Sensor and Simulation Notes

Note 485

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Azimuthally Propagating Modes in an Axially-Truncated, Circular, Coaxial Waveguide

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ABSTRACT

There is considerable interest in transition structures that efficiently transport high power RF fields from transmission lines to antenna apertures. By efficient, we mean ways that minimize the amount of reflected power, and the resulting field enhancement of standing waves, that occurs at the interface of the transmission line and the transition structure. In this note we derive and examine the modal properties of the *Azimuthally Propagating, Truncated, Cylindrical, Coaxial Waveguide*. This guide is defined by an inner and outer radii, and a width. In this note we derive the governing equations for the electromagnetic fields of the TE_z and TM_z azimuthally propagating modes. First, the characteristic equations that define the propagation constants of each mode are derived; then, the electric and magnetic fields are explicitly expressed. With these results the mode impedances are formulated, and the impedance of the propagating mode in the transition section can be calculated. This value of the transition section's impedance can then be used to impedance match the transition section to a transmission line. Finally, as an example, a transmission line transition geometry is defined for which the electric and magnetic fields of the lowest order TE and TM modes are computed and graphed. An Appendix to this note briefly discusses the selection of the form of the vector potentials used to formulate the TE and TM modes.

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1. Introduction

The double-baffled, coaxial waveguide transmission line, shown in Figure 1, is defined by inner and outer radii, and an arc length. In conventional applications, the propagating modes are assumed in the z-direction. The TEz and TMz modes of this geometry, propagation constants, and guide wavelengths were studied in detail in [Ref. 1]. In this memo, the waveguide is truncated in the z-direction, and propagation is assumed in the azimuthal direction. Shown in Figure 2, the ***Azimuthally Propagating, Truncated, Cylindrical, Coaxial Waveguide*** is defined by inner and outer radii, and a depth a in the z-dimension. We are interested in solving for the azimuthally, or ϕ-directed, propagating modes. First, the characteristic equations that define the cut off frequencies of each mode are derived, then the electric fields are explicitly expressed. Finally, an example geometry is defined for which the lowest TE and TM mode cutoff frequencies are computed and graphs of the normalized field components are presented.

2. Geometry

The geometry of the azimuthally propagating, truncated, coaxial waveguide transmission line is shown in Figure 2. The transmission line is defined by is defined by inner (ρ_1) and outer radii (ρ_2), and a depth a in the z-dimension. Propagation is assumed in the azimuthal direction.

3. Wave Equation

The natural coordinate system for the ***Azimuthally Propagating, Truncated, Cylindrical, Coaxial Waveguide*** is the cylindrical coordinate system. The scalar wave equation in cylindrical coordinates is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi(\rho,\varphi,z)}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi(\rho,\varphi,z)}{\partial\varphi^2}+\frac{\partial^2\psi(\rho,\varphi,z)}{\partial z^2}+k^2\psi(\rho,\varphi,z)=0 \tag{1}$$

Using the standard separation of variables technique the wave equation can be written as

$$\rho\frac{d}{d\rho}\left(\rho\frac{dR(\rho)}{d\rho}\right)+\left[\left(k_\rho\rho\right)^2-q^2\right]R(\rho)=0 \tag{2a}$$

$$\frac{d^2}{d\varphi^2}\Phi(\varphi)+q^2\Phi(\varphi)=0 \tag{2b}$$

$$\frac{d^2}{dz^2}Z(z)+n^2Z(z)=0 \tag{2c}$$

where: $\psi(\rho,\varphi,z)=R(\rho)\Phi(\varphi)Z(z)$, the q are dimensionless propagation constants, and $k_\rho^2+n^2=k^2$.

