phi)\right] \vec{1}_\phi \, d\phi = \begin{cases} \pi\left[a_1 \vec{1}_x + b_1 \vec{1}_y\right] & \text{for } m = 1 \\ \vec{0} & \text{for } m > 1 \end{cases} \tag{3.4}$$

With (2.7) we then have

$$\left[\ell^2 + \Psi^2\right]^{-1/2} \int_0^{2\pi} \overleftrightarrow{1}_z \bullet \nabla_s \Phi^{(a)}(\theta,\phi) d\phi = 2\pi\left[\ell^2 + \Psi^2\right]^{-1/2}\left[a_1 \vec{1}_x + b_1 \vec{1}_y\right]$$

$$= 2\pi\left[\ell^2 + \Psi^2\right]^{-1/2} \nabla_s \Phi^{(a)}(0,\phi)$$

$$\int_0^{2\pi} \vec{E}_t(\Psi,\phi;t-\tau(\Psi)) d\phi = -\frac{V_0}{\Delta\Phi^{(s)}} u(t-\tau(\Psi)) 2\pi\left[\ell^2 + \Psi^2\right]^{-1/2} \nabla_s \Phi^{(a)}(0,\phi) \tag{3.5}$$

$$= \frac{2\pi\ell}{\left[\ell^2 + \Psi^2\right]^{1/2}} \vec{E}_t(\vec{0},t-\tau(\Psi))$$

$$\vec{E}_t(\vec{0},t-\tau(\Psi)) = -\frac{V_0}{\ell\Phi^{(s)}} \nabla_s \Phi^{(s)}(0,\phi)\, u(t-\tau(\Psi))$$

Taking the ideal form of the aperture field as a step function, we write

$$\vec{E}_t(\vec{0},t-\tau(\Psi)) \equiv \vec{E}_0\, u(t-\tau(\Psi)) \tag{3.6}$$

Then (3.2) becomes

$$\int_{S_a} \vec{E}_t\left(\vec{r}_a, t - \frac{\tau(r_a)}{c}\right) dS = \vec{E}_0 \int_0^a \frac{2\pi\ell\Psi}{\left[\ell^2 + \Psi^2\right]^{1/2}} u(t-\tau(\Psi)) d\Psi \tag{3.7}$$

and the radiated field takes the form (on the z axis)

$$\vec{E}^{(f)}(z\vec{1}_z,t_r) = \frac{\vec{E}_0 \ell}{r}\frac{\partial}{\partial(ct)}\int_0^a \frac{\Psi}{\left[\ell^2+\Psi^2\right]^{1/2}} u\left(ct-c\tau(\Psi)\right) d\Psi$$

$$= \frac{\vec{E}_0 \ell}{r}\int_0^a \frac{\Psi}{\left[\ell^2+\Psi^2\right]^{1/2}} \delta\left(ct-c\tau(\Psi)\right) d\Psi \tag{3.8}$$

A convenient change of variables has

$$\xi = \frac{1}{2}\Psi^2 , \quad d\xi = \Psi d\Psi \tag{3.9}$$

Rewriting τ as $\tau(\xi)$ then gives

$$\vec{E}^{(f)}(z\vec{1}_z,t_r) = \frac{\vec{E}_0 \ell}{r}\int_0^{\frac{a^2}{2}} \frac{\delta\left(ct-c\tau(\xi)\right)}{\left[\ell^2+2\xi\right]^{1/2}} d\xi \tag{3.10}$$

which can be readily integrated for a specified $\tau(\xi)$.

At this point we can note that one need not use the specific form of spherical TEM wave in (2.1). The choice of ℓ is at our disposal. For $\ell \to \infty$ this corresponds to a planar TEM wave on the aperture (with delay (taper) given by $\tau(\xi)$). In this case (3.10) becomes

$$\vec{E}^{(f)}(z\vec{1}_z,t_r) = \frac{\vec{E}_0}{r}\int_0^{\frac{a^2}{2}} \delta\left(ct-c\tau(\xi)\right) d\xi \tag{3.11}$$

which has a simpler form. The factor $\ell\left[\ell^2+2\xi\right]^{1/2}$, or some other amplitude taper can be reintroduced when desired. In this form it is convenient to define retarded time from the aperture plane on the z axis where we take $\tau(0) = 0$.

