Sensor and Simulation Notes
Note 103
7 May 1970

Sloped Parallel Resistive Rod Terminations for Two-Dimensional Parallel-Plate Transmission Lines

David L. Wright
EG&G, Inc., Albuquerque, New Mexico

Abstract

This note considers parallel resistive rods as a termination for a two-dimensional parallel-plate transmission line. The rods are considered to be small in diameter with respect to a wavelength, and the rod spacing is taken to be uniform and much larger than the rod diameter. Sloping the rods is permitted. The termination is considered as a particular realization of a distributed LR admittance-sheet terminator, and methods are given for designing the rods to have the optimum surface inductance for reducing reflections where the optimum inductance is found from previous work referenced in this note. In certain frequency ranges, reflection coefficients are defined and a diffraction grating effect is discussed.

ACKNOWLEDGMENT

The author wishes to thank Capt. Carl E. Baum for several technical discussions which were of great help to the author. Thanks are also due to Dr. James A. Doran of EG&G, Inc., for initially involving the author in the problem, and to Mr. R. Neiri for preparing the figures and Mrs. M. McGeagh for typing the manuscript.

I. Introduction

Previous Sensor and Simulation Notes[1–3] have examined the concept of vertical and sloped admittance sheets as distributed terminations for a two-dimensional TEM wave parallel-plate transmission line from the point of view of minimizing the high-frequency reflections. The required admittance sheet for the ideal (reflectionless) case will, in general, be of nonuniform surface admittance and may not be completely realizable in terms of lumped passive elements.[2] An admittance sheet of series inductance and resistance can, however, match the ideal admittance in high and low frequency limits.[1] The resistance is chosen so as to terminate the transmission line in its characteristic impedance for low frequencies and the series inductance is chosen so as to minimize the reflection of frequencies with wavelengths of the order of the cross-sectional dimensions of the transmission line.

The method of selecting the appropriate distributed inductance, developed in reference 3, is to calculate the current density response of an ideal reflectionless admittance sheet to a step-function plane wave input, then, while neglecting any reflections, to calculate the corresponding current density response of an LR sheet to the same input with R fixed by the low-frequency matching requirement and L as a parameter. The L which makes the response of the LR sheet most like that of the ideal sheet is the appropriate L. This method is an approximation, rigorously correct only in the limit of small reflections. It is also an approximation in the sense that the assumed geometry includes infinite coplanar conducting flanges which are not actually present. Nevertheless, it is expected that the method predicts a reasonable value for the distributed inductance.

---

1. Capt. Carl E. Baum, Sensor and Simulation Note 53, Admittance Sheets for Terminating High-Frequency Transmission Lines, April, 1968.

2. R. W. Latham and K. S. H. Lee, Sensor and Simulation Note 68, Termination of Two Parallel Semi-Infinite Plates by a Matched Admittance Sheet, January, 1969.

3. Capt. Carl E. Baum, Sensor and Simulation Note 95, A Sloped Admittance Sheet plus Coplanar Conducting Flanges as a Matched Termination of a Two-Dimensional Parallel-Plate Transmission Line, December, 1969.

