Sensor and Simulation Notes

Note 110

July 1970



# Pulse Radiation by an Infinitely Long, Perfectly Conducting, Cylindrical Antenna in Free Space Excited by a Finite Cylindrical Distributed Source Specified by the Tangential Electric Field Associated with a Biconical Antenna

by

1st Lt. Paul R. Barnes
Air Force Weapons Laboratory/Air Force Special Weapons Center

Abstract

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## ABSTRACT

An antenna for the radiation of a fast rising electromagnetic pulse with large peak fields implies a source voltage with high electric fields in the source region. In order to reduce the peak electric fields at the source, the source region can be made larger. In this note, the pulse radiation by an infinite cylindrical antenna excited by a distributed source region is considered. To achieve a fast rising radiated pulse, a distributed source for launching spherical waves is used. The exact expressions for the far zone radiated fields are developed and the time history of the radiation is obtained. Also, the small and large time asymptotic forms of the radiation fields are obtained.

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# CONTENTS

## I. Introduction

One approach to the radiation of pulsed electromagnetic energy is to employ a pulse-radiating electric dipole antenna. Good antenna characteristics for high frequency radiation can be achieved if the central portion of the antenna is a biconical wave launcher as shown in Figure 1. The biconical antenna is driven by a fast rising applied voltage at or near the common apex of the two cones. The initial part of the radiating pulse has the form of a spherical wave with the characteristics appropriate for the radiation of a biconical antenna while the latter part of the pulse has the form of a decaying wave with characteristics appropriate for the radiation by a dipole antenna. To achieve a certain amplitude for the radiated electric or magnetic fields at a particular distance from the antenna, the magnitude of the applied voltage can be adjusted. However, if the applied voltage is made very large, high voltage insulation problems result. The purpose of this paper is to advance a technique to achieve, at least conceptually, a very large amplitude for the radiation fields without high voltage problems. To accomplish this, the source region is made large to reduce the peak electric field there. In order to obtain an exact solution for the radiation fields, the infinite cylindrical antenna with a finite cylindrical source region is used.

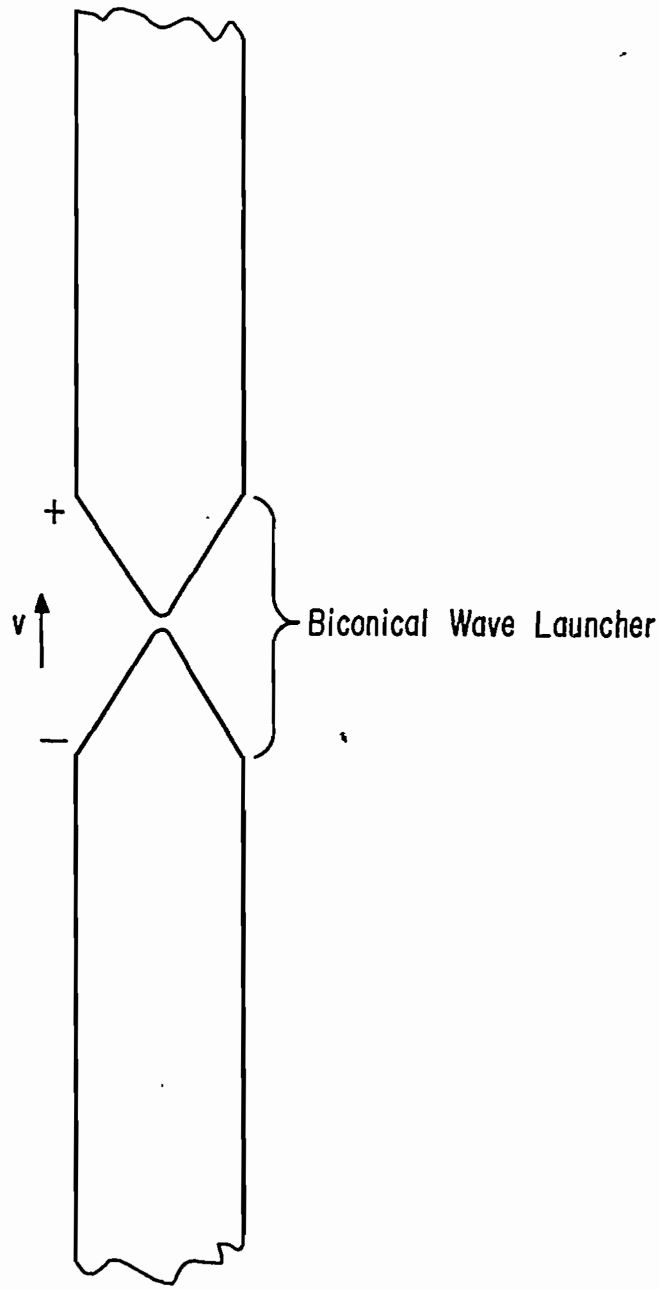


Figure 1. CYLINDRICAL ANTENNA CENTRALLY FED BY A BICONICAL WAVE LAUNCHER.

## II. Pulse Radiation by an Infinite Cylindrical Antenna

Consider an infinitely long cylindrical antenna excited by a source voltage across a circumferential gap of infinitesimal width as shown in Figure 2. The frequency domain expression for the radiated far zone electric field is given, for $0 < \theta < \pi$ with $e^{-i\omega t}$ suppressed, by[1]

$$\vec{E}_\theta(r, \theta, \omega) = \frac{V(\omega)}{i\pi r} \frac{e^{ikr}}{\sin\theta \, H_o^{(1)}\left(k a \sin\theta\right)} \vec{a}_\theta \qquad (1)$$

where $H_o^{(1)}$ is a Hankel function of the first kind of order zero; $V(\omega)$ is the source voltage with dimensions of volts per unit radian frequency; $\vec{E}_\theta$ is the electric field vector in the theta ($\theta$) direction with dimensions of volts per meter per unit radian frequency; and k is the radian wave number with dimension of per meter. The meaning of a, r, and $\theta$ is given in Figure 2.

The magnetic field is given by

$$\vec{H}_\phi(r, \phi, \omega) = \frac{E_\theta(r, \theta, \omega)}{Z_o} \vec{a}_\phi \qquad (2)$$

where $Z_o$ is the free space radiation impedance approximately equal to $120\pi$ ohms and $\vec{H}_\phi$ is the magnetic field vector in the phi ($\phi$) direction with dimensions of ampere per meter per unit radian frequency.

The components of the electric and magnetic fields are related to the electric and magnetic field vectors by

$$\vec{E}_\theta = E_\theta \vec{a}_\theta \quad \text{and} \quad \vec{E}_\phi = E_\phi \vec{a}_\phi$$

where $\vec{a}_\theta$ and $\vec{a}_\phi$ are unit vectors in spherical coordinates.

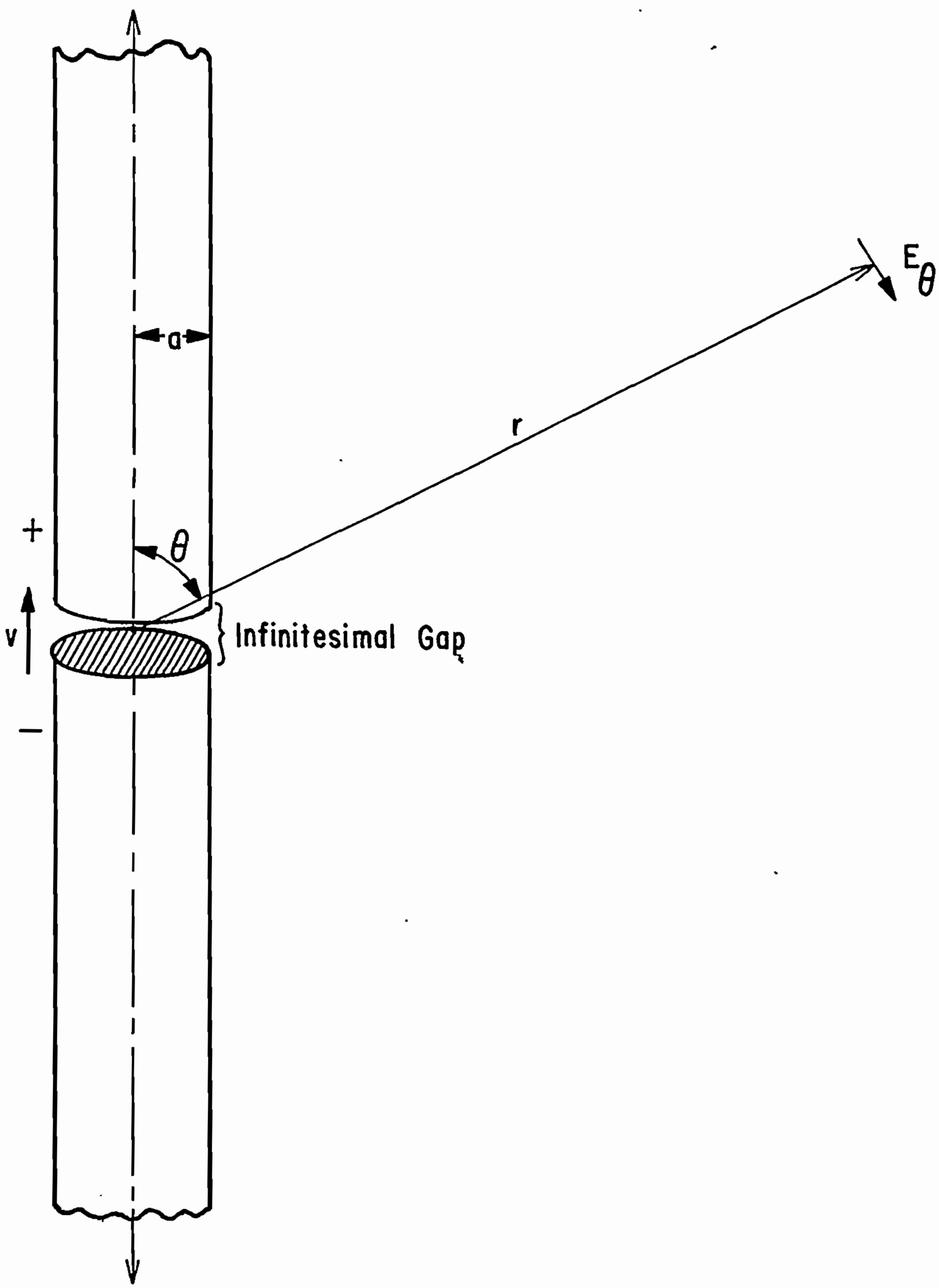


Figure 2. INFINITE CYLINDRICAL ANTENNA EXCITED BY AN INFINITESIMAL GAP VOLTAGE SOURCE.

In terms of the Laplace transform variable p, the magnitude of the electric field becomes[2]

$$E_\theta = V(p) \frac{e^{-pr/c}}{2 \sin\theta \, r K_o\left((pa/c)\sin\theta\right)} \qquad (3)$$

where the relations $k = \omega/c$, $p = -i\omega^*$, and $H_o^{(1)}(ix) = -(2i/\pi) K_o(x)$ have been used. $K_o(x)$ is a modified Bessel function of the second kind of order zero and c is the speed of light.

Let $y = (pa/c) \sin\theta$ for convenience. The time domain electric field is given by the inverse Laplace transform of Eqn. (3) as

$$E_\theta(r, \theta, t) = \frac{1}{2 \sin\theta \, r} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{p(t - r/c)} V(p)}{K_o(y)} dp \qquad (4)$$

where $\gamma$ is chosen to the right of any singularity in the integrand of the integral in Eqn. (4).

Define a retarded time as

$$t^* = t - r/c$$

and a normalized radiation field as

$$\xi(\theta, t^*) = \frac{r E_\theta(r, \theta, t^*)}{V_o} \quad . \qquad (5)$$

For the case $V(p) = V_o/p$, a step-function voltage source, the normalized electric field can be written as

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* $p = -i\omega$ is used to achieve an outward going wave.

$$\xi = \frac{1}{2\sin\theta} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{pt^*}}{p K_o(y)} \, dp \tag{6}$$

Now, make a change of variable from p to y and let $\gamma' = (\gamma a/c)\sin\theta$. The expression becomes

$$\xi = \frac{1}{2\sin\theta} \frac{1}{2\pi i} \int_{\gamma' - i\infty}^{\gamma' + i\infty} \frac{e^{yq\csc\theta}}{y K_o(y)} \, dy \tag{7}$$

where q is a normalized time given by

$$q = \frac{ct^*}{a}$$

Equation (7) may be rewritten for $q \csc\theta > -1$ in the form[2]

$$\xi = \frac{1}{2\sin\theta} \int_0^\infty \frac{e^{-yq\csc\theta} I_o(y)}{y\left[K_o^2(y) + \pi^2 I_o^2(y)\right]} \, dy \tag{8}$$

where $I_o(y)$ is a modified Bessel function of the first kind of order zero.

## III. Distributed Source for Launching Spherical Waves

To minimize high voltage problems, the source region where the pulse radiating antenna is excited can be made arbitrarily large. Consider an arbitrary distributed source surface designated $S_s$ as shown in Figure 3. Quantities on the surface $S_s$ are designated by adding the subscript s, and the normal vector $\vec{n}$ is a unit vector normal to $S_s$. The position vector $\vec{r}$ is referenced from the origin of a convenient coordinate system. For simplicity, the coordinate origin is chosen to be within the source surface.

Let $\vec{E}(\vec{r}, t^*)$ be the electric field radiated by $S_s$ for $\vec{r} \geq \vec{r}_s$ such that $\vec{E}(\vec{r}, t^*)$ satisfies Maxwell's equations and is initially zero at $t^* = 0$. This electric field has a tangential component on $S_s$ designated by $\vec{E}_s(\vec{r}_s, t^*)$, where

$$\vec{E}_s(\vec{r}_s, t^*) = -\left[\vec{E}(\vec{r}_s, t^*) \times \vec{n}\right] \times \vec{n} \qquad (9)$$

as given by Eqn. (1), Reference 3.

If $\vec{E}_s$ is first specified by Eqn. (9) as a function of $r_s$ and $t^*$, then the desired radiated electric field $\vec{E}$ is uniquely determined.[4] Also, $\vec{E}$ satisfies both Maxwell's equations and the initial and boundary conditions by hypothesis. Thus, the distributed source can be specified by first specifying $\vec{E}$, as the radiated field from a particular antenna, and then calculating $\vec{E}_s$.

### Launching Spherical Waves

A distributed source for launching spherical electromagnetic waves can be specified by the tangential component of a spherical TEM wave associated with a biconical antenna. For the purpose of connecting a distributed source to a cylindrical antenna, a cylindrical source surface with the same radius as the antenna is convenient. Consider now the

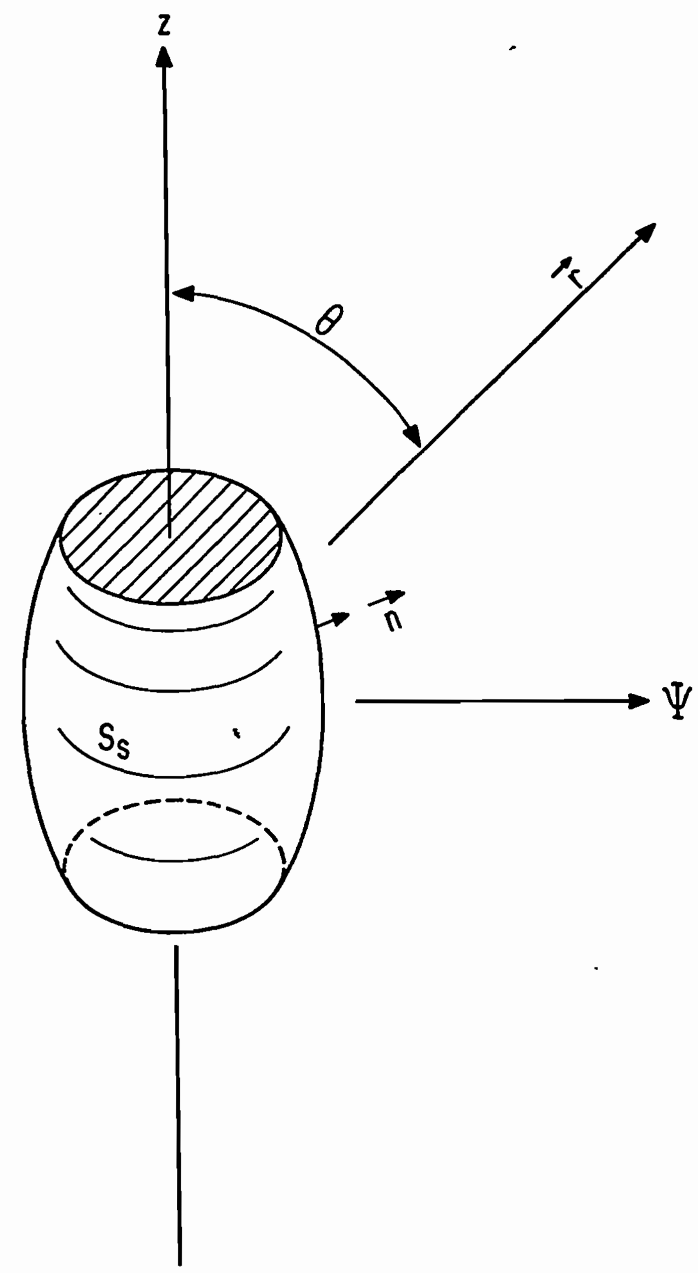


Figure 3. ARBITRARY SOURCE REGION.

problem of specifying the electric field on a cylindrical source surface for launching spherical waves. The cylindrical coordinate system is used to specify the source. However, all radiated fields discussed in this paper are in the more convenient spherical coordinate system. The coordinate origin of the cylindrical ($\Psi$, $\phi$, z) coordinate system is located within the cylindrical source surface which has axial and lengthwise symmetry about the coordinate origin. Now consider the field $\vec{E}_b$ radiated by a biconical antenna of infinite length with the bicone apex located at the coordinate origin. The bicone angle $\theta_o$ is such that the biconical antenna intersects with the ends of the cylindrical source surface at $z = h_s$ and $z = -h_s$ as shown in Figure 4, which depicts the geometry of the problem under consideration. For these conditions the surface electric field is given by

$$\vec{E}_s (\vec{r}_s, t^*) = -\left[\vec{E}_b (\vec{r}_s, t^*) \times \vec{n}\right] \times \vec{n} \tag{10}$$

where $\vec{E}_b$ is the electric field associated with the biconical antenna. $\vec{E}_b$ is given by[5]

$$\vec{E}_b (r_s, t^*) = \frac{V_b (t^*) f_o}{r_s \sin\theta_s} \vec{a}_\theta \quad \text{for} \quad \theta_o < \theta < \pi - \theta_o \tag{11}$$

where $f_o = \left\{2 \ln \left[\cot(\theta_o/2)\right]\right\}^{-1}$.

The normal vector for the circular cylindrical source surface is $\vec{n} = \vec{a}_\Psi$. The surface field can now be written as

$$\vec{E}_s = -\left[E_b \vec{a}_\theta \times \vec{a}_\Psi\right] \times \vec{a}_\Psi$$

$$= - E_b \sin\theta_s \vec{a}_z \tag{12}$$

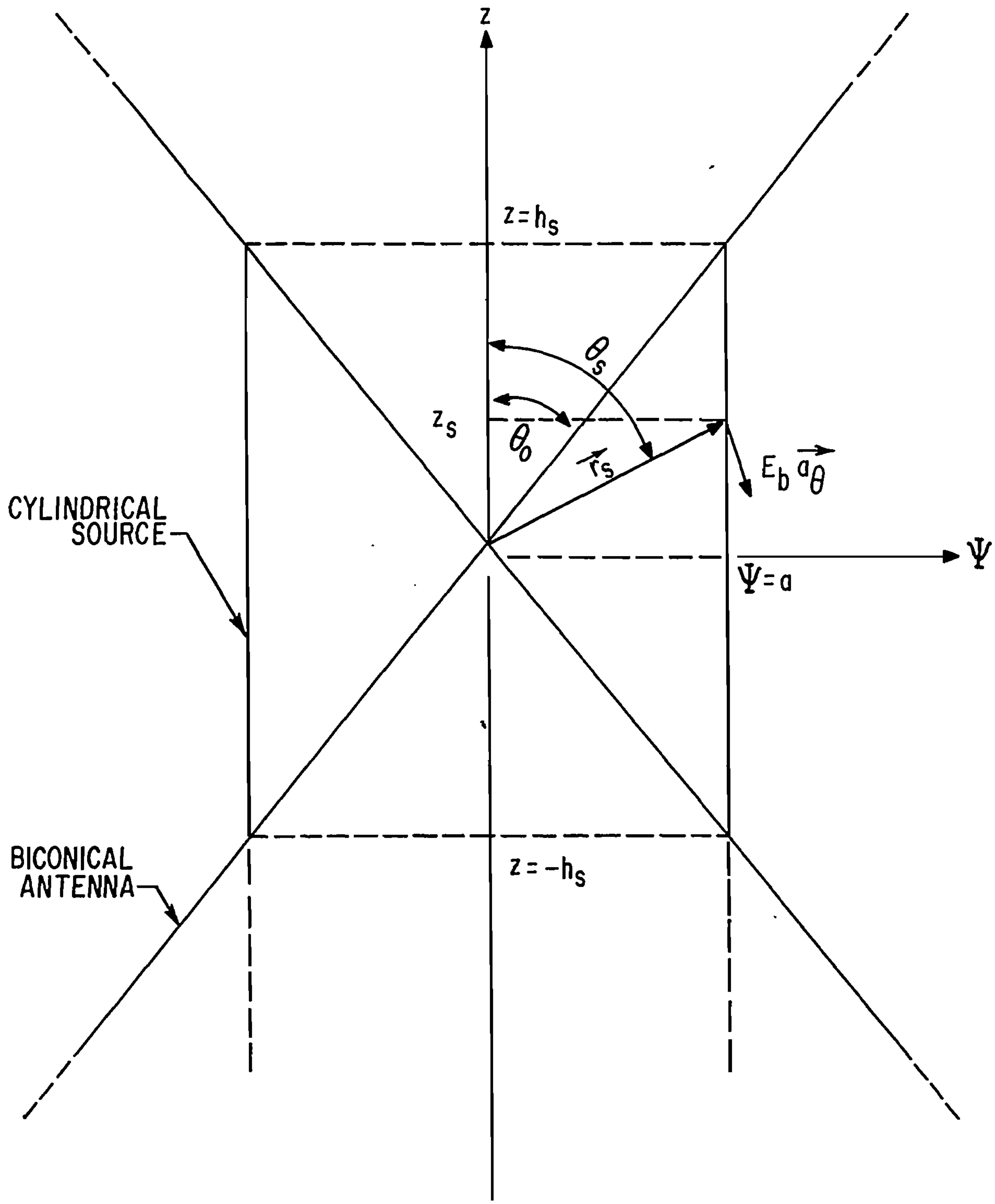


Figure 4. CYLINDRICAL DISTRIBUTED SOURCE SPECIFIED BY AN INFINITELY LONG BICONICAL ANTENNA.

and substituting for $E_b$ gives

$$\vec{E}_s = -\frac{V_b(t^*)\, f_o}{r_s}\, \vec{a}_z$$

$$= -\frac{V_b(t^*)\, f_o}{\sqrt{z_s^2 + a^2}}\, \vec{a}_z \qquad (13)$$

where $r_s$ has been replaced by a function of $z_s$.

Thus far, the magnitude of the tangential electric field on the cylindrical source surface as a function of $z_s$ has been specified. To obtain a spherically expanding wave, the wave front must expand radially about the source origin by definition. Radial expansion of the radiated wave can be achieved if the distributed source elements are turned on at absolute time equal to $r_s/c$. Thus, both the magnitude and time of the $\vec{E}_s$ associated with a biconical antenna can be specified as

$$\vec{E}_s = -\frac{V_b(t^*)\, f_o}{r_s}\, U\left(t^* - r_s/c\right)\, \vec{a}_z \qquad (14)$$

where U(x) is a unit step function equal to one for $x > 0$.

If the bicone voltage is a unit-step function $V_{bo}\, U(t^*)$, then in terms of $z_s$

$$\vec{E}_s = -\frac{V_{bo}\, f_o}{\sqrt{z_s^2 + a^2}}\, U\left(t^* - \frac{1}{c}\sqrt{z_s^2 + a^2}\right) \vec{a}_z. \qquad (15)$$

In terms of the Laplace transform variable p, Eqn. (15) becomes

$$\vec{E}_s\left(\vec{r}_s, p\right) = -\frac{V_{bo}\, f_o}{\sqrt{z_s^2 + a^2}}\; \frac{e^{-p/c \sqrt{z_s^2 + a^2}}}{p}\, \vec{a}_z\,. \qquad (16)$$

The maximum surface field is at $z_s = 0$. The magnitude of the surface field at $z_s = 0$ can be calculated by Eqn. (15) as

$$E_{sm} = \frac{V_{bo} f_o}{a} \quad , \quad t^* \geq a/c \quad . \tag{17}$$

If the surface field is considered as a voltage across a peripheral band of small width $\Delta z$, then the equivalent bicone voltage is

$$V_{bo} = \frac{a V_{sm}}{f_o \Delta z} \tag{18}$$

where $V_{sm}$ is the voltage across $\Delta z$ at $z = 0$. And

$$V_{sm} = \frac{V_{bo} f_o \Delta z}{a} \quad . \tag{19}$$

Since the surface field is continuous after the source elements are turned on, the total voltage across the distributed source can be calculated by integrating the surface field from $z_s = h_s$ to $z_s = -h_s$.* Therefore, after all the source elements are turned on, the total voltage is given by

$$V_{st} = \int_{-h_s}^{h_s} \frac{V_{bo} f_o}{\sqrt{z_s^2 + a^2}} \, dz_s$$

$$= \int_{-h_{s/a}}^{h_{s/a}} \frac{V_{bo} f_o}{\sqrt{z_a^2 + 1}} \, dz_a \tag{20}$$

* For the purpose of integration, $\Delta z$ is considered to be a very small differential quantity dz.

where a change of variable of integration $z_s = a\, z_a$ has been made. Now make another change of variable from $z_a$ to u where $u = \sinh^{-1} z_a$. Then,

$$V_{st} = \int_{-u_o}^{u_o} V_{bo}\, f_o\, du \tag{21}$$

where $u_o = \sinh^{-1}\left(h_{s/a}\right) = \sinh^{-1}\left(\cot\theta_o\right)$. Evaluating the above integral gives

$$\begin{aligned} V_{st} &= 2\, V_{bo}\, f_o\, \sinh^{-1}\left(\cot\theta_o\right) \\ &= V_{bo} \end{aligned} \tag{22}$$

where $2 \sinh^{-1}\left(\cot\theta_o\right) = 1/f_o$.

Thus, the total voltage across the distributed source is the same as the equivalent bicone voltage used to specify the source.

## IV. Infinite Cylindrical Antenna with a Distributed Source

The far zone electric field expression for the infinite cylindrical antenna, Eqn. (3), can be modified for the source voltage located at an arbitrary position along $z_s$. Let R be the distance from the arbitrary source to the observer as shown in Figure 5. For the source located at $z_s$, Eqn. (3) can be written as

$$E_\theta = \frac{V(p)\, e^{-pR/c}}{2 \sin\theta\, R\, K_o(y)} \tag{23}$$

where R is given by

$$R = r - \delta = r - z_s \cos\theta \quad . \tag{24}$$

Since the observer is in the far zone the inverse distance term R in the denominator of Eqn. (23) can be replaced by r; i.e., $R^{-1} = r^{-1} + O(r^{-2})$ where O is the order symbol. However, for the phase factor term in the numerator it is the difference between R and r that is important and the exact value of R must be retained. Thus, the radiated electric field can be written as

$$E_\theta = \frac{V(p)\, e^{-pR/c}}{2 \sin\theta\, r\, K_o(y)} \quad . \tag{25}$$

Consider now an infinite cylindrical antenna excited by several ideal source voltages located at arbitrary positions on $z_s$ within the finite limits $z_s = h_s$ and $z_s = -h_s$. The source voltages are impressed across peripheral bands of infinitesimal width connected by perfectly conducting cylindrical sections. Since the voltage sources are independent and are perfectly conducting, the fields radiated by each voltage source can be added vectorially by superposition to give the total radiated

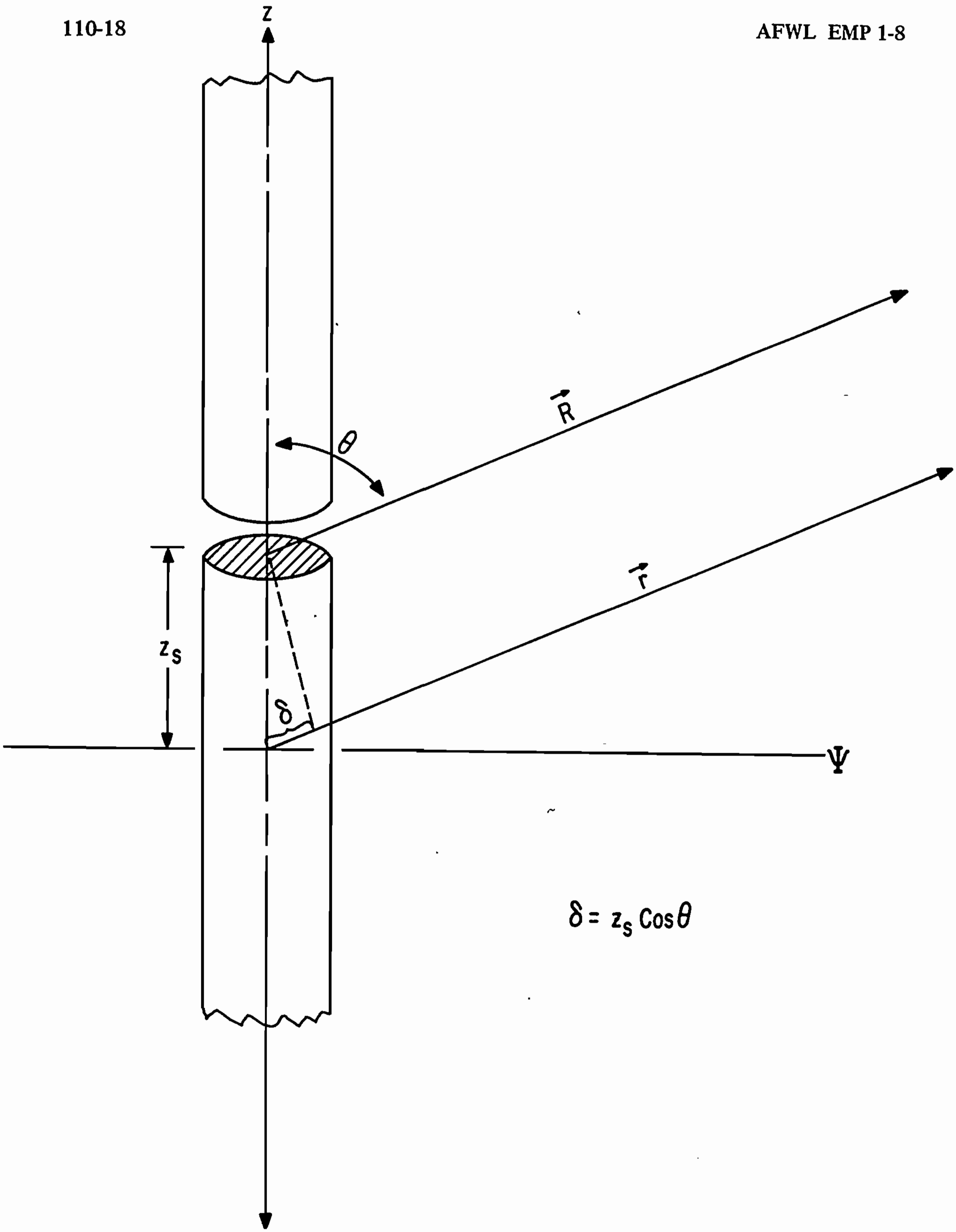


Figure 5. INFINITE CYLINDRICAL ANTENNA WITH THE SOURCE VOLTAGE LOCATED AT AN ARBITRARY POSITION $Z_s$.

field. Hence, the field radiated by an infinite cylindrical antenna with n number of voltage sources as shown in Figure 6 is

$$\vec{E}_i = \sum_{i=0}^{n} \frac{V_{si}\left(\vec{r}_{si}, p\right) e^{-p R_i / c}}{2 \sin\theta \, r \, K_o(y)} \, \vec{a}_i \tag{26}$$

where $\vec{a}_i$ is a unit vector perpendicular to $\vec{R}_i$ and the source voltages are a function of the position vector $\vec{r}_{si}$. For $R_i$ large, $\vec{a}_i$ approaches $\vec{a}_\theta$ and Eqn. (26) becomes

$$\vec{E}_\theta = \sum_{i=0}^{n} \frac{V_{si}\left(\vec{r}_{si}, p\right) e^{-p/c\left(r - z_{si} \cos\theta\right)}}{2 \sin\theta \, r \, K_o(y)} \, \vec{a}_\theta \quad . \tag{27}$$

Now allow the number of peripheral bands to increase and completely fill the region from $z_s = h_s$ to $z_s = -h_s$ as shown in Figure 7.