Making a change of variable gives

$$c\tau(\xi) = \zeta \quad , \quad d\zeta = \frac{c\,d\tau(\xi)}{d\xi}d\xi \quad , \quad \delta(ct-\zeta) = \delta(\zeta-ct) \quad \text{(even function)}$$

$$\int_0^{a\frac{2}{2}} \delta(\text{ct-c}\tau(\xi))\,d\xi = \frac{1}{c}\int_{c\tau(0)}^{c\tau\left(\frac{a^2}{2}\right)} \delta(\zeta-ct)\frac{d\xi}{d\tau}d\zeta = \begin{cases} \left.\frac{1}{c}\frac{d\xi}{d\tau}\right|_{\zeta=ct} & \text{for } \tau(0)<t<\tau\left(\frac{a^2}{2}\right) \\ 0 & \text{otherwise} \end{cases} \tag{3.12}$$

which is a closed-form expression. Again other amplitude tapers can be included. Note for the above result we have constrained τ to be a monotonically nondecreasing function of ξ (and, hence, of Ψ).

4. Spherical TEM Aperture Field

As a first choice for the aperture distribution let us choose

$$c\tau(\xi) \equiv \left[\ell^2 + \Psi^2\right]^{1/2} = \left[\ell^2 + 2\xi\right]^{1/2} \tag{4.1}$$

corresponding to a spherical wave arrival on the aperture plane. In this form (3.10) (spherical TEM waveform) becomes

$$\vec{E}^{(f)}(z\,\vec{1}_z, t_r) = \frac{\vec{E}_0\,\ell}{r}\int_0^{\frac{a^2}{2}} \frac{\delta(ct - c\tau(\xi))}{\left[\ell^2 + 2\xi\right]^{1/2}}\,d\xi \tag{4.2}$$

Modifying (3.12) to include the $\ell\left[\ell^2 + 2\xi\right]^{-1/2}$ factor we have

$$\ell\int_0^{\frac{a^2}{2}} \frac{\delta(ct - c\tau(\xi))}{\left[\ell^2 + 2\xi\right]^{1/2}}\,d\xi = \begin{cases} \ell\left[\left[\ell^2 + 2\xi\right]^{-1/2} \dfrac{1}{c}\dfrac{d\xi}{d\tau}\right]\Bigg|_{\zeta = ct} & u \text{ for } \tau(0) < t < \tau\left(\dfrac{a^2}{2}\right) \\ 0 \text{ otherwise} \end{cases}$$

$$c\tau\left(\frac{a^2}{2}\right) = \left[\ell^2 + a^2\right]^{1/2}, \quad c\tau(0) = \ell \tag{4.3}$$

$$\left.\frac{d\xi}{cd\tau}\right|_{\zeta = ct} = \left[\left.\frac{cd\tau}{d\xi}\right|_{\zeta = ct}\right]^{-1} = \left[\left[\ell^2 + 2\xi\right]^{-1/2}\right]^{-1} = \left[\ell^2 + 2\xi\right]^{1/2}$$

Combining we have

$$\ell\int_0^{\frac{a^1}{2}} \frac{\delta(ct - c\tau(\xi))}{\left[\ell^2 + 2\xi\right]^{1/2}}\,d\xi = u\left(\left[\ell^2 + a^2\right]^{1/2} - ct\right) - u(-ct) = u(ct) - u\left(ct - \left[\ell^2 + a^2\right]^{1/2}\right) \tag{4.4}$$

and shifting to retarded time from $z = 0$

$$\vec{E}^{(f)}(z\,\vec{1}_z, t_r) = \vec{E}_0\,\frac{\ell}{r}\left[u\left(t_r - \frac{1}{c}\left[\ell^2 + a^2\right]^{1/2}\right)\right] \tag{4.5}$$

which agrees exactly with (2.12).