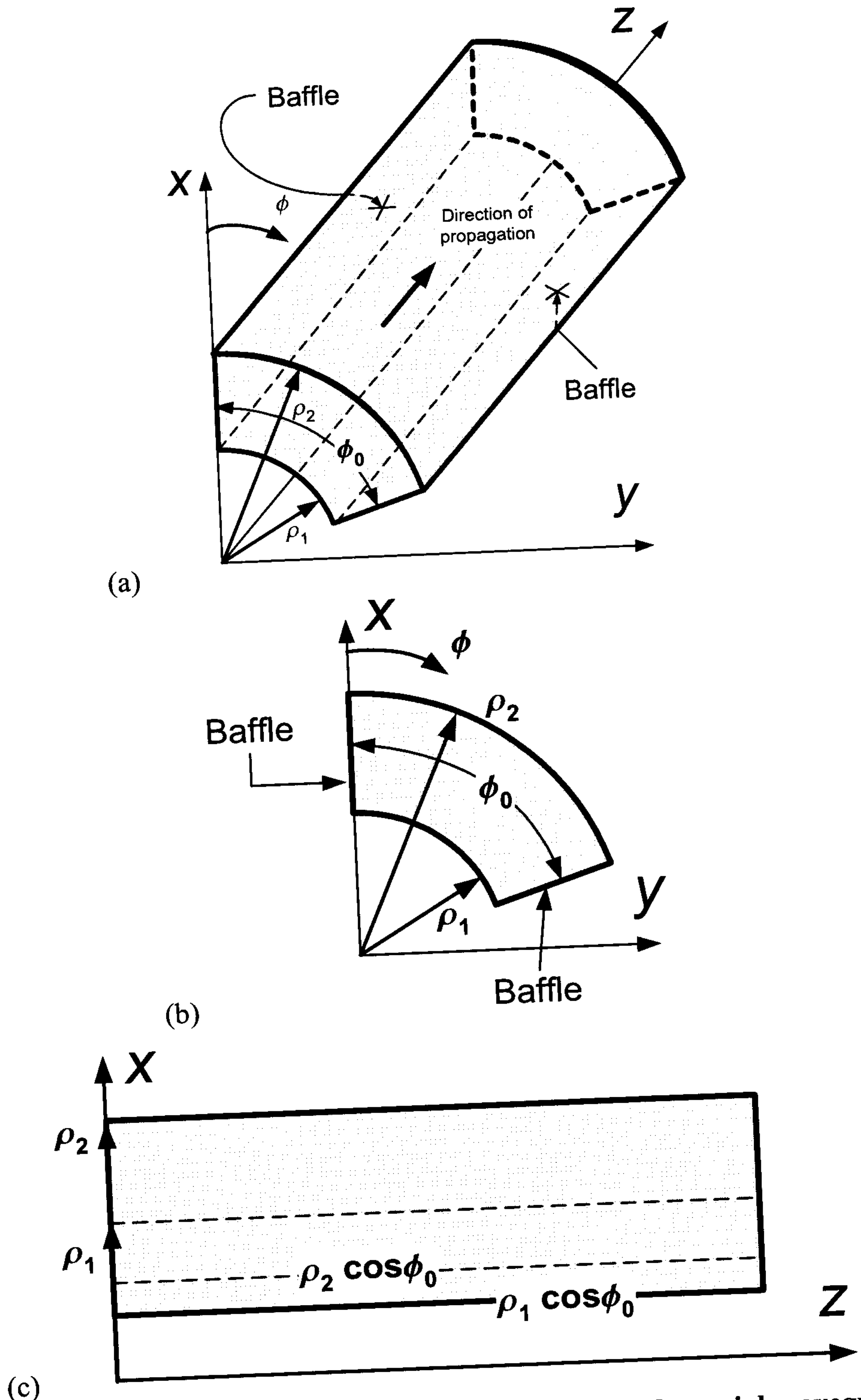

Figure 1. **The geometry of the double baffled cylindrical coaxial waveguide: (a) 3-D perspective drawing; (b) plane view of the xy-plane; and (c) plane view of the xz-plane.**

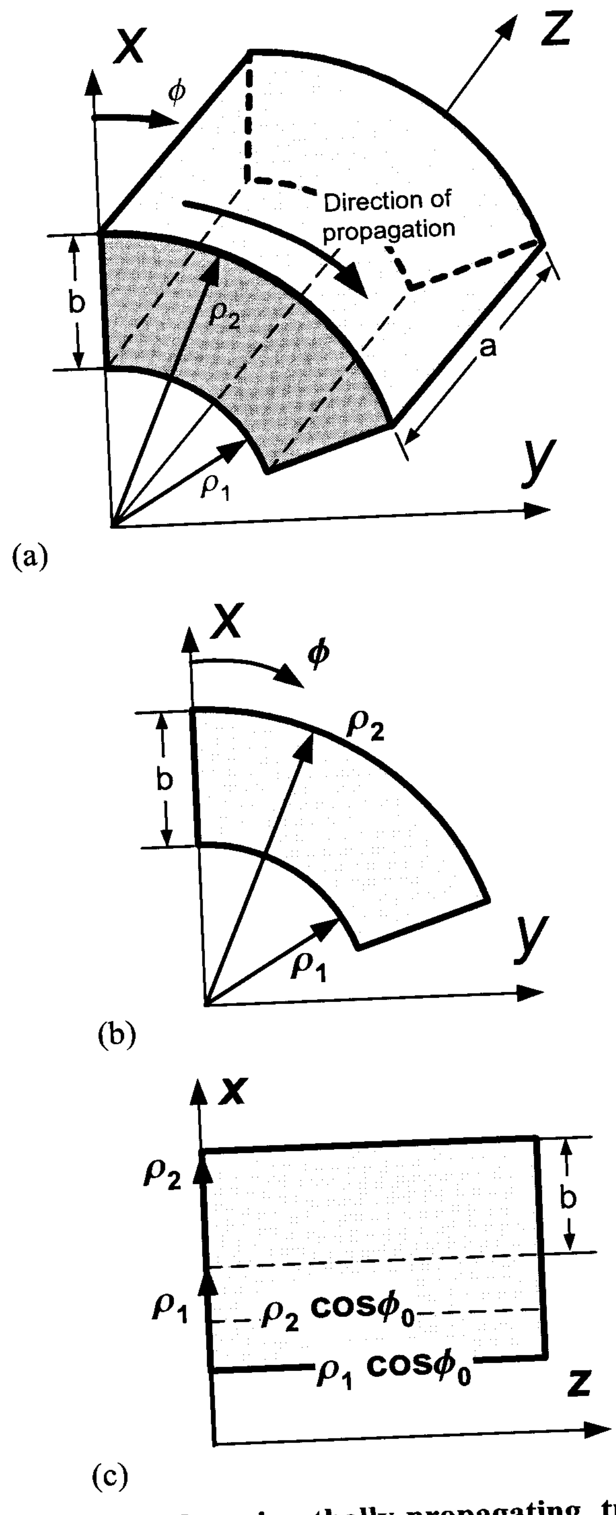

Figure 2. **The geometry of the azimuthally propagating, truncated, cylindrical coaxial waveguide: (a) 3-D perspective drawing; (b) plane view of the xy-plane; and (c) plane view of the xz-plane.**

4. Boundary Conditions

The boundary conditions on the electric field for the geometry are:

$$E_\rho = 0 \text{ for } z = 0, \text{ and } z = a \tag{3a}$$

$$E_\varphi = 0 \text{ for } \rho = \rho_1, \rho = \rho_2, z = 0, \text{ and } z = a \tag{3b}$$

$$E_z = 0 \text{ for } \rho = \rho_1 \text{ and } \rho = \rho_2. \tag{3c}$$

5. Solution of the Separated Wave Equation

The $\Phi(\varphi)$ and $Z(z)$ equations are harmonic equations with harmonic functions as solutions; these will be denoted $h(q\varphi)$ and $h(nz)$.

The equation in $R(\rho)$ is a Bessel equation, and has Bessel function solutions:

$J_q(k_\rho \rho) =$ the Bessel function of the first kind of order q
$N_q(k_\rho \rho) =$ the Bessel function of the second kind of order q
$H_q^{(1)}(k_\rho \rho) =$ the Hankel function of the first kind of order q
$H_q^{(2)}(k_\rho \rho) =$ the Hankel function of the second kind of order q

Let the function $B_q(k_\rho \rho)$ represent the linearly independent combination of two of the above. Then, the general solution to the scalar Helmholtz wave equation is:

$$\psi = B_q(k_\rho \rho) h(q\varphi) h(nz) \tag{4}$$

6. TE_z and TM_z Field Components

The electric and magnetic field components can be written in terms of fields that are TE_z and TM_z. See Appendix I.

