Sensor and Simulation Notes

Note 518

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Extension of the Analytic Results for the Focal Waveform of a Two-Arm Prolate-Spheroidal Impulse-Radiating Antenna (IRA)

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Abstract

This paper is a development of the field waveform at the second focal of a prolate-spheroidal reflector and it is an extension of [1]. We explore the analytic behavior of the waveform at the second focal region.

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1. Introduction

This paper is an extension of a previous paper [1] .IRAs have been developed for the transient far field region but in this paper we focus mainly on the near fields that can be used in some biological applications [2].

In Section 2, we show that the impulsive part of the waveform at the second focus can be described by a delta-like pulse forming for $z < z_0$ and in the limit as $z \to z_0$ gives the required true delta function. This is a physical example of the formation of a delta function. Then, the aperture integral gives the same result (at early time) as the exact incident wave before truncation. This gives confidence in the aperture integration. We can see that the area of the $\delta -$like pulse is the same for both after and before z_0.We illustrate these results with a graphical example.

In Section 3, we continue the analytical evaluation of the prepulse term, E_{p2}, after the impulse, when the aperture integral is truncated by the aperture edge.

Finally in Section 4 from previous and present results, the actual analytical waveform is illustrated, including all the terms.

1.1 . Description of geometry

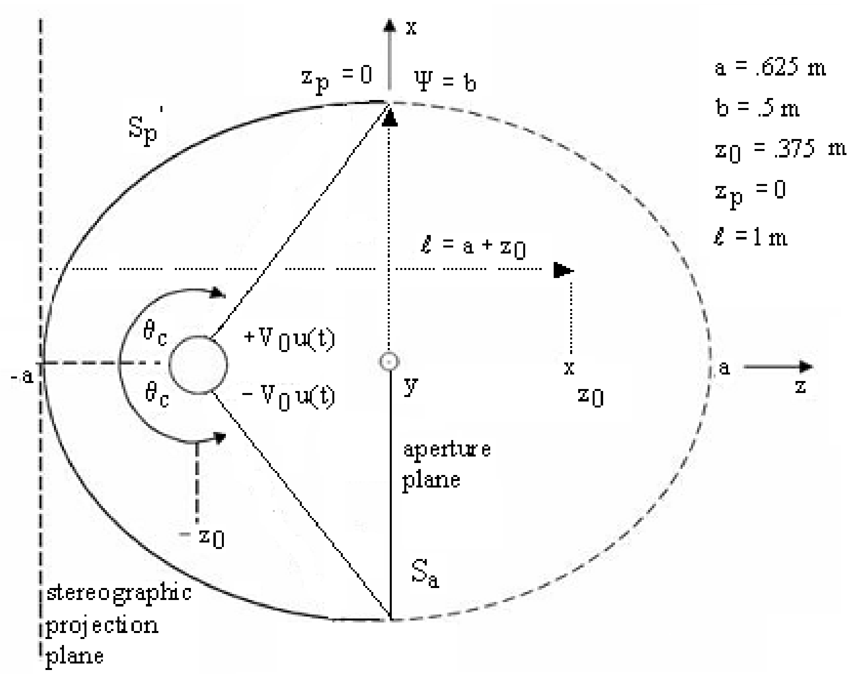

Figure 1.1 IRA Geometry

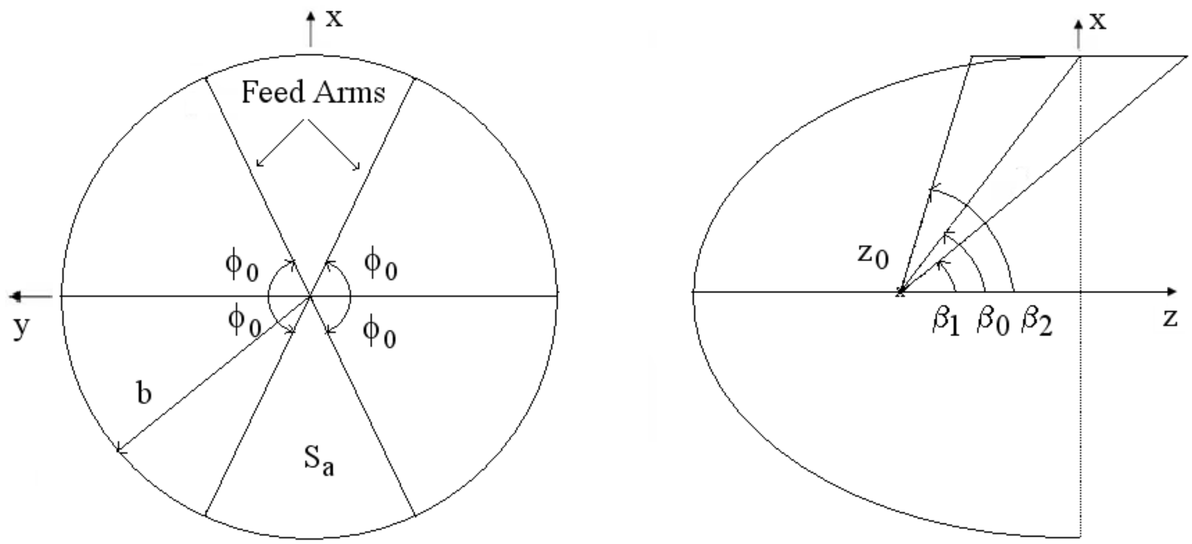

Figure 1.2 Feed Arm Geometry

We choose a special case of the prolate-spheroidal IRA's geometric parameters as [1]

$$z_p = 0,\ b = \Psi_0 = .5\,\text{m},\ a = .625\ \text{m}\ ,\ z_0 = .375\,\text{m}, \ell = 1\,\text{m} \tag{1.1}$$

For our later example calculations, our design has two TEM feed arms and the dimensions of these arms are determined by a $400\,\Omega$ pulse impedance($\phi_0 = 90^\circ$).