5. General Aperture Taper

Within the constraints set previously, we can write a general form for the axial step-response far field as

$$\vec{E}^{(f)}(z\vec{1}_z,t_r) = \frac{\vec{E}_0}{r}\int_0^{\frac{a^2}{2}} g(\xi)\,\delta\big(ct_r - c\tau(\xi)\big)\,d\xi$$

$\tau(\xi) \equiv$ temporal taper function (turn-on retarded time)

$g(\xi) \equiv$ amplitude taper function (5.1)

$\xi = \frac{1}{2}\Psi^2$

where two amplitude taper functions have been used above in (3.11) and (4.2). Following the procedure in (4.3) this becomes

$$\vec{E}(z\vec{1}_z,t_r) = \frac{\vec{E}_0}{r}U(t_r)$$

$$U(t_r) = \begin{cases} \left[g(\xi)\frac{1}{c}\frac{d\xi}{d\tau}\right]\Big|_{\tau=t_r} & \text{for } 0 < t_r < \tau_{\max} \\ 0 & \text{otherwise} \end{cases}$$

$$= \left[g(\xi)\frac{1}{c}\frac{d\xi}{d\tau}\right]\Big|_{\tau=t_r}\big[u(t_r) - u(t_r - \tau_{\max})\big]$$

$$\tau_{\max} = \tau\left(\frac{a^2}{2}\right) \equiv \text{retarded time of last portion of aperture to be excited (aperture edge)} \quad (5.2)$$

Note now that retarded time is defined with respect to propagation parallel to the z axis, not general $\vec{r}$.

Note that

$$\int_{-\infty}^{\infty} U(t_r)\,dt_r = \int_{-\infty}^{\infty}\left[g(\xi)\frac{1}{c}\frac{d\xi}{d\tau}\right]\Big|_{\tau=t_r}\big[u(t_r) - u(t_r - \tau_{\max})\big]\,dt_r$$

$$= \frac{1}{c}\int_0^{\frac{a^2}{2}} g(\xi)\,d\xi \quad (5.3)$$

For the simple special case of $g(\xi) = 1$ this gives

$$\int_{-\infty}^{\infty} U(t_r)dt_r = \frac{a^2}{2c} \tag{5.4}$$

independent of the details of the aperture temporal taper. This is related to the aperture area as one might expect. So, while one may have various waveforms radiated from the aperture, their complete time integrals are constrained by the aperture size, a low-frequency limitation.

The normalized waveform function (scalar) can be defined by

$$W(t_r) \equiv \frac{U(t_r)}{U(0_+)} \tag{5.5}$$

where $U(0_+)$ is assumed nonzero and will be taken as the waveform peak in some applications.

6. Synthesis of Radiated Waveform on Boresight

From (5.2) we are now in a position to synthesize a waveform, i.e., let us specify $W(t_r)$ from (5.3) and compute

$$W(t_r) = \frac{U(t_r)}{U(0_+)} = \frac{\left[g(\xi)\frac{1}{c}\frac{d\xi}{d\tau}\right]\Big|_{\tau=t_r}}{\left[g(\xi)\frac{1}{c}\frac{d\xi}{d\tau}\right]\Big|_{\tau=0_+}}\left[u(t_r) - u(t_r - \tau_{\max})\right] \tag{6.1}$$

The denominator is just a constant for normalization. This is then a differential equation to solve for $\tau(\xi)$, an equation for *aperture synthesis.*

For the special case of $g(\xi) = 1$, corresponding to a planar TEM wave on the aperture plane (before temporal taper) (6.1) reduces to

$$W(t_r) = \frac{U(t_r)}{U(0_+)} = \left[\frac{d\xi}{d\tau}\Big|_{\tau=0_+}\right]^{-1}\frac{d\xi}{d\tau}\Big|_{\tau=t_r} \tag{6.2}$$

This form integrates as

$$\xi\big|_{\tau=t_r} = \left[\frac{d\xi}{d\tau}\Big|_{\tau=0_+}\right]\int_0^{t_r} W(t_r')\,dt_r' \quad \text{for } 0 < t_r < \tau_{\max} \tag{6.3}$$

where we have taken $\xi = 0$ ($\Psi = 0$) at $\tau = t_r = 0$.