6.1 TM_z Field Components

The TM_z field components are found by letting $\mathbf{A} = \mathbf{u}_z \psi$, where $\mathbf{A} =$ the magnetic vector potential, and $\mathbf{u}_z =$ unit vector in the z-direction. Then

$$\mathbf{E} = -j\omega\mathbf{A} + \frac{1}{j\omega\mu\varepsilon}\nabla(\nabla\cdot\mathbf{A}), \tag{5a}$$

and $$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A}. \tag{5b}$$

Since $$\nabla(\nabla\cdot\mathbf{A}) = \nabla(\nabla\cdot\mathbf{u}_z\psi) = \mathbf{u}_\rho\frac{\partial}{\partial\rho}\left[\frac{\partial\psi}{\partial z}\right] + \mathbf{u}_\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\left[\frac{\partial\psi}{\partial z}\right] + \mathbf{u}_z\frac{\partial}{\partial z}\left[\frac{\partial\psi}{\partial z}\right],$$ and

$\nabla\times\mathbf{A} = \nabla\times\mathbf{u}_z\psi = \mathbf{u}_\rho\frac{1}{\rho}\frac{\partial\psi}{\partial\varphi} - \mathbf{u}_\varphi\frac{\partial\psi}{\partial\rho}$, then expanded in cylindrical coordinates the components of the above equations become:

$$E_\rho = \frac{1}{j\omega\mu\varepsilon}\frac{\partial^2\psi}{\partial z\,\partial\rho} \tag{6a}$$

$$E_\varphi = \frac{1}{j\omega\mu\varepsilon}\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} \tag{6b}$$

$$E_z = -j\omega\psi + \frac{1}{j\omega\mu\varepsilon}\frac{\partial^2\psi}{\partial z^2} \tag{6c}$$

$$H_\rho = \frac{1}{\mu}\frac{1}{\rho}\frac{\partial\psi}{\partial\varphi} \tag{6d}$$

$$H_\varphi = -\frac{1}{\mu}\frac{\partial\psi}{\partial\rho} \tag{6e}$$

$$H_z = 0 \tag{6f}$$

6.2 TE$_z$ Field Components

The TE$_z$ field components are found by letting $\mathbf{F} = \mathbf{u}_z\psi$, where $\mathbf{F} =$ the electric vector potential, and $\mathbf{u}_z =$ unit vector in the z-direction. Then

$$\mathbf{E} = -\frac{1}{\varepsilon}\nabla\times\mathbf{F}, \tag{7a}$$

and $$\mathbf{H} = -j\omega\mathbf{F} + \frac{1}{j\omega\mu\varepsilon}\nabla(\nabla\cdot\mathbf{F}). \tag{7b}$$

Since $$\nabla(\nabla\cdot\mathbf{F}) = \nabla(\nabla\cdot\mathbf{u}_z\psi) = \mathbf{u}_\rho\frac{\partial}{\partial\rho}\left[\frac{\partial\psi}{\partial z}\right] + \mathbf{u}_\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\left[\frac{\partial\psi}{\partial z}\right] + \mathbf{u}_z\frac{\partial}{\partial z}\left[\frac{\partial\psi}{\partial z}\right],$$ and

$\nabla\times\mathbf{F} = \nabla\times\mathbf{u}_z\psi = \mathbf{u}_\rho\frac{1}{\rho}\frac{\partial\psi}{\partial\varphi} - \mathbf{u}_\varphi\frac{\partial\psi}{\partial\rho}$, when the above equations are expanded in cylindrical coordinates, the components become:

$$E_\rho = -\frac{1}{\varepsilon}\frac{1}{\rho}\frac{\partial\psi}{\partial\varphi} \tag{8a}$$

$$E_\varphi = \frac{1}{\varepsilon}\frac{\partial\psi}{\partial\rho} \tag{8b}$$

$$H_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2\psi}{\partial z\,\partial\rho} \tag{8d}$$

$$H_\varphi = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2\psi}{\partial\varphi\,\partial z} \tag{8e}$$

$$E_z = 0 \qquad (8c) \qquad\qquad H_z = -j\omega\psi - j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2\psi}{\partial z^2} \qquad (8f)$$

7. Solution of the Separated Wave Equation Subject to the Boundary Conditions of the Generalized Geometry

Propagating waves in the ϕ-direction in the truncated coaxial waveguide give rise to harmonic functions

$$h(q\varphi) = e^{-jq\varphi} \tag{9}$$

and, for $h(nz)$

$$h(nz) = a_n \sin(nz) + b_n \cos(nz) \tag{10}$$

The scalar wave function is then

$$\psi = B_q(k_\rho \rho)e^{-jq\varphi}h(nz) \tag{11}$$

subject to the boundary conditions. The solutions for the TE_z and TM_z modes in the guide are as follows.

7.1 TM_z Field Components

The TM_z electric and magnetic field components in terms of the general wave function are

$$E_\rho = \frac{1}{j\omega\mu\varepsilon}\frac{\partial^2}{\partial z\,\partial\rho}\left\{B_q(k_\rho\rho)e^{-jq\varphi}h(nz)\right\} = \frac{nk_\rho}{j\omega\mu\varepsilon}B_q'(k_\rho\rho)e^{-jq\varphi}h'(nz) \tag{12a}$$

$$E_\varphi = \frac{1}{j\omega\mu\varepsilon}\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\left\{B_q(k_\rho\rho)e^{-jq\varphi}h(nz)\right\} = \frac{(-jq)^2}{j\omega\mu\varepsilon}\frac{1}{\rho^2}B_q(k_\rho\rho)e^{-jq\varphi}h(nz) \tag{12b}$$

$$E_z = \left\{-j\omega + \frac{1}{j\omega\mu\varepsilon}\frac{\partial^2}{\partial z^2}\right\}B_q(k_\rho\rho)e^{-jq\varphi}h(nz) = -j\omega\left(1+\frac{n^2}{k_0^2}\right)B_q(k_\rho\rho)e^{-jq\varphi}h''(nz) \tag{12c}$$

$$H_\rho = -jq\frac{1}{\mu}\frac{1}{\rho}B_q(k_\rho\rho)e^{-jq\varphi}h(nz) \tag{12d}$$

$$H_\varphi = -\frac{1}{\mu}\frac{\partial\psi}{\partial\rho} = -\frac{k_\rho}{\mu}B_q'(k_\rho\rho)e^{-jq\varphi}h(nz) \tag{12e}$$

$$H_z = 0 \tag{12f}$$

Note that $B_q'(k_\rho\rho) = \dfrac{d}{d(k_\rho\rho)}B_q(k_\rho\rho)$. Since

$$E_\varphi = 0 \text{ for } \rho = \rho_1,\ \rho = \rho_2,\ z = 0, \text{ and } z = a$$

then

$$h(nz)\,|_{z=0,a} = (a_n \sin(nz) + b_n \cos(nz))\,|_{z=0,a} = 0$$

is satisfied if: $a_n = 1$, $b_n = 0$, $n = \dfrac{m\pi}{a}$, and $m = 0,1,2,\ldots$ (13)

Note, then, that the cutoff frequencies are defined in the same way as the cutoff frequencies of standard rectangular guide, i.e., the cutoff frequencies of the TM modes of the azimuthally propagating, truncated, cylindrical, coaxial waveguide.