The feed-arm parameters have been previously calculated in the stereographic projection plane as [3]

$$b_0{}^2 = b_1 b_2 \quad , \quad (b_1 - b_2)/b_0 = 0.275 \tag{1.2}$$

from which we find the angles for the two-feed arms as

$$\begin{aligned} &\beta_0 = \arctan(.5/.375) \cong 53.1^\circ \ ,\ \beta_1 = 2\arctan\left[\sqrt{b_1/b_2}\,\tan(\beta_0/2)\right] = 47^\circ \\ &\beta_2 = 2\arctan\left[\sqrt{b_1/b_2}\,\tan(\beta_1/2)\right] = 59.6^\circ \end{aligned} \tag{1.3}$$

β_0 ,β_1 , β_2 are the angles from the z-axis to the electrical center, the first edge and the second edge of the feed arms as in Fig. 1.2 .

From (1.1) and (1.2) and Table 3.1 of [3], one can find the locations and dimensions of the feed arms. The feed arms are symmetric and the upper feed arm has three corners located at

x	y	z
0	0	-37.5
50	0	-8.2
50	0	8.99

Table 1.1 Upper feed arm corner locations in cm

1.2 Fields at second focus

Summarizing, we have [1]

$$E_\delta = \frac{V_0}{\pi f_g c}\frac{a+c}{a-c}\cot\left(\frac{\theta_c}{2}\right)\left[1-\left[1+\left[\frac{\Psi_p}{z_0-z_p}\right]^2\right]^{-1/2}\right]$$

$$E_s = \frac{V_0}{2\pi f_g}\frac{1}{z_0-z_p}\frac{a+z_0}{a-z_0}\cot\left(\frac{\theta_c}{2}\right)\left[1+\left[\frac{z_0-z_p}{\Psi_p}\right]^2\right]^{-1} , \ E_p = \frac{V_0}{2\pi f_g z_0}\tan\left(\frac{\theta_c}{2}\right) \quad (1.4)$$

$$E_{pa} = E_p \Delta t_p = \frac{V_0}{2\pi f_g c}\frac{a-z_0}{z_0}\tan\left(\frac{\theta_c}{2}\right) \ \text{time integral or “area” of prepulse}$$

$$E_0 = \frac{V_0}{\pi f_g}\frac{1}{a-z_0}\cot\left(\frac{\theta_c}{2}\right)$$

where E_δ and E_s is the impulsive and step terms from the reflection in the prolate sphere and E_p is the magnitude of the prepulse wave from first focus.(valid up to the time of aperture truncation)

We can normalize these terms as

$$e_\delta = E_\delta \frac{\pi f_g c}{V_0} \quad \text{impulse} \qquad (1.5)$$

$$e_s = E_s \frac{2\pi f_g \ell}{V_0} \quad \text{postpulse step}$$

$$e_p = E_p \frac{2\pi f_g \ell}{V_0} \quad \text{prepulse (step, negative)}$$

$$e_{pa} = E_p \Delta t_p \frac{2\pi f_g c}{V_0} \quad \text{prepulse integral (area, negative)}$$

1.3 Calculating $\tan(\theta_c / 2)$

We have a simpler form for $\tan(\theta_c / 2)$ than that in [1]. θ_c can be between $0 \le \theta_c \le \pi$. By the geometric construction from Fig. 1.1 we have

$$\tan(\theta_c / 2) = \sin^{-1}(\theta_c)[1-\cos(\theta_c)] \qquad (1.6)$$

$$\sin(\theta_c) = \sin(\pi-\theta_c) = \Psi_p \left[\Psi_p{}^2 + [z_p + z_o]^2\right]^{-1/2} \qquad (1.7)$$

$$\cos(\theta_c) = -\cos(\pi-\theta_c) = -[z_p + z_o]\left[\Psi_p{}^2 + [z_p + z_o]^2\right]^{-1/2} \qquad (1.8)$$

so we have

$$\tan(\frac{\theta_c}{2}) = \frac{1}{\Psi_p}\left[\Psi_p{}^2 + \left[z_p + z_o\right]^2\right]^{1/2}\left[\left[z_p + z_o\right]\left[\Psi_p{}^2 + \left[z_p + z_o\right]^2\right]^{-1/2}\right]$$
$$= \frac{1}{\Psi_p}\left[\Psi_p{}^2 + \left[z_p + z_o\right]^2\right]^{1/2}\frac{\left[z_p + z_o\right]}{\Psi_p} \tag{1.9}$$

Let's try to find $\left[\Psi_p{}^2 + \left[z_p + z_o\right]^2\right]^{1/2}$ in terms of a , z_0 , z_p

$$\Psi_p = b\left[1 - \left[\frac{z_p}{z_o}\right]^2\right]^{1/2} \tag{1.10}$$

$$\left[\Psi_p{}^2 + \left[z_p + z_o\right]^2\right]^{1/2} = \frac{1}{a}\left[b^2\left[a^2 - z_p{}^2\right] + a^2\left[z_p + z_0\right]^2\right]^{1/2} = \frac{1}{a}\left[a^2 + z_0 z_p\right] \tag{1.11}$$

Substituting this in (1.6) so we obtain

$$\tan(\frac{\theta_c}{2}) = \left[\frac{a + z_p}{a - z_p}\right]^{1/2}\frac{a + z_0}{b} \tag{1.12}$$

One can see from (1.1) that, if we take $z_p = 0$, it really simplifies (1.4) and (1.5).

2. E-Field Variation Near z-Axis

2.1 Exact solution of the impulsive term for $z < z_0$ up to aperture truncation of signal

First of all, we will find the exact E_3 for $z < z_0$ (as in Fig 2.1) for times after the pulse arrival when the solution no longer goes to 0.