6.1 Gate-Function Waveform

As a first example let (with $g(\xi) = 1$)

$$W_1(t_r) = u(t_r) - u\left(t_r - \tau_{\max}^{(1)}\right) \tag{6.4}$$

Then we have

$$\int_0^{t_r} W_1(t_r')\,dt_r' = t_r\,u(t) - \left[t_r - \tau_{max}^{(1)}\right]u\left(t - \tau_{max}^{(1)}\right) = \xi(\tau)\Big|_{\tau_1 = t_r}\left[\frac{d\xi_1}{d\tau}\Big|_{\tau=0_+}\right]$$

$$\tau_1 = \xi\left[\frac{d\xi}{d\tau_1}\Big|_{\tau_1=0_+}\right]^{-1} = c_1\frac{\Psi^2}{2} \quad \text{for } 0 < \tau < \tau_{max} \tag{6.5}$$

$$C_1 = \left[\frac{d\xi}{d\tau_1}\Big|_{\tau=0_+}\right]^{-1} > 0 \quad \text{(a constant)}$$

This is a parabolic distribution of τ over the aperture. It is the leading term in the spherical arrival time in (4.1) (delayed to $\tau(0) = 0$) as

$$\begin{aligned} c\tau_1(\xi) &= \left[\ell^2 + 2\xi_1\right]^{1/2} - \ell = \ell\left[1 + \frac{\xi}{\ell^2} + O\left(\frac{\xi^2}{\ell^4}\right)\right] - \ell \\ &= \ell\left[\frac{\Psi^2}{2\ell^2} + O\left(\left(\frac{\Psi}{\ell}\right)^4\right)\right] \quad \text{as} \quad \frac{\Psi}{\ell} \to 0 \end{aligned} \tag{6.6}$$

For small a/ℓ then our form in (6.4) gives the result for a spherical TEM wave on the aperture. Alternately one may consider a planer TEM wave on the aperture with delay in (6.5). In any event this gives us an alternate way to view the radiated field as in (4.5) where the far-field amplitude scales as ℓ/r away from the aperture.

From (5.4) and (5.5) we have

$$\int_{-\infty}^{\infty} W_1(t_r)\,dt_r = C_1\ \xi\left(\tau_{max}^{(1)}\right) = \tau_{max}^{(1)} = C_1\frac{a^2}{2} \tag{6.7}$$

This gives a free parameter to choose, namely C_1 which scales the delay τ_{max} at the aperture edge $\Psi = a$, and in turn *specifies* C_1 as

$$C_1 = \frac{2\tau_{max}^{(1)}}{a^2}\ , \quad \tau_1 = \tau_{max}^{(1)}\left[\frac{\Psi}{a}\right]^2 \tag{6.8}$$

In turn we have

$$U_1(t_r) = U_1(0_+)\, W_1(t_r)$$

$$\int_{-\infty}^{\infty} U_1(t_r)\, dt_r = \frac{a^2}{2c} = U_1(0_+)\, \tau_{\max}^{(1)}$$

$$U_1(0_+) = \frac{a^2}{2c\tau_{\max}^{(1)}} \tag{6.9}$$

$$U_1(t_r) = \frac{a^2}{2c\,\tau_{\max}^{(1)}}\left[u(t_r) - u\left(t_r - \tau_{\max}^{(1)}\right)\right]$$

$$\vec{E}_1^{(f)}\left(z\,\vec{1}_z, t_r\right) = \frac{E_0}{r}\,\frac{a^2}{2c\,\tau_{\max}^{(1)}}\left[u(t_r) - u\left(t_r - \tau_{\max}^{(1)}\right)\right]$$

for the complete solution.

