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Prolate Spheroidal Scatterer for Spherical TEM Waves

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Abstract

This paper considers the scattering of an inhomogeneous spherical TEM wave emitted from one of two foci of a prolate spheroid (special case of an ellipsoid of revolution). The wave scatters on the inside of a perfectly conducting prolate spheroidal surface to produce an exact inhomogeneous spherical TEM wave propagating toward the second focus. This solution is exact for a clear time based on other scattering (e.g., from feed arms) reaching the observer.

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1. Introduction

Previous papers [2-4] have established that inhomogeneous TEM waves in a uniform, isotropic medium (e.g., free space) are exactly transformed by stereographic projection into second such waves in the case of paraboloidal and hyperboloidal scatterers, provided the incident wave is centered on an appropriate focal point (including infinity) of these quadric surfaces. One spherical or planar TEM wave is then transformed into another with an exact matching of the boundary conditions on the (perfectly conducting) reflector. This gives *exact* solutions of the Maxwell equations, valid up until some time related to a signal arriving at the observer from some truncation of the reflector, or from some waveguiding structure used to guide the incident wave (i.e., conical or cylindrical perfectly conducting transmission lines). These “early-time” exact solutions (valid up to some “clear time” are examples of partial geometric symmetries as discussed in [7].

Keeping with bodies of revolution, which give focal *points*, another quadric surface to consider is the prolate spheroid, a special case of an ellipsoid. In this case both focal points are inside the volume enclosed by the surface S_p. So our consideration is to launch an inhomogeneous plane wave from one focus, and reflect it toward the second.

2. Polate Spheroid

The geometry is as discussed in [5]. Summarizing we have S_p defined by

$$\begin{aligned}
&\left[\frac{\Psi}{b}\right]^2 + \left[\frac{z}{a}\right]^2 = 1 \quad , \quad a > b \\
&a \equiv \text{major radius} \\
&b \equiv \text{minor radius} \\
&(\Psi, \phi, z) \equiv \text{cylindrial coordinates} \\
&(x, y, z) \equiv \text{Cartesian coordinates} \\
&x = \Psi\cos(\phi) \quad , \quad y = \Psi\sin(\phi) \quad , \quad \Psi^2 = x^2 + y^2
\end{aligned} \tag{2.1}$$

As in Fig. 2.1 there are two foci at

$$\begin{aligned}
(\Psi, \phi, z) &= (0, \phi, \pm z_0) = \pm \vec{r}_0 = \pm z_0 \vec{1}_z \\
z_0 &= \left[a^2 - b^2\right]^{1/2}
\end{aligned} \tag{2.2}$$

Each ray from $-\vec{r}_0$ to $\vec{r}_0$ travels a distance

$$d_1 + d_2 = 2a = 2\left[z_0^2 + b^2\right]^{1/2} \tag{2.3}$$

At the surface S_p, the angle of incidence equals the angle of reflection.

Of course, only S'_p [5], a portion of S_p is the actual reflector but this need not concern us here. From $-\vec{r}_0$ there is a spherical TEM wave guided by two (or more) perfectly conducting conical conductors toward the reflector. A second wave is reflected toward $\vec{r}_0$. This is a transformation of the first wave.