The boundary conditions are also satisfied if

$$B_q(k_\rho \rho)\,|_{\rho=\rho_1,\rho_2} = 0$$

Let $B_q(k_\rho \rho) = a_q J_q(k_\rho \rho) + b_q N_q(k_\rho \rho)$, then $a_q J_q(k_\rho \rho) + b_q N_q(k_\rho \rho)\,|_{\rho=\rho_1,\rho_2} = 0$. And,

$$a_q J_q(k_\rho \rho_1) + b_q N_q(k_\rho \rho_1) = 0$$
$$a_q J_q(k_\rho \rho_2) + b_q N_q(k_\rho \rho_2) = 0$$

Solving the first equation for a_q: $\quad a_q = b_q \dfrac{N_q(k_\rho \rho_1)}{J_q(k_\rho \rho_1)}$ (14)

Substitution into the second equation yields:

$$a_q J_q(k_\rho \rho_2) + b_q N_q(k_\rho \rho_2) = -b_q \frac{N_q(k_\rho \rho_1)}{J_q(k_\rho \rho_1)} J_q(k_\rho \rho_2) + a_q N_q(k_\rho \rho_2) = 0$$

Or,

$$b_q \left(N_q(k_\rho \rho_2) - \frac{N_q(k_\rho \rho_1)}{J_q(k_\rho \rho_1)} J_q(k_\rho \rho_2) \right) = 0$$

For specific values of k_ρ, ρ_1 and ρ_2, the values of q that solve

$$\frac{N_q(k_\rho \rho_2)}{J_q(k_\rho \rho_2)} = \frac{N_q(k_\rho \rho_1)}{J_q(k_\rho \rho_1)} \tag{15}$$

are the sought after mode numbers that are true for any non-zero value of b_q. Hence,

$b_q = 1$ and $a_q = -\dfrac{N_q(k_\rho \rho_1)}{J_q(k_\rho \rho_1)}$. Finally, the scalar wave function for the TM_z modes is:

$$\psi = \left[N_q(k_\rho \rho) - \frac{N_q(k_\rho \rho_1)}{J_q(k_\rho \rho_1)} J_q(k_\rho \rho) \right] e^{-jq\varphi} \sin(nz) \text{, for } n = \frac{m\pi}{a},\ m = 1,2,3,\ldots \text{, and } k_\rho^2 + n^2 = k^2 .$$

Note that just a single solution for q is possible [Ref. 4]. The TM_z field components are then found explicitly as:

$$E_\rho = \frac{nk_\rho}{j\omega\mu\varepsilon} \left[N_q'(k_\rho \rho) - \frac{N_q(k_\rho \rho_1)}{J_q(k_\rho \rho_1)} J_q'(k_\rho \rho) \right] \cos(nz) e^{-jq\varphi} \tag{16a}$$

$$E_\varphi = -\frac{q^2}{j\omega\mu\varepsilon} \frac{1}{\rho^2} \left[N_q(k_\rho \rho) - \frac{N_q(k_\rho \rho_1)}{J_q(k_\rho \rho_1)} J_q(k_\rho \rho) \right] \sin(nz) e^{-jq\varphi} \tag{16b}$$

$$E_z = j\omega \left(1 + \frac{n^2}{k_0^2} \right) \left[N_q(k_\rho \rho) - \frac{N_q(k_\rho \rho_1)}{J_q(k_\rho \rho_1)} J_q(k_\rho \rho) \right] \sin(nz) e^{-jq\varphi} \tag{16c}$$

$$H_\rho = -jq \frac{1}{\mu} \frac{1}{\rho} \left[N_q(k_\rho \rho) - \frac{N_q(k_\rho \rho_1)}{J_q(k_\rho \rho_1)} J_q(k_\rho \rho) \right] \sin(nz) e^{-jq\varphi} \tag{16d}$$

$$H_\varphi = -\frac{k_\rho}{\mu} \left[N_q'(k_\rho \rho) - \frac{N_q(k_\rho \rho_1)}{J_q(k_\rho \rho_1)} J_q'(k_\rho \rho) \right] \sin(nz) e^{-jq\varphi} \tag{16e}$$

$$H_z = 0 \tag{16f}$$

7.2 TE_z Field Components

The TE_z electric and magnetic field components in terms of the general wave function are:

$$E_\rho = j \frac{q}{\varepsilon} \frac{1}{\rho} B_q(k_\rho \rho) e^{-jq\varphi} h(nz) \tag{17a}$$

$$E_\varphi = \frac{k_\rho}{\varepsilon} B_q'(k_\rho \rho) e^{-jq\varphi} h(nz) \tag{17b}$$

$$E_z = 0 \tag{17c}$$

$$H_\rho = -j \frac{nk_\rho}{\omega\mu\varepsilon} B_q'(k_\rho \rho) e^{-jq\varphi} h'(nz) \tag{17d}$$

$$H_\varphi = -\frac{nq}{\omega\mu\varepsilon} \frac{1}{\rho} B_q(k_\rho \rho) e^{-jq\varphi} h'(nz) \tag{17e}$$

$$H_z = -j\omega\left(1+\frac{n^2}{k_0^2}\right)B_q(k_\rho\rho)e^{-jq\varphi}h''(nz) \quad (17f)$$