We can write (3.13) and (3.10) in [1] as

$$\vec{E}_2 = E_0 \frac{a + z_0}{r_2}\left[\frac{2\cos(\phi_2)}{1+\cos(\theta_2)}\vec{1}_{\theta_2} - \frac{2\sin(\phi_2)}{1+\cos(\theta_2)}\vec{1}_{\theta_2}\right] u\left(t + \frac{r_2}{c} - 2\frac{a}{c}\right) \tag{2.1}$$

this being the tangential electric field on the aperture S_a, from the reflection due to the prolate sphere. On the z-axis $\theta_2 = 0$, ϕ_2 is arbitrary, so let's take $\phi_2 = 0$, giving

$$r_2 = z_0 - z \tag{2.2}$$

$$E_3 = E_0 \frac{a + z_0}{z_0 - z} u\left(t + \frac{r_2}{c} - 2\frac{a}{c}\right) \text{ (oriented in the x direction)}$$

Substitute E_0 and $\cot\left(\frac{\theta_c}{2}\right)$ in E_3

$$E_3 = \frac{V_0}{\pi f_g}\frac{1}{z_0 - z}\left[\frac{a - z_p}{a + z_p}\right]^{1/2}\frac{b}{a - z_0} u\left(t + \frac{z_0 - z}{c} - 2\frac{a}{c}\right) \tag{2.3}$$

We can normalize E_3 as

$$\begin{aligned} e_3 = \frac{\pi f_g \ell}{V_0} E_3 &= -\frac{\ell}{z_0 - z}\left[\frac{a - z_p}{a + z_p}\right]^{1/2}\frac{b}{a - z_0} u\left(t + \frac{z_0 - z}{c} - 2\frac{a}{c}\right) \\ &= e_\delta' \, u\left(t + \frac{z_0 - z}{c} - 2\frac{a}{c}\right) \end{aligned} \tag{2.4}$$

$$e_\delta' = \frac{\ell}{z_0 - z}\left[\frac{a - z_p}{a + z_p}\right]^{1/2}\frac{b}{a - z_0} \tag{2.5}$$

This result applies for the time up until the signal from the truncation of the aperture is seen.

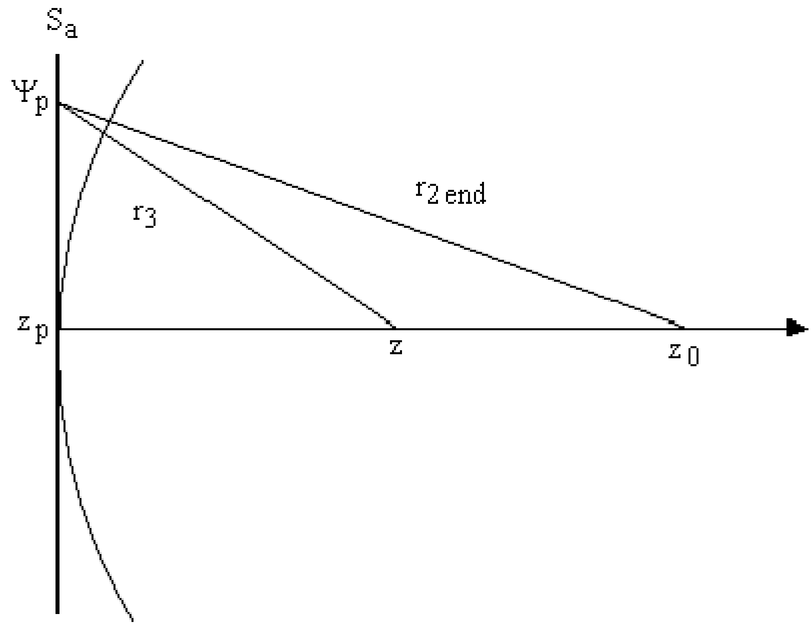

Figure 2.1 z values for $z < z_0$

For convenience we define retarded time t_r such that $t_r = 0$ is the time of arrival of the direct ray along the z-axis. The field from Ψ_p, the aperture truncation, then arrives at the observer on the z-axis at a retarded time

$$
\begin{aligned}
ct_{r\,end} &= -r_{2\,end} + r_3 + [z_0 - z] \\
&= -\left[\Psi_p{}^2 + [z_0 - z_p]^2\right]^{1/2} + \left[\Psi_p{}^2 + [z - z_p]^2\right]^{1/2} + [z_0 - z]
\end{aligned}
\tag{2.6}
$$

For z near z_0 this is approximately

$$
\begin{aligned}
r_3 - r_{2\,end} &= \left[\Psi_p{}^2 + [z - z_p]^2\right]^{1/2} + \left[\Psi_p{}^2 + [z_0 - z_p]^2\right]^{1/2} \\
&= \left[\Psi_p{}^2 + [z_0 - z_p]^2\right]^{1/2} + \frac{[z_0 - z_p][z - z_0]}{\left[\Psi_p{}^2 + [z_0 - z_p]^2\right]^{1/2}} + O\left([z - z_0]^2\right)
\end{aligned}
\tag{2.7}
$$

$r_{2\,end}$ is the r_2 value where $\Psi = \Psi_p$. We can find

$$
\begin{aligned}
r_3 - r_{2\,end} &= \frac{[z_0 - z_p][z - z_0]}{\left[\Psi_p{}^2 + [z_0 - z_p]^2\right]^{1/2}} \\
ct_{r\ end} &= [z_0 - z]\left[1 - \frac{[z_0 - z_p]}{\left[\Psi_p{}^2 + [z_0 - z_p]^2\right]^{1/2}}\right] = [z_0 - z]\left[\frac{r_{2\,end} - z_0 + z_p}{r_{2\,end}}\right]
\end{aligned}
\tag{2.8}
$$

We notice that e_2 is proportional to $[z - z_0]^{-1}$ and ct_r is proportional to $[z - z_0]$. The product gives the "area" under the pulse as

$$E_{3\,mag}\ t_{r\ end} = \frac{V_0}{\pi f_g c}\left[\frac{a - z_p}{a + z_p}\right]^{1/2} \frac{b}{a - z_0}\left[1 - \left[1 + \left[\frac{\Psi_p}{z_0 - z_p}\right]^2\right]^{-1/2}\right] \tag{2.9}$$

This is like an impulse , going to zero width as $z \to z_0$. Let us compare this with (5.1) in [1]. They are exactly same! This shows that the impulsive part of the waveform at the second focus can be described by a delta-like pulse forming for $z < z_0$ and in the limit as $z \to z_0$ gives the required true delta function. This is a physical example of the formation of a delta function.