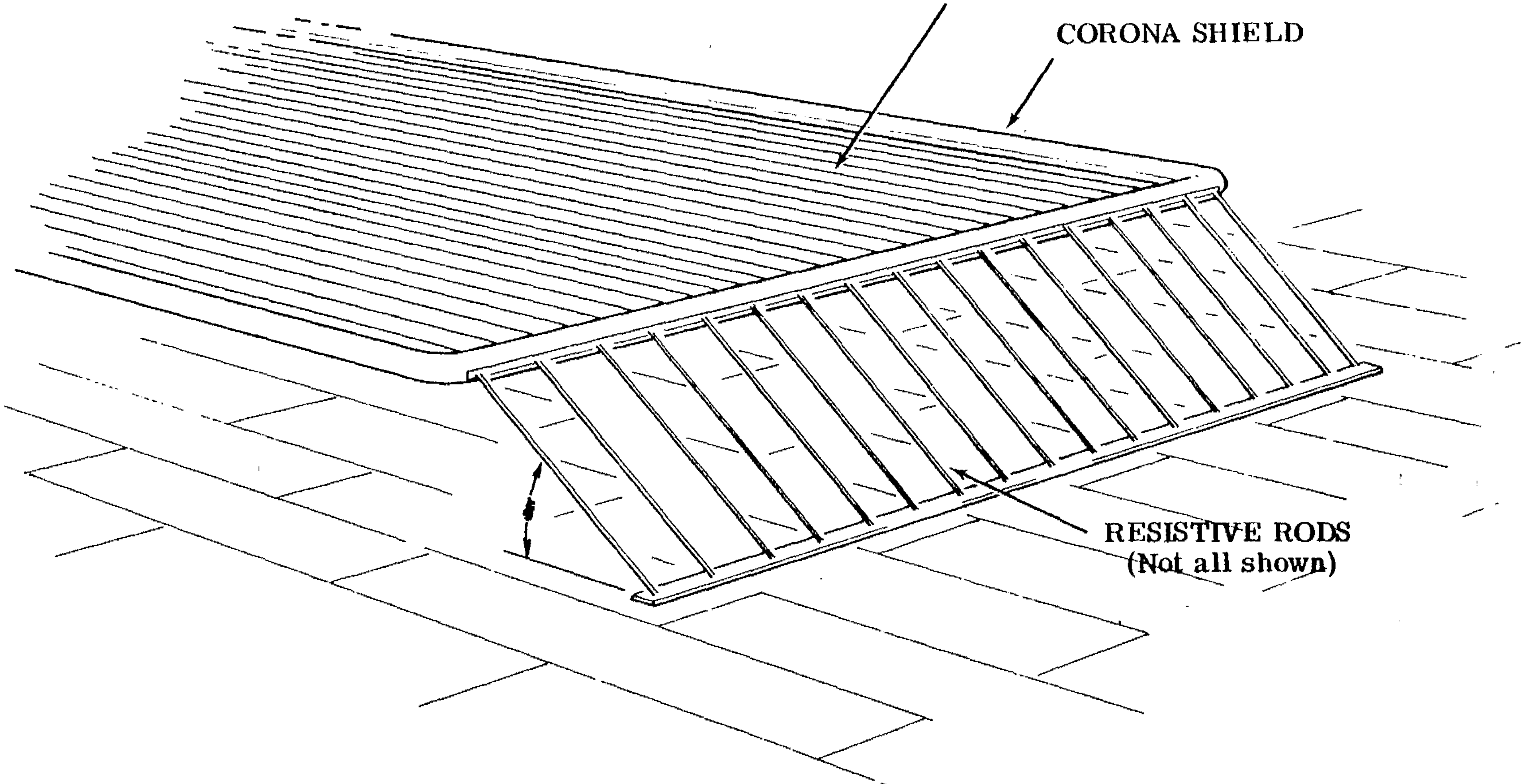

Figure 1. Resistive Rod Termination

The present note considers a particular physical approximation of an LR sheet, namely, a grid of parallel cylindrical resistors or resistive rods (wires). The geometry of a finite termination is illustrated in figure 1. Infinite grids of parallel wires have been the subject of several investigations,[4–6] not all of which are mentioned here, but the one which is best suited for application to the present case appears to be that of J. R. Wait.[7] The following section will give the derivation of the expression for the scattered field from a grid, largely following Wait, although one restriction on the frequency which Wait includes will be relaxed.

II. Fields Scattered from an Infinite Parallel Wire Grid

Before setting forth the assumptions and derivation of the reflection coefficient for the infinite grid, it should be explained that there are really two separate problems here. The one of basic interest is the bounded parallel-plate problem of height h. The second problem, that of the unbounded infinite grid is considered here because for high frequencies it provides a means of

---

4. G. G. MacFarlane, "Surface Impedance of an Infinite Parallel-Wire Grid at Oblique Angles of Incidence," Journal of the Institution of Electrical Engineers (London), Vol. 93, Part IIIA, 1946, pp. 1523-27.

5. Victor Twersky, "On the Scattering of Waves by an Infinite Grating," Institute of Radio Engineers Proceedings on Antennas and Propagation, Vol. 4, 1956, pp. 330-45.

6. E. A. Lewis and J. P. Casey, "Electromagnetic Reflection and Transmission by Gratings of Resistive Wires," Journal of Applied Physics, Vol. 23, No. 6, June, 1952, pp. 605-08.

7. James R. Wait, "Reflection at Arbitrary Incidence from a Parallel Wire Grid," Applied Scientific Research, Vol. 4, Sec. B, 1954, pp. 393-400.

predicting the behavior of the finite termination by enabling one to calculate values for the distributed surface inductance for a wire grid as a function of such things as wire spacing and diameter. In addition, the reflection coefficient for the infinite grid case may be directly applied to the finite case for the case where the cross-sectional dimensions of the grid are large with respect to a wavelength. Although any waveguide must have a finite width, w, we ignore any effects of finite width. The derivation which follows is really that for an infinite grid sloped with respect to an incident TEM wave at the same angle as the termination.

It will be assumed that the rods or wires are uniform circular cylinders of homogeneous material. The wavelength will be assumed long with respect to the radius, a, of the rods, and it will be assumed that $d >> a$ where d is the rod center-to-center spacing. The analysis assumes a grid infinite in two dimensions. The region of applicability to the finite case of the analysis presented here has, therefore, both high and low frequency limits. The high frequency limit is provided by $\lambda >> a$ where λ is the free space wavelength. The low frequency limit is provided by the minimum cross-sectional dimension of the transmission line. If the width, w, of the line is much greater than the height, h, of the line, it will be h which will provide the low frequency limit. It is suggested that if $h << w$, qualitatively correct results may be obtained for wavelengths as long as the height, but the accuracy will improve as λ diminishes from this value. Another way of saying this is that edge effects are being neglected. The edge effects should decrease as the frequency increases.

It is convenient to define a coordinate system with respect to the termination as shown in figure 2. The origin of coordinates is taken to be at the junction of the top plate with the termination. The angle ξ is the termination slope angle measured from the horizontal and lying in the region $0 < \xi \leq (\pi/2)$.

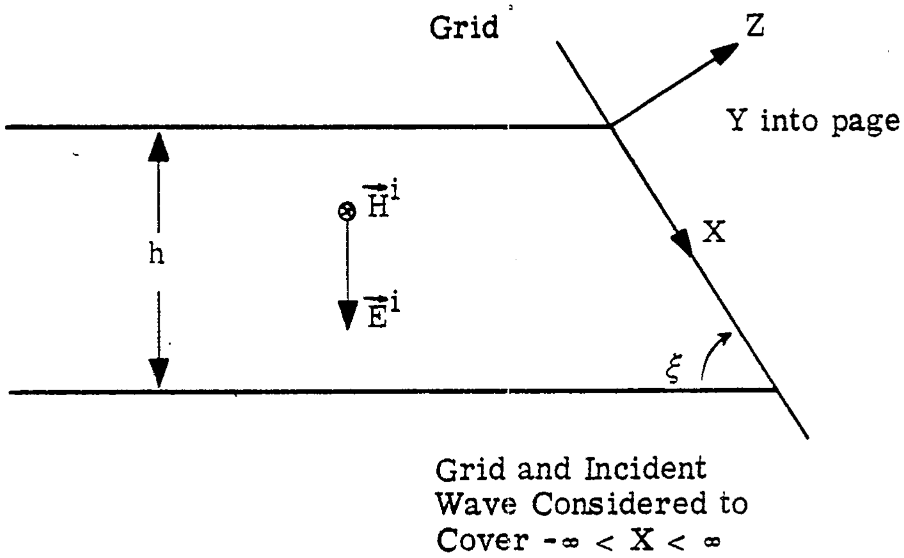

Figure 2. Coordinate System at Infinite Grid.