Since

$$E_\rho = 0 \text{ for } z=0, \text{ and } z=a.$$

then

$$h(nz)|_{z=0,a} = a_n \sin(nz) + b_n \cos(nz)|_{z=0,a}$$

is satisfied if: $a_n = 1$, $b_n = 0$, $n = \dfrac{m\pi}{a}$, and $m = 1,2,3,\ldots$. (18)

From the boundary condition on the E_φ component, the general Bessel function, $B_q(k_\rho\rho)$, also satisfies the boundary conditions if

$$\frac{d}{d(k_\rho\rho)}\left[B_q(k_\rho\rho)\right]|_{\rho=\rho_1,\rho_2} = 0$$

Let $B_q(k_\rho\rho) = a_q J_q(k_\rho\rho) + b_q N_q(k_\rho\rho)$, then $\dfrac{d}{d(k_\rho\rho)}\left\{\left[a_q J_q(k_\rho\rho) + b_q N_q(k_\rho\rho)\right]\right\}|_{\rho=\rho_1,\rho_2} = 0$, or

$$\left[a_q J_q'(k_\rho\rho) + b_q N_q'(k_\rho\rho)\right]|_{\rho=\rho_1,\rho_2} = 0$$

Substituting for the boundaries $\rho = \rho_1$ and $\rho = \rho_2$ give the equations:

$$a_q J_q'(k_\rho\rho_1) + b_q N_q'(k_\rho\rho_1) = 0$$

and

$$a_q J_q'(k_\rho\rho_2) + b_q N_q'(k_\rho\rho_2) = 0.$$

Solving the first equation above for a_q: $$a_q = -b_q\frac{N_q'(k_\rho\rho_1)}{J_q'(k_\rho\rho_1)} \quad (19)$$

Substitution into the second equation yields:

$$-b_q\frac{N_q'(k_\rho\rho_1)}{J_q'(k_\rho\rho_1)}J_q'(k_\rho\rho_2) + b_q N_q'(k_\rho\rho_2) = 0$$

Or,

$$b_q\left[\frac{N'_q(k_\rho\rho_1)}{J'_q(k_\rho\rho_1)}J'_q(k_\rho\rho_2)-N'_q(k_\rho\rho_2)\right] \quad =0$$

For specific values of k_ρ, ρ_1 and ρ_2, the values of q that solve

$$N'_q(k_\rho\rho_1)J'_q(k_\rho\rho_2)=N'_q(k_\rho\rho_2)J'_q(k_\rho\rho_1) \tag{20}$$

are the sought after mode numbers that are true for any non-zero value of b_q. Hence,

$b_q=1$ and $a_q=-b_q\dfrac{N'_q(k_\rho\rho_1)}{J'_q(k_\rho\rho_1)}$ solves the boundary condition for $\rho=\rho_1$, and the value of k_ρ that satisfies Eqn. 20 solves the boundary condition for $\rho=\rho_2$.

Finally, the scalar wave function for the TE_z modes is:

$$B_q(k_\rho\rho)=N_q(k_\rho\rho)-J_q(k_\rho\rho)\frac{N'_q(k_\rho\rho_1)}{J'_q(k_\rho\rho_1)}$$

$$\psi=\left[N_q(k_\rho\rho)-J_q(k_\rho\rho)\frac{N'_q(k_\rho\rho_1)}{J'_q(k_\rho\rho_1)}\right]\sin(nz)e^{-jq\varphi}, \text{ for } n=\frac{m\pi}{a},\ m=0,1,2,\ldots, \text{ and } k_\rho^2+n^2=k^2.$$

Note that again, just a single solution for q is possible [Ref. 4]. The TE_z field components are then found explicitly as:

$$E_\rho=j\frac{q}{\varepsilon}\frac{1}{\rho}\left[N_q(k_\rho\rho)-J_q(k_\rho\rho)\frac{N'_q(k_\rho\rho_1)}{J'_q(k_\rho\rho_1)}\right]\sin(nz)e^{-jq\varphi} \tag{21a}$$

$$E_\varphi=\frac{k_\rho}{\varepsilon}\left[N'_q(k_\rho\rho)-J'_q(k_\rho\rho)\frac{N'_q(k_\rho\rho_1)}{J'_q(k_\rho\rho_1)}\right]\sin(nz)e^{-jq\varphi} \tag{21b}$$

$$E_z=0 \tag{21c}$$

$$H_\rho=-j\frac{nk_\rho}{\omega\mu\varepsilon}\left[N'_q(k_\rho\rho)-J'_q(k_\rho\rho)\frac{N'_q(k_\rho\rho_1)}{J'_q(k_\rho\rho_1)}\right]\cos(nz)e^{-jq\varphi} \tag{21d}$$

$$H_\varphi=-\frac{nq}{\omega\mu\varepsilon}\frac{1}{\rho}\left[N_q(k_\rho\rho)-J_q(k_\rho\rho)\frac{N'_q(k_\rho\rho_1)}{J'_q(k_\rho\rho_1)}\right]\cos(nz)e^{-jq\varphi} \tag{21e}$$

$$H_z = j\omega\left(1+\frac{n^2}{k_0^2}\right)\left[N_q(k_\rho\rho) - J_q(k_\rho\rho)\frac{N'_q(k_\rho\rho_1)}{J'_q(k_\rho\rho_1)}\right]\sin(nz)e^{-jq\varphi} \tag{21f}$$

8. WAVEGUIDE IMPEDANCE

The characteristic wave impedances for the *Azimuthally Propagating, Truncated, Cylindrical, Coaxial Waveguide* are defined in terms of the dominant field components for the TE and TM modes. The wave impedance for the TE mode is given by

$$Z^{TE} = E_\rho / H_z; \tag{22a}$$

while wave impedance for the TM mode is given by

$$Z^{TM} = E_z / H_\rho. \tag{22b}$$

9. EXAMPLE

Determine the fundamental mode and that mode's cutoff frequency of the *Azimuthally Propagating, Truncated, Cylindrical, Coaxial Waveguide* defined by the parameters: $\rho_1 = 5\,in = 0.127\,m$, $\rho_2 = 6\,in = 0.1524\,m$ and $a = 6.5$ inches. Plot the distributions of all field components at $f = 1.3$ GHz, determine the guide wavelength in the propagating direction.