We can find the normalized value of the "area" in (2.9) from (2.4) and (2.8) as

$$e_{3\,mag}\ t_{r\ end} = \frac{b\ell}{c(a - z_0)}\left[\frac{a - z_p}{a + z_p}\right]^{1/2}\left[\frac{r_2 - z_0 + z_p}{r_2}\right] \tag{2.10}$$

2.2 Approximate solution for $z < z_0$ by aperture integration for early time with z near z_0

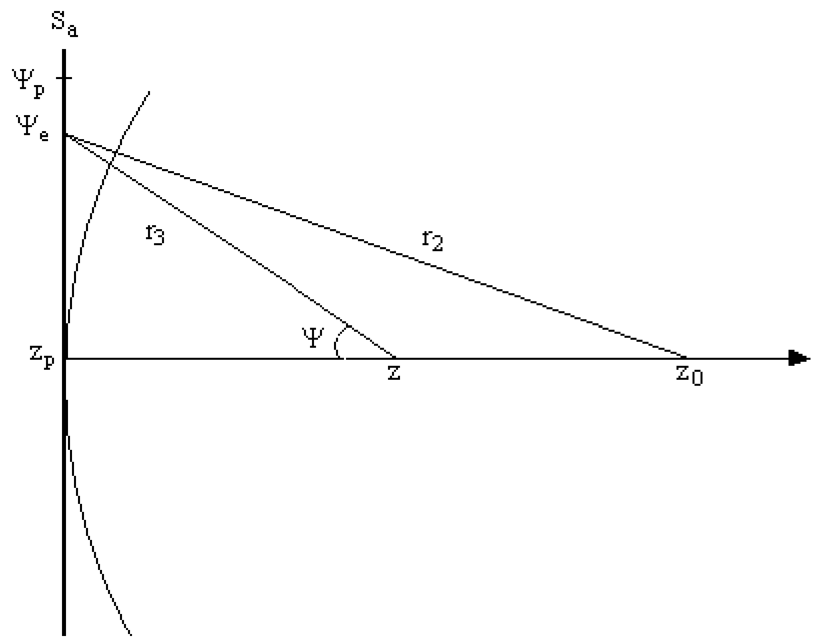

Figure 2.2 z values for $z < z_0$

The field from $\Psi = \Psi_e$ on the aperture isseen at observer at $z\,\vec{1}_z$ is seen at a later retarded time.

$$
\begin{aligned}
& ct_r = [r_3 - [z - z_p]] - [r_2 - [z_0 - z_p]] \\
& r_3 = \left[\Psi_e{}^2 + [z - z_p]^2\right]^{1/2} \quad , \quad r_2 = \left[\Psi_e{}^2 + [z_0 - z_p]^2\right]^{1/2} \\
& ct_r = \left[\left[\Psi_e{}^2 + [z - z_p]^2\right]^{1/2} - [z - z_p]\right] - \left[\left[\Psi_e{}^2 + [z_0 - z_p]^2\right]^{1/2} - [z_0 - z_p]\right]
\end{aligned}
\tag{2.11}
$$

For small Ψ_e we have

$$
ct_r = \frac{1}{2}\Psi_e{}^2\left[\frac{1}{[z - z_p]} - \frac{1}{[z_0 - z_p]}\right] \quad , \quad \Psi_e = \left[\frac{2ct_r}{\dfrac{1}{[z - z_p]} - \dfrac{1}{[z_0 - z_p]}}\right]^{1/2}
\tag{2.12}
$$

If $|z_0 - z|$ is small we have

$$
\begin{aligned}
& ct_r = [r_3 - r_2] + z_0 - z \\
& ct_r - [z_0 - z] = r_3 - r_2 \\
& r_3 - r_2 = \left[\Psi_e{}^2 + z^2\right]^{1/2} - \left[\Psi_e{}^2 + z_0{}^2\right]^{1/2} \cong \frac{[z_0 + z]}{2}\frac{[z_0 - z]}{\Psi_e{}^2 + z_0{}^2}
\end{aligned}
\tag{2.13}
$$

So we have

$$
ct_r - [z_0 - z] \approx + \frac{[z_0 + z]}{2}\frac{[z_0 - z]}{\Psi_e{}^2 + z_0{}^2} \quad , \quad \Psi_e{}^2 = z_0{}^2 + \frac{1}{2}\left[\frac{z_0 + z}{\dfrac{ct_r}{z_0 - z} - 1}\right]
\tag{2.14}
$$

We want to take the surface integral of (4.2 in [1]) to find a new form for E_δ. It does not involve a step-function from S_a. It is now dispersed such that the integration limits can be functions of time, giving

$$
\begin{aligned}
E_\delta &= \frac{E_0}{\pi c}\frac{d}{dt}\int_0^{\Psi_e}\int_0^{2\pi}\frac{z - z_p}{r_3{}^2}\frac{a + z_0}{r_2 + z_0 + z_p}\left[\frac{z_0 - z_p}{r_2}\cos^2(\phi) + \sin^2(\phi)\right]\Psi d\theta d\Psi \\
&= \frac{E_0}{c}\frac{d}{dt}\int_0^{\Psi_e}\frac{z - z_p}{r_3{}^2}\frac{a + z_0}{r_2}\Psi d\Psi
\end{aligned}
\tag{2.15}
$$

Let's take the time derivative of the integral

$$
E_\delta = \frac{d\Psi_e}{dt}\frac{E_0}{c}\frac{z - z_p}{r_3{}^2}\frac{a + z_0}{r_2}\Psi
\tag{2.16}
$$

We can find $d\Psi_e / dt$ from (2.12)

$$c\frac{dt_r}{d\Psi_e} = \Psi_e\left(\frac{1}{[z-z_p]} - \frac{1}{[z_0 - z_p]}\right), \frac{d\Psi_e}{c\,dt_r} = \Psi_e{}^{-1}\left(\frac{1}{[z-z_p]} - \frac{1}{[z_0 - z_p]}\right)^{-1} \qquad (2.17)$$

So (2.15) becomes

$$E_\delta = \Psi_e{}^{-1}\left(\frac{1}{[z-z_p]} - \frac{1}{[z_0 - z_p]}\right)^{-1} \frac{E_0}{c}\frac{z-z_p}{r_3{}^2}\frac{a+z_0}{r_2}\Psi_e \qquad (2.18)$$

$$E_\delta = \frac{V_0}{\pi f_g}\frac{1}{z_0 - z}\left[\frac{a-z_p}{a+z_p}\right]^{1/2}\frac{b}{a-z_0} \qquad (2.19)$$

As we can see it is same as E_3 in (2.3) . This shows that the aperture integral gives the same result (at early time) as the exact incident wave before truncation. This gives confidence in the aperture integration. The reader can note that since the above gives a pulse of width greater than zero, one can add a correction term (zero at zero retarded time) from E_s, also dispersed like E_δ .