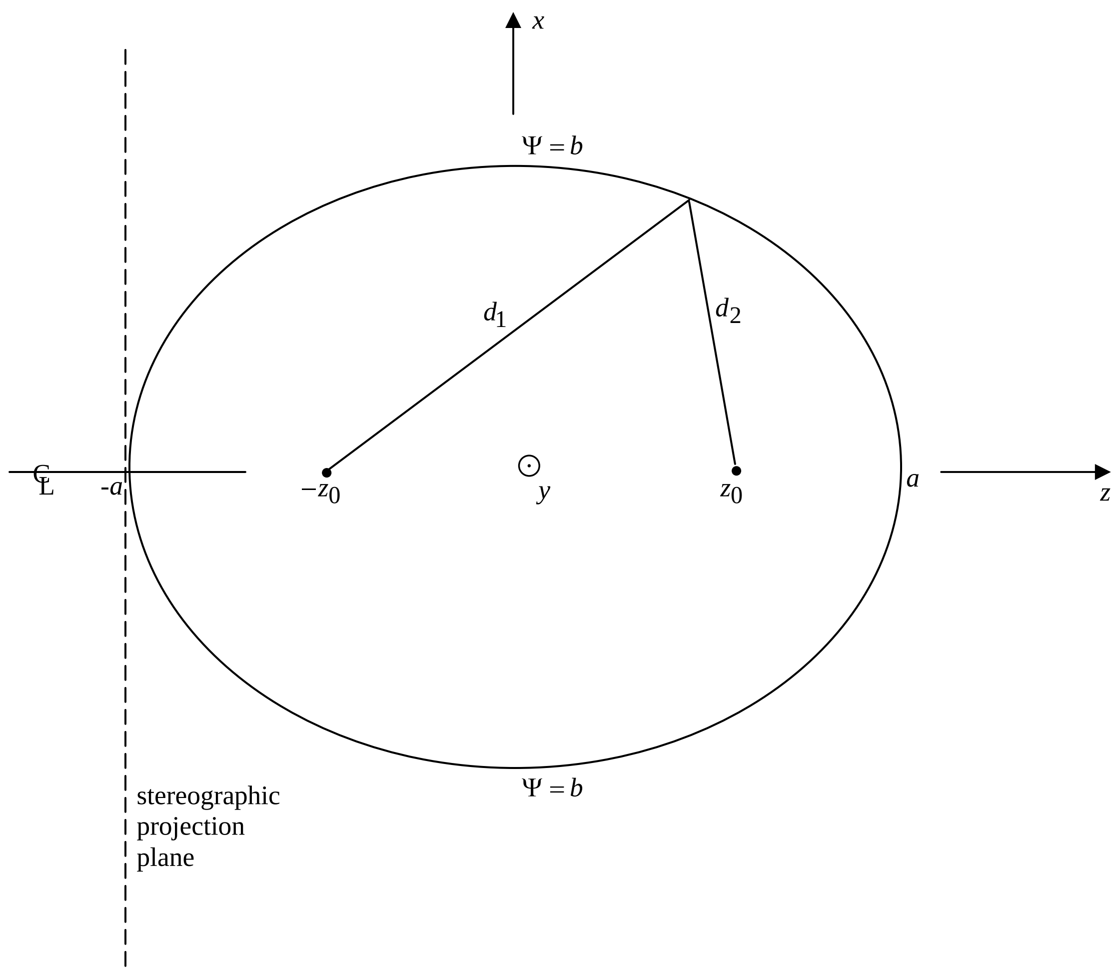

Fig. 2.1 Prolate-Spheroid Cross Section (Ellipse).

3. Matching Spherical TEM Waves

In spherical coordinates (r_1, θ_1, ϕ_1) centered on $-\vec{r}_0$, with $\theta_1 = 0$ pointing along the negative z axis (toward the stereo graphic-projection plane), we have an outward propagating (from $-\vec{r}_0$) inhomogeneous transient TEM wave as

$$\begin{aligned}
\vec{E}_1 &= -\vec{r}_0 \nabla_{\theta_1,\phi_1} V_1(\theta_1,\phi_1)\, f\left(t - \frac{r_1}{c}\right) \\
\nabla_{\theta_1,\phi_1} &= \vec{1}_{\theta_1} \frac{\partial}{\partial\theta_1} + \vec{1}_{\phi_1} \frac{1}{\sin(\theta_1)} \frac{\partial}{\partial\theta_1} \\
\nabla^2_{\theta_1,\phi_1} V_1(\theta_1, \phi) &= 0 \qquad \text{(TEM condition)} \\
\nabla^2_{\theta_1,\phi_1} &= \frac{1}{\sin(\theta_1)} \frac{\partial}{\partial\theta_1} \sin(\theta_1) \frac{\partial}{\partial\theta_1} + \frac{1}{\sin^2(\theta_1)} \frac{\partial^2}{\partial\theta_1^2} \\
c &= [\mu\varepsilon]^{-1/2} \equiv \text{ speed of light in medium}
\end{aligned} \tag{3.1}$$

This is guided by two or more perfectly conducting cones (with apices at $-\vec{r}_0$) toward the reflector.

The stereographic transformation relating spherical TEM waves to cylindrical TEM waves takes the form

$$\Psi_0 = 2[a - z_0]\tan\left(\frac{\theta_1}{2}\right), \ \phi_0 = \phi_1 = -\phi \tag{3.2}$$

The projection plane is here taken as $z = -a$, tangential to the reflecting surface. Note that for purposes of this transform the projection plane extends to ∞, corresponding to $\theta_1 = \pi$. Note that (Ψ_0, ϕ_0) corresponds to a point on the projection plane. In this projection V_1 satisfies the Laplace equation in cylindrical (Ψ_0, ϕ_0) coordinates.