The fundamental mode is the TEz$_{11}$ mode ($m = 1$ and $q = 1$). The cutoff frequency is determined by the standard rectangular guide cutoff frequency $f_c = \frac{c_0}{2a}$, where $c_0 =$ speed of light and $a =$ broad wall dimension. For this case: $f_c = 2.998\times10^8/(2\times0.1651) = 0.9079$ GHz. Shown in Figure 3 is a plot of the numerically determined values of q as a function of frequency for the TEz$_{11}$ mode.

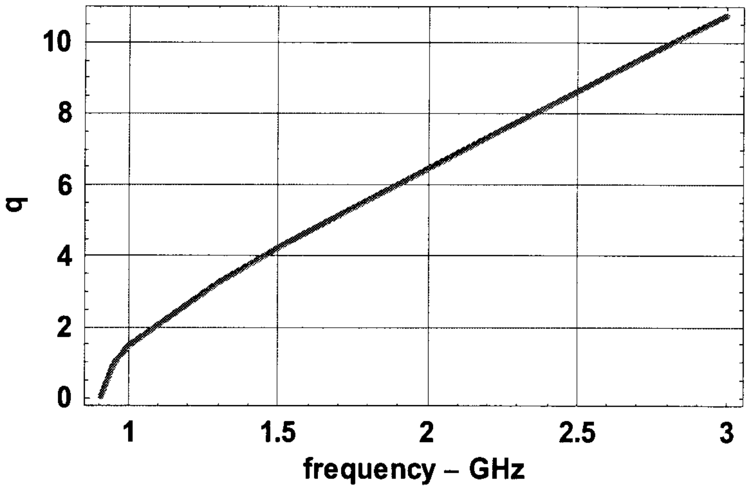

Figure 3. **The values of q for the TEz_{11} mode as a function of frequency for an *Azimuthally Propagating, Truncated, Cylindrical, Coaxial Waveguide* defined by the parameters: $\rho_1 = 5\,in = 0.127\,m$, $\rho_2 = 6\,in = 0.1524\,m$ and $a = 6.5$ inches.**

To compute the field distributions of the TEz_{11} mode at $f = 1.3$ GHz, the value of q must first be determined. Shown in Figure 4 is a graph of the characteristic equation of the TEz11 mode for an *Azimuthally Propagating, Truncated, Cylindrical, Coaxial Waveguide* defined by the parameters: $\rho_1 = 5\,in = 0.127\,m$, $\rho_2 = 6\,in = 0.1524\,m$ and $a = 6.5$ inches. The point at which the curve crosses zero defines the required value of q, in this case $q = 2.72$.

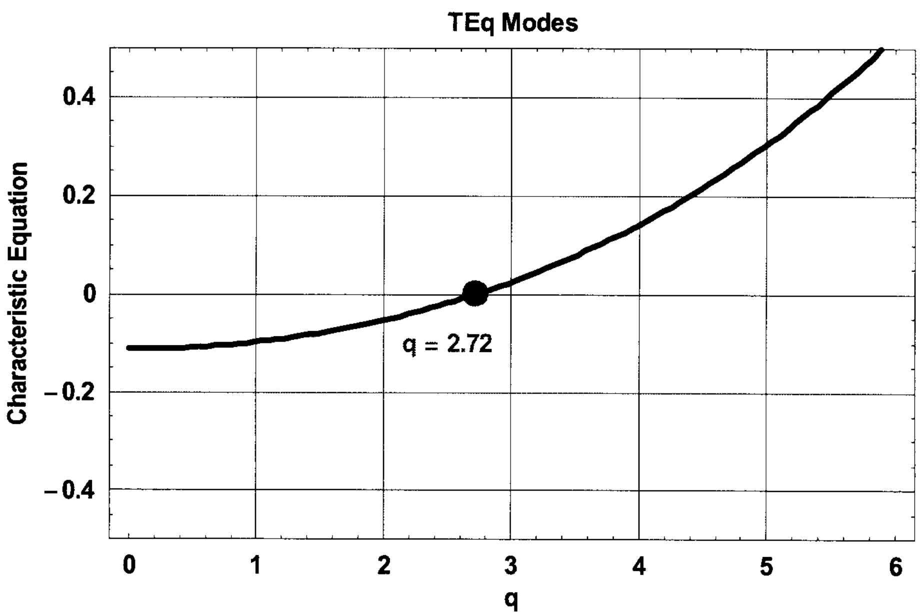

Figure 4. **The characteristic equation of the TEz11 mode for an *Azimuthally Propagating, Truncated, Cylindrical, Coaxial Waveguide* defined by the parameters: $\rho_1 = 5\,in = 0.127\,m$, $\rho_2 = 6\,in = 0.1524\,m$ and $a = 6.5$ inches.**

The field components can now be determined using Equations 21. Shown in Figure 5 are the normalized electromagnetic field distributions of the various field components for the TEz_{11} mode. One notes that E_ρ and H_z are the dominate field components (as expected), since they closely resemble the fundamental mode field distributions in standard rectangular guide. The guide wavelength can be determined numerically by plotting the real component of the phasor E_ρ component of the electric field of the TEz_{11} mode as a function of azimuthal position as shown in Figure 6. The guide wavelength is found to be

$$\lambda_g = \frac{\rho_1 + \rho_2}{2} \times \varphi = \frac{5+6}{2} \times \frac{132}{180} \times \pi = 12.67 \text{ inches}$$

which is just bigger than the free space wavelength at $f = 1.3$ GHz of 9.08 inches, but less than the rectangular guide wavelength of $\lambda_g^{rectangular} = 12.687$ inches. The TE mode impedance can be calculated as $Z^{TE} = E_\rho / H_z$, and in this case is found to be $Z^{TE} = 181\ \Omega$.

As the frequency increases, additional propagating modes become possible. For example, at $f = 2$ GHz, the TEz_{21} mode will propagate. The value of $q = 2.9515$ is found numerically, and the electric field components are shown in Figure 7.

As the frequency further increases, the first TM mode can propagate. For example, at $f = 2.2$ GHz, the TMz_{11} mode will propagate. Note that the cutoff frequencies for the TM modes are not defined by the standard simple rectangular waveguide relations (as was found for the TE modes). Rather, they are determined through Bessel function relations that are solved numerically. The value of $q = 2.998$ is found numerically, and the electric and magnetic field components are shown in Figure 8. The TM mode impedance can be calculated as $Z^{TM} = E_z / H_\rho$, and in this case is found to be $Z^{TM} = 1140\ \Omega$.