2.3 Approximate solution for $z > z_0$ by aperture integration for early time with z near z_0

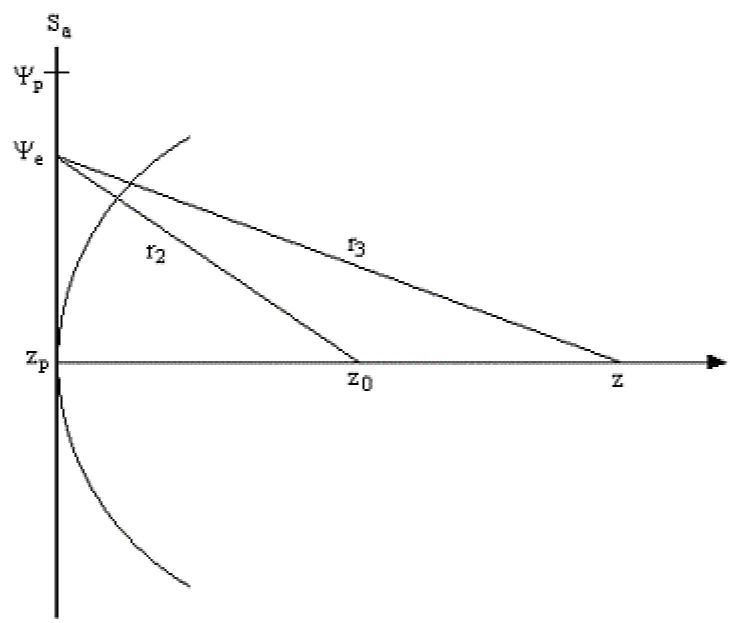

Figure 2.3 z values for $z > z_0$

Let Ψ_e be close to Ψ_p for which

$$r_3 = \left[\Psi_e{}^2 + [z-z_p]^2\right]^{1/2}, \quad r_2 = \left[\Psi_e{}^2 + [z_0 - z_p]^2\right]^{1/2} \qquad (2.20)$$

On S_a fields arrive at the time

$$ct_{r\,a} = \left[z_0 - z_p\right] - \left[\Psi_e{}^2 + \left[z_0 - z_p\right]^2\right]^{1/2} \tag{2.21}$$

which sets $ct_{r\,a} = 0$ on the aperture center . Fields arrive at z at the retarded time

$$ct_{r\,a} + r_3 \tag{2.22}$$

The first fields at z are from $\Psi = \Psi_p$, at the retarded time

$$ct_{r\,\text{first}} = \left[z_0 - z_p\right] - \left[\Psi_p{}^2 + \left[z_0 - z_p\right]^2\right]^{1/2} + \left[\Psi_p{}^2 + \left[z - z_p\right]^2\right]^{1/2} \tag{2.23}$$

The last fields come along the z axis at

$$ct_{r\,\text{last}} = \left[z_0 - z_p\right] - \left[z_0 - z_p\right] + z - z_p = z - z_p \tag{2.24}$$

So define a new retarded time by subtracting $z - z_p$ so the pulse stops at 0 but begins at

$$ct_{r\,\text{begin}} = \left[z - z_p\right] - \left[\Psi_p{}^2 + \left[z_0 - z_p\right]^2\right]^{1/2} + \left[\Psi_p{}^2 + \left[z - z_p\right]^2\right] < 0 \tag{2.25}$$

From an arbitrary point on S_a

$$ct_r = \left[z_0 - z\right] - \left[\Psi^2 + \left[z_0 - z_p\right]^2\right]^{1/2} + \left[\Psi^2 + \left[z - z_p\right]^2\right]^{1/2} \tag{2.26}$$

We can take the derivative of this retarded time, giving

$$c\frac{dt_r}{d\Psi_e} = -\frac{\Psi_e}{\left[\Psi_e{}^2 + \left[z_0 - z_p\right]^2\right]^{1/2}} + \frac{\Psi_e}{\left[\Psi_e{}^2 + \left[z - z_p\right]^2\right]^{1/2}} \tag{2.27}$$

$$c\frac{dt_r}{d\Psi_e}\bigg|_{\Psi_e = \Psi_p} = -\frac{\Psi_p}{\left[\Psi_p{}^2 + \left[z_0 - z_p\right]^2\right]^{1/2}} + \frac{\Psi_p}{\left[\Psi_p{}^2 + \left[z - z_p\right]^2\right]^{1/2}} \tag{2.28}$$

So E_δ can be found as,

$$E_\delta = \frac{E_0}{c}\frac{d\Psi_e}{dt}\bigg|_{\Psi_p} \frac{z - z_p}{r_3{}^2}\frac{a + z_0}{r_2}\Psi_p = \frac{V_0 \ell}{\pi f_g \ell}\frac{1}{a - z_0}\cot\left(\frac{\theta_c}{2}\right)\frac{d\Psi_e}{dt}\bigg|_{\Psi_p} \frac{z - z_p}{r_3{}^2}\frac{a + z_0}{r_2}\Psi_p \tag{2.29}$$

E_δ can be normalized as,

$$e_\delta{}' = \frac{\pi f_g \ell}{V_0} E_\delta = \frac{\ell b}{a - z_0}\left[\frac{a - z_p}{a + z_p}\right]\frac{z - z_p}{r_3}\left[r_3 - r_2\right]^{-1} \tag{2.30}$$