Here we are imagining a wave launched to the "left", to be reflected on S'_p [5] (the portion of S_p on which a reflector is built). The portion to the "right", around the target location, is assumed not used for the reflector, allowing access to the target vicinity. However, the symmetry at the geometry allows one to interchange the roles of source and target.

Let us consider a second spherical TEM wave, centered (incoming) on $\vec{r}_0$ of the form

$$\vec{E}_2 = -\frac{1}{r_2} \nabla_{\theta_2, \phi_2} V_2(\theta_2, \phi_2)\ f\left(t + \frac{r_2}{c} - \frac{2a}{c}\right) \tag{3.3}$$

The projection formula for this wave is

$$\Psi_0 = 2[a + z_0]\tan\left(\frac{\theta_2}{2}\right) , \ \phi_0 = -\phi_2 = -\phi \tag{3.4}$$

The formulas in (3.1) also apply, substituting subscripts 2 for 1.

Now let

$$V_2(\Psi_0 , \phi_0) = -V_1(\Psi_0 , \phi_0) \tag{3.5}$$

on the projection plane. Since V_1 satisfies the Laplace equation there, so does V_2. Going through the projection formula (3.4), then $V_2(\theta_2 , \phi_2)$ satisfies the spherical Laplace equation ((3.1) using 2 subscripts). This double stereographic transformation is of the same general form as in [4 (Section IV)] with a few sign changes. Here we have a diverging wave reflected into a converging wave. Note that the waveforms are the same *f(t)* for these two waves.

We merely need now that $V_2 + V_1 = 0$ (or its tangential derivative, i.e., tangential E-field) on the reflector. On the reflector we have, due to the stereographic transforms

$$V_2(\theta_2 , \phi_2) = -V_1(\theta_1 , \phi_1) \tag{3.6}$$

The two waves match in time as well on the reflector due to (2.3). Differentiating the potential (net zero) on the reflection gives zero tangential electric field, the required boundary condition. This gives an exact solution of the Maxwell equations for times (clear times) before scattering from feed arms, and S_p truncation to S'_p is seen by the observer. Such clear time is observer-position dependent. For analytical convenience we can take the time-domain waveform as a step function

$$f(t) = u(t) \tag{3.7}$$

applying to both transmitted and reflected wave.

The feed arms also fit into the spherical Laplace equation. Their electrical "centers" have been considered in the case of impulse-radiating antennas (IRAs) {2, 6], allowing for placement which in some sense is optimal. Consider that these intersect S_p at some z_p. The double stereographic transform then has "image" feed arms pointing to $\vec{r}_0$ from the intersection at z_p. This leads to an interesting symmetry concept by setting

$$z_p = 0 \tag{3.8}$$

This makes $z = 0$ a symmetry plane between the wave launching side ($z < 0$) and the wave receiving side ($z > 0$). In practice (inverse) feed arms are not included on the receiving side (at least not down to the focal point at $\vec{r}_0$).

4. Separation of Two TEM Waves Incident on Target

The foregoing is limited in that it includes the wave reflected from S'_p. There is also a direct wave from $-\vec{r}_0$ to $\vec{r}_0$. This is a step function in the simple case (3.7). It arrives at a time

$$t_p = 2\frac{z_0}{c} \tag{4.1}$$

It is analogous to the prepulse from a reflector IRA. The reflected wave arrives at $\vec{r}_0$ at a time

$$t_p = \frac{d_1 + d_2}{c} = \frac{2a}{c} \tag{4.2}$$

The length of the prepulse is then

$$\Delta t_p = \frac{2\left[a - z_0\right]}{c} = \frac{2}{c}\left[a - \left[a^2 - b^2\right]^{1/2}\right] \tag{4.3}$$

provided there are no guiding arms for the reflected waves to interfere with this.

Assuming that the prepulse is a negative E_x, corresponding to a positive potential on the upper feed arms (positive x in Fig. 2.1), the reflected pulse reaching toward $\vec{r}_0$ has a positive sign. The waveform, however, is quite different. In [1] the case of the guiding arms is considered by use of an aperture integral. The field at $\vec{r}_0$ has a delta-function pact and a step-function part. A detailed treatment of this may appear in the future.

5. Concluding Remarks

As expected, the spherical TEM wave from one focus sends a second spherical TEM wave toward the other focus. For a clear time (position dependent) the second wave has the same waveform as the first wave. The spatial part is found from the double stereographic transform.

This paper is the third describing an inhomogeneous TEM wave reflecting off a quartic-surface reflector (paraboloid, hyperboloid, and prolate spheroid) to produce exactly a second inhomogeneous TEM wave.

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