The method of attack is to consider that the incident wave will induce in the wires currents which give rise to a reflected or scattered field. The magnitude of the induced currents and, hence of the reflected fields, is governed by a quantity called the internal impedance of the wires. The boundary condition imposed at the wire surface is that the current in a wire times the internal impedance of that wire is equal to the total $\vec{E}$ field tangential to that wire. Express the incident field $\vec{E}^i$ as [8]

$$\vec{E}^i (x, y, z, \omega) = \vec{E}_0 (\omega) e^{+j \left[\omega t - \beta (x \cos \xi + z \sin \xi)\right]} \tag{1}$$

where $\beta = \omega\sqrt{\mu_0 \epsilon_0}$, ω = angular frequency, $E_0(\omega)$ is the amplitude of the incident wave at frequency ω. The phase has been taken to be zero at the coordinate origin. It is assumed that the diameter of the wires, 2a, is small compared with a wavelength. If this is true, it may be assumed that $\vec{E}^i$ is constant over the cross section of the wire, and that the only component of $\vec{E}^i$ which excites currents in the wires is that component which is parallel to the long dimension of the wires.

8. All units are rationalized MKSA.

In the present case, this is the x component of $\vec{E}^i$. The current densities, therefore, may also be taken to be azimuthally symmetric in each wire. The currents in each wire will be in phase with the currents in the other wires, but there will be a variation in phase along the length of each wire given by $\beta \cos \xi$ radians per unit length. The currents in the wires may then be written as

$$I(x) = I_0 e^{-j\beta x \cos \xi} \tag{2}$$

where I_0 is the current in a wire at x = 0. This is treating the current distribution on the wires as that of infinitely long wires.

Since the induced currents flow in the x-direction, the field scattered from a single wire, $\vec{E}^s$, may be derived from an electric Hertz vector having only an x-component π_x^s. The scattered field may then be written as

$$\vec{E}^s = \beta^2 \pi_x^s \hat{u}_x + \nabla \frac{\partial \pi_x^s}{\partial x} . \tag{3}$$

Using the fact that $\partial/\partial x = -j\beta \cos \xi$ the scattered field components may be written as

$$E_x^s = \beta^2 \sin^2 \xi \, \pi_x^s$$

$$E_y^s = -j\beta \cos \xi \frac{\partial \pi_x^s}{\partial y} \tag{4}$$

$$E_z^s = -j\beta \cos \xi \frac{\partial \pi_x^s}{\partial z}$$

So far, we are considering only the scattered field from a single wire. What is ultimately required is an expression for the current I_0 as a function of the incident field. In order to do this, we express the scattered field from each wire as a function of I_0 and then solve for I_0 by setting up a boundary value problem at the surface of the wires. We assume that the fields scattered by each wire look the same as if the wire were infinite in length. Let the current on a wire be given by equation 2.

From the fact that by Ampere's law the circumferential $\vec{H}$ field, H^s_φ, must approach $-I(x)/2\pi\rho$ as $\rho \to a$ and from the fact that H^s_φ must behave as an outgoing cylindrical wave at infinity and satisfy

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2} - \frac{1}{\rho^2} + \beta^2\right] H^s_\varphi = 0 \tag{5}$$

the proper solution for H^s_φ is

$$H^s_\varphi = \frac{j\beta I_0 \sin\xi}{4} e^{-j\beta x\cos(\xi)} H_1^{(2)}(\beta\rho\sin\xi) \tag{6}$$

with the x-component of $\vec{E}^s$ then given by

$$E^s_x = \frac{I_0\mu_0\omega\sin^2\xi}{4} H_0^{(2)}(\beta\rho\sin\xi)e^{-j\beta x\cos\xi} \tag{7}$$

The x-component of the $\vec{E}$ field due to all the wires may be written as

$$E^r_x = \frac{\mu_0\omega I_0\sin^2\xi}{4}\sum_{n=-\infty}^{\infty} e^{-j\beta x\cos\xi} H_0^{(2)}\left[\beta\sin\xi\sqrt{(nd-y)^2+z^2}\right] \tag{8}$$

Wait remarks that the summation over the Hankel functions is a slowly converging one and uses a formula[9] to change the form to the more rapidly converging one

$$E^r_x = \frac{j\mu_0\omega I_0\sin^2\xi}{4\pi} e^{-j\beta x\cos\xi}$$

$$\times \sum_{m=-\infty}^{\infty} e^{-i2\pi my/d} \frac{\exp\left\{-\frac{2\pi|z|}{d}\sqrt{m^2-\left(\frac{d\sin\xi}{\lambda}\right)^2}\right\}}{\sqrt{m^2-\left(\frac{d\sin\xi}{\lambda}\right)^2}} \tag{9}$$

9. Wait, op.cit.

If we assume for the moment that $|z| >> d$ and that the frequency is such that $\lambda > d\sin\xi$, we need retain only the m = 0 term in equation 9. This gives a reflected field with components

$$\left.\begin{aligned} E_x^r &= \frac{\eta_0 I_0 \sin\xi}{2d} \exp(j\beta q) \\ E_y^r &= 0 \\ E_z^r &= \frac{-\eta_0 I_0}{2d} \cos\xi \exp(j\beta q) \end{aligned}\right\} \quad (10)$$

where $q = -|z| \sin\xi - x\cos\xi$

and $\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$

In order to find I_0, the magnitude of I(x) on a wire, we set up the boundary condition

$$E_x^i + E_x^r = -I(x)\ Z_i \quad (11)$$

where we must use the near field expression of equation 8 rather than the far field expression of equation 10 for E_x^r.