Since the source voltages are impressed across peripheral bands of width $dz_s$, the surface field is related to the source voltage by

$$E_s\left(\vec{r}_s, p\right) dz_s = V_s\left(\vec{r}_s, p\right)$$

and the differential source fields on $S_s$ can be summed by an integral. The theta component of the electric field becomes

$$E_\theta = \int_{-h_s}^{h_s} \frac{E_s\left(\vec{r}_s, p\right) e^{-p/c\left(r - z_s \cos\theta\right)}}{2 \sin\theta \, r \, K_o(y)} \, dz_s \quad . \tag{28}$$

## Surface Field for Radiating Spherical Waves

In Eqn. (28) the source surface electric field is a somewhat general function of $\vec{r}_s$. Now allow the source field to be the surface field developed in section III for launching spherical waves. The substitution of Eqn. (16)

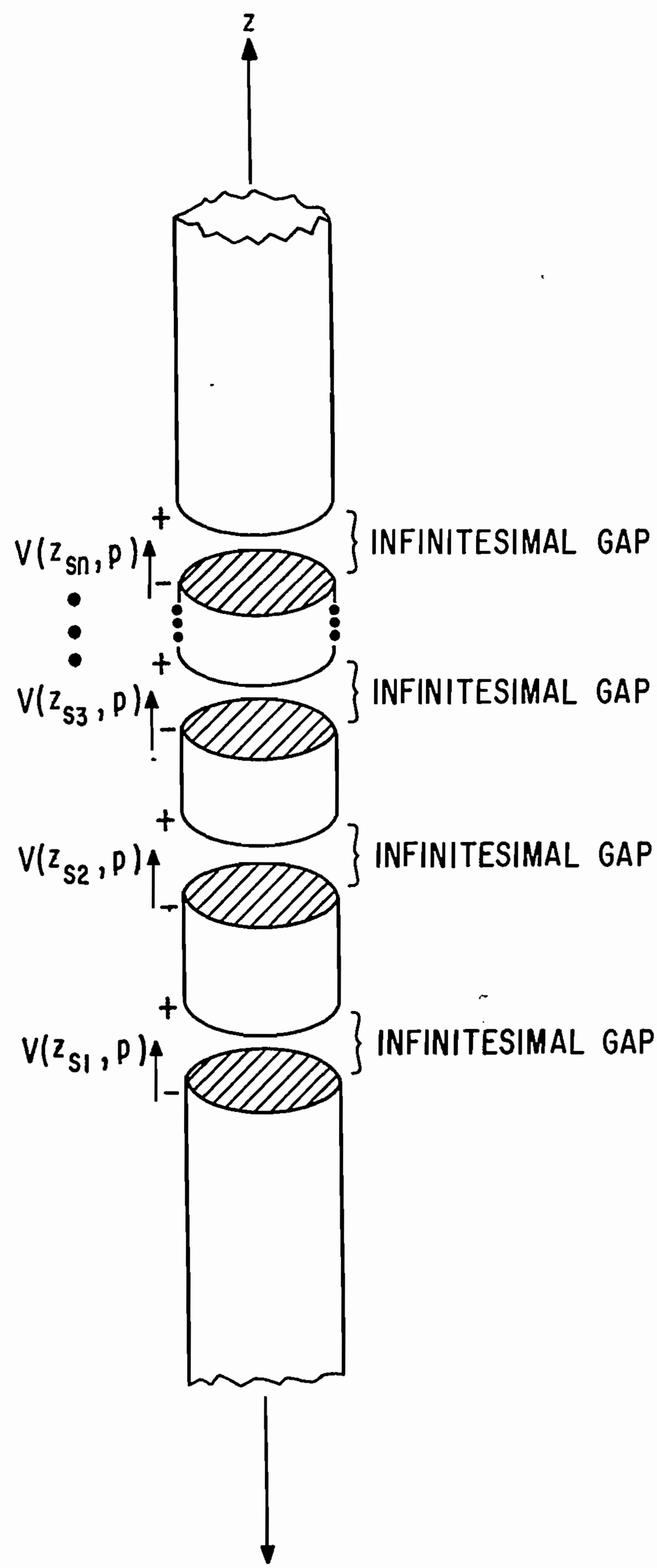


Figure 6. INFINITE CYLINDRICAL ANTENNA WITH n NUMBER OF ARBITRARY VOLTAGE SOURCES.

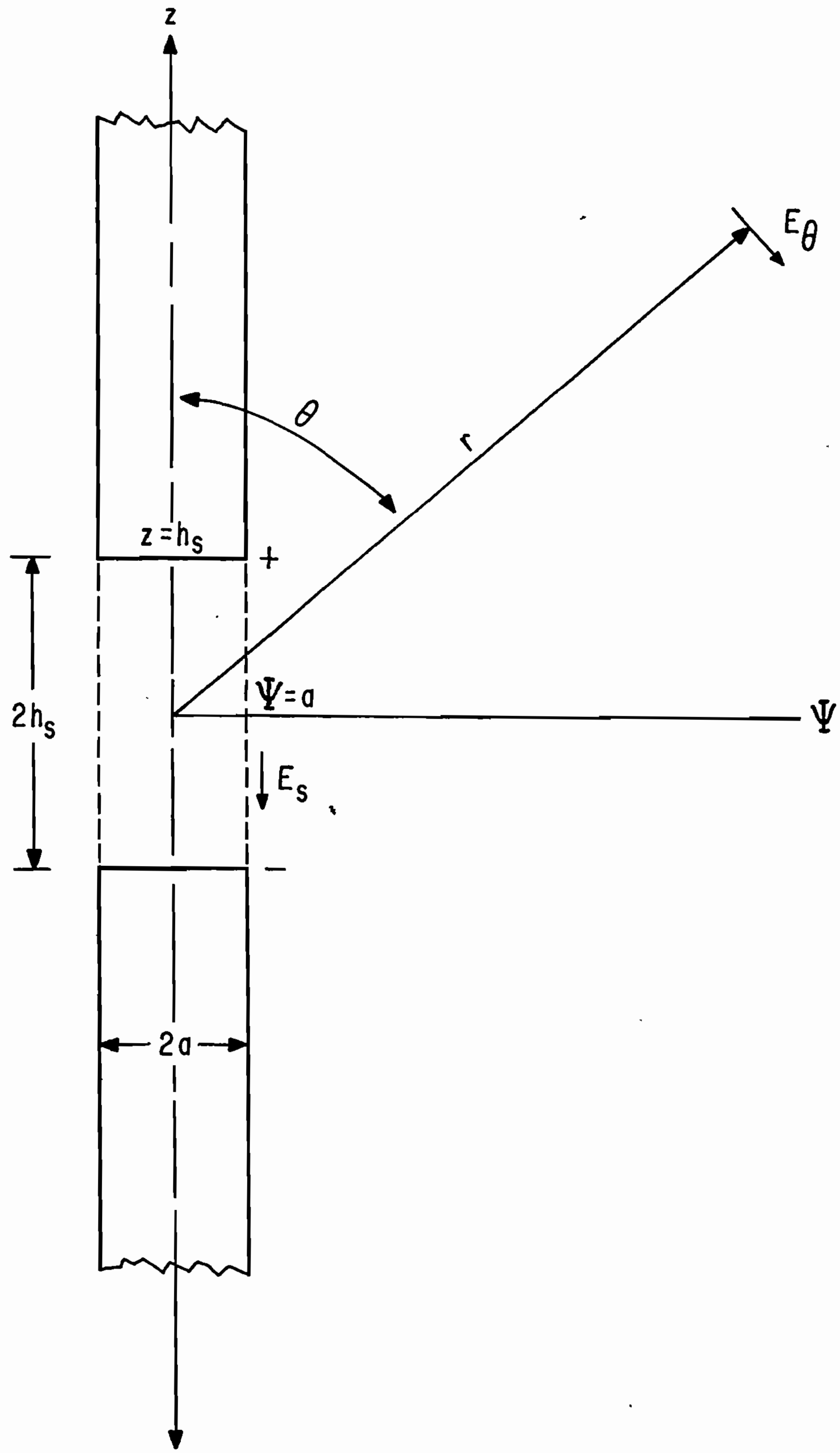


Figure 7. INFINITE CYLINDRICAL ANTENNA WITH A CONTINUOUS DISTRIBUTED SOURCE $E_s$.

into Eqn. (28) gives

$$E_\theta = \int_{-h_s}^{h_s} \frac{V_{bo} f_o e^{-p/c\left(r + \sqrt{z_s^2 + a^2} - z_s \cos\theta\right)}}{2 \sin\theta \, r \, p \sqrt{z_s^2 + a^2} \, K_o(y)} \, dz_s \quad . \tag{29}$$

Now make a change of variable of integration from $z_s$ to $z_a = z_{s/a}$, then

$$E_\theta = \int_{-h_{s/a}}^{h_{s/a}} \frac{V_{bo} f_o e^{-pr/c} e^{-pa/c\left(\sqrt{z_a^2 + 1} - z_a \cos\theta\right)}}{2 \sin\theta \, r \, p \, K_o(y) \sqrt{z_a^2 + 1}} \, dz_a \tag{30}$$

Now make another change of variable $z_a = \sinh u$, Eqn. (30) becomes

$$E_\theta = \int_{-u_o}^{u_o} \frac{V_{bo} f_o e^{-pr/c} e^{pa/c\left(-\cosh u + \sinh u \cos\theta\right)}}{2 \sin\theta \, r \, p \, K_o(y)} \, du \tag{31}$$

where the relation $\sinh^2 u + 1 = \cosh^2 u$ has been used and $u_o = \sinh^{-1}\left(\cot\theta_o\right)$.

For convenience, the electric field can be normalized as in section II. Thus,

$$\xi_b = \frac{r E_\theta}{V_{bo}} \quad . \tag{32}$$

The time domain expression for $\xi_b$ can now be written as

$$\xi_b = \frac{f_o}{2 \sin\theta} \frac{1}{2\pi i} \int_{-u_o}^{u_o} \int_{\gamma' - i\infty}^{\gamma' + i\infty} \frac{e^{pa/c\left(\frac{ct^*}{a} - \cosh u + \sinh u \cos\theta\right)}}{p \, K_o(y)} \, dp \, du \tag{33}$$

As in section II, a change of variable $y = (pa/c)\sin\theta$ yields

$$\xi_b = \frac{f_o}{2\sin\theta}\,\frac{1}{2\pi i}\int_{-u_o}^{u_o}\int_{\gamma'-i\infty}^{\gamma'+i\infty}\frac{e^{y\csc\theta(q-\cosh u+\sinh u\cos\theta)}}{y\,K_o(y)}\,dy\,du \qquad (34)$$

where $q = ct^*/a$.

Consider now the integral expression in Eqn. (7) as a function of $\eta$ given by

$$f(\eta) = \frac{1}{2\pi i}\int_{\gamma'-i\infty}^{\gamma'+i\infty}\frac{e^{y\eta}}{y\,K_o(y)}\,dy \qquad (35)$$

where $q\csc\theta$ has been replaced by $\eta$. This function can also be written for $\eta > -1$ in the form

$$f(\eta) = \int_0^{\infty}\frac{e^{-y\eta}\,I_o(y)}{y\left[K_o^2(y) + \pi^2 I_o^2(y)\right]}\,dy \qquad (36)$$

as developed in Reference 2.

In terms of Laplace transformations, $f(\eta)$ can be written as

$$f(\eta) = \mathop{L^{-1}}_{y\to\eta}\left[\frac{1}{y\,K_o(y)}\right] \qquad (37)$$

and

$$\frac{1}{y\,K_o(y)} = \mathop{L}_{\eta\to y}\left[f(\eta)\right] \qquad (38)$$

where L symbolizes the Laplace transform.

Now let $\eta = \csc\theta\,(q - \cosh u + \sinh u \cos\theta)$. The substitution of the above $\eta$ into Eqns. (35) and (36) and the substitution of Eqn. (35) into Eqn. (34) gives

$$\xi_b = \frac{f_o}{2\sin\theta} \int_{-u_o}^{u_o} \int_0^{\infty} \frac{e^{-y\csc\theta\,(q - \cosh u + \sinh u \cos\theta)}}{y\left[K_o^2(y) + \pi^2 I_o^2(y)\right]} I_o(y)\; dy\, du \quad . \qquad (39)$$

## V. Analysis of the Radiation Fields

The electric field expression, Eqn. (39), developed in section IV, is valid for angles of observation in the range $0 \leq \theta < \pi$. However, it is unnecessary to analyze the electric field over the complete range of $\theta$ since the field is symmetrical for the angles $\theta$ and the supplement of $\theta$ due to the lengthwise symmetry of the distributed source and the cylindrical antenna. Therefore, an analysis of the field for angles between $0^\circ$ and $90^\circ$ $(0 < \theta \leq \pi/2)$ suffices as an analysis for the complete range of $0 < \theta < \pi$.

Recall from section III Eqn. (11) that the expression for the electric field radiated by a biconical antenna is valid only for angles of observation $\theta_o < \theta < \pi - \theta_o$. Since the distributed source in this problem is specified by the electric field as radiated by a biconical antenna, one feels intuitively that the analysis of the electric field for angles $0 < \theta < \theta_o$ will require special attention and that the angle $\theta = \theta_o$ is a special case.

In order to determine how the field behaves for angles in the two ranges, $0 < \theta \leq \theta_o$ and $\theta_o < \theta \leq \pi/2$, rewrite Eqn. (34) as follows

$$\xi_b = \frac{f_o}{2\sin\theta}\frac{1}{2\pi i}\int_{-\infty}^{\infty} G(\theta, q\csc\theta)\,du$$

$$- \frac{f_o}{2\sin\theta}\frac{1}{2\pi i}\int_{u_o}^{\infty} G(\theta, q\csc\theta)\,du$$

$$- \frac{f_o}{2\sin\theta}\frac{1}{2\pi i}\int_{-\infty}^{-u_o} G(\theta, q\csc\theta)\,du \qquad (40)$$

where $G(\theta, q\csc\theta)$ is the integrand of Eqn. (34).

Define the three integral expressions above as $\xi_1$, $\xi_2$, and $\xi_3$ respectively. By exchanging the limits of integration and replacing the variable of integration u by -u, the third integral expression becomes

$$\xi_3 = \frac{f_o}{2\sin\theta}\,\frac{1}{2\pi i}\int_{u_o}^{\infty}\int_{\gamma'-i\infty}^{\gamma'+i\infty}\frac{e^{y\csc\theta(q-\cosh u-\sinh u\cos\theta)}}{y\,K_o(y)}\,dy\,du \quad (41)$$

Since $\cos(\pi-\theta) = -\cos\theta$, it follows that

$$\xi_3(\theta) = \xi_2(\pi-\theta) \quad , \qquad 0 < \theta \leq \pi/2 \quad . \qquad (42)$$

Therefore, an analysis of $\xi_2$ at the angle $\theta$ is also an analysis for $\xi_3$ at the angle $\pi-\theta$.

Now make a change of variable $\nu = u - u_o$, then $d\nu = du$ and $u = \nu + u_o$. The value of $u_o$ is given by

$$u_o = \sinh^{-1}\left(\cot\theta_o\right) = \cosh^{-1}\left(\csc\theta_o\right) \qquad (43)$$

and

$$\cosh u = \cosh\left(\nu + u_o\right) = \cosh\nu\cosh u_o + \sinh\nu\sinh u_o$$

$$= \cosh\nu\csc\theta_o + \sinh\nu\cot\theta_o \qquad (44)$$

also

$$\sinh u = \sinh\left(\nu + u_o\right) = \sinh\nu\cosh u_o + \cosh\nu\sinh u_o$$

$$= \sinh\nu\csc\theta_o + \cosh\nu\cot\theta_o \quad . \qquad (45)$$

After replacing u with $\nu$, the integral expression for $\xi_2$ becomes

$$\xi_2 = \frac{f_o}{2\sin\theta}\,\frac{1}{2\pi i}\int_0^{\infty}\int_{\gamma'-i\infty}^{\gamma'+i\infty}\frac{e^{+y\csc\theta\,\tau_o}}{y\,K_o(y)}\,dy\,d\nu \qquad (46)$$

where $\tau_o = q - \cosh\nu\ \csc\theta_o - \sinh\nu\ \cot\theta_o + \sinh\nu\ \csc\theta_o\cos\theta$
$+ \cosh\nu\ \cot\theta_o\cos\theta.$ (47)

With $\xi_2$ expressed as an inverse Laplace transform, it is clear that $\tau_o$ represents a delay. To determine the minimum delay, let $y \to \infty$. For large y, $K_o(y)$ has the form

$$K_o(y) \sim \sqrt{\frac{\pi}{2y}}\ e^{-y}\left[1 - \frac{1}{8y} + O\left(y^{-2}\right)\right] \qquad (48)$$

as given by Eqn. (9.7.2) in Reference 6. The substitution of Eqn. (48) in Eqn. (46) gives,

$$\xi_2 = \frac{f_o}{2\sin\theta}\,\frac{1}{2\pi i}\int_0^{\infty}\int_{\gamma'-i\infty}^{\gamma'+i\infty}\frac{e^{y\csc\theta\left(\tau_o+\sin\theta\right)}}{\sqrt{\pi/2}\ \sqrt{y}\left[1-\frac{1}{8y}+\cdots\right]}\,dy\,d\nu\ . \qquad (49)$$

The minimum delay $q_o$ occurs when $\tau_o + \sin\theta = 0$. Thus, $q_o$ can be written as

$$q_o = -\sin\theta + \cosh\nu_m\csc\theta_o + \sinh\nu_m\cot\theta_o$$

$$- \sinh\nu_m\csc\theta_o\cos\theta - \cosh\nu_m\cot\theta_o\cos\theta \qquad (50)$$

where $\nu_m$ is the value of $\nu$ at $q = q_o$. To determine $\nu_m$, set $d\tau_o/d\nu = 0$. This gives $\nu_m$ as

$$\nu_m = \tanh^{-1}\left[\frac{\cos\theta - \cos\theta_o}{1 - \cos\theta\cos\theta_o}\right]\ . \qquad (51)$$

Note that if $\theta \geq \theta_o$ then $\nu_m \leq 0$ and if $\theta \leq \theta_o$ then $\nu_m \geq 0$. If $\theta > \theta_o$ and $\nu$ is always positive, then the minimum delay time occurs at $\nu = 0$. However, if $\theta < \theta_o$ and $\nu$ is always positive then the minimum delay time occurs at $\nu_m$ and the values of $0 \leq \nu \leq \nu_m$ corresponds to a delay time larger than the minimum value. To avoid the problem of having the minimum delay occur within the range of integration, express the electric field for $\theta < \theta_o$ such that $\nu$ is always negative. The minimum delay time will now occur at $\nu = 0$. Thus, for $\theta < \theta_o$, define the electric field as

$$\xi_b = \frac{f_o}{2\sin\theta}\,\frac{1}{2\pi i}\int_{-\infty}^{u_o}\int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{y\csc\theta(q-\cosh u+\sinh u\cos\theta)}}{yK_o(y)}\,dy\,du$$

$$-\frac{f_o}{2\sin\theta}\,\frac{1}{2\pi i}\int_{-\infty}^{-u_o}\int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{y\csc\theta(q-\cosh u+\sinh u\cos\theta)}}{yK_o(y)}\,dy\,du \quad (52)$$

where $\nu = u - u_o$ is never positive. The second integral expression has been defined as $\xi_3$. Define the first integral expression as $\xi_4$. After a change of variable of integration from u to $\nu$, $\xi_4$ becomes

$$\xi_4 = \frac{f_o}{2\sin\theta}\,\frac{1}{2\pi i}\int_{-\infty}^{0}\int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{y\csc\theta\,\tau_o}}{yK_o(y)}\,dy\,d\nu$$

$$= \frac{f_o}{2\sin\theta}\,\frac{1}{2\pi i}\int_{0}^{\infty}\int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{y\csc\theta\,\tau_4}}{yK_o(y)}\,dy\,d\nu \quad (53)$$

where $\tau_4 = q - \cosh\nu \csc\theta_o + \sinh\nu \cot\theta_o - \sinh\nu \csc\theta_o \cos\theta + \cosh\nu \cot\theta_o \cos\theta$.

Note that $\tau_4\ (\theta,\ \theta_o) = \tau_o(\pi - \theta,\ \pi - \theta_o)$. Thus, it follows that

$$\xi_4(\theta,\ \theta_o) = \xi_2(\pi - \theta,\ \pi - \theta_o) \tag{54}$$

The analysis of the radiation fields has shown that the angle $\theta$ is an important parameter. If $\theta = \theta_o$ is given special attention, there are three cases to consider:

| | | | |
|---|---|---|---|
| CASE I | $\xi_b = \xi_1 - \xi_2 - \xi_3$ | $\theta_o < \theta \leq \pi/2$ | (55) |
| CASE II | $\xi_b = \xi_4 - \xi_3$ | $\theta = \theta_o$ | (56) |
| CASE III | $\xi_b = \xi_4 - \xi_3$ | $0 < \theta < \theta_o$ . | (57) |

## VI. Analytic Solution of the Electric Field

It was shown in Section V that the electric field can be expressed in terms of $\xi_1$, $\xi_2$, $\xi_3$, and $\xi_4$ by the appropriate Equation (55), (56), or (57). An analytic solution of the electric field can be obtained by finding the analytic solution of $\xi_1$, $\xi_2$, $\xi_3$, and $\xi_4$ if an analytic solution exists.

### Analytic Solution of $\xi_1$

The inverse Laplace transform representation of $\xi_1$ is given by

$$\xi_1 = \frac{f_o}{2\sin\theta}\,\frac{1}{2\pi i}\int_{-\infty}^{\infty}\int_{\gamma'-i\infty}^{\gamma'+i\infty}\frac{e^{y\csc\theta(q-\cosh u+\sinh u\cos\theta)}}{y\,K_o(y)}\,dy\,du \tag{58}$$

The quantity $\xi_1$ represents a distributed source of infinite length along $z_s$. An observer at point P whose coordinates are given by $\phi$, r, and $\theta$ first sees the radiated wave generated by the gap located at $z_s \cot\theta$ since the wave front must expand radially about the source origin. Since the spherical wave is symmetrical with time, a break in the above integral at $u_1 = \sinh^{-1}(\cot\theta)$ may result in two symmetrical expressions. If the integral is broken at $u_1$, $\xi_1$ becomes

$$\xi_1 = \frac{f_o}{2\sin\theta}\,\frac{1}{2\pi i}\int_{u_1}^{\infty}\int_{\gamma'-i\infty}^{\gamma'+i\infty}(\ldots.)\,dy\,du + \frac{f_o}{2\sin\theta}\,\frac{1}{2\pi i}\int_{-\infty}^{u_1}\int_{\gamma'-i\infty}^{\gamma'+i\infty}(\ldots.)\,dy\,du \tag{59}$$

By exchanging the limits of integration and replacing u by -u, the second integral expression becomes

$$\frac{f_o}{2\sin\theta}\,\frac{1}{2\pi i}\int_{-u_1}^{\infty}\int_{\gamma'-i\infty}^{\gamma'+i\infty}\frac{e^{y\csc\theta(q-\cosh u-\sinh u\cos\theta)}}{y\,K_o(y)}\,dy\,du \tag{60}$$

Now make a change of variable $\nu = u - u_1$ and $v = u + u_1$ for the first and second integrals of $\xi_1$ respectively. The quantity $\xi_1$ now becomes

$$\xi_1 = \frac{f_o}{2\sin\theta}\,\frac{1}{2\pi i}\int_0^{\infty}\int_{\gamma'-i\infty}^{\gamma'+i\infty}\frac{e^{y\csc\theta(q+\tau_a)}}{y\,K_o(y)}\,dy\,d\nu$$

$$+\frac{f_o}{2\sin\theta}\,\frac{1}{2\pi i}\int_0^{\infty}\int_{\gamma'-i\infty}^{\gamma'+i\infty}\frac{e^{y\csc\theta(q+\tau_b)}}{y\,K_o(y)}\,dy\,dv \qquad (61)$$

where

$$\begin{aligned}\tau_a &= -\cosh\nu\csc\theta - \sinh\nu\cot\theta + \sinh\nu\cot\theta + \cosh\nu\csc\theta\cos^2\theta\\ &= -\cosh\nu\sin\theta\\ \tau_b &= -\cosh v\csc\theta + \sinh v\cot\theta - \sinh v\cot\theta + \cosh v\csc\theta\cos^2\theta\\ &= -\cosh v\sin\theta \quad .\end{aligned} \qquad (62)$$

The integrals over the variable $\nu$ and $v$ can be expressed as

$$K_o(y) = \int_0^{\infty} e^{-y\cosh\nu}\,d\nu = \int_0^{\infty} e^{-y\cosh v}\,dv \qquad (63)$$

as given by Eqn. 9.6.24 in Reference 6.

Substituting $K_o(y)$ into Eqn. (61) gives

$$\xi_1 = \frac{f_o}{\sin\theta}\,\frac{1}{2\pi i}\int_{\gamma'-i\infty}^{\gamma'+i\infty}\frac{e^{y\csc\theta\, q}}{y}\,dy$$

$$= \frac{f_o}{\sin\theta}\,U(q\csc\theta) = \frac{f_o}{\sin\theta}\,U(t^*) \qquad (64)$$

where $t^*$ is retarded time.

The analytic solution of $\xi_1$ renders precisely the expression for the electric field radiated by a biconical antenna. This is a reasonable result since $\xi_1$ represents a distributed source of infinite length specified by the electric field as radiated by a biconical antenna.

<u>Analytic Solution of $\xi_4$ for $\theta = \theta_o$.</u>

Now consider the special case $\theta = \theta_o$. The quantity $\xi_4$ becomes

$$\xi_4 = \frac{f_o}{2\sin\theta} \int_0^{\infty} \int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{-y\csc\theta_o q}\, e^{-y\cosh\nu}}{y\, K_o(y)}\, dy\, d\nu \tag{65}$$

$$\xi_4 = \frac{f_o}{2\sin\theta}\, \frac{1}{2\pi i} \int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{-y\csc\theta_o q}}{y}\, dy$$

$$= \frac{f_o}{2\sin\theta}\, U(q\csc\theta_o) = \frac{f_o}{2\sin\theta}\, U(t^*)$$

$$= \xi_1/2 \quad . \tag{66}$$

This interesting result reveals that $\xi_4$ for $\theta = \theta_o$ is one-half the value of $\xi_1$ where $\theta > \theta_o$. Since $\xi_2$ and $\xi_3$ both have delays at all angles of $\theta$, it can be concluded that the initial value of the electric field as a function of $\theta$ is discontinuous at $\theta = \theta_o$, i.e., there is a discontinuous jump from $\xi_1$ to $\xi_1/2$. This result is shown in Figure 8.

The other electric field components $\xi_2$, $\xi_3$, and $\xi_4$ have no easy analytic solution.* At this point, for convenience, define a function that

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* The quantity $\xi_4$ has no easy analytic solution at $\theta \neq \theta_o$.

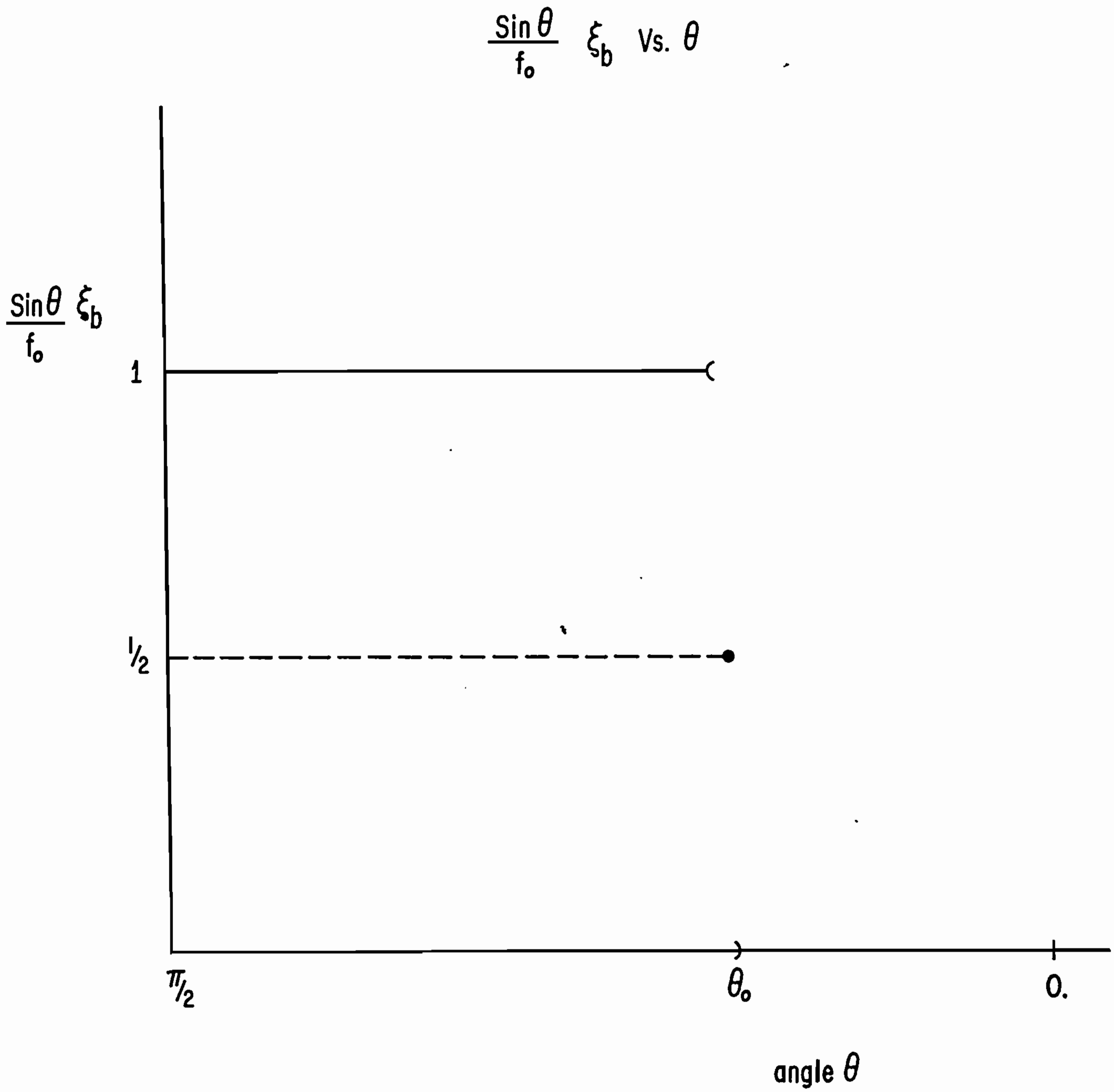


Figure 8. INITIAL VALUE OF THE ELECTRIC FIELD.

renders a solution for $\xi_2$, $\xi_3$, and $\xi_4$ for parameters $\theta$, $\theta_o$, and q such that

$$\xi_2 = \frac{f_o}{2\sin\theta} G_b(\theta, \theta_o, q)$$

$$\xi_3 = \frac{f_o}{2\sin\theta} G_b(\pi - \theta, \theta_o, q)$$

$$\xi_4 = \frac{f_o}{2\sin\theta} G_b(\pi - \theta, \pi - \theta_o, q) \quad . \tag{67}$$

The function $G_b$ is given by

$$G_b(\theta, \theta_o, q) = \frac{1}{2\pi i} \int_0^\infty \int_{\gamma' - i\infty}^{\gamma' + i\infty} \frac{e^{y \csc\theta \, \tau_o}}{y K_o(y)} \, dy \, d\nu \tag{68}$$

where

$$\tau_o = q - \cosh\nu \csc\theta_o - \sinh\nu \cot\theta_o + \sinh\nu \csc\theta_o \cos\theta$$
$$+ \cosh\nu \cot\theta_o \cos\theta \quad .$$

For the general case the solution of the fields radiated by an infinite cylindrical antenna excited by a distributed source can be represented by G functions. A subscript can be used to denote the source distribution. Thus, the fields radiated by an infinite cylindrical antenna excited by a distributed source with a bicone wave distribution can be written in terms of $G_b$.

Time Domain Solution of $G_b$

Equation (68) can be written as

$$G_b = \frac{1}{2\pi i} \int_{u_o}^{\infty} \int_{\gamma' - i\infty}^{\gamma' + i\infty} \frac{e^{y \csc\theta \, (q - \cosh u + \sinh u \cos\theta)}}{y \, K_o(y)} \, dy \, du \qquad (69)$$

where a change of variable $u = \nu + u_o$ has been used and $u_o = \sinh^{-1}\left(\cot\theta_o\right)$. Now make another change of variable $v = u - u_1$ where $u_1 = \sinh^{-1}\left(\cot\theta\right)$. $G_b$ becomes

$$G_b = \frac{1}{2\pi i} \int_{v_o}^{\infty} \int_{\gamma' - i\infty}^{\gamma' + i\infty} \frac{e^{y(q \csc\theta - \cosh v)}}{y \, K_o(y)} \, dy \, dv \qquad (70)$$

where $v_o = u_o - u_1$.

Now let $x = \cosh v$, then $x_o = \cosh(v_o) = \csc\theta_o \csc\theta - \cot\theta_o \cot\theta$. The expression for $G_b$ becomes

$$G_b = \frac{1}{2\pi i} \int_{x_o}^{\infty} \int_{\gamma' - i\infty}^{\gamma' + i\infty} \frac{e^{y(q \csc\theta - x)}}{y \, K_o(y) \sqrt{x^2 - 1}} \, dy \, dx \quad . \qquad (71)$$

If another change of variable $\tau = q \csc\theta + x$ is made, $G_b$ can be written as

$$G_b = \frac{1}{2\pi i} \int_{-\infty}^{b} \int_{\gamma' - i\infty}^{\gamma' + i\infty} \frac{e^{y\tau}}{y \, K_o(y) \sqrt{(\tau - q\csc\theta)^2 - 1}} \, dy \, d\tau \quad . \qquad (72)$$

From the discussion of the pulse radiation by an infinite cylindrical antenna driven by a Dirac delta function gap voltage in Reference 2, Latham and Lee deduced that

$$\int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{y\tau}}{y\,K_o(y)}\,dy = 0 \qquad \text{if} \qquad \tau < -1 \quad . \tag{73}$$

Thus, Eqn. (72) can be expressed as

$$G_b = \frac{1}{2\pi i}\int_{-1}^{b}\int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{y\tau}}{y\,K_o(y)\,\sqrt{(\tau - q\csc\theta)^2 - 1}}\,dy\,d\tau \quad . \tag{74}$$

By the use of Eqns. (35) and (36) where $\eta = \tau$, Eqn. (74) can be written as

$$G_b = \int_{-1}^{b}\int_{0}^{\infty} \frac{e^{-y\tau}\,I_o(y)}{y\left[K_o^{\,2}(y) + \pi^2 I_o^{\,2}(y)\right]\sqrt{(\tau - q\csc\theta)^2 - 1}}\,dy\,d\tau \quad . \tag{75}$$

Now let $\zeta = \tau + 1$, Eqn. (75) becomes

$$G_b\left(\theta, \theta_o, q\right) = \int_{0}^{\zeta_o}\int_{0}^{\infty} \frac{e^{-y(\zeta - 1)}\,I_o(y)}{y\left[K_o^{\,2}(y) + \pi^2 I_o^{\,2}(y)\right]\sqrt{(\zeta - 1 - q\csc\theta)^2 - 1}}\,dy\,d\zeta \tag{76}$$

where $\zeta_o = q\csc\theta - \csc\theta_o\csc\theta + \cot\theta_o\cot\theta + 1$.