We need to expand $r_3 - r_2$ for small $z - z_0$

$$r_3 = \left[\Psi_p{}^2 + \left[[z - z_0] + [z_0 - z_p]\right]^2\right]^{1/2}$$

$$= \left[\Psi_p{}^2 + [z_0 - z_p]^2\right]^{1/2}\left[1 + \frac{2[z - z_0][z_0 - z_p] + [z - z_0]}{\left[\Psi_p{}^2 + [z_0 - z_p]^2\right]} + O([z - z_0])^2\right]^{1/2} \quad (2.31)$$

$$r_2 = \left[\Psi_p{}^2 + [z_0 - z_p]^2\right]^{1/2} \quad (2.32)$$

$$r_3 - r_2 = \frac{[z - z_0]\,[z - z_p]}{r_2} \quad \text{as } \Psi_e \to \Psi_p \text{ and } \frac{r_2}{r_3} = 1 + O([z - z_0]) \cong 1$$

So the normalized field is

$$e_\delta{}' = \frac{\ell b}{a - z_0}\left[\frac{a - z_p}{a + z_p}\right]^{1/2} \frac{1}{z - z_0} \quad (2.33)$$

This is the same as (2.4)

The asymptotic form of ct_{begin} for small $z - z_0$ is

$$ct_{r\,\text{begin}} = [z_0 - z]\left[\frac{r_2 - z_0 + z_p}{r_2}\right](\text{negative}) = -ct_{r\,\text{end}} \quad (\text{as in in } (2.8)) \quad (2.34)$$

The integral (or area) of the pulse is just

$$E_\delta\ t_{r\ \text{begin}} = \frac{1}{c}\frac{\ell b}{a - z_0}\left[\frac{a - z_p}{a + z_p}\right]^{1/2}\left[\frac{-r_2 + z_0 - z_p}{r_3}\right] \quad (2.35)$$

One can see with comparing (2.35) and (2.8) the area of E_δ is same for both z after and before z_0.

2.4 Graphical illustration

In order to illustrate what our results have shown, let us make a graph showing the normalized pulse shape for various $z - z_0$ as one goes from negative values through the second focal point to positive values. For negative values the pulse follows after zero retarded time. For positive values the pulse precedes the zero retarded time. For our example we take the simple case from (1.1) and which is related to the 3,4,5 right triangle.

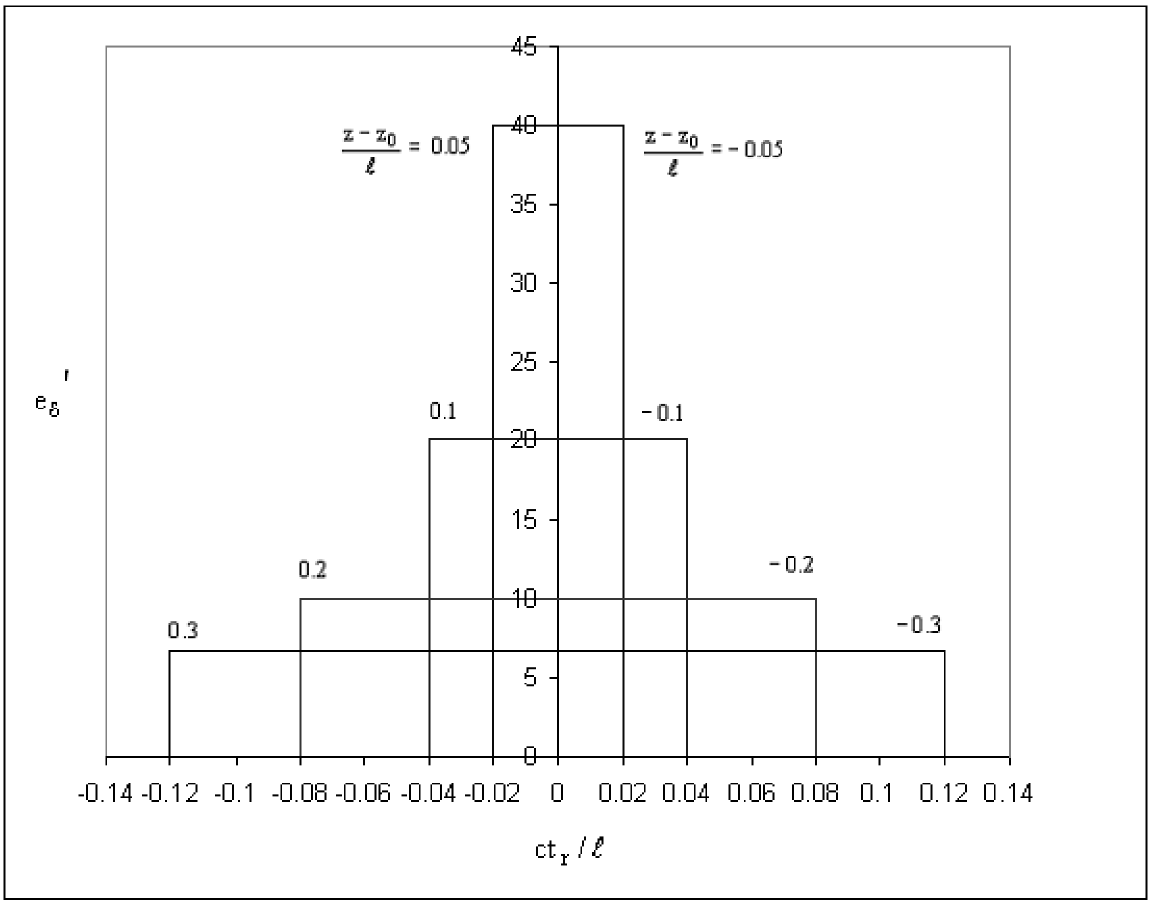

Figure 2.4 Normalized Pulse Shape for the various $\frac{z - z_0}{\ell}$

One can see from Fig. 2.4 the compression of the pulse as $z \to z_0$ and the expansion of the pulse for $z > z_0$ as z increases away from z_0.

3. Prepulse Term E_{p2} after the Impulse

What happens to the prepulse term after the impulse, ie after the truncation at the aperture boundary ($\Psi = \Psi_p$, or b for special case)? Before the aperture truncation the prepulse is given by (1.4).