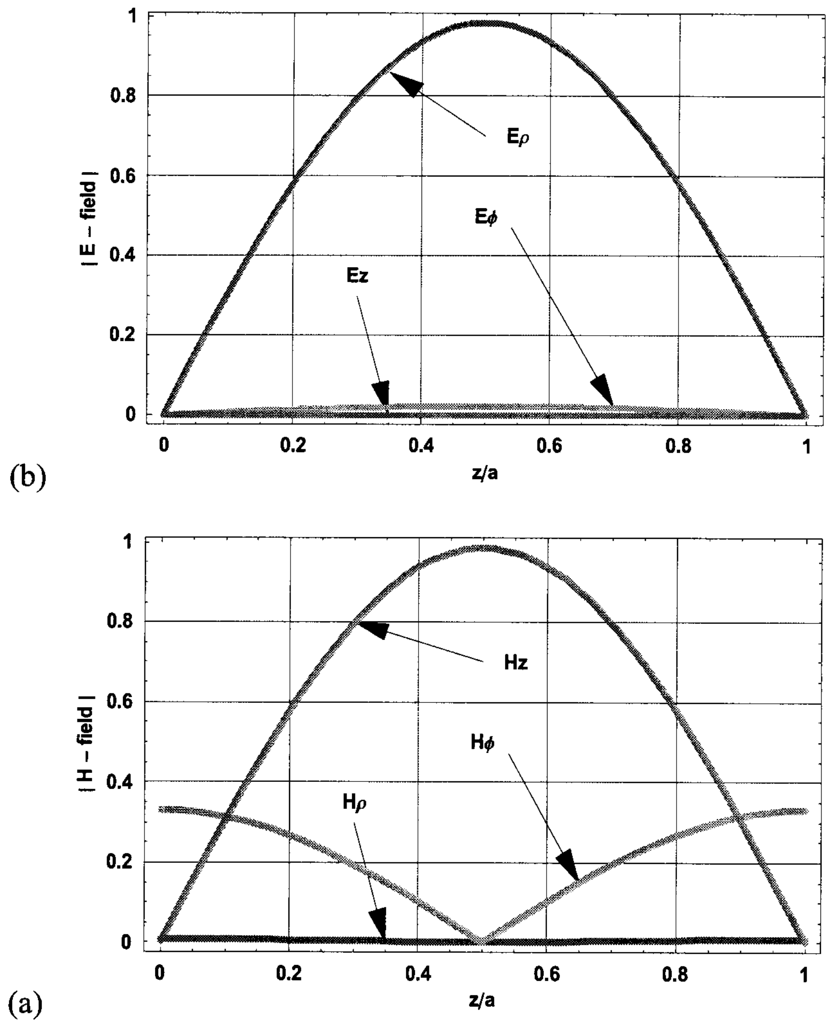

Figure 5. **The electromagnetic field distribution of the various field components in an *Azimuthally Propagating, Truncated, Cylindrical, Coaxial Waveguide* defined by the parameters: $\rho_1 = 5\,in = 0.127\,m$, $\rho_2 = 6\,in = 0.1524\,m$ and $a = 6.5$ inches for the TEz11 mode: (a) normalized electric field; and (b) normalized magnetic field.**

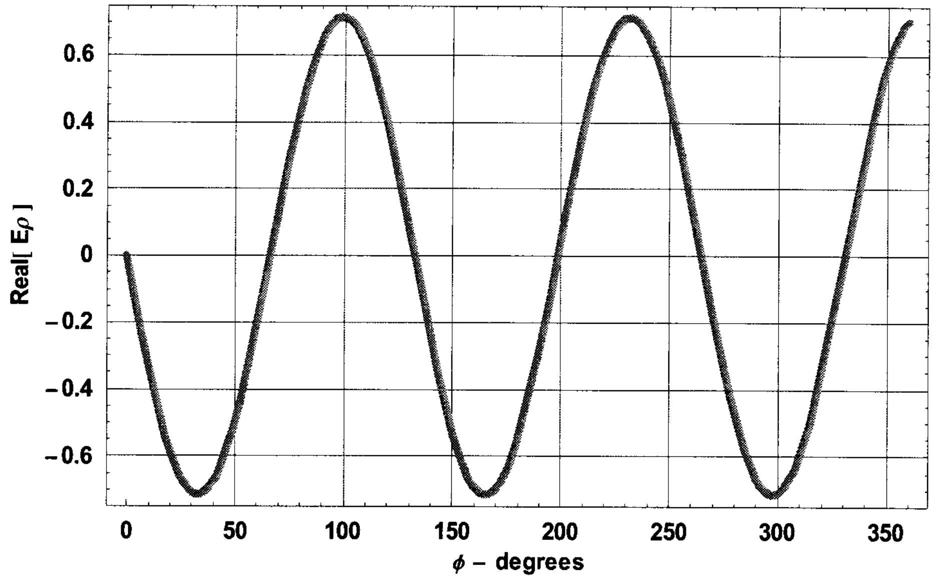

Figure 6. **The value of the E_ρ component of the electric field of the TEz11 mode as a function of azimuthal position.**

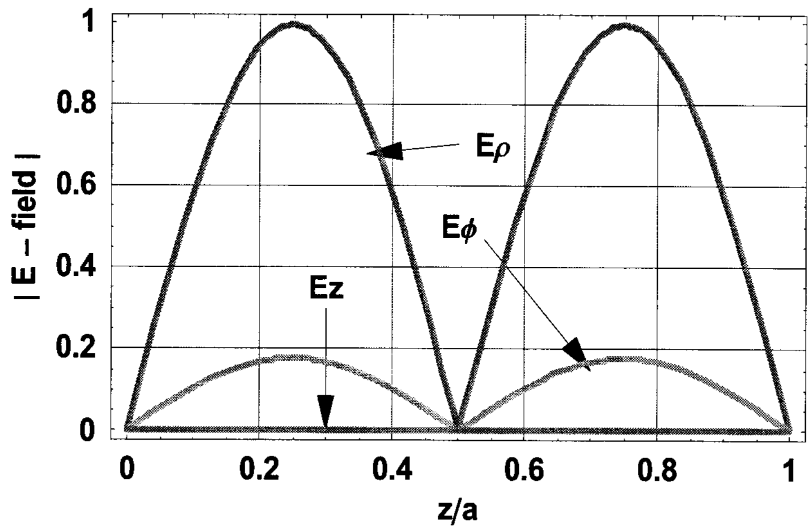

Figure 7. **The electric field distribution of the various field components in an *Azimuthally Propagating, Truncated, Cylindrical, Coaxial Waveguide* defined by the parameters: $\rho_1 = 5\,in = 0.127\,m$, $\rho_2 = 6\,in = 0.1524\,m$ and $a = 6.5$ inches for the TEz21 mode.**