Note that $\csc\theta = \csc(\pi - \theta)$ and that $\zeta_o(\theta, \theta_o) = \zeta_o(\pi - \theta, \pi - \theta_o)$. Thus from Eqn. (76) it may be concluded that

$$G_b(\theta, \theta_o, q) = G_b(\pi - \theta, \pi - \theta_o, q) \quad . \tag{77}$$

With this result, the analytic solution of the electric field can be summarized in terms of $G_b$ as follows:

CASE I: $$\xi_b = \frac{f_o}{2 \sin \theta} \left[ 2 - G_b(\theta, \theta_o, q) - G_b(\pi - \theta, \theta_o, q) \right] \quad \theta_o < \theta \leq \pi/2 \tag{78}$$

CASE II: $$\xi_b = \frac{f_o}{2 \sin \theta_o} \left[ 1 - G_b(\pi - \theta_o, \theta_o, q) \right] \quad \theta = \theta_o \tag{79}$$

CASE III: $$\xi_b = \frac{f_o}{2 \sin \theta} \left[ G_b(\theta, \theta_o, q) - G_b(\pi - \theta, \theta_o, q) \right] \quad 0 < \theta < \theta_o \quad . \tag{80}$$

## VII. Asymptotic Forms of the Radiation Fields

The asymptotic forms of the radiation fields will give an insight into the behavior of the fields at early and late times. These expressions also will be helpful with the numerical calculations of the radiated fields.

### Large Time Behavior

To find the asymptotic form of $\xi_b$ for $q \csc\theta \to \infty$ write Eqn. (34) as

$$\xi_b = \frac{f_o}{2\sin\theta}\,\frac{1}{2\pi i}\int_{-u_o}^{u_o}\int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{y\eta}}{y\,K_o(y)}\,dy\,du \tag{81}$$

where $\eta = \csc\theta\,(q - \cosh u + \sinh u \cos\theta)$. By virtue of Eqns. (35) and (36), Eqn. (81) can be written as

$$\xi_b = \frac{f_o}{2\sin\theta}\int_{-u_o}^{u_o}\int_{0}^{\infty} \frac{e^{-y\eta}\,I_o(y)}{y\left[K_o^2(y) + \pi^2 I_o^2(y)\right]}\,dy\,du$$

$$= \frac{f_o}{2\sin\theta}\int_{-u_o}^{u_o} G(\eta)\,du \quad . \tag{82}$$

The function $G(\eta)$ for $\eta \to \infty$ has the form

$$G(\eta) = \frac{1}{\ell n(2\eta/\Gamma)} + O\left(\ell n^{-2}\,(2\eta/\Gamma)\right) \tag{83}$$

where O is the order symbol, and $\Gamma$ is the exponential of Euler's constant,

and

$$G(\eta) \sim \frac{1}{\ln(2\eta/\Gamma)} \qquad \text{for } \eta \to \infty \quad . \tag{84}$$

This result is developed in detail in Appendix A. The substitution of Eqn. (83) into Eqn. (82) gives

$$\xi_b = \frac{f_o}{2\sin\theta}\int_{-u_o}^{u_o}\left\{\frac{1}{\ln(2\eta/\Gamma)} + O\left(\ln^{-2}(2\eta/\Gamma)\right)\right\} du$$

$$= \frac{f_o}{2\sin\theta}\int_{-u_o}^{u_o}\left\{\frac{1 + O\left(\ln^{-1}(2\eta/\Gamma)\right)}{\ln[2\csc\theta(q - \cosh u + \sinh u\cos\theta)/\Gamma]}\right\} du$$

$$= \frac{f_o}{2\sin\theta\,\ln(2q\csc\theta/\Gamma)}\int_{-u_o}^{u_o}\left\{\left[1 + O\left(\ln^{-2}(2\eta/\Gamma)\right)\right]\right.$$

$$\left.\left[1 + \frac{\ln\left[1 - \frac{1}{q}(\cosh u - \sinh u\cos\theta)\right]}{\ln(2q\csc\theta/\Gamma)}\right]^{-1}\right\} du \quad . \tag{85}$$

Thus, for $q\csc\theta$ very large, $\xi_b$ becomes

$$\xi_b = \frac{f_o}{2\sin\theta\,\ln(2q\csc\theta/\Gamma)}\left[\int_{-u_o}^{u_o} du + O\left(\ln^{-2}(2\eta/\Gamma)\right)\right] du$$

$$= \frac{1}{2\sin\theta\,\ln(2q\csc\theta/\Gamma)} + O\left(\ln^{-2}(2q\csc\theta/\Gamma)\right) \tag{86}$$

and

$$\xi_b \sim \frac{1}{2 \sin\theta \, \ln(2q \csc\theta/\Gamma)} \quad . \tag{87}$$

Also, the asymptotic form of $G_b$ for $q \csc\theta \to \infty$ can be written as

$$G_b \sim 1 - \frac{1}{2 f_o \, \ln(2q \csc\theta/\Gamma)} \quad . \tag{88}$$

Small Time Behavior

The initial radiation fields for $\theta > \theta_o$ are exactly the fields that would be radiated by a biconical antenna and their small time behavior is trivial. However, the small time behavior of the radiation fields associated with the surface field discontinuities at the ends of the distributed source is worthy of special consideration.

The small time behavior of $\xi_b$ can be determined by obtaining the small time behavior of $G_b$. Eqn. (80) can be written as

$$G_b = \int_0^{\zeta_o} \frac{F(\zeta)\, d\zeta}{\sqrt{(\zeta - 1 - q \csc\theta)^2 - 1}} \tag{89}$$

where

$$F(\zeta) = \int_0^{\infty} \frac{e^{-y(\zeta - 1)} I_o(y)}{y\left[K_o^2(y) + \pi^2 I_o^2(y)\right]} \, dy \tag{90}$$

$F(\zeta)$ exists for $\zeta > 0$. To obtain the small time behavior of $G_b$, let $\zeta \rightarrow 0$ (but not equal zero). The asymptotic form of $F(\zeta)$ for $\zeta \rightarrow 0$ is given in Appendix B as

$$F(\zeta) = \frac{\sqrt{2}}{\pi\sqrt{\zeta}} + O\left(\zeta_o^{3/2}\right) \tag{91}$$

Now write Eqn. (89) in the form

$$G_b = \int_0^{\zeta_o} \frac{F(\zeta)}{\sqrt{(\zeta - \zeta_o - x_o)^2 - 1}} \, d\zeta$$

$$= \int_0^{\zeta_o} \frac{F(\zeta)}{\sqrt{x_o^2 - 1}} \left[1 - \frac{2(\zeta - \zeta_o)\, x_o}{x_o^2 - 1} + \frac{(\zeta - \zeta_o)^2}{x_o^2 - 1}\right]^{-1/2} d\zeta$$

$$= \int_0^{\zeta_o} \frac{F(\zeta)}{\sqrt{x_o^2 - 1}} \left[1 + (\zeta - \zeta_o)\frac{x_o}{x_o^2 - 1} + O\left((\zeta - \zeta_o)^2\right)\right] d\zeta \tag{92}$$

As $\zeta \rightarrow 0$, Eqn. (92) becomes

$$G_b = \frac{\sqrt{2}}{\pi} \frac{1}{\sqrt{x_o^2 - 1}} \int_0^{\zeta_o} \left(\frac{1}{\sqrt{\zeta}} + O\left(\sqrt{\zeta}\right)\right) d\zeta$$

$$= \frac{2\sqrt{2}}{\pi} \frac{1}{\sqrt{x_o^2 - 1}} \left[\sqrt{\zeta_o} + O\left(\zeta_o^{3/2}\right)\right] U(\zeta_o) \tag{93}$$

Now it is clear that $G_b$ becomes non-zero for $\zeta_o > 0$. This occurs at time

$$q_o \csc\theta = \csc\theta_o \csc\theta - \cot\theta \cot\theta_o - 1 \quad (94)$$

and

$$t_o^* = \frac{a}{c}\left[\csc\theta_o - \cos\theta \cot\theta_o - \sin_\theta\right] \quad (95)$$

$\zeta_o$ can be written in the form

$$\zeta_o = \csc\theta\,(q - q_o) = \csc\theta\, q^* \quad (96)$$

where $q^*$ is a normalized retarded time, i.e., $q^* = q - q_o$.

In terms of $q^*$, Eqn. (93) becomes

$$G_b = \frac{2\sqrt{2}}{\pi} \frac{1}{\sqrt{x_o^2 - 1}} \left[\sqrt{\csc\theta\, q^*} + O\left(q^{*\ 3/2}\right)\right] U(q^*) \quad (97)$$

The term $x_o$ is given by $x_o = \cosh v_o$ and

$$\frac{1}{\sqrt{x_o^2 - 1}} = \frac{1}{\sinh v_o} = \frac{\sin\theta \sin\theta_o}{(\cos\theta_o - \cos\theta)} \quad (98)$$

The substitution of Eqn. (98) in (97) gives

$$G_b \sim \frac{2\sin\theta_o \sqrt{2\sin\theta}}{\pi(\cos\theta_o - \cos\theta)} \sqrt{q^*}\ U(q^*) \quad (99)$$

and

$$G_b \sim A(\theta, \theta_o) \sqrt{q^*} \; U(q^*) \quad , \tag{100}$$

where

$$A(\theta, \theta_o) = \frac{2 \sin \theta_o \sqrt{2 \sin \theta}}{\pi (\cos \theta_o - \cos \theta)} \tag{101}$$

<u>Behavior of the Radiation Fields at $\theta_o = \pi/2$</u>

The asymptotic form of the radiation fields for $\theta_o \to \pi/2$ can be obtained from Eqn. (39) where $\xi_b$ is given by

$$\xi_b = \frac{f_o}{2 \sin \theta} \int_{-u_o}^{u_o} \int_0^{\infty} \frac{e^{-y\eta} I_o(y)}{y\left[K_o^2(y) + \pi^2 I_o^2(y)\right]} \, dy \, du \tag{102}$$

where $\eta = \csc \theta \, (q - \cosh u + \sinh u \cos \theta)$.

As $\theta_o \to \pi/2$, $u_o \to 0$. $\eta$ can be expanded about $u = 0$ as

$$\eta = \csc \theta \, (q - 1 - u^2/2 - \cdots + u \cos \theta + u^3 \cos \theta / 6 + \cdots)$$

$\xi_b$ can now be written for small u as

$$\xi_b = \frac{1}{2 \sin \theta} \int_0^{\infty} \frac{e^{-y \csc \theta (q-1)} I_o(y)}{y\left[K_o^2(y) + \pi^2 I_o^2(y)\right]} \, dy$$

$$- f_o \frac{u_o^3 (1 - 2 \cos \theta)}{6 \sin \theta} \int_0^{\infty} \frac{e^{-y \csc \theta (q-1)} I_o(y)}{y\left[K_o^2(y) + \pi^2 I_o^2(y)\right]} \, dy + O\left(f_o u_o^5\right) \tag{103}$$

Define a G function for the delta gap source distribution given by

$$G_d(\theta, q) = \int_0^\infty \frac{e^{-y\csc\theta(q-1)}\, I_o(y)}{y\left[K_o^2(y) + \pi^2 I_o^2(y)\right]}\, dy \tag{104}$$

Eqn. (102) can now be written for $\theta_o \to \pi/2$ as

$$\xi_b = \frac{1}{2\sin\theta}\, G_d(\theta, q) + O(u_o^2) \tag{105}$$

where $2u_o f_o = 1$.

The value of $G_b$ at $\theta_o = \pi/2$ can be determined from Eqn. (68). At $\theta_o = \pi/2$, $\tau_o$ becomes

$$\tau_o' = q\cosh\nu + \sinh\nu\, \cos\theta \tag{106}$$

and

$$G_b(\theta, \pi/2, q) = \frac{1}{2\pi i}\int_0^\infty \int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{y\csc\theta\, \tau_o'}}{y\, K_o(y)}\, dy\, d\nu \tag{107}$$

To calculate $\xi_b$ at $\theta_o = \pi/2$, Eqn. (78) is applicable only for $\theta = \pi/2$. The value of $G_b$ in the general case $\theta = \theta_o$ can be obtained from Eqn. (68) as

$$G_b(\theta_o, \theta_o, q) = \frac{1}{2\pi i}\int_0^\infty \int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{y\csc\theta_o(q - \sin\theta_o\cosh\nu)}}{y\, K_o(y)}\, dy\, d\nu$$

$$= \frac{1}{2\pi i}\int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{y\csc\theta_o q}}{y}\, dy \tag{108}$$

or

$$G_b(\theta_o, \theta_o, q) = U(q \csc \theta_o) = U(t^*) \tag{109}$$

Thus, $G_b(\pi/2, \pi/2, q) = 1$ for all $q > 0$. This is a reasonable result since $2\xi_b = G_d(\pi/2, q)$ at $\theta = \theta_o = \pi/2$ and $G_d(\pi/2, q)$ is singular only at $q = 0$. But $\xi_b$ as defined by Eqn. (78) is singular for all q at $\theta_o = \pi/2$, therefore the value of $2 - 2G_b(\pi/2, \pi/2, q)$ must equal zero. For $\theta < \theta_o$ Eqn. (80) is applicable and it can be concluded that at $\theta_o = \pi/2$

$$G_b(\theta, \pi/2, q) = G_b(\pi - \theta, \pi/2, q) \tag{110}$$

This result is easily verified by Eqn. (76).

Behavior of $G_b$ at $\theta_o = 0$

The value of $G_b(\theta, 0, q)$ can be determined from Eqn. (76). For $q \csc \theta$ finite, $\zeta_o \rightarrow -\infty$ as $\theta_o \rightarrow 0$. But for $\zeta_o < 0$, $G_b$ is identically equal to zero. Thus, for $\theta \neq 0$ the value of $G_b$ is given by

$$G_b(\theta, 0, q) \equiv 0 \qquad \text{for } \theta \neq 0 \tag{111}$$

The behavior of $G_b$ as both $\theta$ and $\theta_o$ approach zero at the same rate can be determined by setting $\theta = \theta_o$ and allow $\theta_o \rightarrow 0$. For this case Eqn. (109) applies and $G_b$ can be written for $q > 0$ as

$$G_b(0, 0, q) = 1 \qquad \text{for } q > 0 \tag{112}$$

and

$$G_b\,(0,\ 0,\ 0) = 0 \qquad \text{for } q \le 0 \tag{113}$$

<u>Behavior of $G_b$ for $\theta \to 0$ and $\theta \to \pi$</u>

Equation (70) can be written as

$$G_b = 2\,U(q\csc\theta) - \frac{1}{2\pi i}\int_{-\infty}^{v_o}\int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{y(q\csc\theta - \cosh v)}}{y\,K_o(y)}\,dy\,dv$$

$$= 1 + \frac{1}{2\pi i}\int_{0}^{-v_o}\int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{y(q\csc\theta - \cosh v)}}{y\,K_o(y)}\,dy\,dv \tag{114}$$

where $q > 0$ and $v_o \to -\infty$ as $\theta \to 0$. Now make a change of variable $\eta = q - \sin\theta\cosh v$. Eqn. (114) becomes

$$G_b = 1 - \frac{1}{2\pi i}\int_{q-\sin\theta}^{q^*-\sin\theta}\int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{y\csc\theta\eta}}{y\,K_o(y)\,\sqrt{(q-\eta)^2 - \sin^2\theta}}\,dy\,d\eta$$

$$= 1 + I(\theta,\ \theta_o) \tag{115}$$

For $\theta \to \pi$, $v_o \to \infty$ and Eqn. (70) can be written as

$$G_b = U(q\csc\theta) - \frac{1}{2\pi i}\int_{0}^{v_o}\int_{\gamma-i\infty}^{\gamma'+i\infty} \frac{e^{y(q\csc\theta - \cosh v)}}{y\,K_o(y)}\,dy\,dv$$

$$= 1 - I(\theta,\ \theta_o) \tag{116}$$

For $\eta > -1$, $I(\theta, \theta_o)$ can be written as

$$I = -\int_{q-\sin\theta}^{q^*-\sin\theta} \int_0^\infty \frac{e^{-y\csc\theta\eta}\, I_o(y)}{y\left[K_o^2(y) + \pi^2 I_o^2(y)\right] \sqrt{(q-\eta)^2 - \sin^2\theta}}\, dy\, d\eta \qquad (117)$$

For $q > 0$, the asymptotic form of $G(\eta \csc\theta)$ for $\eta\csc\theta \rightarrow \infty$ as developed in Appendix A can be used. Thus, as $\theta \rightarrow 0$ or $\theta \rightarrow \pi$ Eqn. (117) becomes

$$I = -\int_{q-\sin\theta}^{q^*-\sin\theta} \frac{1}{\sqrt{(q-\eta)^2 - \sin^2\theta}}$$

$$\left[\frac{1}{\ell n(2\eta\csc\theta/\Gamma)} + O\left(\ell n^{-2}(2\eta\csc\theta/\Gamma)\right)\right] d\eta \qquad (118)$$

Now make another change of variable $\kappa = q - \eta$, Eqn. (118) becomes

$$I = \int_{\sin\theta}^{q_o+\sin\theta} \frac{1}{\sqrt{\kappa^2 - \sin^2\theta}} \left[\frac{1}{\ell n\left[2\csc\theta(q-\kappa)/\Gamma\right]}\right.$$

$$\left. + O\left(\ell n^{-2}\left[2\csc\theta(q-\kappa)/\Gamma\right]\right)\right] d\kappa \qquad (119)$$

Eqn. (119) can be rewritten as

$$I = \frac{1}{\ell n(2\csc\theta/\Gamma)} \int_{\sin\theta}^{q_o+\sin\theta} \frac{1}{\sqrt{\kappa^2 - \sin^2\theta}}$$

$$\left[1 + O\left(\frac{\ell n(q-\kappa)}{\ell n(2\csc\theta/\Gamma)}\right)\right] d\kappa$$

$$= \ell n\left[q_o \csc\theta + 1 + \sqrt{(q_o \csc\theta + 1)^2 - 1}\right]\left[\frac{1}{\ell n(2\csc\theta/\Gamma)}\right.$$

$$\left. + O\left(\ell n^{-1}(2\csc\theta/\Gamma)\right)\right] \tag{120}$$

Eqn. (120) can be reduced to

$$I = \ell n\left(x_o + \sqrt{x_o^2 - 1}\right)\left[\frac{1}{\ell n(2\csc\theta/\Gamma)} + O\left(\ell n^{-1}(2\csc\theta/\Gamma)\right)\right]$$

$$= \frac{v_o}{\ell n(2\csc\theta/\Gamma)}\left[1 + O\left(\ell n^{-1}(2\csc\theta/\Gamma)\right)\right] \tag{121}$$

The substitution of $v_o = u_o - u_1$ in Eqn. (121) gives

$$I = \left[\frac{u_o}{\ell n(2\csc\theta/\Gamma)} - \frac{u_1}{\ell n(2\csc\theta/\Gamma)}\right]\left[1 + O\left(\ell n^{-1}(2\csc\theta/\Gamma)\right)\right]$$

$$= \left[\frac{u_o}{\ell n(2\csc\theta/\Gamma)} - \kappa_1\right]\left[1 + O\left(\ell n^{-1}(2\csc\theta/\Gamma)\right)\right] \tag{122}$$

where

$$\kappa_1 = \frac{u_1}{\ell n(2\csc\theta/\Gamma)} = \frac{\ell n(\cot\theta + \csc\theta)}{\ell n(2\csc\theta/\Gamma)} \tag{123}$$

The limit of $\kappa_1$ as $\theta \to 0$ is

$$\lim_{\theta \to 0} \kappa_1 = 1 \tag{124}$$

and the limit of $\kappa_1$ as $\theta \to \pi$ is

$$\lim_{\theta \to \pi} \kappa_1 = -1 \tag{125}$$

Eqn. (115) can now be written for $\theta \to 0$ as

$$G_b = \left[(1 - \kappa_1) + \frac{u_o}{\ell n(2 \csc \theta / \Gamma)}\right] \left[1 + O\left(\ell n^{-1}(2 \csc \theta / \Gamma)\right)\right] \tag{126}$$

and as $\theta \to \pi$, Eqn. (116) can be written as

$$G_b = \left[(1 + \kappa_1) - \frac{u_o}{\ell n(2 \csc \theta / \Gamma)}\right] \left[1 + O\left(\ell n^{-1}(2 \csc \theta / \Gamma)\right)\right] \tag{127}$$

Thus, $G_b \to 0$ for $\theta \to 0$, $\theta \to \pi$ as shown in Eqns. (126) and (127).

## VIII. Numerical Solution of $G_b$

A solution of $G_b$ involves numerical integration of a double integral, Eqn. (89). Since $G_b$ is zero until $q = q_o$, the function can be defined in terms of $q^* = q - q_o$ as

$$G_b\left(\theta, \theta_o, q^*\right) = \int_0^{q^* \csc\theta} \frac{F(\zeta)}{\sqrt{\left(\zeta - q^* \csc\theta - x_o\right)^2 - 1}} \, d\zeta \tag{128}$$

or

$$G_b\left(\theta, \theta_o, q^*\right) = \int_0^{q^* \csc\theta} H(\zeta) \, d\zeta \tag{129}$$

where $H(\zeta)$ is the integrand of Eqn. (128). $H(\zeta)$ has an integrable singularity at $\zeta = 0$. This singularity caused no numerical problems for numerical integration by Gaussian Quadrature techniques.

The integrand of $F(\zeta)$ has a singularity at $y = 0$. To remove this singularity write

$$F(\zeta) = \int_0^{\infty} \frac{e^{-y(\zeta - 1)} I_o(y)}{y\left[K_o^2(y) + \pi^2 I_o^2(y)\right]} \, dy$$

$$= \int_0^{\epsilon} (\cdots)\, dy + \int_{\epsilon}^{m} (\cdots)\, dy + \int_m^{\infty} (\cdots)\, dy$$

$$= I_1 + I_2 + R_m \tag{130}$$

where $\epsilon$ is an arbitrary number greater than zero and m is some large number to truncate the integration. Define a function $\Phi(y)$ as

$$\Phi(y) = \frac{I_o(y)}{y\left[K_o^2(y) + \pi^2 I_o^2(y)\right]} \tag{131}$$

For $y \to 0$, $\Phi(y)$ has an asymptotic expansion given by

$$\Phi(y) = \frac{1}{y\left[\ell n^2(y\Gamma/2) + \pi^2\right]} \left[1 + O(y)\right] \tag{132}$$

Now write $I_1$ in the form

$$I_1 = \int_0^{\epsilon} \frac{1}{y\left[\ell n^2(y\Gamma/2) + \pi^2\right]} dy$$

$$+ \int_0^{\epsilon} \left\{ \frac{e^{-y(\zeta - 1)} I_o(y)}{K_o^2(y) + \pi^2 I_o^2(y)} - \frac{1}{\ell n^2(y\Gamma/2) + \pi^2} \right\} \frac{dy}{y} \tag{133}$$

Choose $\epsilon = 2/\Gamma$ , then

$$\int_0^{2/\Gamma} \frac{1}{y\left[\ell n^2(y\Gamma/2) + \pi^2\right]} dy = \int_{-\infty}^{0} \frac{1}{x^2 + \pi^2} dx$$

$$= 1/2 \tag{134}$$

and $I_1$ becomes

$$I_1 = 1/2 + \int_0^{2/\Gamma} \left\{ \frac{e^{-y(\zeta-1)} I_o(y)}{K_o^2(y) + \pi^2 I_o^2(y)} - \frac{1}{\ell n^2(y\Gamma/2) + \pi^2} \right\} \frac{dy}{y} \tag{135}$$

Clearly the singularity has been removed from $I_1$. The analytic part of $F(\zeta)$, $I_2$, can be integrated numerically. The remainder, $R_m$, can be approximated for large m. For large m, $R_m$ can be written as

$$R_m = \frac{\sqrt{2}}{\pi\sqrt{\pi}} \int_m^{\infty} \frac{e^{-y\zeta}}{\sqrt{y}} \left[1 - \frac{1}{8y} + O\left(y^{-2}\right)\right] dy$$

$$= \frac{\sqrt{2}}{\pi} \left[\left(\frac{4+\zeta}{4\sqrt{\zeta}}\right) \operatorname{erfc}\left(\sqrt{m\zeta}\right) - \frac{e^{-\zeta m}}{4\sqrt{\pi m}} \left(1 + O(m^{-2})\right)\right] \tag{136}$$

The results of Eqns. (130), (135), and (136) can be collected to give

$$F(\zeta) = 1/2 + \int_0^{2/\Gamma} \left\{ \frac{e^{-y(\zeta-1)} I_o(y)}{K_o^2(y) + \pi^2 I_o^2(y)} - \frac{1}{\ell n^2(y\Gamma/2) + \pi^2} \right\} \frac{dy}{y}$$

$$+ \int_{2/\Gamma}^{m} \frac{e^{-y(\zeta-1)} I_o(y)}{y\left[K_o^2(y) + \pi^2 I_o^2(y)\right]} dy + R_m \tag{137}$$

There was no significant change in the value of $F(\zeta)$ for m greater than 5.

Approximation of $G_b$

The function $G_b$ can be written exactly as

$$G_b(\theta, \theta_o, q^*) = \int_0^{q^* \csc\theta} H(\zeta)\ d\zeta \tag{138}$$

To reduce the computation time required for $G_b$, the function $F(\zeta)$ can be approximated by a series given by

$$F(\zeta) \simeq P(\zeta) = \sum_{m=0}^{n} a_m\, \zeta^{m+k} \tag{139}$$

where k = 0 or k = - 1/2 depending on the range of $\zeta$.

The approximation of $F(\zeta)$ is developed in detail in Appendix C. $H(\zeta)$ can be approximated by

$$H(\zeta) \simeq \frac{P(\zeta)}{\sqrt{\left(\zeta - q^* \csc\theta - x_o\right)^2 - 1}} \tag{140}$$

The substitution of Eqn. (140) into Eqn. (138) gives an approximation for $G_b$ for numerical evaluation.

Accuracy of the Numerical Solution

Error can be introduced in the numerical solution of $G_b$ by the numerical calculations, the truncation of $F(\zeta)$ and the approximation of $F(\zeta)$. The relative error resulting from the numerical calculations is less than $10^{-4}$. If the maximum truncation error of $F(\zeta)$ is $\Delta_t$ and the maximum approximation error is $\Delta_m$, the $G_b$ function can be expressed as

$$G_b(1 \pm \Delta_b) = \int_0^{q^* \csc\theta} \frac{F(\zeta)\left(1 \pm \Delta_t\right)\left(1 \pm \Delta_m\right)}{\sqrt{\left(\zeta - q^* \csc\theta - x_o\right)^2 - 1}} \, d\zeta \qquad (141)$$

where $\Delta_b$ is the maximum relative error of $G_b$. The maximum relative error of $G_b$ can be estimated as

$$\Delta_b = \left(\Delta_t + \Delta_m + \Delta_c + \Delta_c\right) \qquad (142)$$

where $\Delta_c$ is the error due to numerical integration.

The maximum truncation error can be approximated by the next term in the series given in Eqn. (136), thus

$$\Delta_t \leq \frac{\sqrt{2}}{\pi\sqrt{\pi}} \frac{9}{128} \frac{1}{F(\zeta)} \int_m^{\infty} \frac{e^{-y\zeta}}{y^2 \sqrt{y}} \, dy$$

$$= \frac{\sqrt{2}}{\pi\sqrt{\pi}} \frac{3}{(64)\,F(\zeta)} \left[\frac{e^{-m\zeta}}{m\sqrt{m}} - \frac{\zeta}{2} \frac{e^{-m\zeta}}{\sqrt{m}} + \zeta^2 \frac{\sqrt{\zeta\pi}}{2} \operatorname{erfc}\left(\sqrt{m\zeta}\right)\right] \qquad (143)$$

The truncation error function $\Delta_t(\zeta)$ was calculated for selected values of $\zeta$ in the range of $0 \leq \zeta \leq 10{,}000.0$. The maximum value of $\Delta_t = 3.6 \times 10^{-4}$ occurred at $\zeta = 0.06$. The maximum $\Delta_m = 3.5 \times 10^{-4}$ is given in Appendix C. From Eqn. (142) it is seen that $\Delta_b$ is in the order of $10^{-3}$. This value of $\Delta_b$ compares favorably with the observed maximum $\Delta_b = 1.1 \times 10^{-3}$ for $G_b\left(\theta_o, \theta_o, q\right)$ compared with the theoretical value of $G_b\left(\theta_o, \theta_o, q\right) = 1.0$.

## IX. Results

Equation (128) was numerically evaluated for a wide range of $\theta$, $\theta_o$, and $q^*$. The resulting values of $G_b$ are tabulated in Tables 1 through 12. Tables 1 through 9 define $G_b$ for $0 < \theta < \pi$ and $0 < \theta_o < \pi/2$. Tables 10, 11, and 12 define $G_b$ for ratios of $h_s/a$ with $\theta = \pi/2$, $\pi/3$, and $\pi/18$, respectively.

The equivalent bicone voltage $V_{bo}$ as given in Eqn. (17) is a function of $\theta_o$. The equivalent bicone voltage normalized by the product of the maximum surface electric field and the radius is shown in Figure 9 for a wide range of $\theta_o$.

Figure 10 shows the variation in $q_o$, the normalized retarded time that the radiated field is distorted by the ends of the distributed source, as a function of $\theta$ and the parameter $\theta_o$.

Figure 11 shows the relative magnitude of the field deviation from the initial time-independent field at normalized retarded time just greater than $q_o$. The relative magnitude $A(\theta, \theta_o)$ is plotted as a function of $\theta$ and the parameter $\theta_o$.

The normalized radiated field $\sin\theta\ \xi_b$ is presented in Figures 12 through 20 for a wide range of $\theta$ and $\theta_o$. Each figure is divided into small time and intermediate time plots for clarity. The small time asymptotes* for the first distortion in the radiated field associated with the source surface field discontinuities are indicated by broken lines except where the actual field and the asymptotes are indistinguishable. The normalized radiated field at late time is independent of $\theta_o$ as seen by Eqn. (87). In fact, Eqn. (87) is the same asymptotic form developed for the delta gap source distribution in Reference 2. The plots for $\sin\theta\ \xi_b$ at late time

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* The small time asymptotes were calculated from Eqn. (78), (79), or (80) with the substitution of Eqn. (100) for $G_b$.

along with the late time asymptotes indicated by broken lines are presented in Figure 21. The late time asymptotes were calculated from Eqn. (87).

For the application of $\xi_b$ to the un-normalized radiated field, $E_\theta$, one must keep in mind that $V_{bo}$ is a function of $\theta_o$. For $\theta \geq \theta_o$, $E_\theta$ is independent of $\theta_o$ initially whereas the late time $E_\theta$ is dependent on $\theta_o$ considering the source field and the source radius held fixed. This is the opposite functional relationship of $\xi_b$ and $\theta_o$. In order to get a feel for the behavior of $E_\theta$, an example problem is considered where $E_{sm}$ = one megavolt per meter, a = 5 meters, $h_s$ = 10 meters, and $V_{bo}$ = 14.4 megavolts. The values of $rE_\theta$ are presented in Figure 22 for both small and late time.

## X. Summary

In this note, the concept of driving an infinitely long cylindrical antenna with a cylindrical distributed source region has been considered. In particular, the finite distributed source for radiating a fast rising spherical TEM wave was specified by the tangential components of a spherical wave associated with a biconical antenna with a step-function applied voltage. The exact expressions for the far zone fields radiated by an infinitely long cylindrical antenna with the above specified distributed source were developed. It was shown that the time history of the radiation fields for $\theta_o \leq \theta \leq \pi - \theta_o$ is initially the exact time history of the fields radiated by the biconical antenna used to specify the distributed source. It was found that the late time behavior of the radiation fields is inversely proportional to the logarithm of time. Also, it was found that the small time behavior of the radiation fields associated with the surface field discontinuities at the ends of the distributed source decays proportionally to the square root of time.

As an extension to this note, one could consider near zone fields, antenna current, an approximation of the radiated fields for a finite length antenna, and the effects of a distributed source consisting of an array of capacitors and switches. Also, the distributed source driving a cylindrical antenna concept could be extended to include other source field distributions in magnitude and time.