Let E_{pt} = tangential E field (x component) on S_a due to the prepulse wave

Then we have[1,4]

$$E_{p1} = \frac{1}{2\pi c}\frac{\partial}{\partial t}\int_{S_a}\frac{z_0 - z_p}{{r_2}^2}E_{pt}\,dS\ ,\ \ E_{p2} = \frac{1}{2\pi}\int_{S_a}\frac{z_0 - z_p}{{r_2}^2}E_{pt}\,dS \tag{3.1}$$

After we see the edge of S_a, neglecting diffraction terms from this edge, and approximating E_{pt} by the negative of the TEM prepulse wave out to this edge(for a positive parameter as in [1]) we have, for step-function excitation, time-independent prepulse field on S_a,

$$E_{p1} = 0 \text{ the derivative being zero after the aperture edge is seen} \tag{3.2}$$

$$E_{p2} = \frac{1}{2\pi}\int\frac{z_0 - z_p}{{r_2}^3}E_{pt}\,dS = \text{constant , i.e. a step term.}$$

So we need the static E_{pt}. As before, since we are confining ourself to the z-axis we can use a uniform field on the projection plane to give E_{pt} in the above integral.

From (2.11) of [1] at $r_1 = z_0$ (aperture plane center)

$$E_{pa0} \cong \frac{V_0}{z_0\pi f_g}\tan\left(\frac{\theta_c}{2}\right) \tag{3.3}$$

We need this extended over S_a, since, as we have seen , for the z axis only the uniform field terms (on the projection plane) need be considered (by symmetry).

On the projection plane at $z = -a$ [1 (2.7)]

$$V(x_0, y_0) = \frac{V_0}{2}\operatorname{arccosh}^{-1}\left(\frac{\Psi_{c0}}{r_{w0}}\right)\ln\left(\frac{\left(\frac{\Psi_{c0}}{r_{w0}}\right)^2 + 2\frac{\Psi_{c0}}{r_{w0}}\cos(\theta_1) + 1}{\left(\frac{\Psi_{c0}}{r_{w0}}\right)^2 - 2\frac{\Psi_{c0}}{r_{w0}}\cos(\theta_1) + 1}\right) \tag{3.4}$$

$$\Psi_0 = 2[a - z_0]\tan\left(\frac{\theta_1}{2}\right)$$

On the projection plane at $\Psi_0 = 0$ we need the uniform field component (x-directed)

$$V \cong \frac{V_0}{2}\operatorname{arccosh}^{-1}\left(\frac{\Psi_{c0}}{r_{w0}}\right)\ln\frac{2\frac{x}{\Psi_{c0}}+1}{-2\frac{x}{\Psi_{c0}}+1} \cong \frac{V_0}{2}\operatorname{arccosh}^{-1}\left(\frac{\Psi_{c0}}{r_{w0}}\right)2\frac{x}{\Psi_{c0}} \tag{3.5}$$

$$E_{4,0} = \frac{2V_0}{\Psi_{c0}}\operatorname{arccosh}^{-1}\left(\frac{\Psi_{c0}}{r_{w0}}\right) = \frac{2V_0}{\pi f_g}\frac{1}{\Psi_{c0}} \text{ (taken as positive in our convention[1])}$$

Choose a potential (uniform)

$$V = E_{4,0}\, x = -\frac{2V_0}{\Psi_{c0}}\operatorname{arccosh}^{-1}\left(\frac{\Psi_{c0}}{r_{w0}}\right)\Psi_0 \cos(\Phi_0) = -E_{40}\,\Psi_0\cos(\Phi_0) \tag{3.6}$$

Map this back onto the $\vec{r}_1$ system

$$V = E_0\, 2[a - z_0]\tan\left(\frac{\theta_1}{2}\right)\cos(\phi)$$

$$\Psi_{c0} = 2[a - z_0]\tan\left(\frac{\theta_c}{2}\right) = 2[a - z_0]\left[\frac{a + z_p}{a - z_p}\right]^{1/2}\frac{a + z_0}{b} \tag{3.7}$$

$$V = E_{4,0}\, 2[a - z_0]\tan\left(\frac{\theta_1}{2}\right)\cos(\phi)$$

Now on Sa we have

$$\vec{E}_4 = \frac{1}{r_1}\nabla_{\theta_1,\phi_1} V = -E_{4,0}\, 2\frac{[a - z_0]}{r_1}\left(\sec^2\left(\frac{\theta_1}{2}\right)\frac{1}{2}\cos(\phi)\,\vec{1}_{\theta_1} - \frac{\tan\left(\frac{\theta_1}{2}\right)}{\sin(\theta_1)}\sin(\phi)\,\vec{1}_{\phi_1}\right) \tag{3.8}$$

The tangential part is

$$\vec{E}_{4t} = E_{4,0}\frac{[a - z_0]}{r_1}\left(\sec^2\left(\frac{\theta_1}{2}\right)\frac{1}{2}\cos(\theta_1)\cos(\phi_1)\,\vec{1}_{\Psi} - 2\frac{\tan\left(\frac{\theta_1}{2}\right)}{\sin(\theta_1)}\sin(\phi)\,\vec{1}_{\phi_1}\right)$$

$$= E_{4,0}\frac{[a - z_0]}{r_1}\left(\frac{2\cos(\theta_1)\cos(\phi_1)}{1 + \cos(\theta_1)}\vec{1}_{\Psi} - 2\frac{\sin(\phi_1)}{1 + \cos(\theta_1)}\vec{1}_{\phi_1}\right) \tag{3.9}$$

$$r_1\left[1+\cos(\theta_1)\right] = r_1 + z_0 + z_p$$

$$r_1 = \left[\Psi_0{}^2 + \left[z_0 + z_p\right]^2\right]^{1/2} \;,\; \cos(\theta_1) = \frac{z_0 + z_p}{r_1} \tag{3.10}$$

The x component is

$$E_{4x} = E_{4,0}\frac{\left[a - z_0\right]}{r_1 + z_0 + z_p}\left[2\frac{z_0 + z_p}{r_1}\cos^2(\theta_a) + 2\sin^2(\phi_a)\right]$$

$$E_{p2} = E_{40}\left[z_0 - z_p\right]\left[a - z_0\right]\int_0^{\Psi_p}\frac{1}{r_2{}^3\, r_1}d\Psi \tag{3.11}$$