The quantity Z_i is the internal impedance of each wire. If $a << d$, the assumption that the field is uniform around the wire may be used and for a homogeneous wire the impedance is given by[10]

10. E. C. Jordan and K. G. Balmain, Electromagnetic Waves and Radiating Systems, 2d. ed., Prentice-Hall, Inc., 1968, p. 562.

$$Z_i = \frac{\eta I_0(\gamma a)}{2\pi a I_1(\gamma a)} \tag{12}$$

where $\eta = \left[j\mu_1\omega/(\sigma + j\omega\epsilon_1)\right]^{1/2}$, $\gamma = \left[j\mu_1\omega(\sigma + j\omega\epsilon_1)\right]^{1/2}$

μ_1, σ, and ϵ_1 are the permeability, conductivity, and permittivity of the wire. $I_0(\gamma a)$ and $I_1(\gamma a)$ are modified Bessel functions of order zero and one. If equations 8 and 11 are used with the condition that a << d, the boundary condition is given by

$$-I_0 Z_i = E_{0x} + \frac{\mu_0 \omega I_0 \sin^2\xi}{4}$$

$$\times \left[2\left\{\sum_{n=1}^{\infty} H_0^{(2)}(n\beta d \sin\xi)\right\} + H_0^{(2)}(\beta a \sin\xi)\right] \tag{13}$$

As in equation 8, the summation in equation 13 does not converge rapidly and may be made more rapidly converging by using the formula

$$2\pi \sum_{n=1}^{\infty} \cos(2\pi kn)\, H_0^{(2)}(2\pi n\alpha) = -\pi + 2j\ln\frac{\Gamma\alpha}{2}$$

$$+ j(k^2 - \alpha^2)^{-1/2} + j \sum_{m=1}^{\infty}\left[\left[(m+k)^2 - \alpha^2\right]^{-1/2}\right.$$

$$\left. + \left[(m-k)^2 - \alpha^2\right]^{-1/2} - 2m^{-1}\right] \tag{14}$$

where k and α are real and $\ell n\Gamma$ is Euler's number (=0.5773 ⋯). For the frequency spectrum under consideration $\beta a << 1$. For this case

$$H_0^{(2)}(\beta a \sin\xi) \simeq 1 - j\frac{2}{\pi}\,\ell n\frac{\Gamma\beta a \sin\xi}{2} \tag{15}$$

It is now possible to write I_0 as

$$I_0 = \frac{-E_{0x} d}{(\eta_0 \sin\xi/2) + (j\mu\omega d/2\pi)\sin^2\xi \left| \ln(d/2\pi a) + F \right| + Z_i d \sin^2\xi} \qquad (16)$$

where E_{0x} is the x-component of $\vec{E}_0(\omega)$ of equation 1, and

$$F\left(\frac{d\sin\xi}{\lambda}\right) = \sum_{m=1}^{\infty} \left[\left[m^2 - (d\sin\xi/\lambda)^2 \right]^{-1/2} - m^{-1} \right] \qquad (17)$$

This factor F is discussed by MacFarlane[11] for the case $\xi = \pi/2$ and $\sigma = \infty$.

One may write an intrinsic impedance of the incident wave in the direction normal to the grid as

$$Z = \frac{E_x^i}{H_y^i} = \eta_0 \sin\xi \qquad (18)$$

If the surface impedance of the grid is Z_s, the equivalent circuit looks like figure 3.

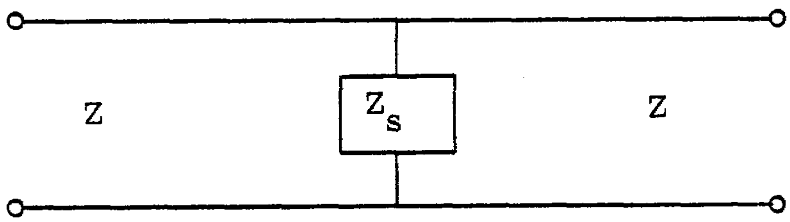

Figure 3. Equivalent Circuit

11. McFarlane, op. cit.

which is equivalent, as far as reflected fields are concerned, to figure 4.

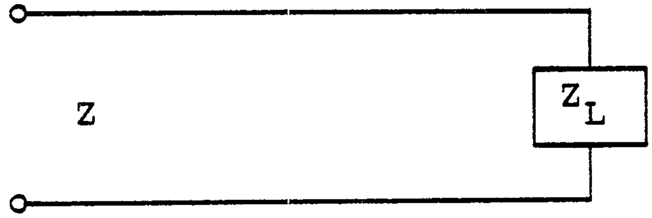

Figure 4. Equivalent Circuit for Reflections

where $$Z_L = \frac{Z_s Z}{Z_s + Z}$$

The voltage reflection coefficient is

$$\rho = \frac{Z_L - Z}{Z_L + Z} = \frac{-1}{1 + 2Z_s/Z} \tag{19}$$

Z_s, the equivalent shunt impedance of the grid, is given by

$$\frac{Z_s}{Z} = j(d/\lambda)\sin\xi \left[\ell n\left(\frac{d}{2\pi a}\right) + F\left(\frac{d\sin\xi}{\lambda}\right)\right] + \frac{Z_i d}{\eta_0 \sin\xi} \tag{20}$$

We now remove the restriction $\lambda > d\sin\xi$ which permitted dropping all but the $m = 0$ term in equation 9. If $m\lambda < d\sin\xi$ where m is an integer, the grid no longer reflects a single wave, but additional "side waves" will also be scattered from the grid as shown in figure 5 which, for simplicity, is drawn for $\xi = (\pi/2)$ and $m = 1$.