Table 1a. Values of $G_b$ for $2\theta_o/\pi = 0.1$.

| $q^*$ \ $\theta/\pi$ | .1 | .2 | .3 | .4 | .5 |
|---|---|---|---|---|---|
| .00010 | .0213 | .0060 | .0032 | .0020 | .0014 |
| .00015 | .0261 | .0074 | .0039 | .0025 | .0017 |
| .00020 | .0302 | .0085 | .0045 | .0029 | .0020 |
| .00030 | .0369 | .0105 | .0055 | .0035 | .0025 |
| .00050 | .0476 | .0135 | .0071 | .0045 | .0032 |
| .00070 | .0563 | .0160 | .0084 | .0053 | .0038 |
| .00100 | .0672 | .0191 | .0100 | .0064 | .0045 |
| .00150 | .0822 | .0234 | .0123 | .0078 | .0055 |
| .00200 | .0947 | .0270 | .0142 | .0090 | .0064 |
| .00300 | .1154 | .0330 | .0173 | .0111 | .0078 |
| .00500 | .1478 | .0426 | .0224 | .0143 | .0101 |
| .00700 | .1734 | .0503 | .0265 | .0169 | .0119 |
| .01000 | .2048 | .0601 | .0316 | .0202 | .0142 |
| .01500 | .2459 | .0734 | .0387 | .0247 | .0174 |
| .02000 | .2787 | .0845 | .0446 | .0286 | .0201 |
| .03000 | .3296 | .1030 | .0546 | .0350 | .0247 |
| .05000 | .3999 | .1317 | .0702 | .0451 | .0318 |
| .07000 | .4481 | .1544 | .0828 | .0533 | .0376 |
| .10000 | .4992 | .1820 | .0985 | .0635 | .0450 |
| .15000 | .5550 | .2180 | .1196 | .0775 | .0550 |
| .20000 | .5921 | .2465 | .1370 | .0892 | .0634 |
| .30000 | .6396 | .2905 | .1653 | .1085 | .0775 |
| .50000 | .6909 | .3504 | .2073 | .1383 | .0995 |
| .70000 | .7195 | .3913 | .2389 | .1615 | .1170 |
| 1.0000 | .7458 | .4346 | .2752 | .1894 | .1386 |
| 1.5000 | .7714 | .4821 | .3193 | .2252 | .1673 |
| 2.0000 | .7872 | .5140 | .3516 | .2530 | .1902 |
| 3.0000 | .8066 | .5558 | .3974 | .2947 | .2262 |
| 5.0000 | .8272 | .6027 | .4534 | .3498 | .2763 |
| 7.0000 | .8388 | .6300 | .4882 | .3861 | .3111 |
| 10.000 | .8496 | .6561 | .5227 | .4237 | .3486 |
| 15.000 | .8606 | .6825 | .5585 | .4644 | .3907 |
| 20.000 | .8674 | .6993 | .5818 | .4915 | .4195 |
| 30.000 | .8762 | .7206 | .6117 | .5269 | .4581 |
| 50.000 | .8858 | .7438 | .6447 | .5668 | .5025 |
| 70.000 | .8914 | .7574 | .6641 | .5904 | .5291 |
| 100.00 | .8968 | .7704 | .6827 | .6132 | .5550 |
| 150.00 | .9024 | .7837 | .7016 | .6366 | .5818 |
| 200.00 | .9060 | .7923 | .7139 | .6518 | .5992 |
| 300.00 | .9106 | .8034 | .7297 | .6713 | .6216 |
| 500.00 | .9159 | .8159 | .7475 | .6932 | .6469 |
| 700.00 | .9191 | .8233 | .7580 | .7062 | .6619 |
| 1000.0 | .9222 | .8306 | .7683 | .7189 | .6766 |

Table 1b. Values of $G_b$ for $2\,\theta_o/\pi = 0.1$.

| q* \ θ/π | .5 | .6 | .7 | .8 | .9 |
|---|---|---|---|---|---|
| .00010 | .0014 | .0011 | .0008 | .0006 | .0004 |
| .00015 | .0017 | .0013 | .0010 | .0007 | .0005 |
| .00020 | .0020 | .0015 | .0011 | .0008 | .0006 |
| .00030 | .0025 | .0018 | .0014 | .0010 | .0007 |
| .00050 | .0032 | .0024 | .0018 | .0013 | .0009 |
| .00070 | .0038 | .0028 | .0021 | .0016 | .0011 |
| .00100 | .0045 | .0033 | .0025 | .0019 | .0013 |
| .00150 | .0055 | .0041 | .0031 | .0023 | .0016 |
| .00200 | .0064 | .0047 | .0036 | .0027 | .0018 |
| .00300 | .0078 | .0058 | .0044 | .0033 | .0022 |
| .00500 | .0101 | .0075 | .0057 | .0042 | .0029 |
| .00700 | .0119 | .0089 | .0067 | .0050 | .0034 |
| .01000 | .0142 | .0106 | .0080 | .0060 | .0040 |
| .01500 | .0174 | .0130 | .0098 | .0074 | .0050 |
| .02000 | .0201 | .0150 | .0114 | .0085 | .0057 |
| .03000 | .0247 | .0183 | .0139 | .0104 | .0070 |
| .05000 | .0318 | .0237 | .0180 | .0135 | .0091 |
| .07000 | .0376 | .0280 | .0213 | .0160 | .0108 |
| .10000 | .0450 | .0335 | .0255 | .0191 | .0130 |
| .15000 | .0550 | .0410 | .0313 | .0235 | .0161 |
| .20000 | .0634 | .0474 | .0362 | .0273 | .0187 |
| .30000 | .0775 | .0580 | .0444 | .0336 | .0233 |
| .50000 | .0995 | .0749 | .0576 | .0439 | .0308 |
| .70000 | .1170 | .0886 | .0684 | .0524 | .0373 |
| 1.0000 | .1386 | .1056 | .0821 | .0634 | .0457 |
| 1.5000 | .1673 | .1287 | .1009 | .0787 | .0577 |
| 2.0000 | .1902 | .1477 | .1167 | .0917 | .0681 |
| 3.0000 | .2262 | .1782 | .1425 | .1135 | .0859 |
| 5.0000 | .2763 | .2225 | .1813 | .1472 | .1142 |
| 7.0000 | .3111 | .2547 | .2104 | .1730 | .1364 |
| 10.000 | .3486 | .2903 | .2435 | .2032 | .1630 |
| 15.000 | .3907 | .3317 | .2830 | .2400 | .1961 |
| 20.000 | .4195 | .3608 | .3115 | .2672 | .2211 |
| 30.000 | .4581 | .4007 | .3512 | .3058 | .2572 |
| 50.000 | .5025 | .4476 | .3991 | .3533 | .3027 |
| 70.000 | .5291 | .4762 | .4288 | .3832 | .3317 |
| 100.00 | .5550 | .5043 | .4582 | .4131 | .3613 |
| 150.00 | .5818 | .5335 | .4891 | .4449 | .3930 |
| 200.00 | .5992 | .5526 | .5094 | .4659 | .4141 |
| 300.00 | .6216 | .5774 | .5358 | .4935 | .4420 |
| 500.00 | .6469 | .6053 | .5658 | .5249 | .4742 |
| 700.00 | .6619 | .6220 | .5838 | .5438 | .4937 |
| 1000.0 | .6766 | .6382 | .6013 | .5624 | .5129 |

Table 2a. Values of $G_b$ for $2\,\theta_o/\pi = 0.2$.

| q* \ θ/π | .1 | .2 | .3 | .4 | .5 |
|---|---|---|---|---|---|
| .00010 | 1.0000 | .0150 | .0069 | .0042 | .0029 |
| .00015 | 1.0000 | .0184 | .0084 | .0052 | .0036 |
| .00020 | 1.0000 | .0212 | .0097 | .0060 | .0041 |
| .00030 | 1.0000 | .0260 | .0119 | .0073 | .0051 |
| .00050 | 1.0000 | .0335 | .0154 | .0094 | .0065 |
| .00070 | 1.0000 | .0396 | .0182 | .0112 | .0077 |
| .00100 | 1.0000 | .0473 | .0218 | .0133 | .0092 |
| .00150 | 1.0000 | .0579 | .0266 | .0163 | .0113 |
| .00200 | 1.0000 | .0668 | .0307 | .0189 | .0131 |
| .00300 | 1.0000 | .0816 | .0376 | .0231 | .0160 |
| .00500 | 1.0000 | .1050 | .0485 | .0298 | .0207 |
| .00700 | 1.0000 | .1237 | .0574 | .0353 | .0244 |
| .01000 | 1.0000 | .1469 | .0684 | .0421 | .0292 |
| .01500 | 1.0000 | .1780 | .0836 | .0515 | .0357 |
| .02000 | 1.0000 | .2035 | .0962 | .0594 | .0412 |
| .03000 | 1.0000 | .2444 | .1172 | .0726 | .0504 |
| .05000 | 1.0000 | .3043 | .1496 | .0932 | .0649 |
| .07000 | 1.0000 | .3482 | .1752 | .1097 | .0766 |
| .10000 | 1.0000 | .3975 | .2061 | .1302 | .0911 |
| .15000 | 1.0000 | .4555 | .2463 | .1575 | .1109 |
| .20000 | 1.0000 | .4964 | .2778 | .1797 | .1272 |
| .30000 | 1.0000 | .5521 | .3259 | .2152 | .1537 |
| .50000 | 1.0000 | .6158 | .3904 | .2665 | .1935 |
| .70000 | 1.0000 | .6528 | .4337 | .3037 | .2237 |
| 1.0000 | 1.0000 | .6875 | .4786 | .3452 | .2588 |
| 1.5000 | 1.0000 | .7216 | .5269 | .3935 | .3020 |
| 2.0000 | 1.0000 | .7425 | .5589 | .4277 | .3340 |
| 3.0000 | 1.0000 | .7682 | .6000 | .4743 | .3800 |
| 5.0000 | 1.0000 | .7952 | .6453 | .5288 | .4371 |
| 7.0000 | 1.0000 | .8103 | .6713 | .5616 | .4729 |
| 10.000 | 1.0000 | .8243 | .6960 | .5933 | .5087 |
| 15.000 | 1.0000 | .8383 | .7205 | .6256 | .5461 |
| 20.000 | 1.0000 | .8470 | .7361 | .6463 | .5704 |
| 30.000 | 1.0000 | .8580 | .7557 | .6727 | .6018 |
| 50.000 | 1.0000 | .8700 | .7770 | .7015 | .6365 |
| 70.000 | 1.0000 | .8769 | .7894 | .7182 | .6567 |
| 100.00 | 1.0000 | .8835 | .8012 | .7342 | .6762 |
| 150.00 | 1.0000 | .8903 | .8132 | .7505 | .6961 |
| 200.00 | 1.0000 | .8947 | .8209 | .7610 | .7089 |
| 300.00 | 1.0000 | .9004 | .8308 | .7745 | .7254 |
| 500.00 | 1.0000 | .9067 | .8420 | .7896 | .7439 |
| 700.00 | 1.0000 | .9105 | .8486 | .7986 | .7548 |
| 1000.0 | 1.0000 | .9141 | .8551 | .8073 | .7655 |

Table 2b. Values of $G_b$ for $2\ \theta_o/\pi = 0.2$.

| q* \ θ/π | .5 | .6 | .7 | .8 | .9 |
|---|---|---|---|---|---|
| .00010 | .0029 | .0022 | .0016 | .0012 | .0008 |
| .00015 | .0036 | .0026 | .0020 | .0015 | .0010 |
| .00020 | .0041 | .0030 | .0023 | .0017 | .0011 |
| .00030 | .0051 | .0037 | .0028 | .0021 | .0014 |
| .00050 | .0065 | .0048 | .0036 | .0027 | .0018 |
| .00070 | .0077 | .0057 | .0043 | .0032 | .0021 |
| .00100 | .0092 | .0068 | .0051 | .0038 | .0026 |
| .00150 | .0113 | .0083 | .0063 | .0047 | .0031 |
| .00200 | .0131 | .0096 | .0073 | .0054 | .0036 |
| .00300 | .0160 | .0118 | .0089 | .0066 | .0045 |
| .00500 | .0207 | .0152 | .0115 | .0086 | .0057 |
| .00700 | .0244 | .0180 | .0136 | .0101 | .0068 |
| .01000 | .0292 | .0215 | .0162 | .0121 | .0081 |
| .01500 | .0357 | .0263 | .0199 | .0148 | .0100 |
| .02000 | .0412 | .0304 | .0230 | .0171 | .0115 |
| .03000 | .0504 | .0372 | .0281 | .0210 | .0141 |
| .05000 | .0649 | .0479 | .0363 | .0271 | .0183 |
| .07000 | .0766 | .0566 | .0429 | .0321 | .0217 |
| .10000 | .0911 | .0675 | .0512 | .0384 | .0261 |
| .15000 | .1109 | .0824 | .0626 | .0470 | .0321 |
| .20000 | .1272 | .0947 | .0722 | .0544 | .0373 |
| .30000 | .1537 | .1151 | .0881 | .0666 | .0462 |
| .50000 | .1935 | .1464 | .1129 | .0861 | .0606 |
| .70000 | .2237 | .1707 | .1325 | .1018 | .0725 |
| 1.0000 | .2588 | .1999 | .1565 | .1214 | .0878 |
| 1.5000 | .3020 | .2370 | .1880 | .1477 | .1088 |
| 2.0000 | .3340 | .2655 | .2129 | .1690 | .1263 |
| 3.0000 | .3800 | .3081 | .2512 | .2026 | .1546 |
| 5.0000 | .4371 | .3637 | .3032 | .2501 | .1961 |
| 7.0000 | .4729 | .4002 | .3387 | .2834 | .2261 |
| 10.000 | .5087 | .4376 | .3760 | .3194 | .2594 |
| 15.000 | .5461 | .4778 | .4172 | .3601 | .2979 |
| 20.000 | .5704 | .5045 | .4451 | .3882 | .3250 |
| 30.000 | .6018 | .5392 | .4820 | .4259 | .3622 |
| 50.000 | .6365 | .5783 | .5241 | .4697 | .4061 |
| 70.000 | .6567 | .6014 | .5491 | .4961 | .4330 |
| 100.00 | .6762 | .6236 | .5735 | .5220 | .4596 |
| 150.00 | .6961 | .6464 | .5987 | .5489 | .4876 |
| 200.00 | .7089 | .6612 | .6150 | .5665 | .5060 |
| 300.00 | .7254 | .6802 | .6361 | .5893 | .5301 |
| 500.00 | .7439 | .7015 | .6599 | .6151 | .5575 |
| 700.00 | .7548 | .7142 | .6740 | .6305 | .5741 |
| 1000.0 | .7655 | .7265 | .6879 | .6457 | .5904 |

Table 3a. Values of $G_b$ for $2\,\theta_o/\pi = 0.3$.

| q* \ θ/π | .1 | .2 | .3 | .4 | .5 |
|---|---|---|---|---|---|
| .00010 | .0378 | .0381 | .0121 | .0068 | .0046 |
| .00015 | .0462 | .0467 | .0148 | .0084 | .0056 |
| .00020 | .0533 | .0539 | .0171 | .0097 | .0065 |
| .00030 | .0652 | .0659 | .0210 | .0119 | .0079 |
| .00050 | .0840 | .0849 | .0271 | .0153 | .0102 |
| .00070 | .0992 | .1002 | .0320 | .0181 | .0121 |
| .00100 | .1181 | .1193 | .0383 | .0216 | .0145 |
| .00150 | .1438 | .1452 | .0468 | .0265 | .0177 |
| .00200 | .1650 | .1667 | .0540 | .0306 | .0205 |
| .00300 | .1998 | .2019 | .0661 | .0374 | .0251 |
| .00500 | .2522 | .2549 | .0850 | .0483 | .0324 |
| .00700 | .2922 | .2954 | .1003 | .0570 | .0383 |
| .01000 | .3391 | .3428 | .1194 | .0681 | .0457 |
| .01500 | .3969 | .4015 | .1452 | .0831 | .0559 |
| .02000 | .4401 | .4453 | .1666 | .0957 | .0644 |
| .03000 | .5019 | .5083 | .2013 | .1166 | .0787 |
| .05000 | .5780 | .5860 | .2533 | .1490 | .1010 |
| .07000 | .6248 | .6340 | .2925 | .1744 | .1189 |
| .10000 | .6702 | .6808 | .3380 | .2054 | .1409 |
| .15000 | .7155 | .7279 | .3935 | .2456 | .1702 |
| .20000 | .7436 | .7571 | .4342 | .2773 | .1940 |
| .30000 | .7776 | .7926 | .4916 | .3257 | .2317 |
| .50000 | .8121 | .8286 | .5606 | .3910 | .2858 |
| .70000 | .8306 | .8477 | .6021 | .4349 | .3245 |
| 1.0000 | .8472 | .8646 | .6420 | .4805 | .3674 |
| 1.5000 | .8632 | .8806 | .6817 | .5298 | .4166 |
| 2.0000 | .8728 | .8901 | .7064 | .5623 | .4510 |
| 3.0000 | .8846 | .9015 | .7368 | .6042 | .4975 |
| 5.0000 | .8970 | .9132 | .7687 | .6501 | .5512 |
| 7.0000 | .9040 | .9198 | .7865 | .6765 | .5831 |
| 10.000 | .9105 | .9258 | .8030 | .7012 | .6139 |
| 15.000 | .9170 | .9317 | .8193 | .7260 | .6451 |
| 20.000 | .9211 | .9354 | .8296 | .7417 | .6650 |
| 30.000 | .9263 | .9401 | .8424 | .7613 | .6902 |
| 50.000 | .9320 | .9452 | .8562 | .7826 | .7178 |
| 70.000 | .9354 | .9481 | .8642 | .7949 | .7337 |
| 100.00 | .9386 | .9509 | .8719 | .8066 | .7490 |
| 150.00 | .9419 | .9538 | .8796 | .8185 | .7645 |
| 200.00 | .9441 | .9556 | .8846 | .8262 | .7745 |
| 300.00 | .9468 | .9580 | .8910 | .8360 | .7873 |
| 500.00 | .9500 | .9607 | .8982 | .8471 | .8017 |
| 700.00 | .9519 | .9623 | .9025 | .8536 | .8101 |
| 1000.0 | .9537 | .9638 | .9066 | .8599 | .8184 |

Table 3b. Values of $G_b$ for $2\ \theta_o/\pi = 0.3$.

| q* \ θ/π | .5 | .6 | .7 | .8 | .9 |
|---|---|---|---|---|---|
| .00010 | .0046 | .0033 | .0025 | .0018 | .0012 |
| .00015 | .0056 | .0041 | .0030 | .0023 | .0015 |
| .00020 | .0065 | .0047 | .0035 | .0026 | .0017 |
| .00030 | .0079 | .0057 | .0043 | .0032 | .0021 |
| .00050 | .0102 | .0074 | .0056 | .0041 | .0028 |
| .00070 | .0121 | .0088 | .0066 | .0049 | .0033 |
| .00100 | .0145 | .0105 | .0079 | .0058 | .0039 |
| .00150 | .0177 | .0129 | .0096 | .0071 | .0048 |
| .00200 | .0205 | .0148 | .0111 | .0082 | .0055 |
| .00300 | .0251 | .0182 | .0136 | .0101 | .0068 |
| .00500 | .0324 | .0234 | .0176 | .0130 | .0087 |
| .00700 | .0383 | .0277 | .0208 | .0154 | .0103 |
| .01000 | .0457 | .0331 | .0248 | .0184 | .0123 |
| .01500 | .0559 | .0405 | .0304 | .0225 | .0151 |
| .02000 | .0644 | .0468 | .0351 | .0260 | .0175 |
| .03000 | .0787 | .0572 | .0429 | .0319 | .0214 |
| .05000 | .1010 | .0736 | .0553 | .0411 | .0277 |
| .07000 | .1189 | .0867 | .0652 | .0486 | .0328 |
| .10000 | .1409 | .1031 | .0777 | .0580 | .0393 |
| .15000 | .1702 | .1253 | .0947 | .0709 | .0484 |
| .20000 | .1940 | .1434 | .1088 | .0817 | .0560 |
| .30000 | .2317 | .1729 | .1319 | .0996 | .0690 |
| .50000 | .2858 | .2165 | .1670 | .1274 | .0896 |
| .70000 | .3245 | .2492 | .1940 | .1492 | .1064 |
| 1.0000 | .3674 | .2866 | .2258 | .1757 | .1273 |
| 1.5000 | .4166 | .3318 | .2656 | .2098 | .1552 |
| 2.0000 | .4510 | .3648 | .2958 | .2365 | .1776 |
| 3.0000 | .4975 | .4113 | .3398 | .2766 | .2125 |
| 5.0000 | .5512 | .4679 | .3958 | .3298 | .2605 |
| 7.0000 | .5831 | .5028 | .4316 | .3650 | .2935 |
| 10.000 | .6139 | .5372 | .4680 | .4016 | .3286 |
| 15.000 | .6451 | .5729 | .5064 | .4412 | .3676 |
| 20.000 | .6650 | .5961 | .5317 | .4677 | .3942 |
| 30.000 | .6902 | .6256 | .5644 | .5025 | .4297 |
| 50.000 | .7178 | .6583 | .6010 | .5420 | .4707 |
| 70.000 | .7337 | .6773 | .6225 | .5654 | .4953 |
| 100.00 | .7490 | .6955 | .6433 | .5881 | .5195 |
| 150.00 | .7645 | .7141 | .6646 | .6116 | .5448 |
| 200.00 | .7745 | .7262 | .6783 | .6269 | .5613 |
| 300.00 | .7873 | .7416 | .6961 | .6467 | .5828 |
| 500.00 | .8017 | .7589 | .7160 | .6689 | .6073 |
| 700.00 | .8101 | .7691 | .7278 | .6823 | .6221 |
| 1000.0 | .8184 | .7791 | .7394 | .6953 | .6365 |

Table 4a. Values of $G_b$ for $2\theta_o/\pi = 0.4$.

| q* \ θ/π | .1 | .2 | .3 | .4 | .5 |
|---|---|---|---|---|---|
| .00010 | .0207 | 1.0000 | .0215 | .0103 | .0065 |
| .00015 | .0253 | 1.0000 | .0263 | .0126 | .0080 |
| .00020 | .0292 | 1.0000 | .0304 | .0146 | .0092 |
| .00030 | .0358 | 1.0000 | .0372 | .0179 | .0113 |
| .00050 | .0462 | 1.0000 | .0480 | .0230 | .0146 |
| .00070 | .0546 | 1.0000 | .0567 | .0273 | .0173 |
| .00100 | .0652 | 1.0000 | .0677 | .0326 | .0207 |
| .00150 | .0796 | 1.0000 | .0828 | .0399 | .0253 |
| .00200 | .0918 | 1.0000 | .0954 | .0460 | .0292 |
| .00300 | .1119 | 1.0000 | .1163 | .0563 | .0357 |
| .00500 | .1434 | 1.0000 | .1490 | .0725 | .0461 |
| .00700 | .1683 | 1.0000 | .1750 | .0856 | .0545 |
| .01000 | .1988 | 1.0000 | .2068 | .1020 | .0650 |
| .01500 | .2390 | 1.0000 | .2487 | .1243 | .0794 |
| .02000 | .2711 | 1.0000 | .2822 | .1428 | .0915 |
| .03000 | .3211 | 1.0000 | .3346 | .1731 | .1115 |
| .05000 | .3905 | 1.0000 | .4074 | .2192 | .1426 |
| .07000 | .4384 | 1.0000 | .4580 | .2545 | .1671 |
| .10000 | .4893 | 1.0000 | .5121 | .2962 | .1969 |
| .15000 | .5454 | 1.0000 | .5719 | .3482 | .2359 |
| .20000 | .5827 | 1.0000 | .6120 | .3873 | .2667 |
| .30000 | .6308 | 1.0000 | .6638 | .4440 | .3141 |
| .50000 | .6829 | 1.0000 | .7197 | .5147 | .3785 |
| .70000 | .7121 | 1.0000 | .7507 | .5586 | .4222 |
| 1.0000 | .7391 | 1.0000 | .7789 | .6017 | .4680 |
| 1.5000 | .7653 | 1.0000 | .8056 | .6454 | .5178 |
| 2.0000 | .7814 | 1.0000 | .8217 | .6729 | .5509 |
| 3.0000 | .8014 | 1.0000 | .8410 | .7070 | .5937 |
| 5.0000 | .8224 | 1.0000 | .8609 | .7430 | .6408 |
| 7.0000 | .8344 | 1.0000 | .8718 | .7631 | .6679 |
| 10.000 | .8455 | 1.0000 | .8819 | .7818 | .6934 |
| 15.000 | .8568 | 1.0000 | .8917 | .8002 | .7190 |
| 20.000 | .8638 | 1.0000 | .8980 | .8118 | .7351 |
| 30.000 | .8728 | 1.0000 | .9056 | .8262 | .7554 |
| 50.000 | .8826 | 1.0000 | .9140 | .8419 | .7774 |
| 70.000 | .8884 | 1.0000 | .9188 | .8509 | .7900 |
| 100.00 | .8940 | 1.0000 | .9234 | .8594 | .8021 |
| 150.00 | .8997 | 1.0000 | .9280 | .8681 | .8144 |
| 200.00 | .9034 | 1.0000 | .9310 | .8737 | .8223 |
| 300.00 | .9082 | 1.0000 | .9348 | .8808 | .8324 |
| 500.00 | .9136 | 1.0000 | .9391 | .8888 | .8437 |
| 700.00 | .9169 | 1.0000 | .9417 | .8936 | .8504 |
| 1000.0 | .9201 | 1.0000 | .9442 | .8982 | .8569 |

Table 4b. Values of $G_b$ for $2\theta_o/\pi = 0.4$.

| q* \ θ/π | .5 | .6 | .7 | .8 | .9 |
|---|---|---|---|---|---|
| .00010 | .0065 | .0046 | .0034 | .0025 | .0017 |
| .00015 | .0080 | .0056 | .0042 | .0031 | .0020 |
| .00020 | .0092 | .0065 | .0048 | .0035 | .0024 |
| .00030 | .0113 | .0080 | .0059 | .0043 | .0029 |
| .00050 | .0146 | .0103 | .0076 | .0056 | .0037 |
| .00070 | .0173 | .0122 | .0090 | .0066 | .0044 |
| .00100 | .0207 | .0146 | .0108 | .0079 | .0053 |
| .00150 | .0253 | .0179 | .0132 | .0097 | .0065 |
| .00200 | .0292 | .0206 | .0152 | .0112 | .0075 |
| .00300 | .0357 | .0252 | .0186 | .0137 | .0091 |
| .00500 | .0461 | .0326 | .0241 | .0177 | .0118 |
| .00700 | .0545 | .0385 | .0284 | .0209 | .0140 |
| .01000 | .0650 | .0460 | .0340 | .0250 | .0167 |
| .01500 | .0794 | .0562 | .0416 | .0306 | .0205 |
| .02000 | .0915 | .0648 | .0480 | .0354 | .0236 |
| .03000 | .1115 | .0792 | .0586 | .0433 | .0290 |
| .05000 | .1426 | .1016 | .0754 | .0557 | .0374 |
| .07000 | .1671 | .1195 | .0889 | .0658 | .0443 |
| .10000 | .1969 | .1417 | .1057 | .0784 | .0530 |
| .15000 | .2359 | .1711 | .1283 | .0955 | .0650 |
| .20000 | .2667 | .1950 | .1468 | .1098 | .0751 |
| .30000 | .3141 | .2328 | .1768 | .1331 | .0920 |
| .50000 | .3785 | .2869 | .2210 | .1684 | .1184 |
| .70000 | .4222 | .3257 | .2539 | .1955 | .1395 |
| 1.0000 | .4680 | .3684 | .2916 | .2274 | .1651 |
| 1.5000 | .5178 | .4175 | .3367 | .2672 | .1982 |
| 2.0000 | .5509 | .4517 | .3695 | .2972 | .2241 |
| 3.0000 | .5937 | .4980 | .4156 | .3407 | .2630 |
| 5.0000 | .6408 | .5514 | .4713 | .3957 | .3143 |
| 7.0000 | .6679 | .5831 | .5055 | .4307 | .3483 |
| 10.000 | .6934 | .6136 | .5393 | .4660 | .3833 |
| 15.000 | .7190 | .6447 | .5743 | .5034 | .4214 |
| 20.000 | .7351 | .6645 | .5969 | .5279 | .4468 |
| 30.000 | .7554 | .6895 | .6258 | .5597 | .4803 |
| 50.000 | .7774 | .7170 | .6578 | .5953 | .5185 |
| 70.000 | .7900 | .7329 | .6764 | .6163 | .5413 |
| 100.00 | .8021 | .7480 | .6944 | .6366 | .5635 |
| 150.00 | .8144 | .7635 | .7127 | .6575 | .5866 |
| 200.00 | .8223 | .7735 | .7245 | .6710 | .6017 |
| 300.00 | .8324 | .7863 | .7398 | .6885 | .6213 |
| 500.00 | .8437 | .8006 | .7568 | .7081 | .6436 |
| 700.00 | .8504 | .8091 | .7670 | .7199 | .6570 |
| 1000.0 | .8569 | .8174 | .7769 | .7314 | .6702 |

Table 5a. Values of $G_b$ for $2\ \theta_o/\pi = 0.5$.

| q* \ θ/π | .1 | .2 | .3 | .4 | .5 |
|---|---|---|---|---|---|
| .00010 | .0145 | .0478 | .0479 | .0156 | .0090 |
| .00015 | .0177 | .0584 | .0586 | .0191 | .0110 |
| .00020 | .0205 | .0674 | .0675 | .0220 | .0127 |
| .00030 | .0251 | .0824 | .0826 | .0270 | .0156 |
| .00050 | .0324 | .1060 | .1062 | .0348 | .0201 |
| .00070 | .0383 | .1249 | .1252 | .0412 | .0238 |
| .00100 | .0457 | .1485 | .1488 | .0492 | .0284 |
| .00150 | .0559 | .1802 | .1806 | .0601 | .0348 |
| .00200 | .0645 | .2063 | .2067 | .0694 | .0402 |
| .00300 | .0788 | .2484 | .2489 | .0848 | .0491 |
| .00500 | .1013 | .3105 | .3111 | .1090 | .0633 |
| .00700 | .1194 | .3565 | .3573 | .1284 | .0748 |
| .01000 | .1417 | .4091 | .4100 | .1525 | .0892 |
| .01500 | .1717 | .4717 | .4728 | .1848 | .1088 |
| .02000 | .1962 | .5168 | .5181 | .2112 | .1251 |
| .03000 | .2354 | .5792 | .5808 | .2537 | .1520 |
| .05000 | .2926 | .6526 | .6546 | .3157 | .1931 |
| .07000 | .3344 | .6961 | .6985 | .3612 | .2251 |
| .10000 | .3812 | .7374 | .7402 | .4124 | .2632 |
| .15000 | .4359 | .7779 | .7812 | .4724 | .3115 |
| .20000 | .4745 | .8026 | .8062 | .5148 | .3485 |
| .30000 | .5267 | .8321 | .8364 | .5725 | .4033 |
| .50000 | .5867 | .8617 | .8666 | .6384 | .4735 |
| .70000 | .6219 | .8774 | .8825 | .6766 | .5183 |
| 1.0000 | .6553 | .8911 | .8965 | .7122 | .5632 |
| 1.5000 | .6885 | .9040 | .9095 | .7467 | .6095 |
| 2.0000 | .7093 | .9116 | .9172 | .7678 | .6391 |
| 3.0000 | .7353 | .9209 | .9263 | .7932 | .6762 |
| 5.0000 | .7630 | .9303 | .9357 | .8194 | .7157 |
| 7.0000 | .7788 | .9356 | .9407 | .8339 | .7379 |
| 10.000 | .7936 | .9404 | .9455 | .8471 | .7586 |
| 15.000 | .8085 | .9452 | .9500 | .8602 | .7790 |
| 20.000 | .8179 | .9481 | .9529 | .8684 | .7919 |
| 30.000 | .8299 | .9519 | .9565 | .8785 | .8079 |
| 50.000 | .8430 | .9560 | .9603 | .8895 | .8253 |
| 70.000 | .8508 | .9584 | .9625 | .8958 | .8353 |
| 100.00 | .8581 | .9606 | .9647 | .9018 | .8447 |
| 150.00 | .8658 | .9629 | .9668 | .9078 | .8544 |
| 200.00 | .8707 | .9644 | .9682 | .9118 | .8606 |
| 300.00 | .8772 | .9663 | .9700 | .9168 | .8686 |
| 500.00 | .8844 | .9684 | .9719 | .9224 | .8775 |
| 700.00 | .8888 | .9697 | .9731 | .9257 | .8827 |
| 1000.0 | .8931 | .9709 | .9743 | .9289 | .8878 |

Table 5b. Values of $G_b$ for $2\ \theta_o/\pi = 0.5$.

| q* \ θ/π | .5 | .6 | .7 | .8 | .9 |
|---|---|---|---|---|---|
| .00010 | .0090 | .0061 | .0044 | .0032 | .0021 |
| .00015 | .0110 | .0075 | .0054 | .0039 | .0026 |
| .00020 | .0127 | .0086 | .0062 | .0045 | .0030 |
| .00030 | .0156 | .0106 | .0077 | .0056 | .0037 |
| .00050 | .0201 | .0136 | .0099 | .0072 | .0048 |
| .00070 | .0238 | .0161 | .0117 | .0085 | .0056 |
| .00100 | .0284 | .0193 | .0140 | .0102 | .0067 |
| .00150 | .0348 | .0236 | .0171 | .0125 | .0083 |
| .00200 | .0402 | .0273 | .0197 | .0144 | .0095 |
| .00300 | .0491 | .0334 | .0242 | .0176 | .0117 |
| .00500 | .0633 | .0431 | .0312 | .0227 | .0151 |
| .00700 | .0748 | .0509 | .0369 | .0269 | .0178 |
| .01000 | .0892 | .0608 | .0441 | .0321 | .0213 |
| .01500 | .1088 | .0743 | .0539 | .0393 | .0261 |
| .02000 | .1251 | .0855 | .0621 | .0453 | .0302 |
| .03000 | .1520 | .1043 | .0759 | .0554 | .0369 |
| .05000 | .1931 | .1335 | .0974 | .0713 | .0476 |
| .07000 | .2251 | .1565 | .1146 | .0841 | .0563 |
| .10000 | .2632 | .1847 | .1358 | .1000 | .0673 |
| .15000 | .3115 | .2217 | .1641 | .1214 | .0822 |
| .20000 | .3485 | .2510 | .1871 | .1390 | .0948 |
| .30000 | .4033 | .2964 | .2236 | .1675 | .1155 |
| .50000 | .4735 | .3589 | .2759 | .2098 | .1474 |
| .70000 | .5183 | .4017 | .3135 | .2414 | .1722 |
| 1.0000 | .5632 | .4470 | .3552 | .2776 | .2018 |
| 1.5000 | .6095 | .4968 | .4033 | .3213 | .2390 |
| 2.0000 | .6391 | .5302 | .4370 | .3532 | .2672 |
| 3.0000 | .6762 | .5737 | .4828 | .3981 | .3086 |
| 5.0000 | .7157 | .6221 | .5360 | .4527 | .3613 |
| 7.0000 | .7379 | .6501 | .5678 | .4865 | .3951 |
| 10.000 | .7586 | .6766 | .5986 | .5199 | .4294 |
| 15.000 | .7790 | .7033 | .6300 | .5547 | .4660 |
| 20.000 | .7919 | .7201 | .6501 | .5773 | .4901 |
| 30.000 | .8079 | .7413 | .6756 | .6063 | .5217 |
| 50.000 | .8253 | .7643 | .7036 | .6385 | .5573 |
| 70.000 | .8353 | .7776 | .7199 | .6574 | .5784 |
| 100.00 | .8447 | .7903 | .7355 | .6756 | .5989 |
| 150.00 | .8544 | .8032 | .7514 | .6943 | .6203 |
| 200.00 | .8606 | .8116 | .7617 | .7064 | .6342 |
| 300.00 | .8686 | .8222 | .7748 | .7221 | .6522 |
| 500.00 | .8775 | .8341 | .7896 | .7396 | .6727 |
| 700.00 | .8827 | .8412 | .7984 | .7501 | .6850 |
| 1000.0 | .8878 | .8481 | .8070 | .7604 | .6971 |