To solve this integral consider the special case $z_p = 0, \Psi_p = b, r_1 = r_2$. Then from (4.2) and (4.3) of [1]

$$E_{p_2} = \frac{V_0}{\pi f_g}\frac{1}{\Psi_{c0}}\frac{\left[a - z_0\right]}{z_0}\left[1+\left[\frac{z_0}{b}\right]^2\right]^{-1}$$

$$\Psi_{c0} = 2\left[a - z_0\right]\tan\left(\frac{\theta_c}{2}\right) = 2\left[a - z_0\right]\frac{\left[a + z_0\right]}{b} \tag{3.12}$$

At the end we obtain

$$E_{p_2} = \frac{V_0}{2\pi f_g}\frac{1}{z_0}\frac{b}{a + z_0}\left[1+\left[\frac{z_0}{b}\right]^2\right]^{-1} \tag{3.13}$$

The normalized e_{p2} is

$$e_{p_2} = \frac{2\pi f_g \ell}{V_0}E_{p2} = \frac{\ell}{z_0}\frac{b}{a + z_0}\left[1+\left[\frac{z_0}{b}\right]^2\right]^{-1} \tag{3.14}$$

Note that this is actually the negative of the prepulse (to give a positive parameter) by convention in [1]. Let’s find the ratio of $\frac{e_{p2}}{e_p}$ from (1.5) and (3.14)

$$\frac{e_{p2}}{e_p} = \left(\frac{b}{a + z_0}\right)^2\left[1+\left[\frac{z_0}{b}\right]^2\right]^{-1}\left[\frac{a - z_p}{a + z_p}\right]^{1/2} \tag{3.15}$$

For our case (1.1)

$$\frac{e_{p2}}{e_p} = \left(\frac{b}{a + z_0}\right)^2\left[1+\left[\frac{z_0}{b}\right]^2\right]^{-1} = 0.16 < 1 \tag{3.16}$$

as expected

4. The Actual Analytical Waveform

We take the simple example case in (1.1) to illustrate the analytical waveform. The excitation is a 1 Volt ($V_0 = .5$ Volt) step, rising as a ramp function lasting 100 ps.

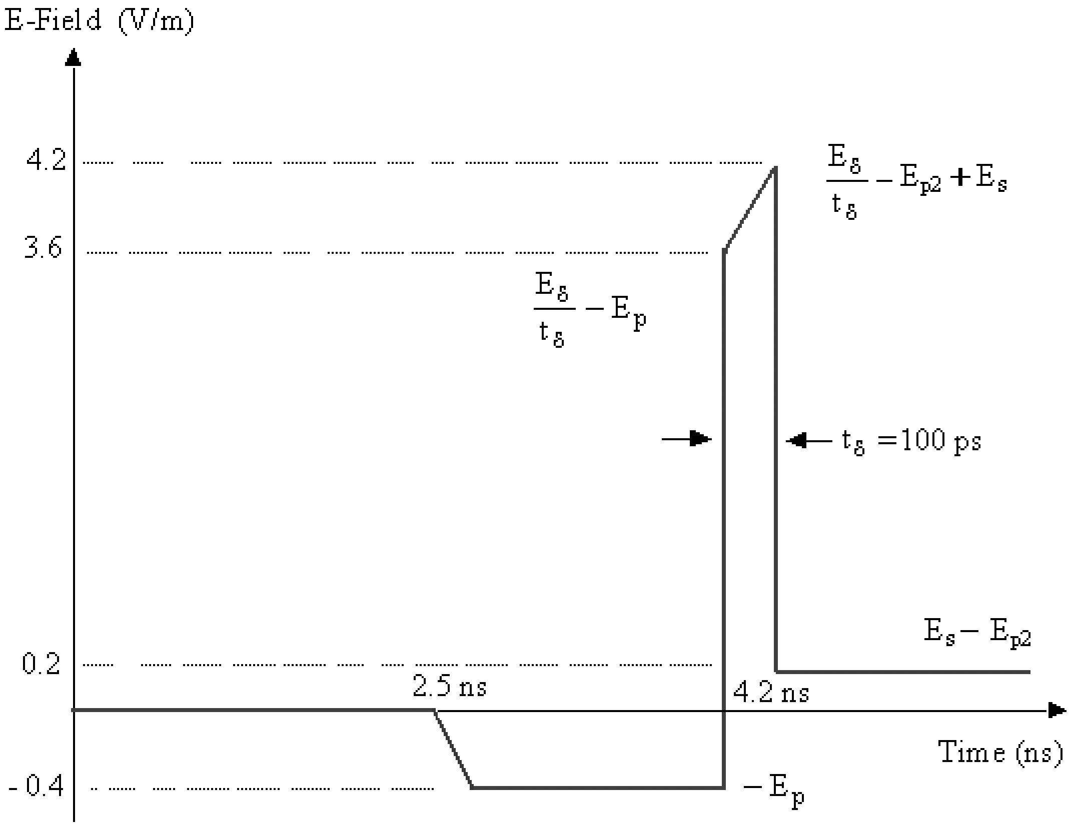

Figure 4.1 Analytic Waveform at the Second Focus

Where

$$
\begin{aligned}
E_p &= 0.4\,\mathrm{V/m} \quad \text{(negative prepulse)} \\
E_i &= \frac{E_\delta}{t_\delta} = 4\,\mathrm{V/m} \\
E_s &= 0.26\,\mathrm{V/m} \\
E_{p_2} &= 0.06\,\mathrm{V/m} \quad \text{(negative prepulse)}
\end{aligned}
\tag{4.1}
$$

5. Concluding Remarks

We now have an approximate analytic waveform at the focus for the interesting (and useful) case of $z_p = 0$. This includes the "post-pulse" which we should not extend very far to later time, since other scattering begins to have an effect. This can be used for comparison to experimental results and more general numerical computations.

Analytical errors can be classified into two groups. First of all, analytical calculations do not account for the feed arm width, and it is a little different from [5]. Secondly, when calculating the aperture integrals we have used the uniform-field part all the way to $\Psi = \Psi_p$ but the feed arms intersect partly into S_a for $\Psi < \Psi_p$. So the aperture integrals are correct up to $\Psi < \Psi_p$ [feed arm incursion]. The analytical waveform, while simple, is still good, but not perfect.

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