Note the tradeoff between peak field (proportional to $\tau_{\max}^{-1}$) and pulse width (proportional to $\tau_{\max}$). This is also related to the beamwidth through the approximate relation (from (2.9) and (6.6))

$$c\,\tau_{\max} = \frac{a^2}{2\ell} \;,\quad \theta_a = \arctan\left(\frac{a}{\ell}\right) = \arctan\left(\frac{2c\,t_{\max}}{a}\right) \tag{6.10}$$

6.2 Sawtooth Waveform

Let us now modify the waveform to the same one considered in [4]. In particular let us look for a $\tau(\xi)$ which gives

$$W_2(t_r) = \left[1 - \frac{t_r}{\tau_{\max}^{(2)}}\right]\left[u(t) - u\left(t_r - \tau_{\max}^{(2)}\right)\right] \tag{6.11}$$

This has a step discontinuity at $t_r = 0$, but only a slope discontinuity at $t_r = \tau_{\max}$. For simplicity let us refer to this as a sawtooth waveform.

Following the previous procedure we have

$$\int_0^{t_r} W_2(t_r')\,dt_r' = \left[t_r - \frac{t_r^2}{2\,\tau_{\max}}\right]u(t_r) + \frac{\left[t_r - \tau_{\max}^{(2)}\right]^2}{2\,\tau_{\max}^{(2)}}u\left(t_r - \tau_{\max}^{(2)}\right) = \xi(\tau_2)\Big|_{\tau_2 = t_r}\left[\frac{d\xi}{d\tau}\Big|_{\tau=0_+}\right]^{-1}$$

$$\tau_2 - \frac{\tau_2^2}{2\,\tau_{max}^{(2)}} = \xi \left[\left. \frac{d\xi}{d\tau_2} \right|_{\tau_2 = 0_+} \right]^{-1} = C_2 \frac{\Psi^2}{2}$$

$$C_2 = \left[\left. \frac{d\xi}{d\tau_2'} \right|_{\tau_2 = 0_+} \right]^{-1} > 0 \quad \text{(a constant)} \tag{6.12}$$

$$\int_{-\infty}^{\infty} W_2(t_r)dt_r = \frac{\tau_{max}^{(2)}}{2} = C_2 \xi\left(\tau_{max}^{(2)}\right) = C_2 \frac{a^2}{2}$$

Solving for $\tau_2(\xi)$ gives

$$\frac{\tau_2^2}{2\,\tau_{max}^{(2)}} - \tau_2 + C_2(\xi) = 0$$

$$\tau_2 = \left[1 - \left[1 - \frac{2C_2(\xi)}{\tau_{max}} \right]^{1/2} \right] \tau_{max}$$

$$\tau_2(0) = 0 \quad , \quad \tau_2 = \tau_{max}^{(2)} \text{ at } \xi = \frac{a^2}{2} \quad , \quad C_2 = \frac{\tau_{max}^{(2)}}{a^2} \tag{6.13}$$

$$\tau_2 = \left[1 - \left[1 - \left[\frac{\Psi}{a} \right]^2 \right]^{1/2} \right] \tau_{max}^{(2)}$$

Then the solution for the boresight radiated field is

$$U_2(t_r) = U_2(0_+)W_2(t_r)$$

$$\int_{-\infty}^{\infty} U_2(t_r)dt_r = \frac{a^2}{2c} = U_2(0_+)\frac{\tau_{max}^{(2)}}{2}$$

$$U_2(0_+) = \frac{a^2}{c\,\tau_{max}^{(2)}} \tag{6.14}$$

$$U_2(t_r) = \frac{a^2}{c\,\tau_{max}^{(2)}} \left[1 - \frac{t_r}{\tau_{max}^{(2)}} \right] \left[u(t_r) - u\left(t_r - \tau_{max}^{(2)}\right) \right]$$

$$E_2^{(f)}\left(z \vec{1}_z, t_r \right) = \frac{E_0}{r} \frac{a^2}{c\tau_{max}^{(2)}} \left[1 - \frac{t_r}{\tau_{max}^{(1)}} \right] \left[u(t_r) - u\left(t_r - \tau_{max}^{(2)}\right) \right]$$