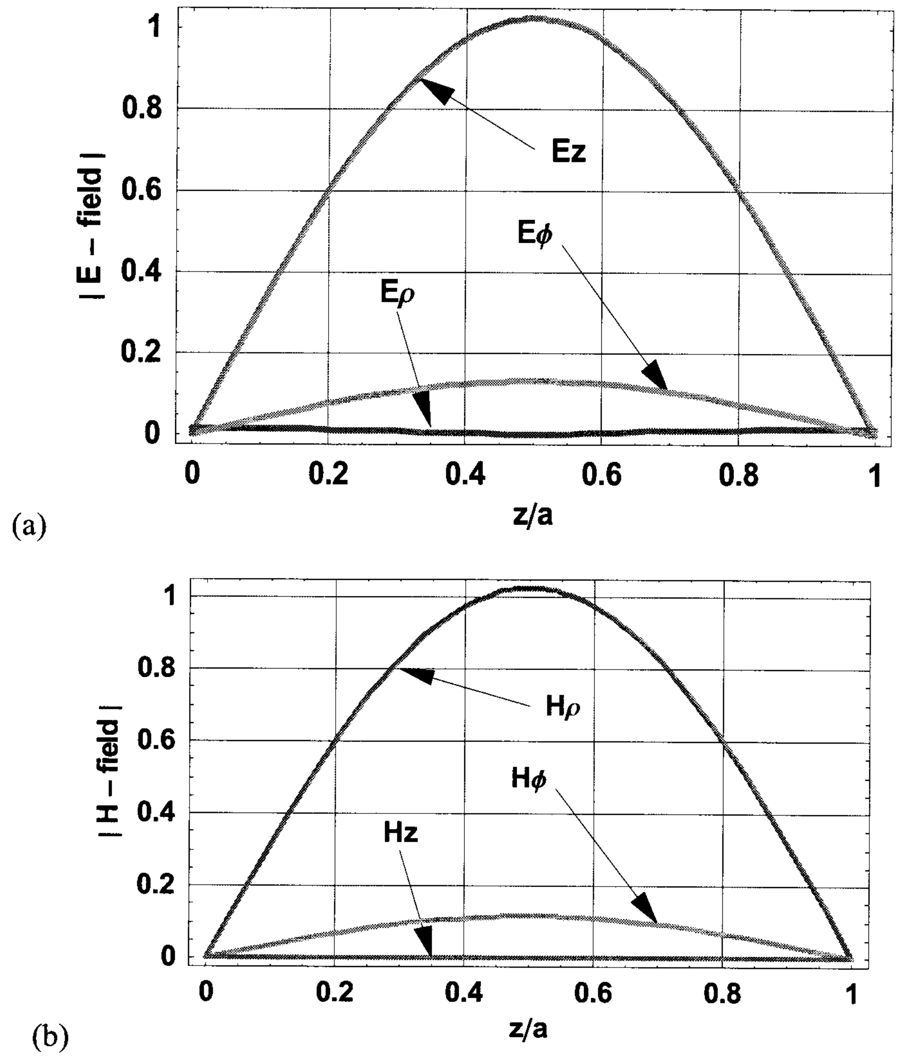

Figure 8. **The electromagnetic field distribution of the various field components for the TMz_{11} mode: (a) normalized electric field; and (b) normalized magnetic field.**

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APPENDIX I – About the Wave Equation in Cylindrical Coordinates[1]

The natural coordinate system for the ***Azimuthally Propagating, Truncated, Cylindrical, Coaxial Waveguide*** is the cylindrical coordinate system. And one might be tempted to propose solutions that TE_ϕ and TM_ϕ. This would not be wise.

The VECTOR wave equation is

$$\left(\underline{\nabla}^2 + k_0^2\right)\boldsymbol{\Psi} = 0 \tag{1}$$

where the vector operator $\underline{\nabla}^2() = \nabla(\nabla \cdot ()) - \nabla \times \nabla()$. Now this equation reduces to the scalar wave equation for Cartesian coordinates, since the unit vectors are constants. This is not so in cylindrical coordinates, except for the unit vector in the z-direction. When written out, Equation No. 1 becomes

$$\left(\nabla^2\Psi_\rho - \frac{1}{\rho^2}\Psi_\rho - \frac{2}{\rho^2}\frac{\partial}{\partial\varphi}\Psi_\varphi\right)\mathbf{u}_\rho + \left(\nabla^2\Psi_\varphi - \frac{1}{\rho^2}\Psi_\varphi + \frac{2}{\rho^2}\frac{\partial}{\partial\varphi}\Psi_\rho\right)\mathbf{u}_\varphi + \left(\nabla^2\Psi_z\right)\mathbf{u}_z + k_0^2\boldsymbol{\Psi} = 0 \tag{2}$$

Separating this into its components yields three equations

$$\nabla^2\Psi_\rho - \frac{1}{\rho^2}\Psi_\rho - \frac{2}{\rho^2}\frac{\partial}{\partial\varphi}\Psi_\varphi + k_0^2\Psi_\rho = 0 \tag{3a}$$

$$\nabla^2\Psi_\varphi - \frac{1}{\rho^2}\Psi_\varphi + \frac{2}{\rho^2}\frac{\partial}{\partial\varphi}\Psi_\rho + k_0^2\Psi_\varphi = 0 \tag{3b}$$

$$\nabla^2\Psi_z + k_0^2\Psi_z = 0 \tag{3c}$$

Only Equation 3c is a scalar wave equation. In forming the solutions for the potentials, it is important that we propose solutions that are TE_z and TM_z, so that solutions to the scalar wave equation can be postulated.

[1] From Prof. Chalmers Butler's ECE 831 class notes, Clemson University, 1988.