Table 6a. Values of $G_b$ for $2\ \theta_o/\pi = 0.6$.

| q* \ θ/π | .1 | .2 | .3 | .4 | .5 |
|---|---|---|---|---|---|
| .00010 | .0111 | .0252 | 1.0000 | .0254 | .0124 |
| .00015 | .0136 | .0309 | 1.0000 | .0312 | .0152 |
| .00020 | .0157 | .0356 | 1.0000 | .0360 | .0175 |
| .00030 | .0193 | .0436 | 1.0000 | .0440 | .0214 |
| .00050 | .0249 | .0562 | 1.0000 | .0568 | .0277 |
| .00070 | .0294 | .0665 | 1.0000 | .0671 | .0327 |
| .00100 | .0352 | .0793 | 1.0000 | .0801 | .0391 |
| .00150 | .0430 | .0969 | 1.0000 | .0978 | .0478 |
| .00200 | .0497 | .1115 | 1.0000 | .1126 | .0552 |
| .00300 | .0607 | .1359 | 1.0000 | .1372 | .0675 |
| .00500 | .0782 | .1736 | 1.0000 | .1752 | .0869 |
| .00700 | .0922 | .2032 | 1.0000 | .2052 | .1025 |
| .01000 | .1097 | .2393 | 1.0000 | .2416 | .1220 |
| .01500 | .1334 | .2861 | 1.0000 | .2890 | .1484 |
| .02000 | .1529 | .3230 | 1.0000 | .3263 | .1702 |
| .03000 | .1847 | .3794 | 1.0000 | .3834 | .2056 |
| .05000 | .2322 | .4554 | 1.0000 | .4606 | .2587 |
| .07000 | .2680 | .5065 | 1.0000 | .5125 | .2988 |
| .10000 | .3093 | .5595 | 1.0000 | .5665 | .3452 |
| .15000 | .3595 | .6163 | 1.0000 | .6246 | .4017 |
| .20000 | .3964 | .6533 | 1.0000 | .6626 | .4432 |
| .30000 | .4483 | .7000 | 1.0000 | .7107 | .5016 |
| .50000 | .5111 | .7493 | 1.0000 | .7616 | .5718 |
| .70000 | .5494 | .7762 | 1.0000 | .7894 | .6140 |
| 1.0000 | .5867 | .8004 | 1.0000 | .8143 | .6544 |
| 1.5000 | .6248 | .8234 | 1.0000 | .8378 | .6945 |
| 2.0000 | .6491 | .8372 | 1.0000 | .8518 | .7192 |
| 3.0000 | .6796 | .8540 | 1.0000 | .8685 | .7496 |
| 5.0000 | .7127 | .8713 | 1.0000 | .8855 | .7811 |
| 7.0000 | .7316 | .8809 | 1.0000 | .8948 | .7986 |
| 10.000 | .7494 | .8898 | 1.0000 | .9033 | .8147 |
| 15.000 | .7674 | .8986 | 1.0000 | .9116 | .8306 |
| 20.000 | .7788 | .9041 | 1.0000 | .9168 | .8405 |
| 30.000 | .7933 | .9110 | 1.0000 | .9232 | .8529 |
| 50.000 | .8093 | .9185 | 1.0000 | .9302 | .8663 |
| 70.000 | .8186 | .9229 | 1.0000 | .9342 | .8739 |
| 100.00 | .8276 | .9270 | 1.0000 | .9379 | .8812 |
| 150.00 | .8369 | .9313 | 1.0000 | .9418 | .8886 |
| 200.00 | .8429 | .9340 | 1.0000 | .9443 | .8933 |
| 300.00 | .8507 | .9376 | 1.0000 | .9474 | .8994 |
| 500.00 | .8596 | .9415 | 1.0000 | .9510 | .9062 |
| 700.00 | .8649 | .9439 | 1.0000 | .9531 | .9103 |
| 1000.0 | .8700 | .9462 | 1.0000 | .9551 | .9141 |

Table 6b. Values of $G_b$ for $2\ \theta_o/\pi = 0.6$.

| $q^*$ \ $\theta/\pi$ | .5 | .6 | .7 | .8 | .9 |
|---|---|---|---|---|---|
| .00010 | .0124 | .0079 | .0056 | .0040 | .0026 |
| .00015 | .0152 | .0097 | .0068 | .0049 | .0032 |
| .00020 | .0175 | .0112 | .0079 | .0056 | .0037 |
| .00030 | .0214 | .0137 | .0096 | .0069 | .0046 |
| .00050 | .0277 | .0177 | .0124 | .0089 | .0059 |
| .00070 | .0327 | .0209 | .0147 | .0106 | .0070 |
| .00100 | .0391 | .0250 | .0176 | .0126 | .0083 |
| .00150 | .0478 | .0306 | .0216 | .0155 | .0102 |
| .00200 | .0552 | .0353 | .0249 | .0179 | .0118 |
| .00300 | .0675 | .0433 | .0305 | .0219 | .0144 |
| .00500 | .0869 | .0558 | .0393 | .0282 | .0186 |
| .00700 | .1025 | .0659 | .0465 | .0334 | .0220 |
| .01000 | .1220 | .0786 | .0555 | .0398 | .0263 |
| .01500 | .1484 | .0959 | .0678 | .0487 | .0322 |
| .02000 | .1702 | .1104 | .0781 | .0562 | .0371 |
| .03000 | .2056 | .1343 | .0953 | .0687 | .0454 |
| .05000 | .2587 | .1711 | .1220 | .0882 | .0585 |
| .07000 | .2988 | .1999 | .1432 | .1038 | .0691 |
| .10000 | .3452 | .2346 | .1692 | .1231 | .0824 |
| .15000 | .4017 | .2790 | .2034 | .1490 | .1004 |
| .20000 | .4432 | .3136 | .2307 | .1700 | .1154 |
| .30000 | .5016 | .3655 | .2733 | .2036 | .1399 |
| .50000 | .5718 | .4337 | .3325 | .2522 | .1769 |
| .70000 | .6140 | .4783 | .3736 | .2875 | .2051 |
| 1.0000 | .6544 | .5237 | .4177 | .3270 | .2379 |
| 1.5000 | .6945 | .5716 | .4668 | .3732 | .2782 |
| 2.0000 | .7192 | .6027 | .5001 | .4059 | .3081 |
| 3.0000 | .7496 | .6421 | .5441 | .4509 | .3508 |
| 5.0000 | .7811 | .6846 | .5937 | .5039 | .4036 |
| 7.0000 | .7986 | .7087 | .6227 | .5359 | .4367 |
| 10.000 | .8147 | .7313 | .6503 | .5671 | .4699 |
| 15.000 | .8306 | .7538 | .6782 | .5993 | .5048 |
| 20.000 | .8405 | .7679 | .6960 | .6199 | .5276 |
| 30.000 | .8529 | .7856 | .7183 | .6464 | .5573 |
| 50.000 | .8663 | .8048 | .7428 | .6755 | .5905 |
| 70.000 | .8739 | .8159 | .7570 | .6926 | .6102 |
| 100.00 | .8812 | .8264 | .7706 | .7090 | .6293 |
| 150.00 | .8886 | .8371 | .7844 | .7258 | .6490 |
| 200.00 | .8933 | .8440 | .7934 | .7367 | .6619 |
| 300.00 | .8994 | .8529 | .8048 | .7507 | .6786 |
| 500.00 | .9062 | .8627 | .8176 | .7665 | .6976 |
| 700.00 | .9103 | .8686 | .8253 | .7759 | .7090 |
| 1000.0 | .9141 | .8743 | .8327 | .7851 | .7201 |

Table 7a. Values of $G_b$ for $2\,\theta_o/\pi = 0.7$.

| q* \ $\theta/\pi$ | .1 | .2 | .3 | .4 | .5 |
|---|---|---|---|---|---|
| .00010 | .0090 | .0173 | .0538 | .0538 | .0176 |
| .00015 | .0110 | .0212 | .0658 | .0658 | .0216 |
| .00020 | .0127 | .0245 | .0758 | .0759 | .0250 |
| .00030 | .0155 | .0300 | .0927 | .0927 | .0306 |
| .00050 | .0200 | .0386 | .1191 | .1191 | .0394 |
| .00070 | .0237 | .0457 | .1402 | .1403 | .0466 |
| .00100 | .0283 | .0546 | .1664 | .1665 | .0557 |
| .00150 | .0347 | .0668 | .2016 | .2017 | .0681 |
| .00200 | .0400 | .0770 | .2302 | .2304 | .0785 |
| .00300 | .0489 | .0940 | .2761 | .2763 | .0959 |
| .00500 | .0630 | .1207 | .3429 | .3431 | .1231 |
| .00700 | .0744 | .1420 | .3916 | .3919 | .1449 |
| .01000 | .0887 | .1684 | .4462 | .4465 | .1719 |
| .01500 | .1080 | .2036 | .5099 | .5103 | .2078 |
| .02000 | .1241 | .2321 | .5550 | .5554 | .2370 |
| .03000 | .1505 | .2775 | .6161 | .6167 | .2834 |
| .05000 | .1904 | .3427 | .6865 | .6872 | .3502 |
| .07000 | .2211 | .3895 | .7276 | .7284 | .3982 |
| .10000 | .2574 | .4411 | .7661 | .7671 | .4513 |
| .15000 | .3026 | .5002 | .8036 | .8048 | .5123 |
| .20000 | .3368 | .5411 | .8264 | .8278 | .5546 |
| .30000 | .3865 | .5953 | .8537 | .8552 | .6109 |
| .50000 | .4491 | .6558 | .8808 | .8827 | .6740 |
| .70000 | .4887 | .6903 | .8951 | .8971 | .7098 |
| 1.0000 | .5284 | .7221 | .9077 | .9097 | .7428 |
| 1.5000 | .5698 | .7530 | .9193 | .9215 | .7746 |
| 2.0000 | .5966 | .7719 | .9262 | .9284 | .7937 |
| 3.0000 | .6308 | .7949 | .9343 | .9366 | .8167 |
| 5.0000 | .6682 | .8189 | .9246 | .9448 | .8404 |
| 7.0000 | .6897 | .8323 | .9472 | .9494 | .8533 |
| 10.000 | .7102 | .8448 | .9514 | .9534 | .8651 |
| 15.000 | .7309 | .8571 | .9554 | .9575 | .8768 |
| 20.000 | .7440 | .8648 | .9581 | .9600 | .8840 |
| 30.000 | .7608 | .8746 | .9612 | .9631 | .8931 |
| 50.000 | .7792 | .8852 | .9646 | .9664 | .9028 |
| 70.000 | .7900 | .8913 | .9666 | .9684 | .9084 |
| 100.00 | .8004 | .8971 | .9685 | .9702 | .9137 |
| 150.00 | .8112 | .9032 | .9704 | .9720 | .9191 |
| 200.00 | .8181 | .9070 | .9717 | .9732 | .9225 |
| 300.00 | .8272 | .9120 | .9732 | .9747 | .9269 |
| 500.00 | .8374 | .9176 | .9750 | .9764 | .9319 |
| 700.00 | .8435 | .9209 | .9760 | .9775 | .9348 |
| 1000.0 | .8495 | .9242 | .9771 | .9784 | .9376 |

Table 7b. Values of $G_b$ for $2\,\theta_o/\pi = 0.7$.

| $q^*$ \ $\theta/\pi$ | .5 | .6 | .7 | .8 | .9 |
|---|---|---|---|---|---|
| .00010 | .0176 | .0102 | .0069 | .0049 | .0032 |
| .00015 | .0216 | .0125 | .0085 | .0060 | .0039 |
| .00020 | .0250 | .0145 | .0098 | .0069 | .0045 |
| .00030 | .0306 | .0177 | .0120 | .0084 | .0055 |
| .00050 | .0394 | .0229 | .0155 | .0109 | .0071 |
| .00070 | .0466 | .0271 | .0183 | .0129 | .0084 |
| .00100 | .0557 | .0324 | .0219 | .0154 | .0100 |
| .00150 | .0681 | .0396 | .0268 | .0188 | .0123 |
| .00200 | .0785 | .0457 | .0309 | .0217 | .0142 |
| .00300 | .0959 | .0559 | .0378 | .0266 | .0174 |
| .00500 | .1231 | .0721 | .0488 | .0343 | .0224 |
| .00700 | .1449 | .0851 | .0577 | .0406 | .0265 |
| .01000 | .1719 | .1014 | .0688 | .0485 | .0317 |
| .01500 | .2078 | .1235 | .0840 | .0593 | .0388 |
| .02000 | .2370 | .1419 | .0967 | .0683 | .0447 |
| .03000 | .2834 | .1720 | .1178 | .0834 | .0547 |
| .05000 | .3502 | .2178 | .1504 | .1069 | .0704 |
| .07000 | .3982 | .2530 | .1761 | .1256 | .0830 |
| .10000 | .4513 | .2945 | .2071 | .1486 | .0987 |
| .15000 | .5123 | .3463 | .2474 | .1790 | .1199 |
| .20000 | .5546 | .3854 | .2791 | .2035 | .1374 |
| .30000 | .6109 | .4420 | .3273 | .2420 | .1656 |
| .50000 | .6740 | .5126 | .3920 | .2963 | .2074 |
| .70000 | .7098 | .5566 | .4352 | .3346 | .2386 |
| 1.0000 | .7428 | .5997 | .4801 | .3764 | .2741 |
| 1.5000 | .7746 | .6436 | .5285 | .4238 | .3167 |
| 2.0000 | .7937 | .6712 | .5604 | .4566 | .3475 |
| 3.0000 | .8167 | .7054 | .6014 | .5006 | .3907 |
| 5.0000 | .8404 | .7416 | .6466 | .5511 | .4429 |
| 7.0000 | .8533 | .7618 | .6726 | .5811 | .4750 |
| 10.000 | .8651 | .7806 | .6971 | .6100 | .5068 |
| 15.000 | .8768 | .7991 | .7216 | .6395 | .5399 |
| 20.000 | .8840 | .8108 | .7371 | .6583 | .5614 |
| 30.000 | .8931 | .8253 | .7566 | .6823 | .5892 |
| 50.000 | .9028 | .8410 | .7779 | .7086 | .6203 |
| 70.000 | .9084 | .8500 | .7902 | .7240 | .6386 |
| 100.00 | .9137 | .8586 | .8019 | .7387 | .6563 |
| 150.00 | .9191 | .8673 | .8139 | .7539 | .6747 |
| 200.00 | .9225 | .8730 | .8216 | .7636 | .6866 |
| 300.00 | .9269 | .8802 | .8315 | .7763 | .7022 |
| 500.00 | .9319 | .8882 | .8426 | .7904 | .7197 |
| 700.00 | .9348 | .8930 | .8492 | .7989 | .7303 |
| 1000.0 | .9376 | .8976 | .8556 | .8072 | .7407 |

Table 8a. Values of $G_b$ for $2\ \theta_o/\pi = 0.8$.

| q* \ θ/π | .1 | .2 | .3 | .4 | .5 |
|---|---|---|---|---|---|
| .00010 | .0074 | .0131 | .0276 | 1.0000 | .0277 |
| .00015 | .0091 | .0161 | .0338 | 1.0000 | .0339 |
| .00020 | .0105 | .0185 | .0390 | 1.0000 | .0391 |
| .00030 | .0128 | .0227 | .0477 | 1.0000 | .0479 |
| .00050 | .0166 | .0293 | .0615 | 1.0000 | .0617 |
| .00070 | .0196 | .0347 | .0727 | 1.0000 | .0729 |
| .00100 | .0234 | .0414 | .0867 | 1.0000 | .0870 |
| .00150 | .0287 | .0507 | .1059 | 1.0000 | .1062 |
| .00200 | .0331 | .0585 | .1219 | 1.0000 | .1222 |
| .00300 | .0405 | .0715 | .1483 | 1.0000 | .1488 |
| .00500 | .0522 | .0920 | .1891 | 1.0000 | .1897 |
| .00700 | .0616 | .1084 | .2211 | 1.0000 | .2218 |
| .01000 | .0735 | .1290 | .2598 | 1.0000 | .2606 |
| .01500 | .0896 | .1566 | .3096 | 1.0000 | .3106 |
| .02000 | .1031 | .1794 | .3485 | 1.0000 | .3497 |
| .03000 | .1253 | .2162 | .4074 | 1.0000 | .4088 |
| .05000 | .1594 | .2709 | .4856 | 1.0000 | .4873 |
| .07000 | .1859 | .3117 | .5372 | 1.0000 | .5393 |
| .10000 | .2176 | .3583 | .5902 | 1.0000 | .5926 |
| .15000 | .2580 | .4143 | .6463 | 1.0000 | .6492 |
| .20000 | .2892 | .4547 | .6826 | 1.0000 | .6858 |
| .30000 | .3358 | .5107 | .7280 | 1.0000 | .7318 |
| .50000 | .3966 | .5766 | .7755 | 1.0000 | .7799 |
| .70000 | .4364 | .6157 | .8013 | 1.0000 | .8061 |
| 1.0000 | .4771 | .6529 | .8243 | 1.0000 | .8294 |
| 1.5000 | .5207 | .6898 | .8459 | 1.0000 | .8513 |
| 2.0000 | .5494 | .7128 | .8589 | 1.0000 | .8643 |
| 3.0000 | .5864 | .7411 | .8743 | 1.0000 | .8798 |
| 5.0000 | .6275 | .7710 | .8901 | 1.0000 | .8955 |
| 7.0000 | .6514 | .7878 | .8988 | 1.0000 | .9040 |
| 10.000 | .6742 | .8034 | .9068 | 1.0000 | .9118 |
| 15.000 | .6973 | .8190 | .9146 | 1.0000 | .9195 |
| 20.000 | .7120 | .8287 | .9195 | 1.0000 | .9243 |
| 30.000 | .7308 | .8410 | .9255 | 1.0000 | .9302 |
| 50.000 | .7515 | .8544 | .9321 | 1.0000 | .9366 |
| 70.000 | .7636 | .8622 | .9359 | 1.0000 | .9402 |
| 100.00 | .7753 | .8696 | .9396 | 1.0000 | .9436 |
| 150.00 | .7874 | .8772 | .9432 | 1.0000 | .9472 |
| 200.00 | .7952 | .8821 | .9456 | 1.0000 | .9494 |
| 300.00 | .8054 | .8884 | .9486 | 1.0000 | .9523 |
| 500.00 | .8169 | .8955 | .9520 | 1.0000 | .9556 |
| 700.00 | .8238 | .8997 | .9540 | 1.0000 | .9575 |
| 1000.0 | .8306 | .9038 | .9560 | 1.0000 | .9593 |

Table 8b. Values of $G_b$ for $2\ \theta_o/\pi = 0.8$.

| q* \ θ/π | .5 | .6 | .7 | .8 | .9 |
|---|---|---|---|---|---|
| .00010 | .0277 | .0135 | .0086 | .0059 | .0038 |
| .00015 | .0339 | .0165 | .0105 | .0072 | .0046 |
| .00020 | .0391 | .0191 | .0121 | .0083 | .0053 |
| .00030 | .0479 | .0234 | .0149 | .0102 | .0065 |
| .00050 | .0617 | .0302 | .0192 | .0131 | .0084 |
| .00070 | .0729 | .0357 | .0227 | .0155 | .0100 |
| .00100 | .0870 | .0426 | .0271 | .0185 | .0119 |
| .00150 | .1062 | .0521 | .0332 | .0227 | .0146 |
| .00200 | .1222 | .0602 | .0383 | .0262 | .0169 |
| .00300 | .1488 | .0736 | .0469 | .0321 | .0207 |
| .00500 | .1897 | .0946 | .0604 | .0414 | .0267 |
| .00700 | .2218 | .1116 | .0714 | .0489 | .0315 |
| .01000 | .2606 | .1327 | .0851 | .0584 | .0377 |
| .01500 | .3106 | .1612 | .1038 | .0713 | .0461 |
| .02000 | .3497 | .1847 | .1194 | .0822 | .0531 |
| .03000 | .4088 | .2227 | .1451 | .1002 | .0649 |
| .05000 | .4873 | .2791 | .1845 | .1281 | .0834 |
| .07000 | .5393 | .3213 | .2151 | .1502 | .0982 |
| .10000 | .5926 | .3696 | .2517 | .1772 | .1165 |
| .15000 | .6492 | .4276 | .2982 | .2124 | .1411 |
| .20000 | .6858 | .4696 | .3339 | .2404 | .1611 |
| .30000 | .7318 | .5280 | .3869 | .2836 | .1931 |
| .50000 | .7799 | .5967 | .4553 | .3429 | .2394 |
| .70000 | .8061 | .6374 | .4992 | .3835 | .2732 |
| 1.0000 | .8294 | .6760 | .5434 | .4266 | .3109 |
| 1.5000 | .8513 | .7139 | .5894 | .4742 | .3550 |
| 2.0000 | .8643 | .7372 | .6190 | .5062 | .3863 |
| 3.0000 | .8798 | .7656 | .6563 | .5484 | .4293 |
| 5.0000 | .8955 | .7950 | .6965 | .5958 | .4802 |
| 7.0000 | .9040 | .8113 | .7193 | .6236 | .5111 |
| 10.000 | .9118 | .8264 | .7407 | .6500 | .5413 |
| 15.000 | .9195 | .8412 | .7619 | .6768 | .5726 |
| 20.000 | .9243 | .8504 | .7753 | .6939 | .5928 |
| 30.000 | .9302 | .8619 | .7920 | .7155 | .6188 |
| 50.000 | .9366 | .8744 | .8103 | .7392 | .6478 |
| 70.000 | .9402 | .8816 | .8208 | .7530 | .6648 |
| 100.00 | .9436 | .8883 | .8309 | .7662 | .6813 |
| 150.00 | .9472 | .8952 | .8411 | .7798 | .6984 |
| 200.00 | .9494 | .8997 | .8477 | .7885 | .7095 |
| 300.00 | .9523 | .9054 | .8561 | .7999 | .7239 |
| 500.00 | .9556 | .9117 | .8656 | .8125 | .7402 |
| 700.00 | .9575 | .9155 | .8712 | .8201 | .7500 |
| 1000.0 | .9593 | .9191 | .8767 | .8275 | .7596 |

Table 9a. Values of $G_b$ for 2 $\theta_o/\pi$ = 0.9.

| q* \ $\theta/\pi$ | .1 | .2 | .3 | .4 | .5 |
|---|---|---|---|---|---|
| .00010 | .0062 | .0104 | .0185 | .0566 | .0566 |
| .00015 | .0076 | .0128 | .0227 | .0693 | .0693 |
| .00020 | .0088 | .0148 | .0262 | .0799 | .0799 |
| .00030 | .0108 | .0181 | .0321 | .0976 | .0976 |
| .00050 | .0139 | .0233 | .0414 | .1253 | .1253 |
| .00070 | .0164 | .0276 | .0489 | .1475 | .1475 |
| .00100 | .0196 | .0330 | .0584 | .1750 | .1750 |
| .00150 | .0241 | .0404 | .0714 | .2117 | .2117 |
| .00200 | .0278 | .0466 | .0823 | .2415 | .2416 |
| .00300 | .0340 | .0570 | .1005 | .2891 | .2892 |
| .00500 | .0438 | .0734 | .1290 | .3578 | .3579 |
| .00700 | .0518 | .0866 | .1518 | .4076 | .4076 |
| .01000 | .0618 | .1032 | .1798 | .4628 | .4629 |
| .01500 | .0754 | .1256 | .2171 | .5268 | .5269 |
| .02000 | .0869 | .1442 | .2473 | .5716 | .5717 |
| .03000 | .1057 | .1747 | .2951 | .6319 | .6321 |
| .05000 | .1350 | .2207 | .3633 | .7007 | .7009 |
| .07000 | .1579 | .2558 | .4119 | .7406 | .7409 |
| .10000 | .1857 | .2970 | .4650 | .7779 | .7782 |
| .15000 | .2217 | .3479 | .5255 | .8141 | .8144 |
| .20000 | .2499 | .3860 | .5669 | .8361 | .8364 |
| .30000 | .2930 | .4406 | .6216 | .8623 | .8627 |
| .50000 | .3509 | .5080 | .6821 | .8884 | .8889 |
| .70000 | .3900 | .5496 | .7163 | .9022 | .9027 |
| 1.0000 | .4310 | .5903 | .7477 | .9142 | .9148 |
| 1.5000 | .4758 | .6317 | .7777 | .9254 | .9260 |
| 2.0000 | .5058 | .6580 | .7959 | .9319 | .9325 |
| 3.0000 | .5452 | .6908 | .8178 | .9397 | .9403 |
| 5.0000 | .5894 | .7258 | .8404 | .9476 | .9482 |
| 7.0000 | .6153 | .7457 | .8529 | .9519 | .9525 |
| 10.000 | .6402 | .7642 | .8645 | .9557 | .9563 |
| 15.000 | .6655 | .7828 | .8758 | .9596 | .9601 |
| 20.000 | .6817 | .7944 | .8829 | .9620 | .9625 |
| 30.000 | .7023 | .8092 | .8917 | .9649 | .9655 |
| 50.000 | .7252 | .8252 | .9012 | .9681 | .9686 |
| 70.000 | .7386 | .8345 | .9067 | .9700 | .9704 |
| 100.00 | .7515 | .8434 | .9120 | .9716 | .9721 |
| 150.00 | .7649 | .8525 | .9173 | .9734 | .9739 |
| 200.00 | .7735 | .8584 | .9208 | .9746 | .9750 |
| 300.00 | .7848 | .8660 | .9252 | .9760 | .9764 |
| 500.00 | .7975 | .8745 | .9301 | .9776 | .9780 |
| 700.00 | .8051 | .8796 | .9330 | .9786 | .9790 |
| 1000.0 | .8126 | .8845 | .9359 | .9795 | .9799 |

Table 9b. Values of $G_b$ for 2 $\theta_o/\pi$ = 0.9.

| q* \ $\theta/\pi$ | .5 | .6 | .7 | .8 | .9 |
|---|---|---|---|---|---|
| .00010 | .0566 | .0186 | .0107 | .0071 | .0045 |
| .00015 | .0693 | .0228 | .0131 | .0086 | .0055 |
| .00020 | .0799 | .0263 | .0152 | .0100 | .0063 |
| .00030 | .0976 | .0322 | .0186 | .0122 | .0077 |
| .00050 | .1253 | .0416 | .0240 | .0158 | .0100 |
| .00070 | .1475 | .0491 | .0284 | .0187 | .0118 |
| .00100 | .1750 | .0587 | .0339 | .0223 | .0141 |
| .00150 | .2117 | .0718 | .0415 | .0273 | .0173 |
| .00200 | .2416 | .0827 | .0479 | .0315 | .0199 |
| .00300 | .2892 | .1010 | .0586 | .0386 | .0244 |
| .00500 | .3579 | .1297 | .0755 | .0497 | .0315 |
| .00700 | .4076 | .1525 | .0891 | .0588 | .0372 |
| .01000 | .4629 | .1807 | .1061 | .0701 | .0444 |
| .01500 | .5269 | .2182 | .1292 | .0856 | .0543 |
| .02000 | .5717 | .2486 | .1484 | .0985 | .0626 |
| .03000 | .6321 | .2966 | .1797 | .1199 | .0765 |
| .05000 | .7009 | .3652 | .2271 | .1529 | .0981 |
| .07000 | .7409 | .4141 | .2633 | .1788 | .1153 |
| .10000 | .7782 | .4677 | .3058 | .2101 | .1365 |
| .15000 | .8144 | .5286 | .3584 | .2504 | .1645 |
| .20000 | .8364 | .5704 | .3977 | .2818 | .1872 |
| .30000 | .8627 | .6257 | .4542 | .3295 | .2229 |
| .50000 | .8889 | .6870 | .5238 | .3928 | .2735 |
| .70000 | .9027 | .7215 | .5666 | .4349 | .3094 |
| 1.0000 | .9148 | .7532 | .6084 | .4783 | .3488 |
| 1.5000 | .9260 | .7836 | .6506 | .5250 | .3938 |
| 2.0000 | .9325 | .8019 | .6771 | .5557 | .4251 |
| 3.0000 | .9403 | .8238 | .7100 | .5954 | .4675 |
| 5.0000 | .9482 | .8464 | .7448 | .6392 | .5165 |
| 7.0000 | .9525 | .8587 | .7642 | .6645 | .5460 |
| 10.000 | .9563 | .8701 | .7824 | .6885 | .5745 |
| 15.000 | .9601 | .8812 | .8004 | .7126 | .6039 |
| 20.000 | .9625 | .8882 | .8117 | .7279 | .6228 |
| 30.000 | .9655 | .8968 | .8258 | .7472 | .6470 |
| 50.000 | .9686 | .9061 | .8411 | .7683 | .6740 |
| 70.000 | .9704 | .9115 | .8499 | .7806 | .6898 |
| 100.00 | .9721 | .9166 | .8584 | .7924 | .7051 |
| 150.00 | .9739 | .9217 | .8669 | .8045 | .7209 |
| 200.00 | .9750 | .9251 | .8725 | .8122 | .7312 |
| 300.00 | .9764 | .9293 | .8795 | .8223 | .7445 |
| 500.00 | .9780 | .9341 | .8875 | .8335 | .7596 |
| 700.00 | .9790 | .9369 | .8922 | .8403 | .7687 |
| 1000.0 | .9799 | .9396 | .8968 | .8468 | .7776 |

Table 10a. Values of $G_b$ for $\theta = \pi/2$.

| $q^*$ \ $h_s/5a$ | .1 | .2 | .3 | .4 | .5 |
|---|---|---|---|---|---|
| .00010 | .0180 | .0090 | .0060 | .0045 | .0036 |
| .00015 | .0220 | .0110 | .0073 | .0055 | .0044 |
| .00020 | .0254 | .0127 | .0085 | .0064 | .0051 |
| .00030 | .0311 | .0156 | .0104 | .0078 | .0062 |
| .00050 | .0402 | .0201 | .0134 | .0101 | .0080 |
| .00070 | .0475 | .0238 | .0159 | .0119 | .0095 |
| .00100 | .0567 | .0284 | .0190 | .0142 | .0114 |
| .00150 | .0694 | .0348 | .0232 | .0174 | .0139 |
| .00200 | .0800 | .0402 | .0268 | .0201 | .0161 |
| .00300 | .0977 | .0491 | .0328 | .0246 | .0197 |
| .00500 | .1254 | .0633 | .0423 | .0318 | .0254 |
| .00700 | .1476 | .0748 | .0500 | .0376 | .0301 |
| .01000 | .1750 | .0892 | .0597 | .0449 | .0359 |
| .01500 | .2115 | .1088 | .0730 | .0549 | .0439 |
| .02000 | .2411 | .1251 | .0841 | .0633 | .0507 |
| .03000 | .2881 | .1520 | .1025 | .0772 | .0620 |
| .05000 | .3555 | .1931 | .1312 | .0992 | .0797 |
| .07000 | .4039 | .2251 | .1539 | .1167 | .0939 |
| .10000 | .4573 | .2632 | .1817 | .1383 | .1116 |
| .15000 | .5183 | .3115 | .2182 | .1672 | .1354 |
| .20000 | .5605 | .3485 | .2473 | .1906 | .1548 |
| .30000 | .6165 | .4033 | .2924 | .2278 | .1863 |
| .50000 | .6790 | .4735 | .3545 | .2813 | .2326 |
| .70000 | .7145 | .5183 | .3973 | .3198 | .2668 |
| 1.0000 | .7471 | .5632 | .4428 | .3623 | .3058 |
| 1.5000 | .7783 | .6095 | .4929 | .4115 | .3524 |
| 2.0000 | .7972 | .6391 | .5265 | .4458 | .3860 |
| 3.0000 | .8199 | .6762 | .5705 | .4924 | .4330 |
| 5.0000 | .8431 | .7157 | .6195 | .5464 | .4893 |
| 7.0000 | .8558 | .7379 | .6479 | .5785 | .5237 |
| 10.000 | .8675 | .7586 | .6747 | .6095 | .5574 |
| 15.000 | .8789 | .7790 | .7017 | .6410 | .5922 |
| 20.000 | .8860 | .7919 | .7187 | .6611 | .6146 |
| 30.000 | .8949 | .8079 | .7401 | .6866 | .6432 |
| 50.000 | .9045 | .8253 | .7634 | .7145 | .6746 |
| 70.000 | .9100 | .8353 | .7769 | .7306 | .6929 |
| 100.00 | .9152 | .8447 | .7897 | .7460 | .7104 |
| 150.00 | .9205 | .8544 | .8027 | .7617 | .7282 |
| 200.00 | .9239 | .8606 | .8111 | .7718 | .7397 |
| 300.00 | .9282 | .8686 | .8219 | .7848 | .7545 |
| 500.00 | .9331 | .8775 | .8339 | .7993 | .7710 |
| 700.00 | .9359 | .8827 | .8410 | .8079 | .7808 |
| 1000.0 | .9387 | .8878 | .8479 | .8163 | .7904 |

Table 10b. Values of $G_b$ for $\theta = \pi/2$.