6.3 Comparison of Solutions

If we compare the radiated field for the two waveforms, gate and sawtooth we have a tradeoff between peak field and pulse width on boresight. Equating the two for peak far field gives

$$\tau_{max}^{(2)} = 2\tau_{max}^{(1)} \tag{6.15}$$

This should not be surprising This gives the same time integral (low-frequency content) to the waveforms. The fact that the early-time peaks are the same is associated with the same curvature of the wavefront near the center of the aperture ($\Psi = 0$). In frequency domain, the sawtooth waveform gives comparable high- and low-frequency performance, but with the absence of the notch behavior of the gate waveform.

Looking in greater detail at $\tau(\xi)$ for the two waveforms we have

$$\begin{aligned}
\tau_2(\xi) - \tau_1(\xi) &= \left[1-\left[1-\left[\frac{\Psi}{a}\right]^2\right]^{1/2}\right]\tau_{max}^{(2)} - \left[\frac{\Psi}{a}\right]^2 \tau_{max}^{(1)} \\
&= \left[2-2\left[1-\left[\frac{\Psi}{a}\right]^2\right]^{1/2} - \left[\frac{\Psi}{a}\right]^2\right]\tau_{max}^{(1)} \\
&= \left[2-2\left[1-\frac{1}{2}\left[\frac{\Psi}{a}\right]^2 - \frac{1}{8}\left[\frac{\Psi}{a}\right]^4 + O\left(\left[\frac{\Psi}{a}\right]^6\right)\right] - \left[\frac{\Psi}{a}\right]^2\right]\tau_{max}^{(1)} \\
&= -\frac{1}{4}\left[\frac{\Psi}{a}\right]^4 \tau_{max}^{(1)} + O\left(\left[\frac{\Psi}{a}\right]^6\right) \quad \text{as } \frac{\Psi}{a} \to 0
\end{aligned}$$

$$\tau_2(\xi) - \tau_1(\xi) > 0 \text{ for } 0 < \frac{\Psi}{a} \le 1 \tag{6.16}$$

So the difference between the two delay functions is quite small near the z axis, but increases to $\tau_{max}^{(1)}$ for $\Psi = a$.

7. Concluding Remarks

Having developed an aperture-delay taper for the sawtooth waveform we can compare the present results to those in [4] for the same waveform produced by a spherical (or parabolic) delay taper with a special resistive loading giving the appropriate amplitude taper. For a given aperture radius, a, the present results have $\tau_{max}^{(2)}$ equal to twice the pulse width with the same peak amplitude as the resistively loaded case in [4]. Thus, the boresight response in the present case has the same high frequencies, but twice the low frequencies, thereby giving a more efficient design. Essentially, the resistive losses have been removed.

The present paper introduces an aperture synthesis procedure to shape pulse waveforms and their associated frequency spectra. While the emphasis has been on the delay taper (delay in turning on the aperture field as a function of radius), amplitude taper is also included in the general expressions. This opens the possibility of the synthesis of various boresight waveforms. At this point let us also mention some other related considerations of aperture distributions for pulses [5, 6].

While the present paper considers the aperture field for synthesis, it is another matter to physically construct an antenna with the desired aperture distribution. One can forsee various possible approaches to this problem, including special lenses and special reflector shapes which modify the designs of IRAs to accommodate the special nonimpulse waveforms. One can begin with a TEM horn and add additional delay by dielectric near the aperture edge for an aperture limited to the region between the conical plates. Alternately one can start with a hyperboloidal reflector designed for a given θ_a and deform the hyperboloid near the reflector rim for the additional delay to the aperture plane.

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