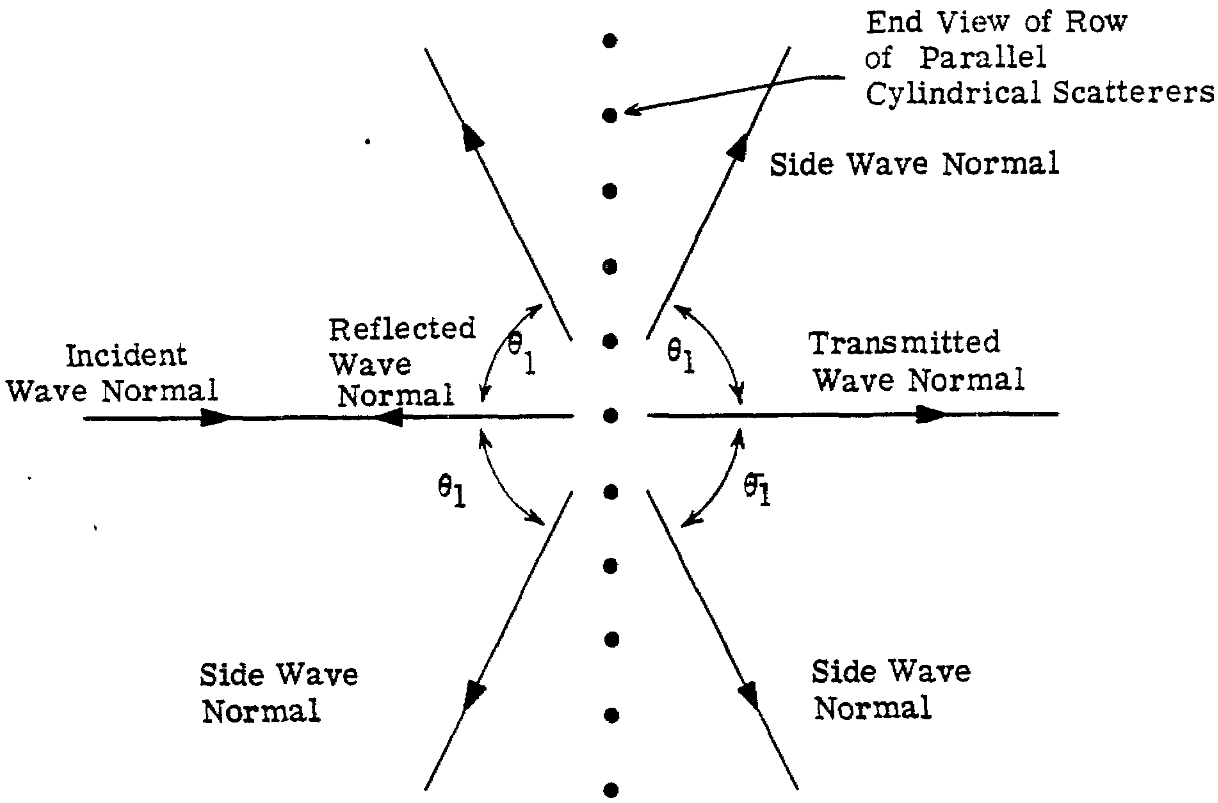

Figure 5. End View of Parallel Cylindrical Scatterers Showing the Wave Normals of Incident, Reflected, Transmitted, and Side Waves for Normally Incident Wave and One Set of Side Waves

Side waves will occur under this condition because planes of equal phase may be found whose normals do not lie in the plane of incidence. For sufficiently high frequencies, side waves may be found for many values of m. The side waves come off symmetrically, as shown, at angles measured from the normal to the grid in a plane perpendicular to the plane of incidence, given by

$$\theta_m = \arc\sin \left(\pm \frac{m\lambda}{d \sin \xi} \right) \quad . \tag{21}$$

As is seen from this relation, each frequency which produces side waves produces them at certain discrete angles. This is the spectrum grating effect familiar from optics.

The modifications to Wait's analysis which are needed to allow frequencies such that $\lambda \leq d \sin \xi$ to be considered are straightforward. The condition $m\lambda = d \sin \xi$ for $m = 1, 2, 3, \cdots$ will be referred to as the mth resonance, and the angle θ_m will be referred to as the scattering angle of the mth side wave. If, and only if, $m\lambda < d \sin \xi$ a reflection coefficient for the mth side wave may be defined as

$$\rho_m = \left. \frac{E^r_{xm}}{E^i_x} \right|_{x = 0} \tag{22}$$

For the $m = 0$ term in equation 9, this is identical to the reflection coefficient defined by equations 19 and 20. For each value of m a reflection coefficient may be calculated using equation 22 and the scattering angle may be computed using equation 21.

The behavior of ρ_m at frequencies near the mth resonance is quite interesting. Consider that the frequency is being increased from a value just less than the mth resonance. As $\lambda \rightarrow m\, d \sin \xi$ from above (frequency from below), $|\rho_m| \rightarrow 0$ for $n = 0, 1, 2, \cdots, m-1$. When the frequency just equals the mth resonance, a new set of side waves is generated with a $|\rho_m| = 1$, and a scattering angle of 90 degrees. As the frequency continues to increase, θ_m decreases, but $|\rho_m|$ does also. Figure 6 illustrates this behavior for a hypothetical termination terminating a transmission line 50 meters wide by 3 meters high. The resistivity of the rods is chosen to match the low-frequency characteristic impedance. The spacing may not be optimum, but the salient features are illustrated. Figure 6 is calculated from equations 21 and 22.