| $q^*$ \ $h_s/5a$ | .5 | .6 | .7 | .8 | .9 |
|---|---|---|---|---|---|
| .00010 | .0036 | .0030 | .0026 | .0022 | .0020 |
| .00015 | .0044 | .0037 | .0031 | .0028 | .0024 |
| .00020 | .0051 | .0042 | .0036 | .0032 | .0028 |
| .00030 | .0062 | .0052 | .0045 | .0039 | .0035 |
| .00050 | .0080 | .0067 | .0057 | .0050 | .0045 |
| .00070 | .0095 | .0079 | .0068 | .0059 | .0053 |
| .00100 | .0114 | .0095 | .0081 | .0071 | .0063 |
| .00150 | .0139 | .0116 | .0100 | .0087 | .0077 |
| .00200 | .0161 | .0134 | .0115 | .0101 | .0089 |
| .00300 | .0197 | .0164 | .0141 | .0123 | .0109 |
| .00500 | .0254 | .0212 | .0182 | .0159 | .0141 |
| .00700 | .0301 | .0251 | .0215 | .0188 | .0167 |
| .01000 | .0359 | .0299 | .0257 | .0225 | .0200 |
| .01500 | .0439 | .0366 | .0314 | .0275 | .0245 |
| .02000 | .0507 | .0423 | .0363 | .0318 | .0282 |
| .03000 | .0620 | .0517 | .0444 | .0389 | .0346 |
| .05000 | .0797 | .0666 | .0572 | .0501 | .0446 |
| .07000 | .0939 | .0785 | .0675 | .0591 | .0526 |
| .10000 | .1116 | .0934 | .0804 | .0705 | .0628 |
| .15000 | .1354 | .1136 | .0979 | .0860 | .0767 |
| .20000 | .1548 | .1303 | .1124 | .0989 | .0882 |
| .30000 | .1863 | .1574 | .1362 | .1201 | .1073 |
| .50000 | .2326 | .1980 | .1723 | .1525 | .1368 |
| .70000 | .2668 | .2287 | .2000 | .1777 | .1598 |
| 1.0000 | .3058 | .2643 | .2326 | .2076 | .1875 |
| 1.5000 | .3524 | .3079 | .2733 | .2457 | .2231 |
| 2.0000 | .3860 | .3402 | .3040 | .2748 | .2507 |
| 3.0000 | .4330 | .3864 | .3489 | .3181 | .2923 |
| 5.0000 | .4893 | .4434 | .4057 | .3742 | .3474 |
| 7.0000 | .5237 | .4791 | .4421 | .4107 | .3838 |
| 10.000 | .5574 | .5146 | .4788 | .4481 | .4216 |
| 15.000 | .5922 | .5518 | .5176 | .4881 | .4624 |
| 20.000 | .6146 | .5759 | .5430 | .5146 | .4897 |
| 30.000 | .6432 | .6069 | .5759 | .5491 | .5254 |
| 50.000 | .6746 | .6412 | .6126 | .5877 | .5657 |
| 70.000 | .6929 | .6612 | .6341 | .6104 | .5894 |
| 100.00 | .7104 | .6804 | .6547 | .6323 | .6124 |
| 150.00 | .7282 | .7001 | .6759 | .6548 | .6360 |
| 200.00 | .7397 | .7127 | .6896 | .6693 | .6513 |
| 300.00 | .7545 | .7290 | .7071 | .6880 | .6710 |
| 500.00 | .7710 | .7473 | .7268 | .7089 | .6930 |
| 700.00 | .7808 | .7581 | .7385 | .7213 | .7061 |
| 1000.0 | .7904 | .7686 | .7498 | .7334 | .7189 |

Table 11a. Values of $G_b$ for $\theta = \pi/3$.

| q* \ $h_s/5a$ | .1 | .2 | .3 | .4 | .5 |
|---|---|---|---|---|---|
| .00010 | .1396 | .0286 | .0140 | .0095 | .0073 |
| .00015 | .1696 | .0350 | .0171 | .0116 | .0089 |
| .00020 | .1943 | .0404 | .0198 | .0134 | .0103 |
| .00030 | .2344 | .0494 | .0242 | .0164 | .0126 |
| .00050 | .2942 | .0637 | .0312 | .0212 | .0162 |
| .00070 | .3390 | .0753 | .0370 | .0251 | .0192 |
| .00100 | .3906 | .0898 | .0441 | .0300 | .0229 |
| .00150 | .4530 | .1096 | .0540 | .0367 | .0281 |
| .00200 | .4986 | .1261 | .0623 | .0424 | .0324 |
| .00300 | .5626 | .1534 | .0762 | .0518 | .0397 |
| .00500 | .6394 | .1954 | .0980 | .0668 | .0512 |
| .00700 | .6858 | .2283 | .1155 | .0789 | .0604 |
| .01000 | .7305 | .2680 | .1373 | .0940 | .0721 |
| .01500 | .7753 | .3189 | .1667 | .1146 | .0881 |
| .02000 | .8031 | .3585 | .1909 | .1317 | .1014 |
| .03000 | .8370 | .4182 | .2299 | .1599 | .1234 |
| .05000 | .8719 | .4969 | .2876 | .2028 | .1575 |
| .07000 | .8907 | .5486 | .3305 | .2360 | .1842 |
| .10000 | .9075 | .6013 | .3794 | .2753 | .2165 |
| .15000 | .9233 | .6568 | .4377 | .3249 | .2583 |
| .20000 | .9326 | .6926 | .4796 | .3625 | .2910 |
| .30000 | .9436 | .7372 | .5375 | .4176 | .3406 |
| .50000 | .9543 | .7838 | .6051 | .4873 | .4066 |
| .70000 | .9599 | .8090 | .6448 | .5313 | .4505 |
| 1.0000 | .9648 | .8315 | .6823 | .5749 | .4957 |
| 1.5000 | .9692 | .8525 | .7191 | .6197 | .5440 |
| 2.0000 | .9720 | .8651 | .7417 | .6481 | .5757 |
| 3.0000 | .9751 | .8801 | .7691 | .6837 | .6162 |
| 5.0000 | .9783 | .8953 | .7978 | .7216 | .6605 |
| 7.0000 | .9800 | .9037 | .8136 | .7429 | .6859 |
| 10.000 | .9817 | .9114 | .8283 | .7628 | .7097 |
| 15.000 | .9832 | .9189 | .8427 | .7825 | .7335 |
| 20.000 | .9842 | .9236 | .8517 | .7949 | .7486 |
| 30.000 | .9854 | .9294 | .8630 | .8104 | .7674 |
| 50.000 | .9867 | .9357 | .8752 | .8272 | .7880 |
| 70.000 | .9875 | .9393 | .8822 | .8368 | .7998 |
| 100.00 | .9882 | .9428 | .8889 | .8461 | .8111 |
| 150.00 | .9889 | .9463 | .8957 | .8555 | .8226 |
| 200.00 | .9894 | .9486 | .9001 | .8615 | .8301 |
| 300.00 | .9899 | .9514 | .9056 | .8693 | .8395 |
| 500.00 | .9904 | .9546 | .9119 | .8780 | .8502 |
| 700.00 | .9906 | .9566 | .9156 | .8831 | .8565 |
| 1000.0 | .9907 | .9585 | .9193 | .8881 | .8627 |

Table 11b. Values of $G_b$ for $\theta = \pi/3$.

| $q^*$ \ $h_s/5a$ | .5 | .6 | .7 | .8 | .9 |
|---|---|---|---|---|---|
| .00010 | .0073 | .0059 | .0050 | .0043 | .0038 |
| .00015 | .0089 | .0072 | .0061 | .0053 | .0047 |
| .00020 | .0103 | .0083 | .0070 | .0061 | .0054 |
| .00030 | .0126 | .0102 | .0086 | .0075 | .0066 |
| .00050 | .0162 | .0132 | .0111 | .0097 | .0085 |
| .00070 | .0192 | .0156 | .0132 | .0114 | .0101 |
| .00100 | .0229 | .0186 | .0158 | .0137 | .0121 |
| .00150 | .0281 | .0228 | .0193 | .0167 | .0148 |
| .00200 | .0324 | .0264 | .0223 | .0193 | .0170 |
| .00300 | .0397 | .0323 | .0273 | .0236 | .0209 |
| .00500 | .0512 | .0416 | .0352 | .0305 | .0269 |
| .00700 | .0604 | .0492 | .0416 | .0361 | .0319 |
| .01000 | .0721 | .0587 | .0497 | .0431 | .0381 |
| .01500 | .0881 | .0718 | .0607 | .0527 | .0466 |
| .02000 | .1014 | .0827 | .0700 | .0607 | .0537 |
| .03000 | .1234 | .1008 | .0854 | .0742 | .0656 |
| .05000 | .1575 | .1291 | .1096 | .0953 | .0843 |
| .07000 | .1842 | .1514 | .1288 | .1121 | .0993 |
| .10000 | .2165 | .1788 | .1524 | .1329 | .1180 |
| .15000 | .2583 | .2146 | .1838 | .1608 | .1429 |
| .20000 | .2910 | .2432 | .2090 | .1833 | .1633 |
| .30000 | .3406 | .2875 | .2487 | .2193 | .1961 |
| .50000 | .4066 | .3486 | .3050 | .2711 | .2440 |
| .70000 | .4505 | .3906 | .3447 | .3085 | .2791 |
| 1.0000 | .4957 | .4353 | .3880 | .3500 | .3188 |
| 1.5000 | .5440 | .4847 | .4372 | .3982 | .3657 |
| 2.0000 | .5757 | .5180 | .4711 | .4321 | .3992 |
| 3.0000 | .6162 | .5616 | .5164 | .4782 | .4456 |
| 5.0000 | .6605 | .6103 | .5681 | .5321 | .5008 |
| 7.0000 | .6859 | .6386 | .5987 | .5643 | .5343 |
| 10.000 | .7097 | .6656 | .6280 | .5955 | .5670 |
| 15.000 | .7335 | .6926 | .6577 | .6274 | .6006 |
| 20.000 | .7486 | .7098 | .6767 | .6478 | .6223 |
| 30.000 | .7674 | .7314 | .7006 | .6737 | .6498 |
| 50.000 | .7880 | .7550 | .7268 | .7021 | .6802 |
| 70.000 | .7998 | .7687 | .7419 | .7186 | .6978 |
| 100.00 | .8111 | .7817 | .7565 | .7344 | .7147 |
| 150.00 | .8226 | .7950 | .7713 | .7505 | .7320 |
| 200.00 | .8301 | .8036 | .7808 | .7609 | .7432 |
| 300.00 | .8395 | .8145 | .7930 | .7742 | .7575 |
| 500.00 | .8502 | .8268 | .8068 | .7892 | .7736 |
| 700.00 | .8565 | .8341 | .8149 | .7980 | .7831 |
| 1000.0 | .8627 | .8413 | .8229 | .8067 | .7924 |

Table 12a. Values of $G_b$ for $\theta = \pi/18$.

| $q^*$ \ $h_s/5a$ | .1 | .2 | .3 | .4 | .5 |
|---|---|---|---|---|---|
| .00010 | .0062 | .0095 | .0136 | .0185 | .0247 |
| .00015 | .0076 | .0117 | .0167 | .0227 | .0302 |
| .00020 | .0088 | .0135 | .0192 | .0262 | .0349 |
| .00030 | .0108 | .0165 | .0236 | .0321 | .0427 |
| .00050 | .0139 | .0213 | .0304 | .0414 | .0551 |
| .00070 | .0165 | .0252 | .0360 | .0489 | .0651 |
| .00100 | .0197 | .0301 | .0429 | .0584 | .0777 |
| .00150 | .0241 | .0369 | .0525 | .0714 | .0948 |
| .00200 | .0279 | .0426 | .0606 | .0823 | .1092 |
| .00300 | .0341 | .0521 | .0740 | .1004 | .1329 |
| .00500 | .0440 | .0670 | .0951 | .1287 | .1696 |
| .00700 | .0519 | .0791 | .1121 | .1512 | .1984 |
| .01000 | .0620 | .0942 | .1330 | .1788 | .2333 |
| .01500 | .0756 | .1146 | .1611 | .2152 | .2784 |
| .02000 | .0871 | .1315 | .1841 | .2444 | .3137 |
| .03000 | .1059 | .1590 | .2208 | .2901 | .3672 |
| .05000 | .1351 | .2004 | .2743 | .3540 | .4385 |
| .07000 | .1579 | .2318 | .3133 | .3985 | .4857 |
| .10000 | .1854 | .2685 | .3571 | .4463 | .5341 |
| .15000 | .2209 | .3135 | .4082 | .4995 | .5854 |
| .20000 | .2487 | .3470 | .4444 | .5353 | .6187 |
| .30000 | .2907 | .3951 | .4937 | .5822 | .6606 |
| .50000 | .3469 | .4547 | .5511 | .6341 | .7054 |
| .70000 | .3845 | .4921 | .5853 | .6638 | .7303 |
| 1.0000 | .4238 | .5292 | .6181 | .6918 | .7535 |
| 1.5000 | .4666 | .5679 | .6514 | .7195 | .7762 |
| 2.0000 | .4952 | .5930 | .6725 | .7370 | .7903 |
| 3.0000 | .5327 | .6250 | .6991 | .7587 | .8078 |
| 5.0000 | .5750 | .6602 | .7279 | .7821 | .8266 |
| 7.0000 | .5999 | .6806 | .7445 | .7955 | .8373 |
| 10.000 | .6239 | .7001 | .7602 | .8081 | .8474 |
| 15.000 | .6484 | .7199 | .7762 | .8209 | .8576 |
| 20.000 | .6642 | .7326 | .7864 | .8291 | .8642 |
| 30.000 | .6845 | .7488 | .7994 | .8395 | .8724 |
| 50.000 | .7070 | .7669 | .8138 | .8511 | .8816 |
| 70.000 | .7203 | .7775 | .8223 | .8579 | .8870 |
| 100.00 | .7333 | .7878 | .8306 | .8645 | .8923 |
| 150.00 | .7467 | .7985 | .8391 | .8713 | .8977 |
| 200.00 | .7554 | .8055 | .8447 | .8758 | .9013 |
| 300.00 | .7668 | .8145 | .8519 | .8816 | .9059 |
| 500.00 | .7798 | .8249 | .8602 | .8882 | .9111 |
| 700.00 | .7877 | .8311 | .8651 | .8922 | .9143 |
| 1000.0 | .7954 | .8372 | .8700 | .8961 | .9174 |

Table 12b. Values of $G_b$ for $\theta = \pi/18$.

| $q^*$ \ $h_s/5a$ | .5 | .6 | .7 | .8 | .9 |
|---|---|---|---|---|---|
| .00010 | .0247 | .0328 | .0442 | .0618 | .0936 |
| .00015 | .0302 | .0401 | .0540 | .0756 | .1143 |
| .00020 | .0349 | .0463 | .0623 | .0871 | .1315 |
| .00030 | .0427 | .0567 | .0762 | .1063 | .1599 |
| .00050 | .0551 | .0730 | .0981 | .1364 | .2035 |
| .00070 | .0651 | .0862 | .1156 | .1604 | .2376 |
| .00100 | .0777 | .1027 | .1375 | .1899 | .2786 |
| .00150 | .0948 | .1252 | .1670 | .2292 | .3311 |
| .00200 | .1092 | .1439 | .1913 | .2608 | .3717 |
| .00300 | .1329 | .1746 | .2306 | .3108 | .4327 |
| .00500 | .1696 | .2212 | .2890 | .3819 | .5126 |
| .00700 | .1984 | .2572 | .3327 | .4323 | .5648 |
| .01000 | .2333 | .2998 | .3827 | .4872 | .6179 |
| .01500 | .2784 | .3532 | .4427 | .5495 | .6737 |
| .02000 | .3137 | .3937 | .4862 | .5921 | .7097 |
| .03000 | .3672 | .4527 | .5468 | .6483 | .7547 |
| .05000 | .4385 | .5269 | .6182 | .7105 | .8017 |
| .07000 | .4857 | .5735 | .6606 | .7456 | .8273 |
| .10000 | .5341 | .6192 | .7006 | .7777 | .8501 |
| .15000 | .5854 | .6656 | .7398 | .8083 | .8714 |
| .20000 | .6187 | .6947 | .7637 | .8266 | .8839 |
| .30000 | .6606 | .7303 | .7925 | .8483 | .8987 |
| .50000 | .7054 | .7673 | .8217 | .8701 | .9134 |
| .70000 | .7303 | .7876 | .8375 | .8817 | .9212 |
| 1.0000 | .7535 | .8062 | .8520 | .8923 | .9283 |
| 1.5000 | .7762 | .8243 | .8659 | .9025 | .9351 |
| 2.0000 | .7903 | .8355 | .8745 | .9088 | .9393 |
| 3.0000 | .8078 | .8493 | .8851 | .9165 | .9444 |
| 5.0000 | .8266 | .8641 | .8964 | .9248 | .9500 |
| 7.0000 | .8373 | .8725 | .9029 | .9294 | .9531 |
| 10.000 | .8474 | .8805 | .9089 | .9338 | .9560 |
| 15.000 | .8576 | .8885 | .9150 | .9383 | .9590 |
| 20.000 | .8642 | .8936 | .9189 | .9411 | .9609 |
| 30.000 | .8724 | .9001 | .9239 | .9447 | .9632 |
| 50.000 | .8816 | .9073 | .9294 | .9487 | .9659 |
| 70.000 | .8870 | .9115 | .9326 | .9511 | .9675 |
| 100.00 | .8923 | .9156 | .9357 | .9533 | .9690 |
| 150.00 | .8977 | .9199 | .9390 | .9557 | .9705 |
| 200.00 | .9013 | .9227 | .9411 | .9573 | .9716 |
| 300.00 | .9059 | .9263 | .9438 | .9592 | .9729 |
| 500.00 | .9111 | .9304 | .9470 | .9615 | .9744 |
| 700.00 | .9143 | .9329 | .9489 | .9629 | .9752 |
| 1000.0 | .9174 | .9353 | .9507 | .9642 | .9758 |

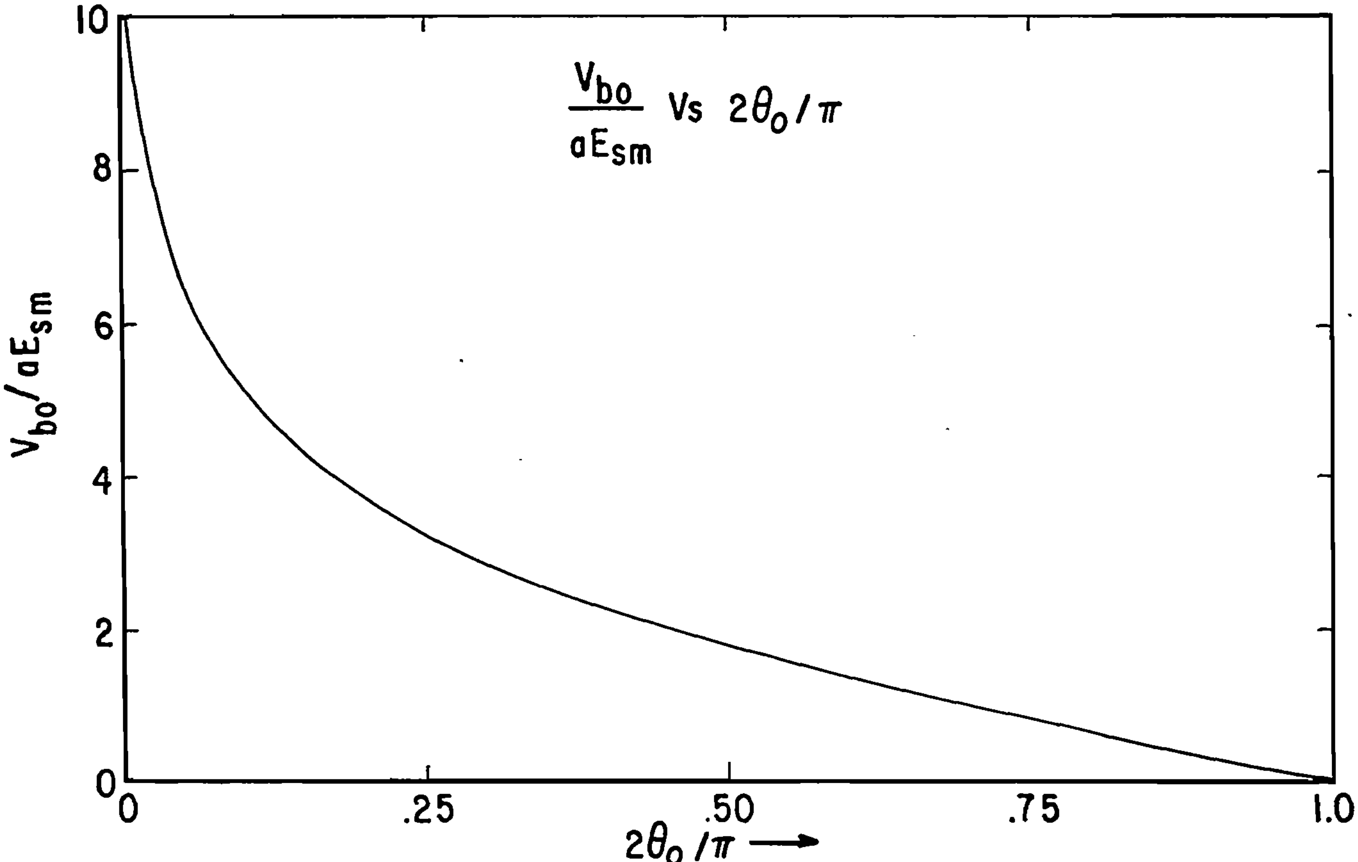



FIGURE 9. NORMALIZED EQUIVALENT BICONE VOLTAGE

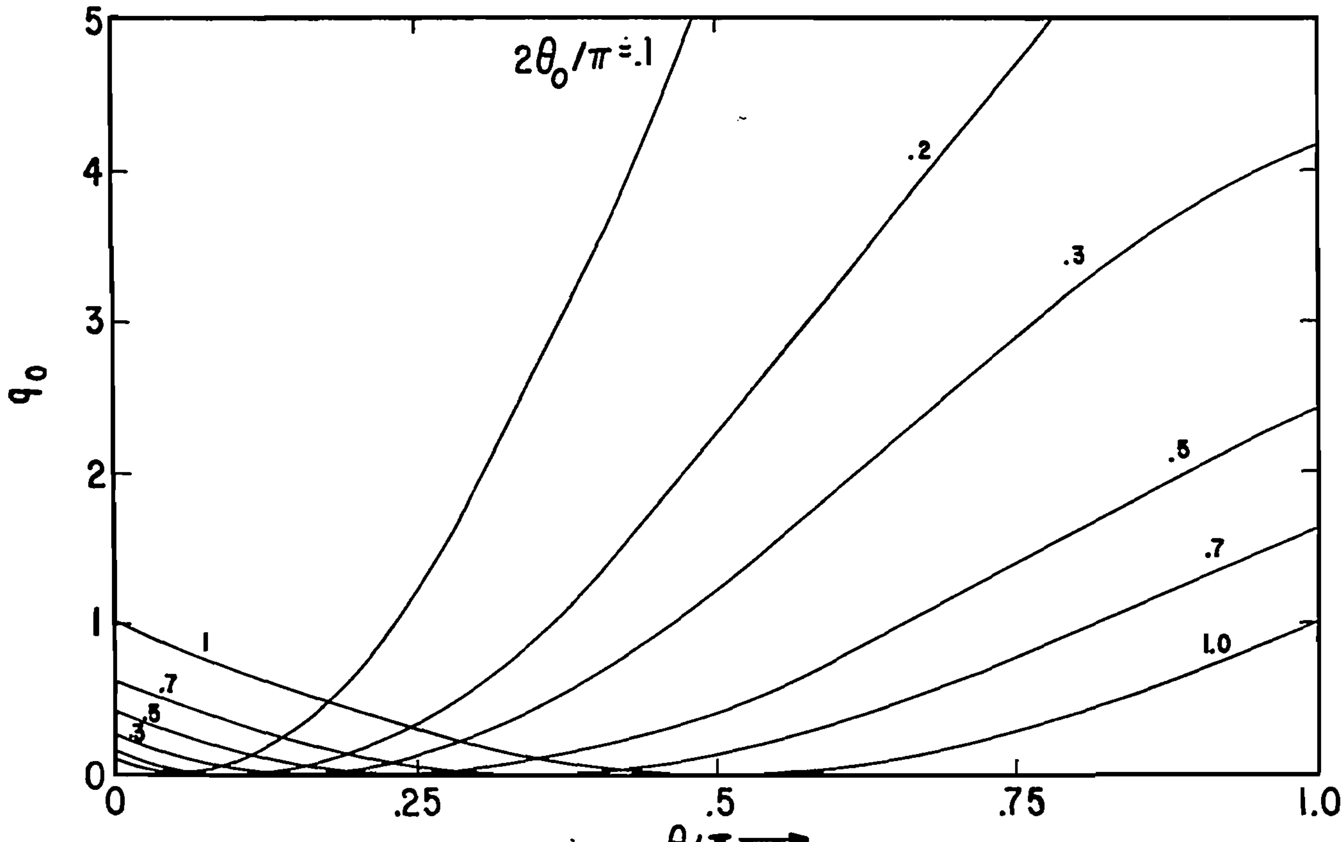



FIGURE 10. NORMALIZED RETARDED DISTORTION TIME WITH $2\theta_o/\pi$ AS A PARAMETER

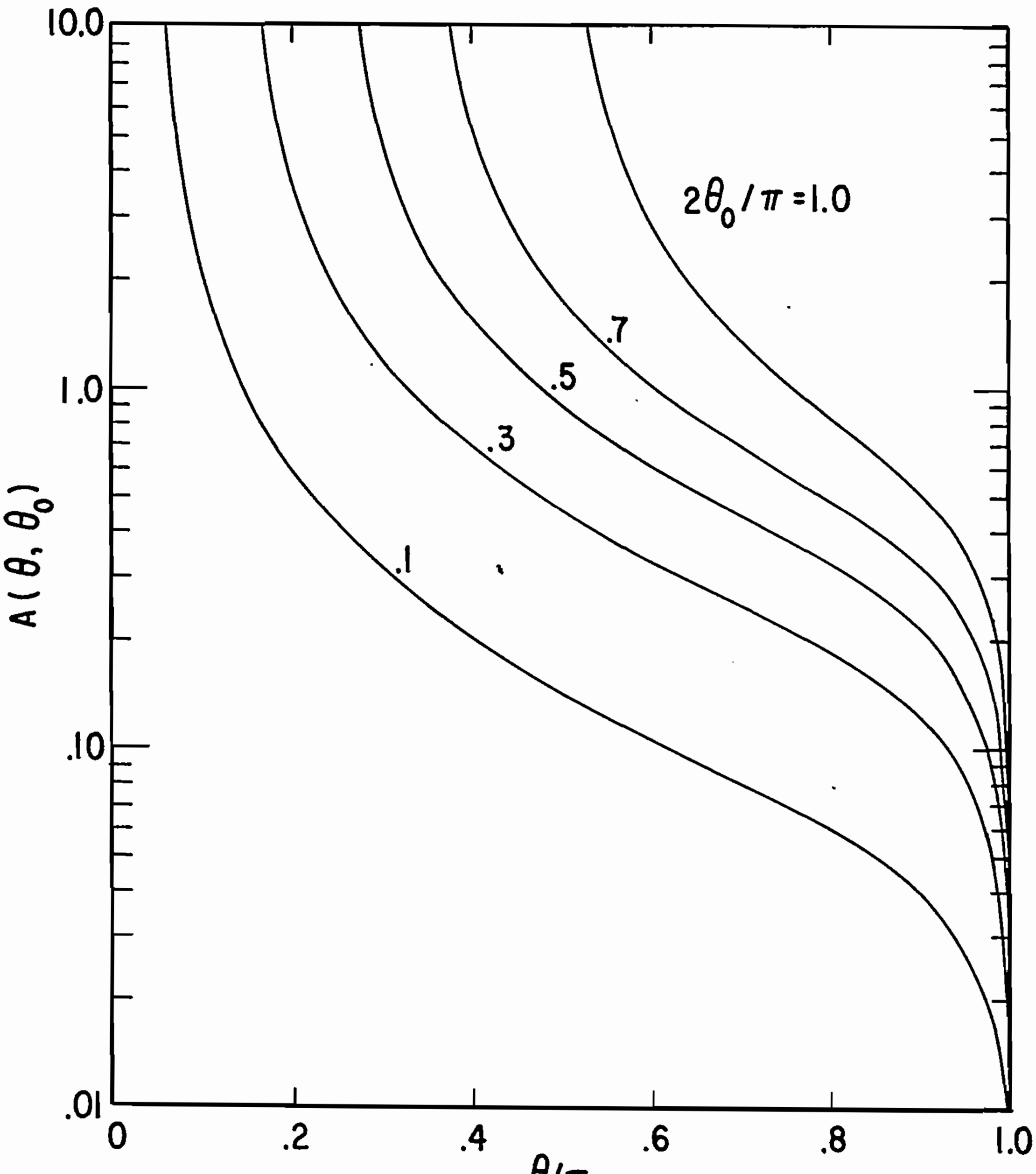
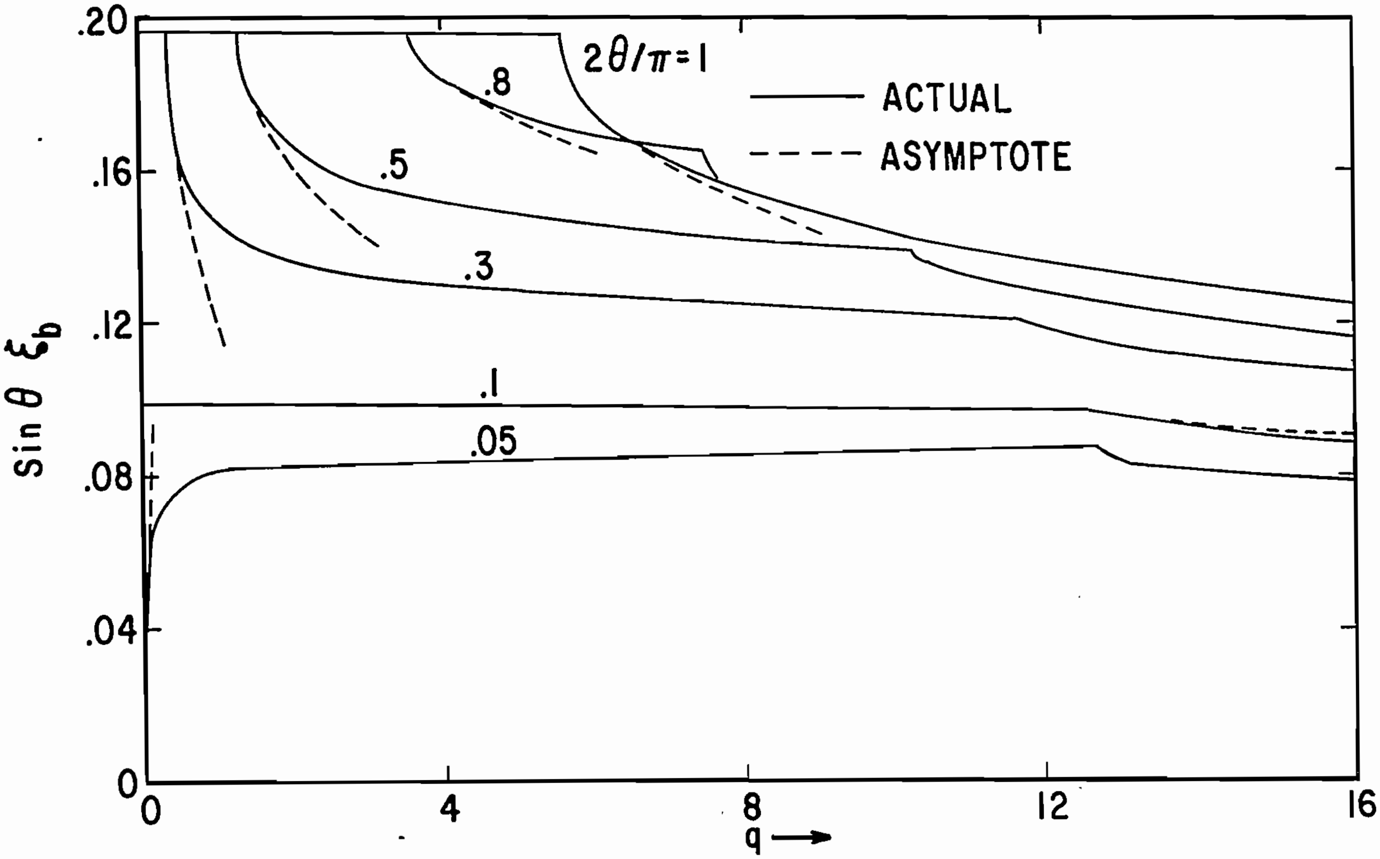
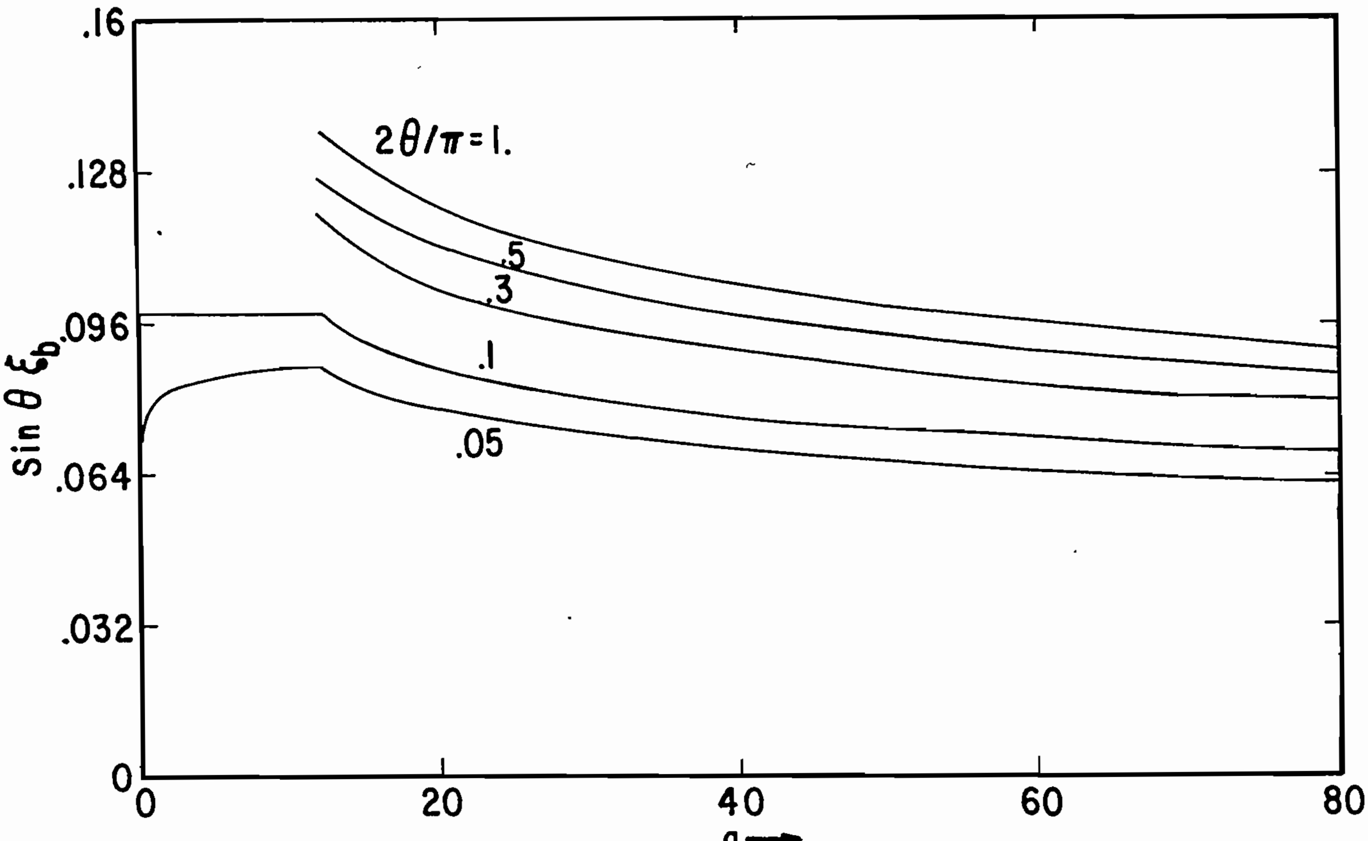