To close the present section it may be remarked that the reflection coefficients of equations 19 or 22 are "first bounce" reflection coefficients. For $\xi \leq (\pi/4)$ and frequencies such that $\lambda <<$ minimum cross-sectional dimension, ray tracing will indicate that the incident wave will have to strike the termination more than once before being returned down the line. This consideration argues for small slope angles.

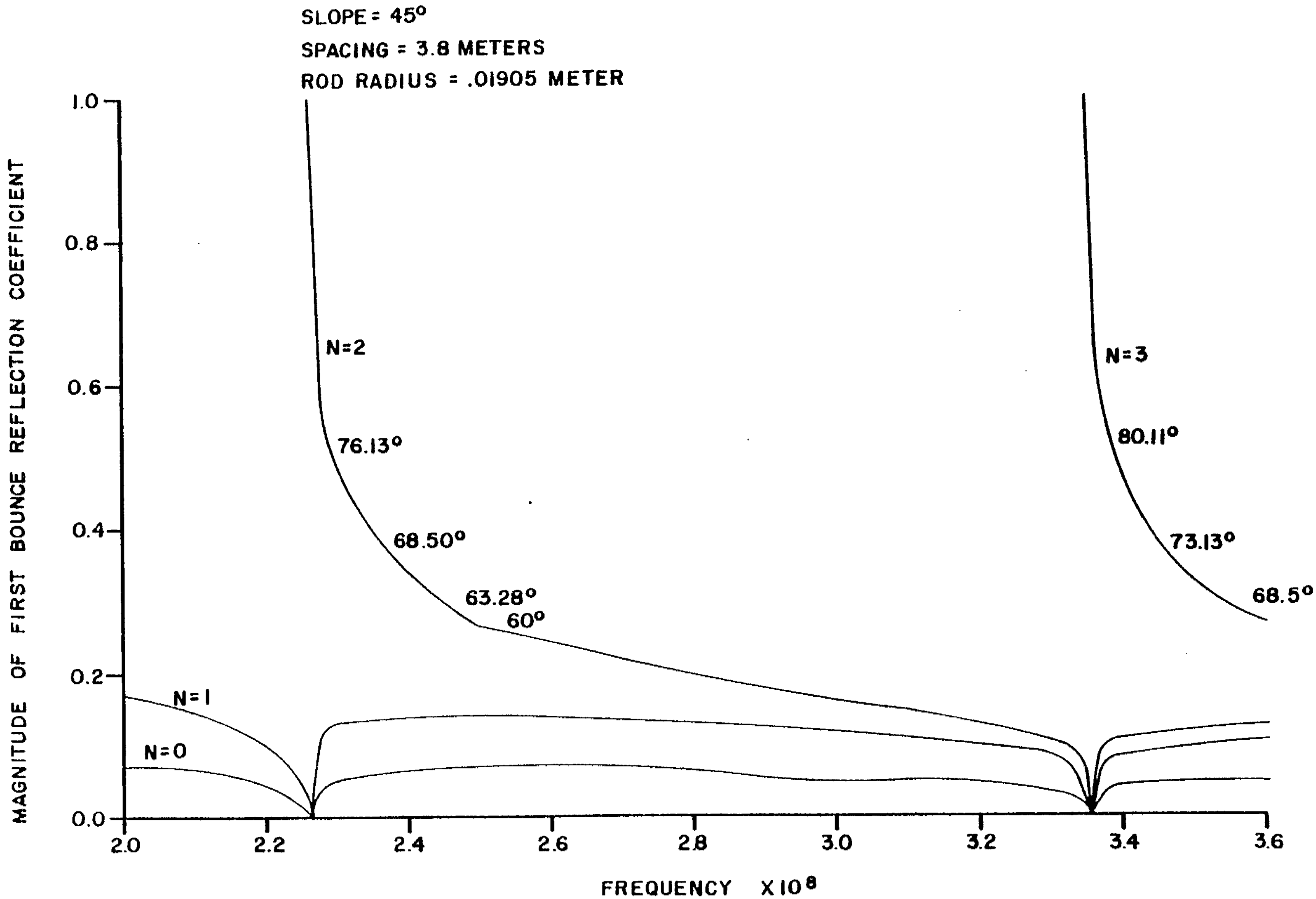

Figure 6. Reflection Coefficient vs Frequency

III. Low-Frequency Expansion of Reflection Coefficient

For frequencies such that $\lambda > d \sin \xi$ the reflection coefficient is given by equation 19 with supplementary relations in equations 20, 17, and 12 where equation 12 assumes a solid rod of homogeneous material. The low-frequency form for Z_i may be found by expanding $I_0(\gamma a)$ and $I_1(\gamma a)$ using the assumptions that $\sigma >> j\omega\epsilon$, and that $\gamma a << 1$. The latter two assumptions are "low frequency" assumptions, but are still valid for some frequencies too high for transmission line theory to be useful.

$$I_0(\gamma a) = 1 + \frac{(\gamma a)^2}{4} + 0(\gamma a)^4 \tag{23}$$

$$I_1(\gamma a) = \frac{\gamma a}{2} + \frac{(\gamma a)^3}{16} + 0(\gamma a)^5 \tag{24}$$

Using $\eta \equiv \left[j\omega\mu_1/\sigma \right]^{1/2}$ and $\gamma \equiv \left[j\omega\mu_1\sigma \right]^{1/2}$ and retaining only the first two terms in equations 23 and 24, one arrives at a form for the low-frequency value of Z_i to first order in ω given by

$$Z_i = \frac{1}{\pi a^2 \sigma} + j\omega \frac{\mu_1}{8\pi} + 0(\omega^2) \tag{25}$$

which we may write as

$$Z_i \equiv R_i + j\omega L_i \tag{26}$$

neglecting terms higher than first power in ω. The value for R_i is the familiar dc value for resistance per unit length and $L_i = (\mu/8\pi)$ is also a well known low-frequency value for inductance per unit length. This value is .05 μH/meter which for many cases may be ignored. If the rod is a cylindrical shell rather than a solid, L_i is in fact zero.

We now turn our attention to the quantity F, defined by equation 17, which for low frequencies is a small correction to the

$\ell n(d/2\pi a)$ term in equation 20. Rewriting F in terms of ω with a $(1/m)$ factored out gives

$$F\left(\frac{\omega d \sin\xi}{2\pi c}\right) = \sum_{m=1}^{\infty} \frac{1}{m} \left\{\left[1 - \left(\frac{\omega d \sin\xi}{2\pi mc}\right)^2\right]^{-1/2} - 1\right\} \tag{27}$$

Expanding the first term in the braces of equation 27 gives

$$F\left(\frac{\omega d \sin\xi}{2\pi c}\right) = \sum_{m=1}^{\infty} \frac{1}{m} \left\{ 1 + \frac{1}{2}\left(\frac{\omega d \sin\xi}{2\pi mc}\right)^2 + 0(\omega^4) - 1 \right\} \tag{28}$$

which in turn may be written as

$$F\left(\frac{\omega d \sin\xi}{2\pi c}\right) = \frac{\zeta(3)}{2}\left(\frac{\omega d \sin\xi}{2\pi c}\right)^2 + 0(\omega^4) \tag{29}$$

where $\zeta(n) = \sum_{m=1}^{\infty} m^{-n}$ is the Riemann Zeta function.[12]

We now rewrite equation 19, incorporating equations 26 and 29 which gives

$$\rho_w = -Z\Big/\left\{Z + 2(R_i + j\omega L_i)d + \frac{j\omega d\mu_0 \sin^2(\xi)}{\pi}\left[\ell n\left(\frac{d}{2\pi a}\right) + \frac{\zeta(3)}{2}\left(\frac{\omega d \sin\xi}{2\pi c}\right)^2 + 0(\omega^4)\right]\right\} \tag{30}$$

12. M. Abramowitz and I. A. Stegun, eds., Handbook of Mathematical Functions, National Bureau of Standards, AMS-55, 1964, p. 811.

where the subscript "w" indicates "wire" and $Z = \eta_0 \sin\xi$. The reflection coefficient for an LR sheet considered to be in parallel with free space and oriented at the same angle with respect to an incident plane wave as the wire grid, would have a reflection coefficient given by

$$\rho_s = -Z\left\{Z + 2(R_s + j\omega L_s)\right\}^{-1} \tag{31}$$

Suppose R_s in equation 31 is chosen to properly terminate the transmission line for d.c., and L_s has been chosen optimally according to the analysis of C. E. Baum.[13] Then ρ_s has been minimized and it is desired to equate ρ_w to ρ_s. In an attempt to cancel the first term in the expansion of F in the denominator of equation 30, it was supposed that the wire grid was physically displaced from the LR sheet by a distance δ' as shown in figure 7. The additional distance traveled by a wave striking the LR sheet over that of a wave reflected from the wire grid is

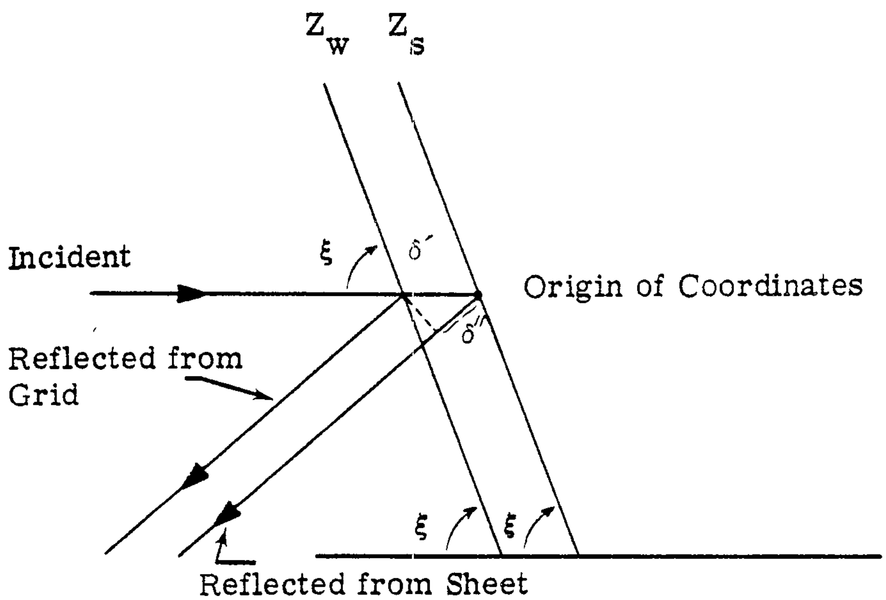

Figure 7. Reflections from Wire Grid and Displaced LR Sheet

$$\delta = \delta' + \delta'' = 2\delta' \sin^2\xi \tag{32}$$

13. Baum, Sensor and Simulator Note No. 95.

If the reflections from the LR sheet are taken as having zero phase, then the reflection coefficient in equation 30 for the wire grid must be multiplied by $\exp(-j(\omega/c)\delta)$ in the denominator to account for the displacement, giving

$$\rho_w = -Z \Bigg/ \left\{ \exp(-j\frac{\omega}{c}\delta)\left[Z + 2(R_i + j\omega L_i)d + \frac{j\omega d\mu_0 \sin^2\xi}{\pi}\left[\ell n\left(\frac{d}{2\pi a}\right) + \frac{\zeta(3)}{2}\left(\frac{\omega d \sin\xi}{2\pi c}\right)^2 + 0(\omega^4)\right]\right]\right\} \tag{33}$$