FIGURE 11. RELATIVE MAGNITUDE OF THE FIELD DISTORTION WITH $2\theta_0/\pi$ AS A PARAMETER




A. EARLY NORMALIZED RETARDED TIME




B. INTERMEDIATE NORMALIZED RETARDED TIME

FIGURE 12. NORMALIZED ELECTRIC FIELD WITH $2\theta/\pi$ AS A PARAMETER FOR $2\theta_0/\pi = 0.1$

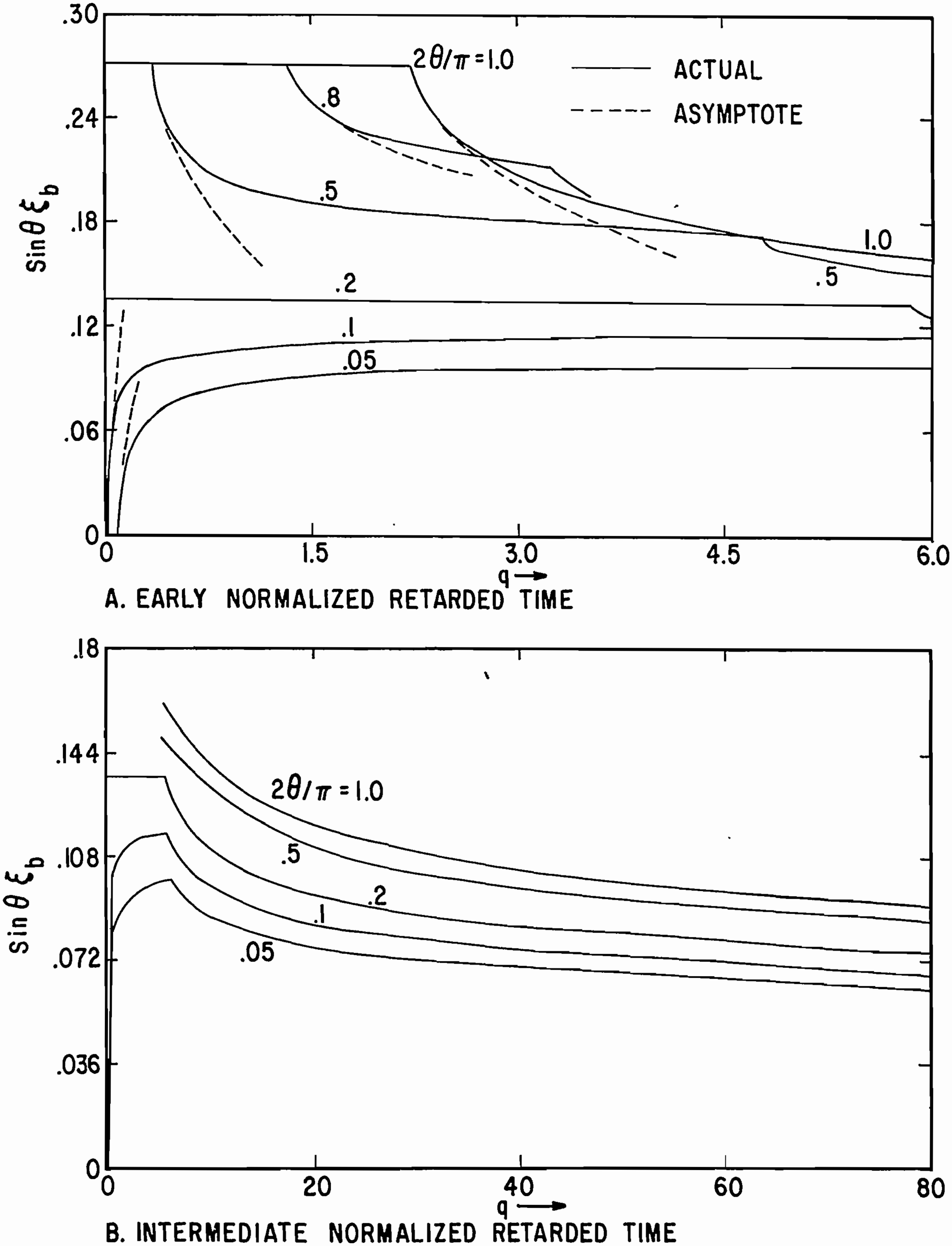
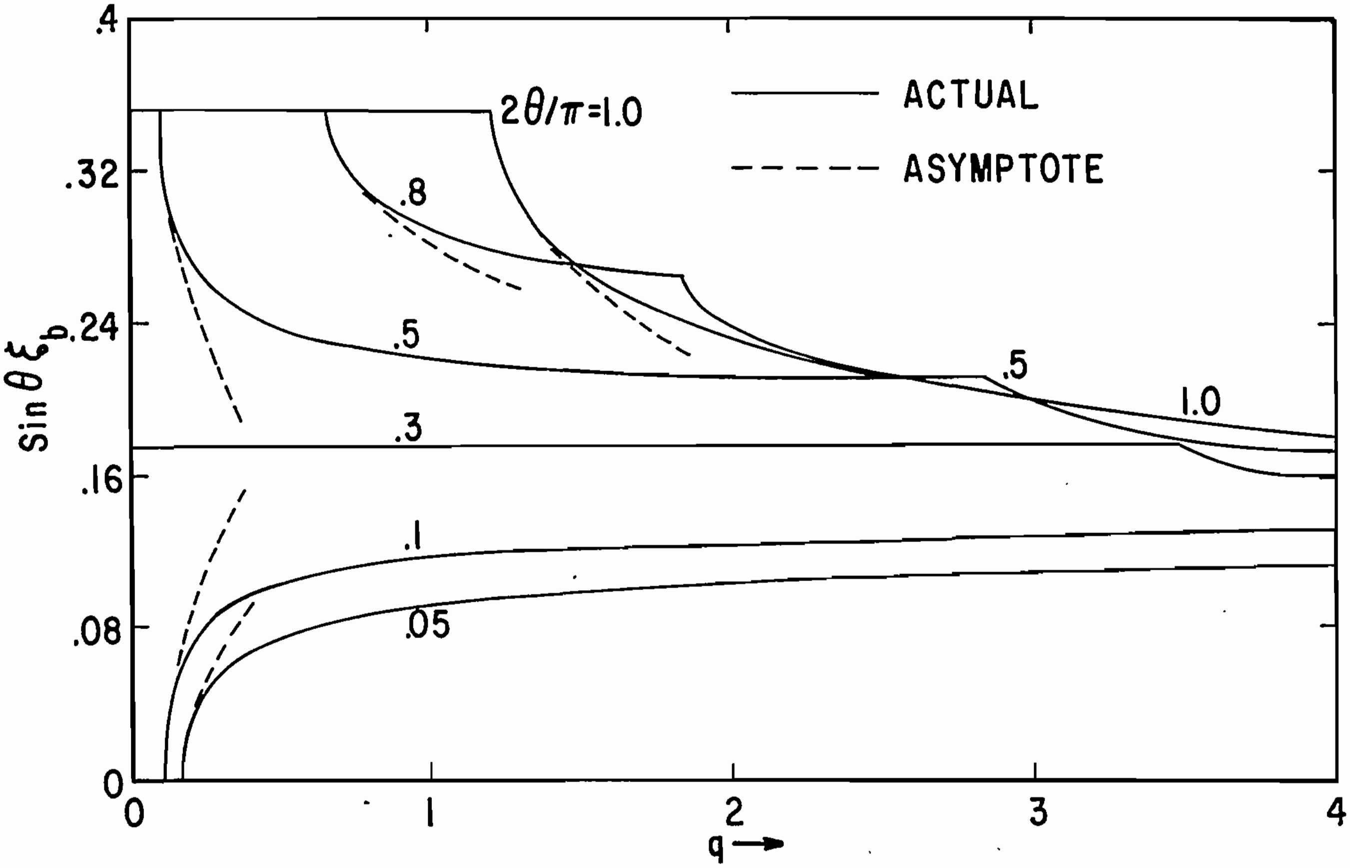
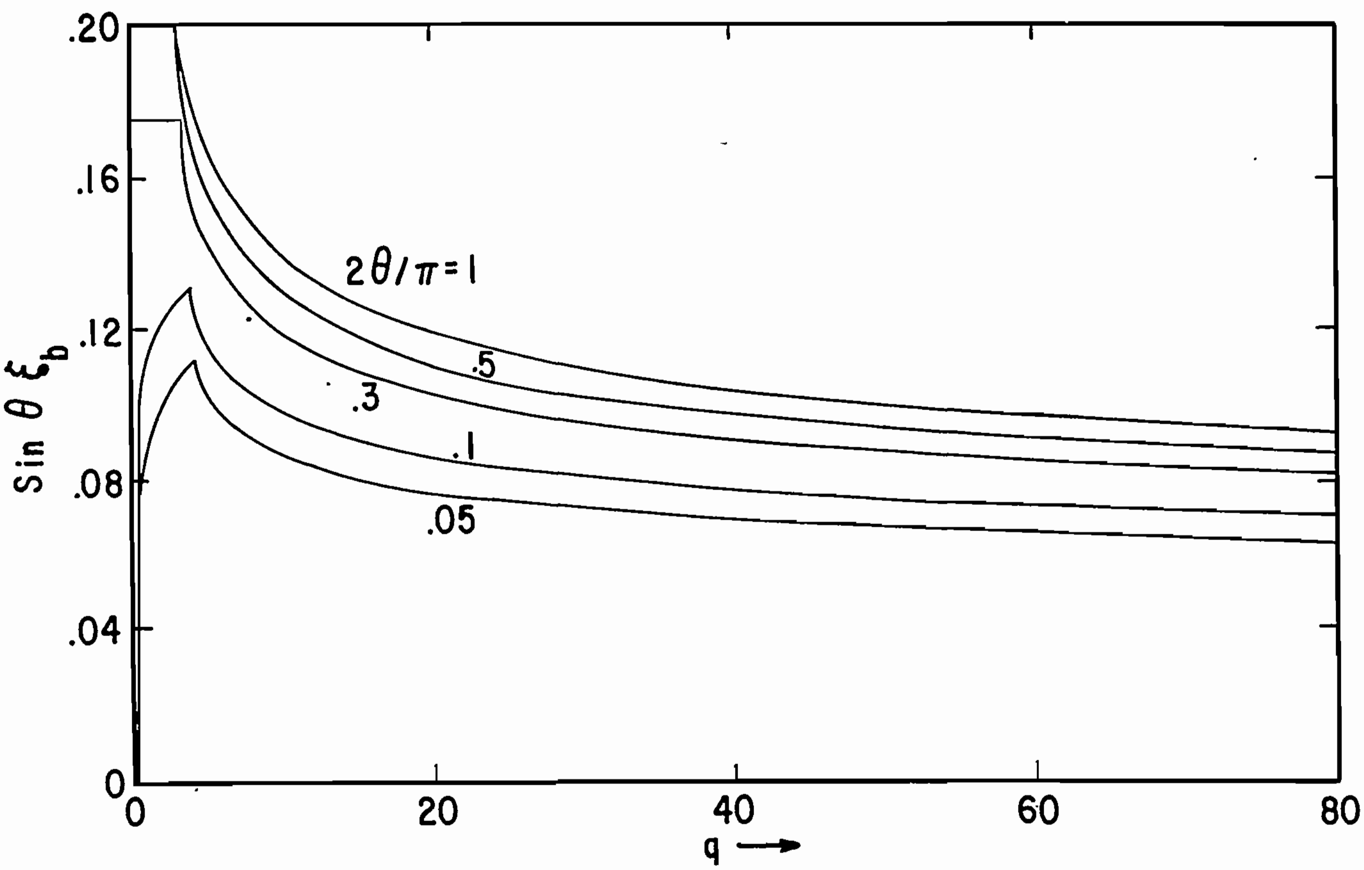



FIGURE 13. NORMALIZED ELECTRIC FIELD WITH $2\theta/\pi$ AS A PARAMETER FOR $2\theta_0/\pi = 0.2$

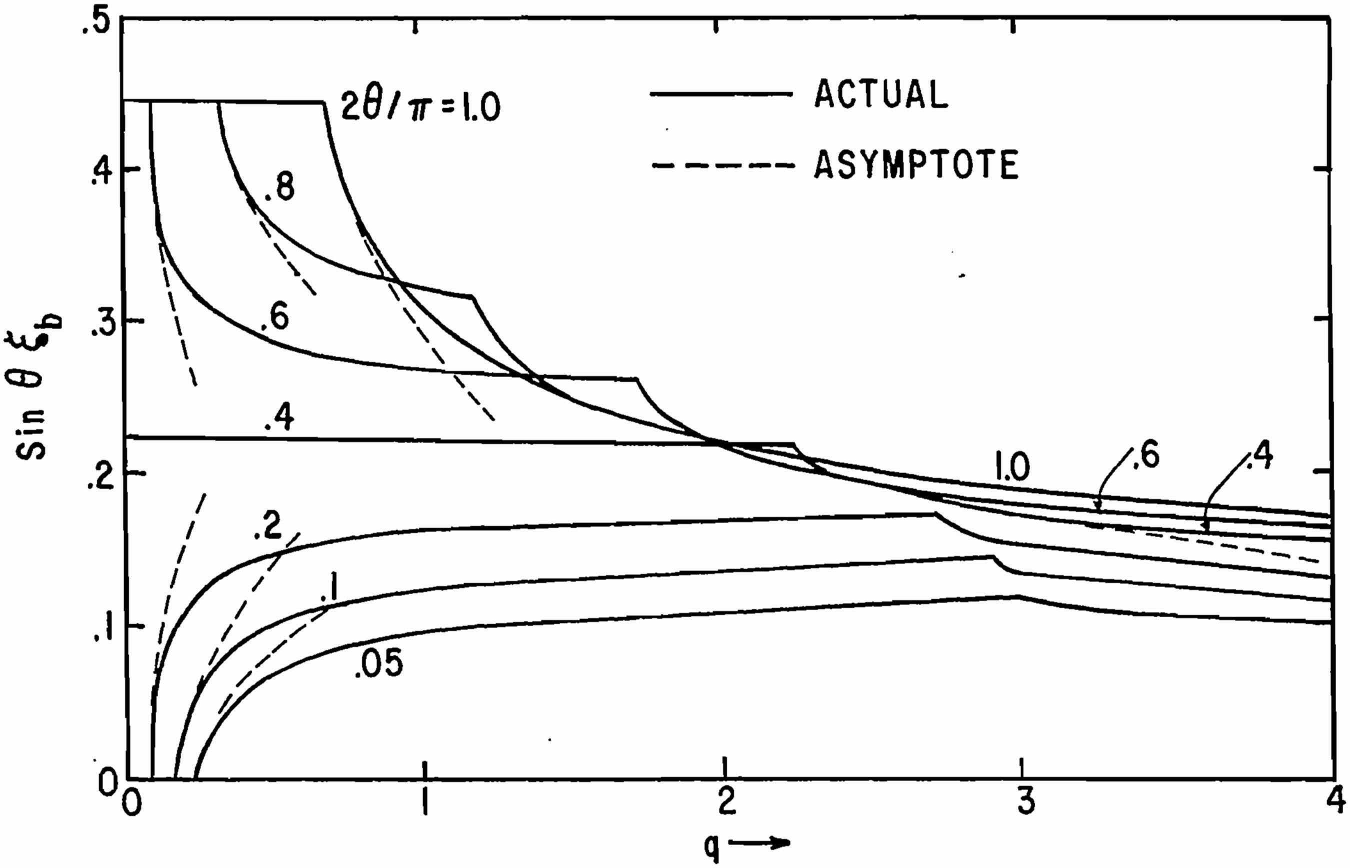



A. EARLY NORMALIZED RETARDED TIME

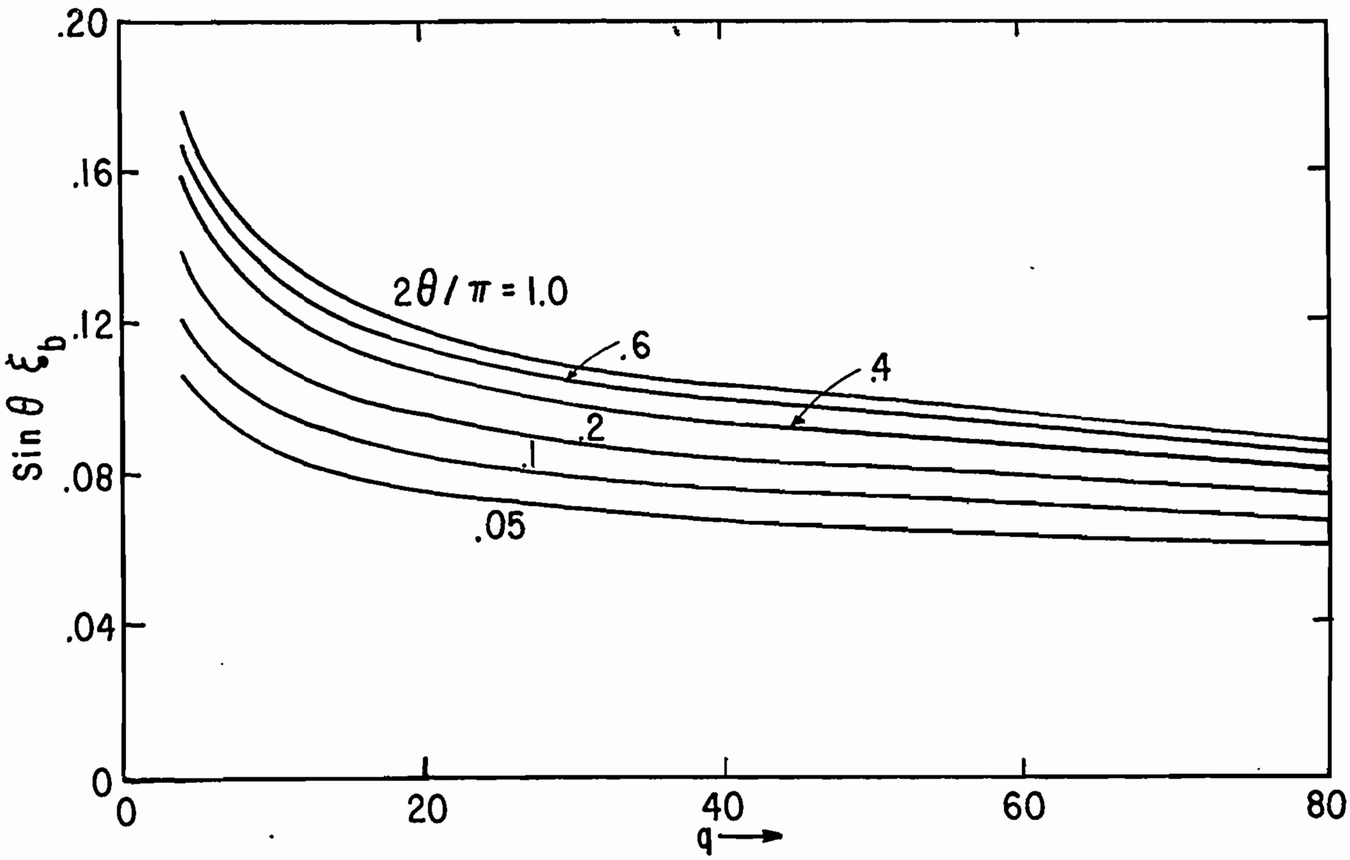



B. INTERMEDIATE NORMALIZED RETARDED TIME

FIGURE 14. NORMALIZED ELECTRIC FIELD WITH $2\theta/\pi$ AS A PARAMETER FOR $2\theta_0/\pi = 0.3$




A. EARLY NORMALIZED RETARDED TIME




B. INTERMEDIATE NORMALIZED RETARDED TIME

FIGURE 15. NORMALIZED ELECTRIC FIELD WITH $2\theta/\pi$ AS A PARAMETER FOR $2\theta_0/\pi = 0.4$

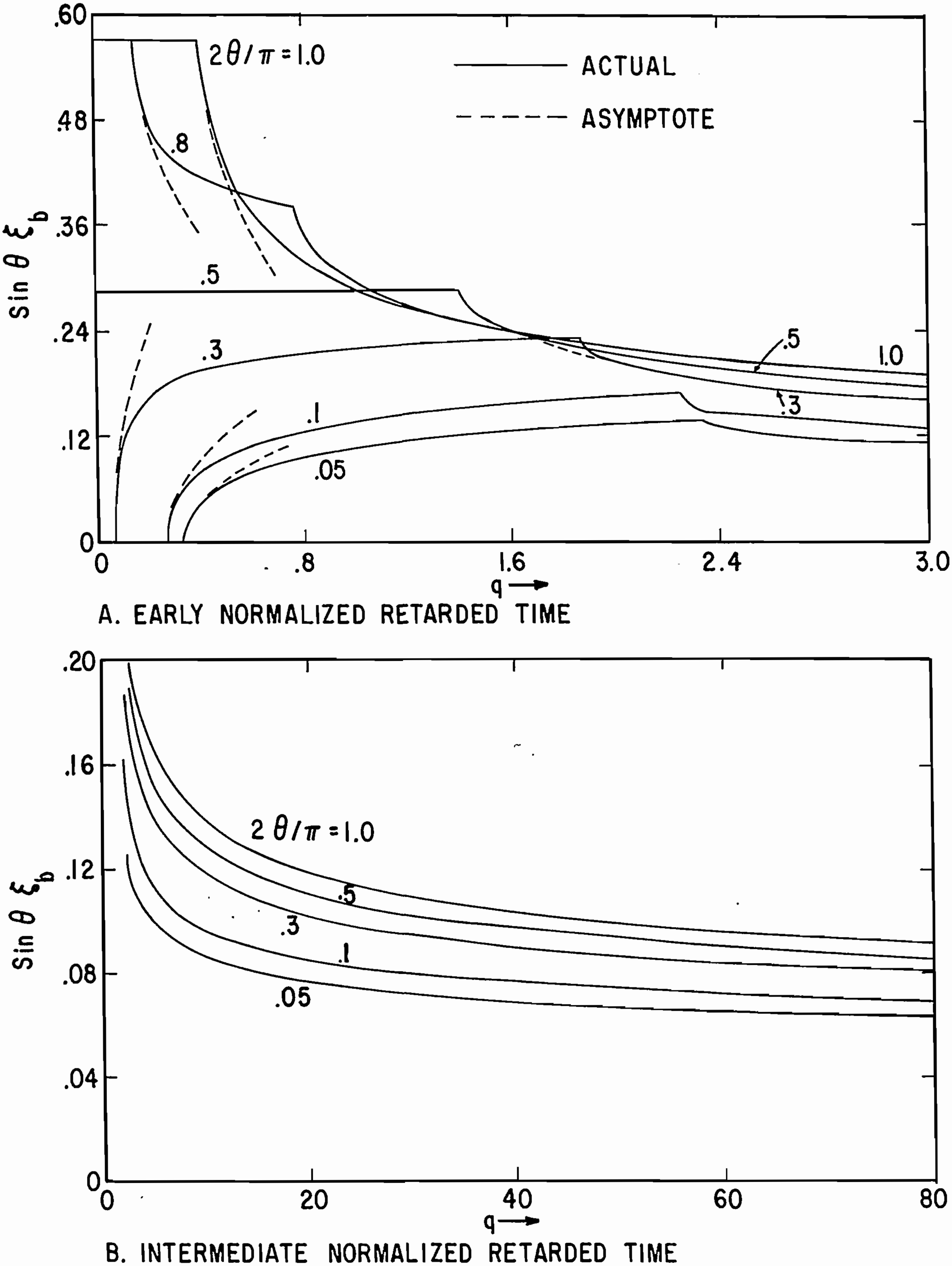
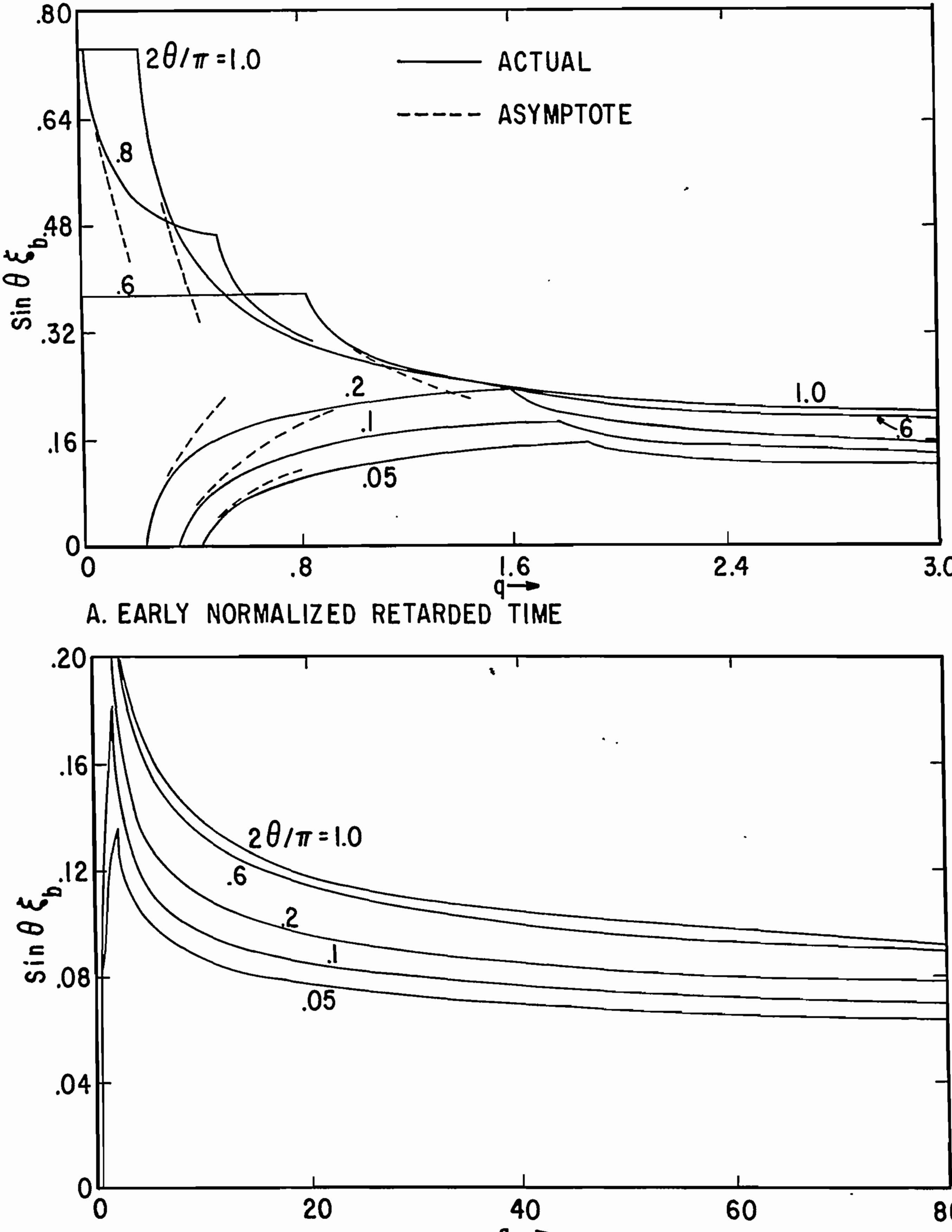



FIGURE 16. NORMALIZED ELECTRIC FIELD WITH $2\theta/\pi$ AS A PARAMETER FOR $2\theta_0/\pi = 0.5$




A. EARLY NORMALIZED RETARDED TIME



B. INTERMEDIATE NORMALIZED RETARDED TIME

FIGURE 17. NORMALIZED ELECTRIC FIELD WITH $2\theta/\pi$ AS A PARAMETER FOR $2\theta_0/\pi = 0.6$