We now equate equation 33 to equation 31 using a power series expansion of the exponential phase shift in equation 33. If the denominator of equation 33 is multiplied out using the first few terms of the power series expansion of the phase shift, arranging in ascending powers of ω and equating the result to the denominator of equation 31, the result is that δ must be zero and none of the terms in the expansion of F may be eliminated. Thus the low-frequency form of ρ_w becomes simply

$$\rho_w \cong -Z\left\{Z + 2(R_i + j\omega L_i)d + \frac{j\omega\mu_0 \sin^2\xi}{\pi}\left[\ell n\left(\frac{d}{2\pi a}\right)\right]\right\}^{-1} \tag{34}$$

Equating equation 34 to 31 leads to the relations

$$R_s = R_i d \tag{35}$$

$$L_s = L_i d + \frac{\mu_0 d \sin^2\xi}{2\pi}\ \ell n\left(\frac{d}{2\pi a}\right) \tag{36}$$

If R_S and L_S are optimally chosen, equations 35 and 36 give the prescription for designing a wire grid termination with the least reflections in the frequency range where the wavelength is of the order of the cross-sectional dimensions of the transmission line even though a reflection coefficient per se is not available in this frequency region. For lower frequencies, simple transmission line theory provides a reflection coefficient. For higher frequencies up to the point where $\lambda = d\sin\xi + \epsilon$ where ϵ is an arbitrarily small real positive quantity, equation 19 gives a reflection coefficient. For still higher frequencies where $\lambda < d\sin\xi$, side waves occur, but a series of reflection coefficients for the various reflected waves may still be defined by equation 22 as displayed in figure 6.

It is interesting to note in equation 36 that if $L_i d$ is zero or sufficiently small with respect to the second term, then the surface inductance of the wire grid may be interpreted as being due to an "effective displacement," Δ, as $L_s = \mu_0 \Delta \sin^2 \xi$ where the Δ is exactly the Δ of equation 88 in Sensor and Simulator Note 21[14] if "a" is replaced by "c," and "d" by "2d" to account for differences of notation, and the $\sin^2 \xi$ accounts for the effect of sloping.

IV. Selection of Optimum Surface Inductance

The selection of the best L_s is effected by using the results of the previously referenced Note 95.[5] By studying figures 3 through 9 in that note with the intent of choosing the curves which best match the ideal current density for early times and using equation 27 of that note, it appears that a reasonable value for L_s is

$$L_s = \mu_0 h \sin^2 \xi \tag{37}$$

where h is the height of the termination.

If equation 37 is compared with equation 36, it is seen that if $L_i = 0$, and with a given rod radius, the spacing, d, is not a function of slope angle. If $L_i \neq 0$, d does vary as a function of ξ.

V. Addition of Internal Inductance

As has been previously mentioned, if the rods are of homogeneous material, and if skin effect is negligible, the quantity L_i is $(\mu_0/8\pi)$ henrys per meter. If the current flows on a thin cylindrical shell or tube, $L_i = 0$. It may be desirable to increase the internal inductance per unit length, L_i, in order to decrease the spacing, d, between rods. As an example, for a hypothetical transmission line of 50-meter width and 3-meter height, a termination of solid homogeneous rods with $\mu = \mu_0$ and $\epsilon = \epsilon_0$ of radius 3.175 cm (1.25 inches) and with a slope angle $\xi = (\pi/6)$ should have a spacing, d, of about

14. 1/Lt. Carl E. Baum, Sensor and Simulation Note 21, Impedances and Field Distributions for Parallel Plate Transmission Line Simulators, June, 1966.

4.6 meters. This is rather a wide spacing which would give a total of about 12 resistive rods in the 50-meter width of the hypothetical simulator. The theory assumes an infinite number of rods, and it might be questioned that twelve rods behave approximately as an infinity of rods. It also might be desirable to reduce the spacing (increase the number of rods) in order to reduce the currents flowing in any single rod. In order to do this, additional inductance per unit length may be added in the rods. If it is desired, for example, to reduce the spacing of the rods to 1 meter, while retaining the desired L_s and without changing a or ξ, an additional internal inductance of $.812 \times 10^{-6}$ H/meter would be required. The internal inductance might be increased by replacing the homogeneous rods by rods of another type such as wirewound resistors whose internal inductance might be adjusted to provide the desired L_s at reasonable values of ξ, a, and d. It may be noted that L_i has the units of H/meter, whereas L_s is surface inductance per square and has the units of henrys. Thus $L_i d$ is of the correct dimensionality.

VI. Summary

In this note we have considered a sloped parallel resistive rod termination as a possible physical realization of a distributed LR termination studied in previous notes. Equations are given which permit selecting compatible spacing, internal inductance, diameter, and termination slope angle for realizing the desired distributed inductance. In addition, for a parallel grid it is possible to quantitatively predict reflections for certain low- and high-frequency ranges. The derivation for high-frequency reflections is given, together with a discussion of a "diffraction grating" effect. Low-frequency reflections may be found from simple transmission line theory and are governed by resistive mismatch.