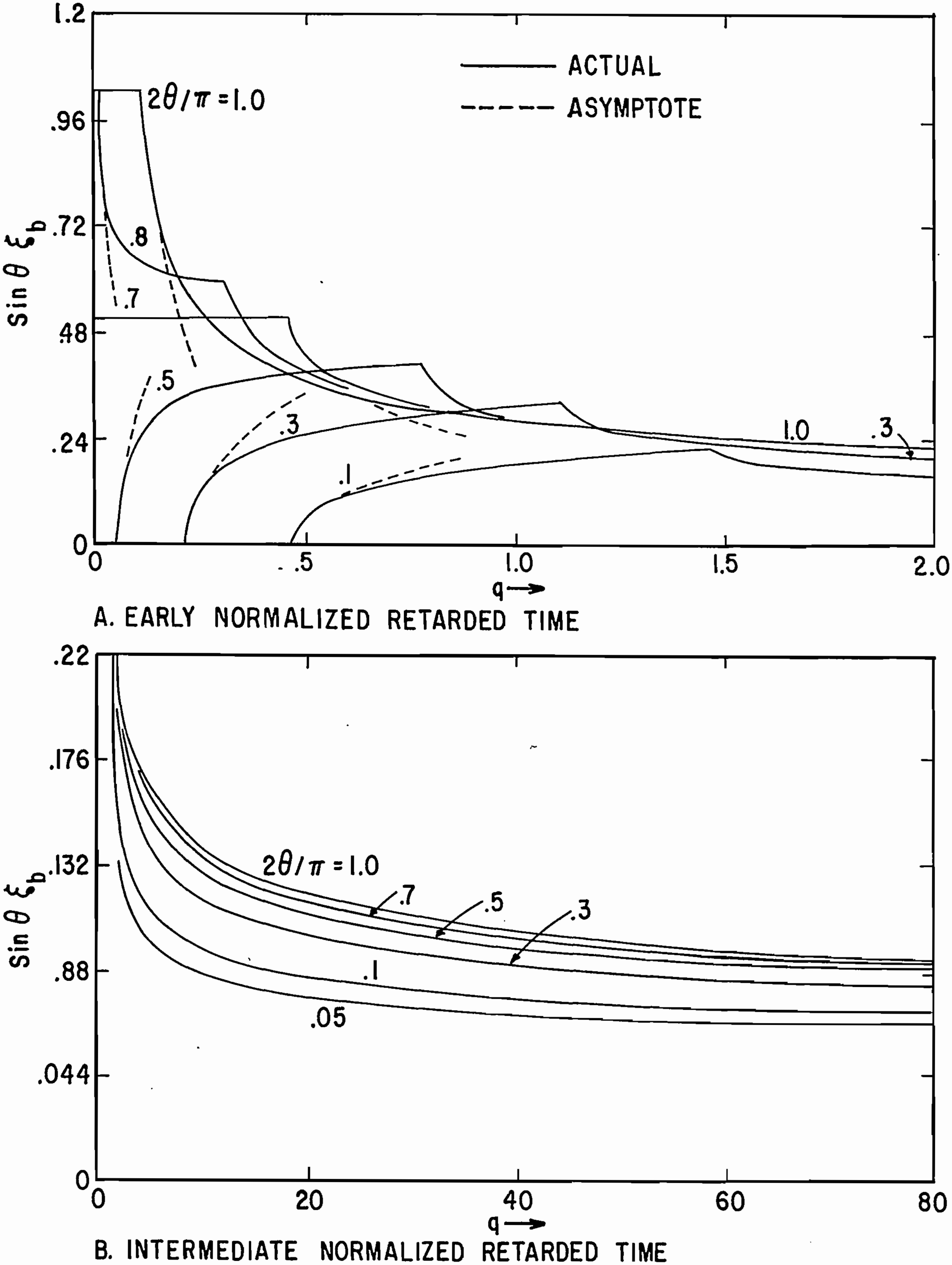
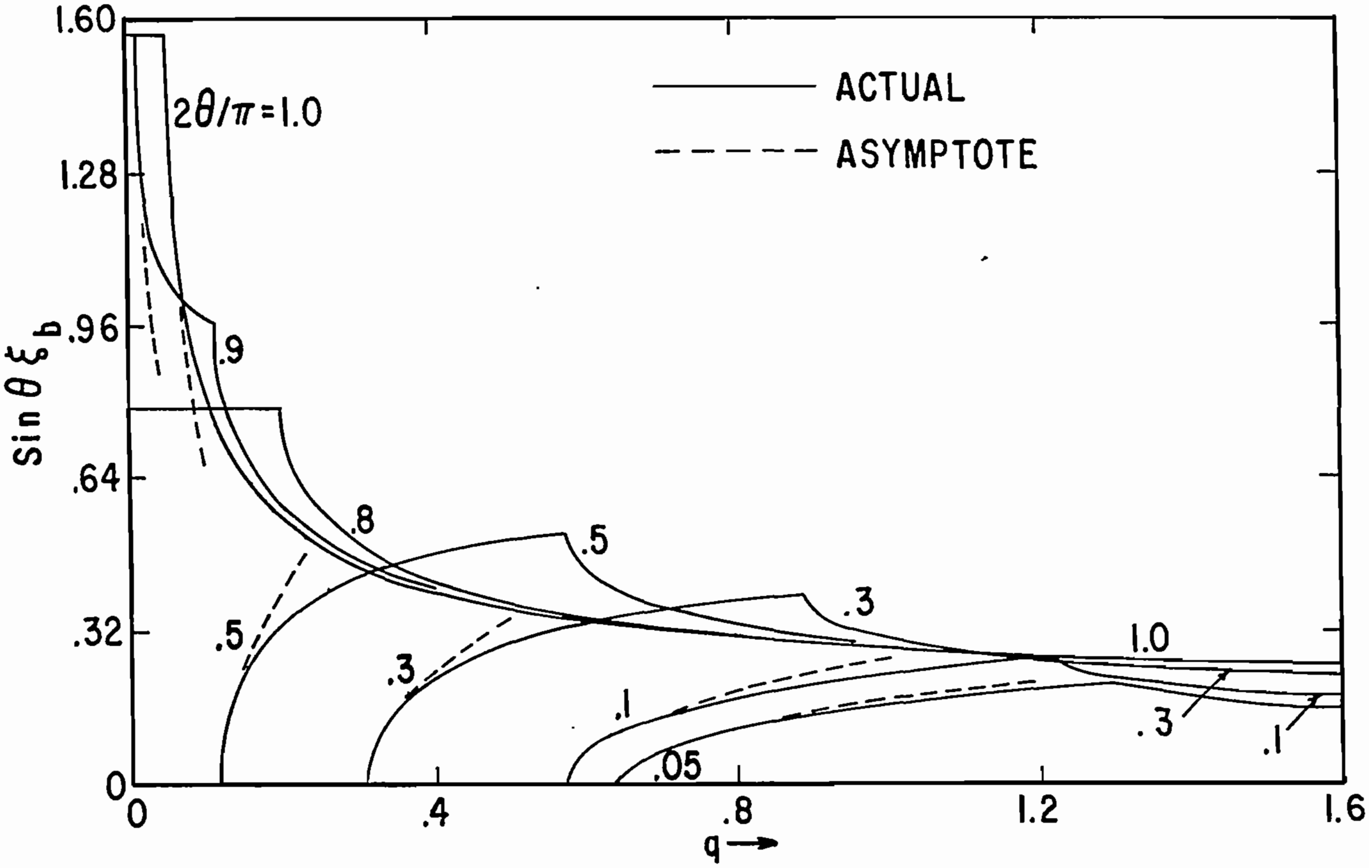
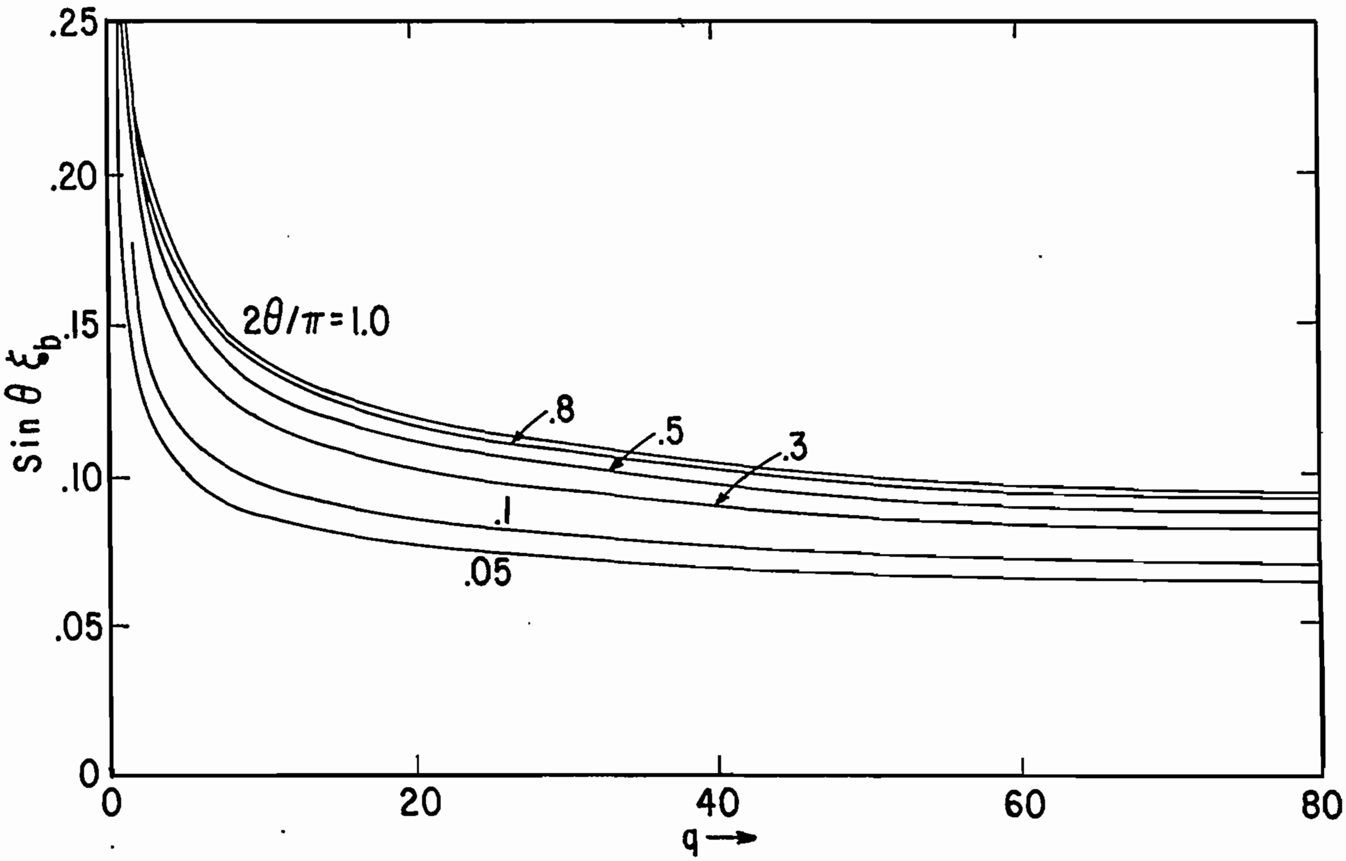



FIGURE 18. NORMALIZED ELECTRIC FIELD WITH $2\theta/\pi$ AS A PARAMETER FOR $2\theta_0/\pi = 0.7$

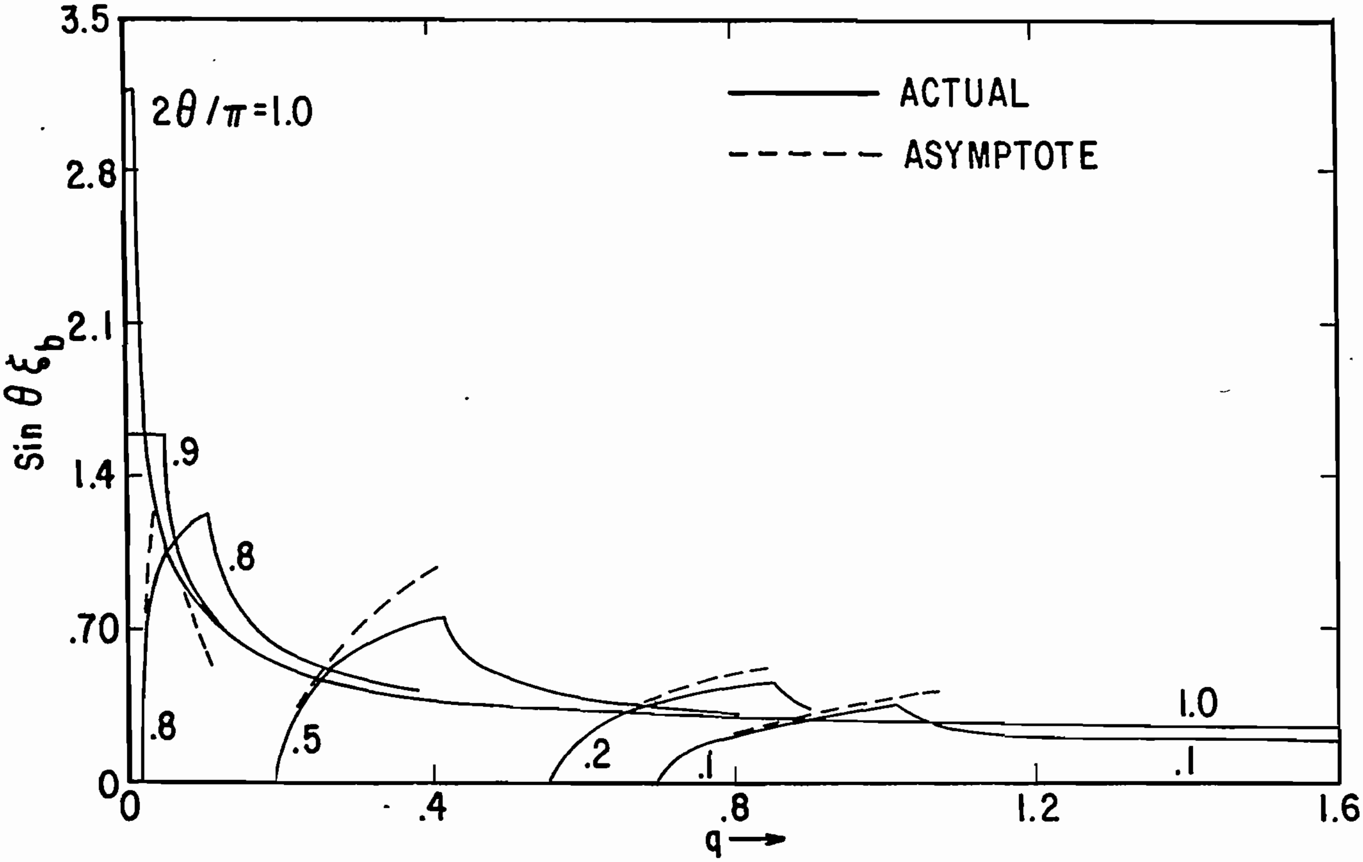



A. EARLY NORMALIZED RETARDED TIME

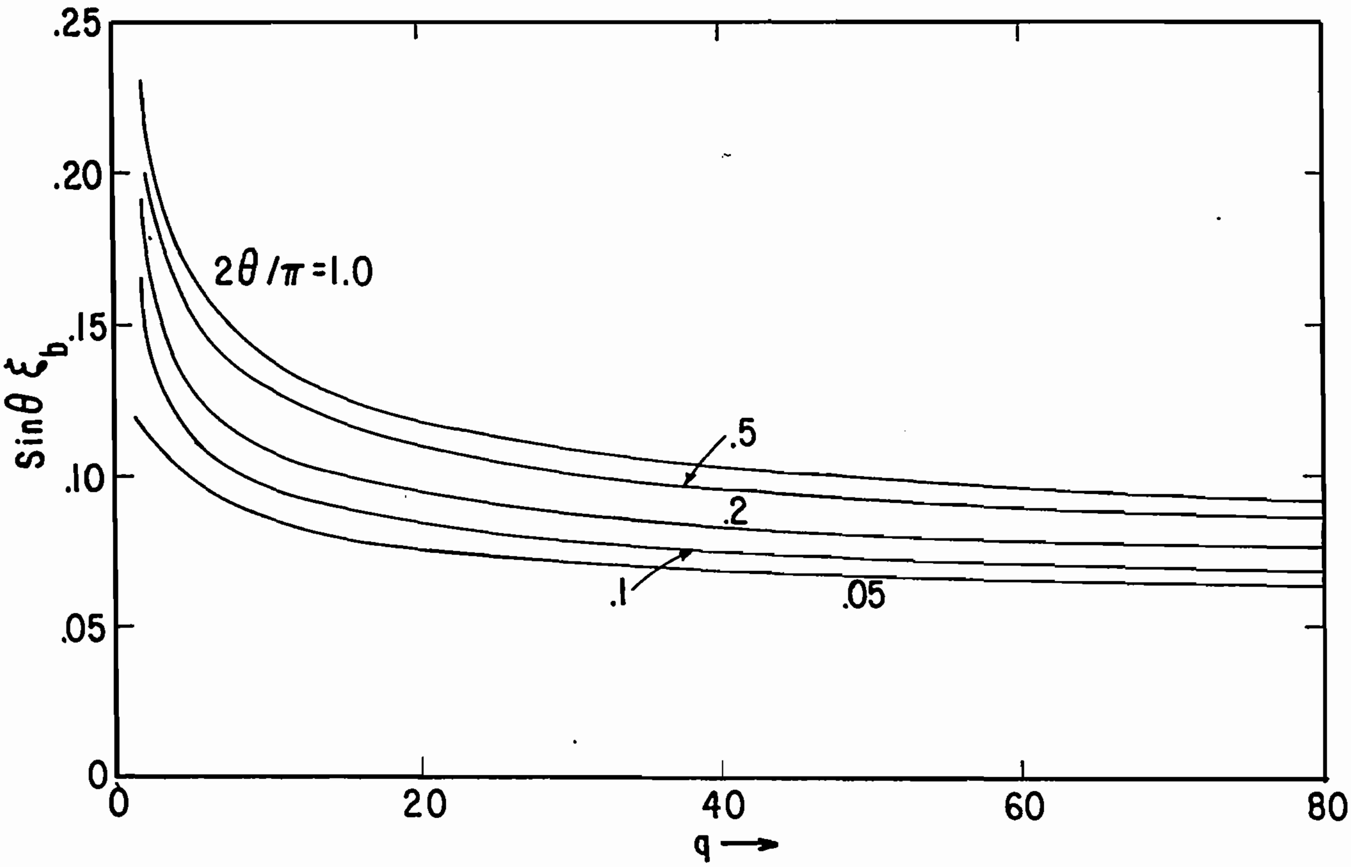



B. INTERMEDIATE NORMAIZED RETARDED TIME

FIGURE 19. NORMALIZED ELECTRIC FIELD WITH $2\theta/\pi$ AS A PARAMETER FOR $2\theta_0/\pi = 0.8$




A. EARLY NORMALIZED RETARDED TIME




B. INTERMEDIATE NORMALIZED RETARDED TIME

FIGURE 20. NORMALIZED ELECTRIC FIELD WITH $2\theta/\pi$ AS A PARAMETER FOR $2\theta/\pi = 0.9$

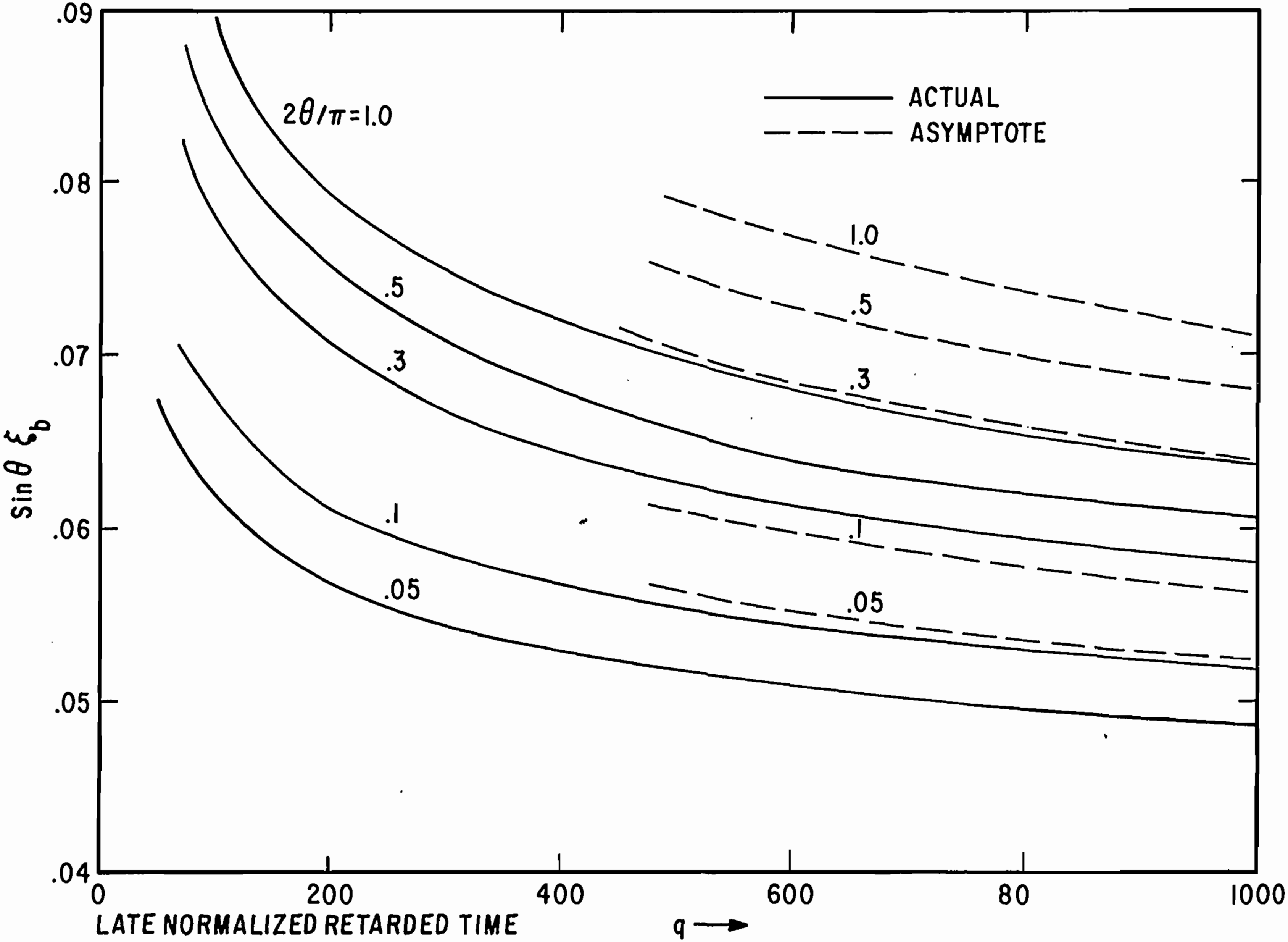
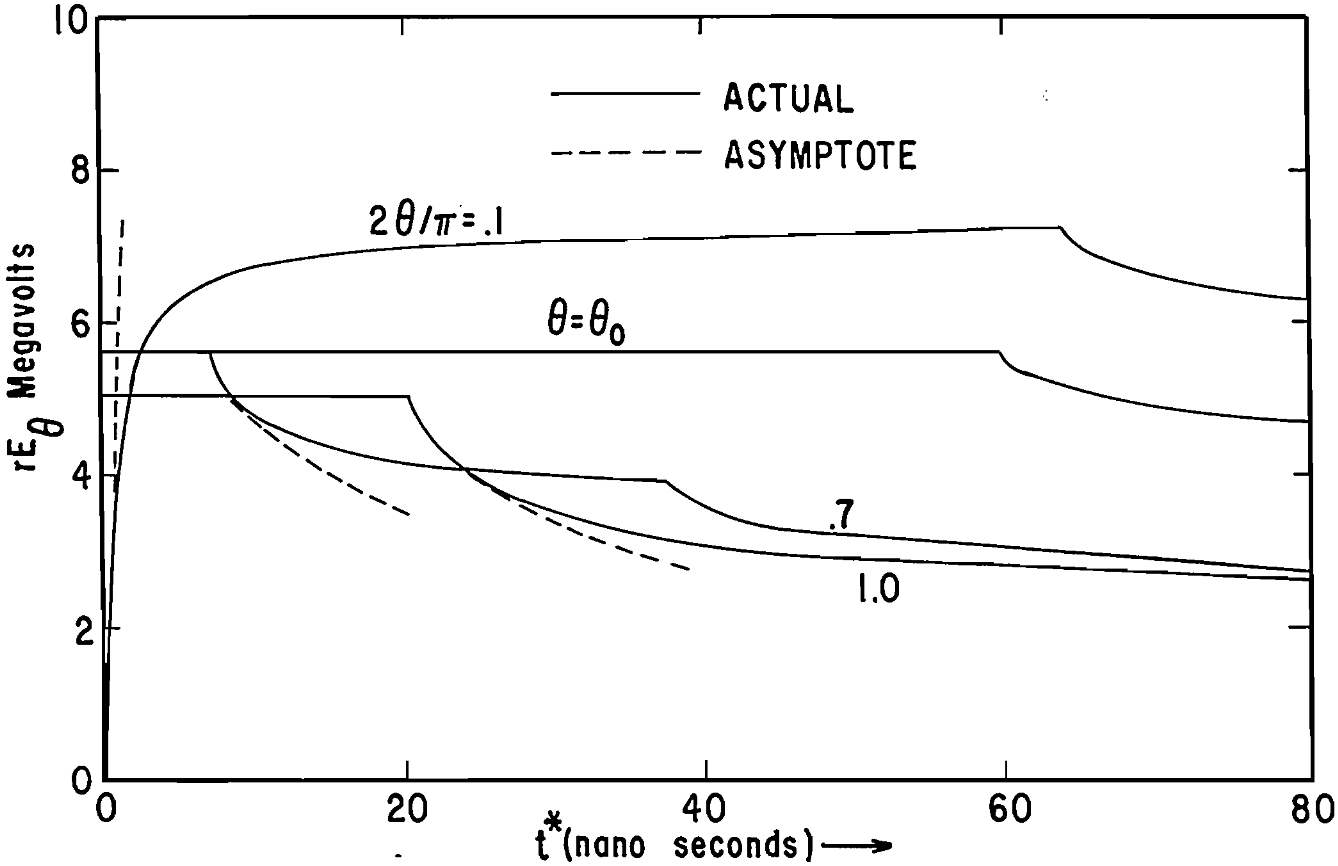
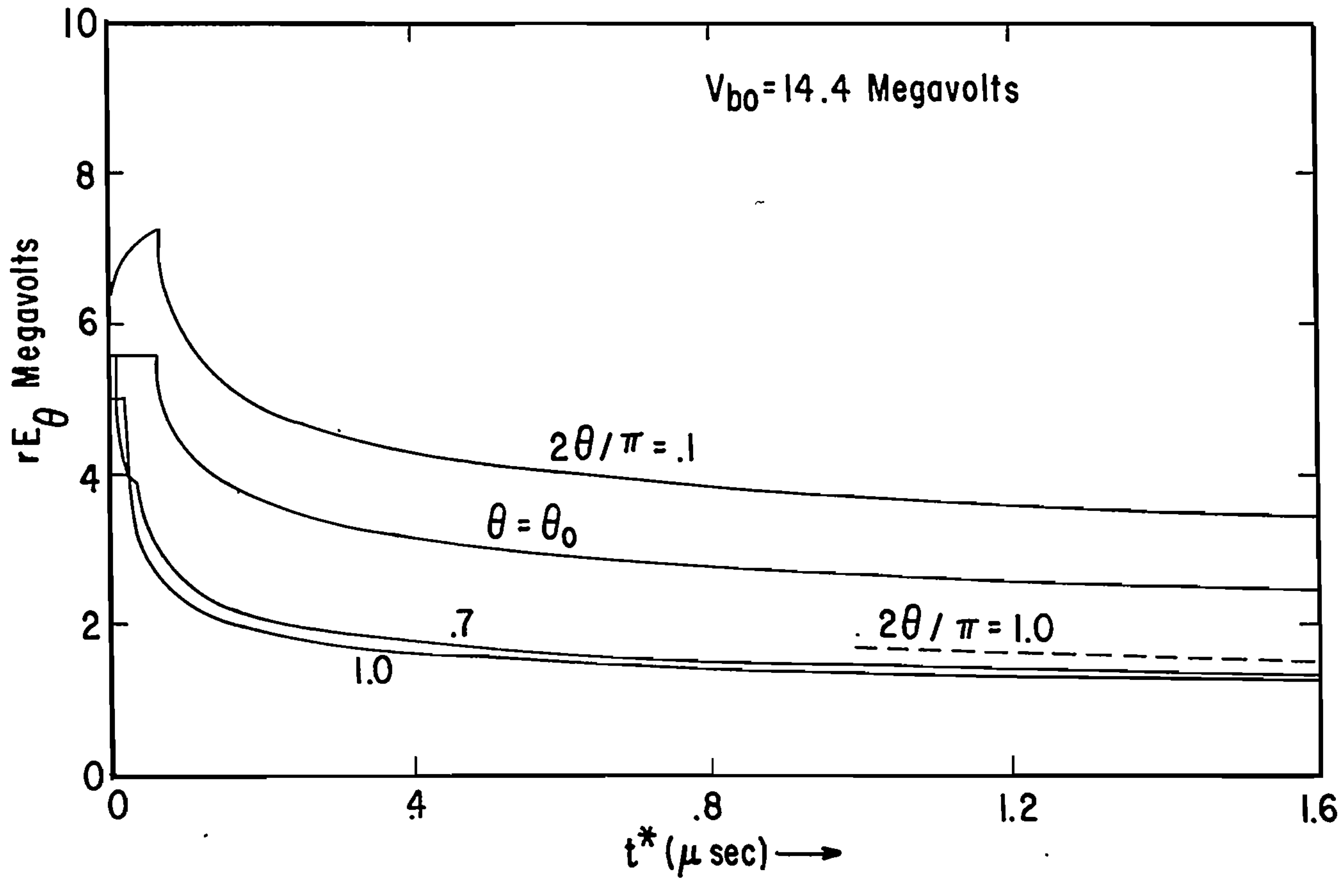



FIGURE 21. NORMALIZED ELECTRIC FIELD WITH $2\theta/\pi$ AS A PARAMETER FOR $\theta_0 > 0.0$




A. EARLY RETARDED TIME




B. LATE RETARDED TIME

FIGURE 22. NORMALIZED ELECTRIC FIELD WITH $2\theta/\pi$ AS A PARAMETER FOR $h_s/a = 2.0$ AND $E_{sm} = 10^6$ VOLTS PER METER

## Appendix A. Asymptotic Expansion of $G(\eta)$ for $\eta \rightarrow \infty$

The function $G(\eta)$ can be written in the form

$$G(\eta) = \int_0^{\infty} e^{-y\eta}\, \Phi(y)\ dy \tag{A1}$$

where

$$\Phi(y) = \frac{I_o(y)}{y\left[K_o^{\,2}(y) + \pi^2\, I_o^{\,2}(y)\right]} \tag{A2}$$

The function $\Phi(y)$ has an asymptotic expansion for very small y given by

$$\Phi(y) = \frac{1}{y\left[\ell n^2(y\,\Gamma/2) + \pi^2\right]}\left(1 + O(y^2)\right) \tag{A3}$$

where $\Gamma = 1.7810 \cdots$, the exponential of Euler's constant.

Define the first term of Eqn. ( A 3) as

$$\phi(y) = \frac{1}{y\left[\ell n^2(y\,\Gamma/2) + \pi^2\right]} \tag{A4}$$

Then, there exist a $\delta$ and an $\epsilon > 0$ such that

$$\left|\Phi(y) - \phi(y)\right| < \epsilon \qquad 0 < y \leq \delta \tag{A5}$$

Choose $\delta$ such that $0 < \delta < 2/\Gamma$.

Also, $\Phi(y)$ and $\phi(y)$ are bounded for $y > \delta$, thus,

$$\left|\Phi(y) - \phi(y)\right| < \epsilon' \qquad 0 < y \leq \infty \tag{A6}$$

where $\epsilon'$ is a finite constant.

To obtain the asymptotic form of $G(\eta)$ for $\eta \to \infty$, write[7]

$$G(\eta) = \int_0^\infty e^{-y\eta}\phi(y)\, dy + \int_0^\infty e^{-y\eta}\left[\Phi(y) - \phi(y)\right]\, dy \tag{A7}$$

By condition (A6), the second integral can be bounded.

$$\int_0^\infty e^{-y\eta}\left[\Phi(y) - \phi(y)\right]\, dy = O(\eta^{-1}) \tag{A8}$$

The first integral can be written as

$$\int_0^\infty e^{-y\eta}\phi(y)\, dy = \int_0^\delta e^{-y\eta}\phi(y)\, dy + \int_\delta^\infty e^{-y\eta}\phi(y)\, dy \tag{A9}$$

The function $\phi(y)$ is bounded for $y > \delta$, thus

$$\int_\delta^\infty e^{-y\eta}\phi(y)\, dy \doteq O\left(\frac{e^{-\delta\eta}}{\eta}\right) \tag{A10}$$

The first integral on the right side of Eqn. (A9) can be written as

$$\int_0^{\delta} e^{-y\eta}\phi(y)\, dy = \int_0^{\delta} \frac{e^{-y\eta}}{y\left[\ell n^2(y\Gamma/2) + \pi^2\right]}\, dy$$

$$= \int_0^{\delta} \frac{e^{-y\eta}}{y\,\ell n^2(y\Gamma/2)\left[1 + \frac{\pi^2}{\ell n^2(y\Gamma/2)}\right]}\, dy \qquad \text{(A11)}$$

and

$$\int_0^{\delta} e^{-y\eta}\phi(y)\, dy = \int_0^{\delta} \frac{e^{-y\eta}}{y\,\ell n^2(y\Gamma/2)}\left[1 + O\left(\ell n^{-2}(y\Gamma/2)\right)\right] dy \qquad \text{(A12)}$$

Integration by parts gives

$$\int_0^{\delta} e^{-y\eta}\phi(y)\, dy = \frac{e^{-\delta\eta}}{\ell n(\delta\Gamma/2)} + \eta\int_0^{\delta} \frac{e^{-y\eta}}{\ell n(y\Gamma/2)}\left[1 + O\left(\ell n^{-2}(y\Gamma/2)\right)\right] dy$$

$$+ O(e^{-\delta\eta}) + O\left(\frac{e^{-\delta\eta}}{\eta}\right) \qquad \text{(A13)}$$

Now let $u = y\eta$, Eqn. (A13) becomes

$$\int_0^{\delta} e^{-y\eta}\phi(y)\, dy = O\left(e^{-\delta\eta}\right) - \int_0^{\delta\eta} \frac{e^{-u}}{\left[\ell n(2\eta/\Gamma) - \ell nu\right]}\left[1 + O\left(\ell n^{-2}(u\Gamma/2\eta)\right)\right] du$$

$$= \frac{-1}{\ell n(2\eta/\Gamma)} \int_0^{\delta\eta} \frac{e^{-u}}{\left[1 - \frac{\ell nu}{\ell n(2\eta/\Gamma)}\right]}\left[1 + O\left(\ell n^{-2}(u\Gamma/2\eta)\right)\right] du$$

$$+ O(e^{-\delta\eta}) \qquad \text{(A14)}$$

Evaluating the integrals gives

$$\int_0^{\delta} e^{-y\eta}\phi(y)\,dy = \frac{1}{\ln(2\eta/\Gamma)} + O\left(\frac{1}{\ln^2(2\eta/\Gamma)}\right) \tag{A15}$$

The results of conditions (A8), (A10), and (A15) show that the main contribution to $G(\eta)$ for $\eta \rightarrow \infty$ comes from Eqn. (A15). Thus,*

$$G(\eta) = \frac{1}{\ln(2\eta/\Gamma)} + O\left(\ln^{-2}(2\eta/\Gamma)\right)$$

and

$$G(\eta) \sim \frac{1}{\ln(2\eta/\Gamma)} \qquad \text{as } \eta \rightarrow \infty \tag{A16}$$

* This development is an extension of the asymptotic form for $G(\eta)$ developed in Reference 2.

## Appendix B. Asymptotic Expansion of $F(\zeta)$ for $\zeta \to 0$

The Laplace transform of $F(\zeta)$ is given by

$$f(y) = \frac{e^{-y}}{y K_o(y)} \tag{B1}$$

where y is the normalized transform variable. For large y with $|\arg y| < 3\pi/2$, the asymptotic expansion of f(y) is

$$f(y) = \frac{e^{-y}}{y\sqrt{\frac{\pi}{2y}}\, e^{-y}\left\{1 - \frac{1}{8y} + \frac{9}{128\,y^2} + O\left(y^{-3}\right)\right\}}$$

$$= \sqrt{\frac{2}{\pi y}}\left\{1 + \frac{1}{8y} + O\left(y^{-2}\right)\right\} \tag{B2}$$

where the asymptotic expansion of $K_o(y)$ for large y with $|\arg y| < 3\pi/2$ has been used.

To get a solution of $F(\zeta)$ for small $\zeta$, $F(\zeta)$ can be written down by the term by term inverse Laplace transformation of f(y) as given in Eqn. (B2). This procedure for obtaining the asymptotic expansion for small argument is justified by the theorem developed on the following pages. Thus, as $\zeta \to 0$,

$$F(\zeta) = \frac{\sqrt{2}}{\pi\sqrt{\zeta}}\left\{1 + \zeta/4 + O(\zeta^2)\right\} \tag{B3}$$

THEOREM: If $f(\xi)$, the Laplace transformation of F(q), has the asymptotic expansion for $\xi \to \infty$ with $R_e(\xi) > \xi_a$ where $\xi_a$ is some real number

$$f(\xi) = g(\xi) + O\left(\xi^{-\mu}\right) \tag{B4}$$

where $\mu > 0$ and there exists G(q) the inverse Laplace transform of $g(\xi)$, then as $q \rightarrow 0$

$$F(q) = G(q) + O\left(q^{\mu - 1}\right) \tag{B5}$$

To prove the theorem, write $f(\xi)$ as

$$f(\xi) = g(\xi) + r(\xi) \tag{B6}$$

where $r(\xi)$ is bounded for $|\xi| > \xi_o > \xi_a$ by

$$\left|r(\xi)\right| < C\left|\xi^{-\mu}\right| \tag{B7}$$

and C is a constant. F(q) can be written as

$$F(q) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left\{g(\xi) + r(\xi)\right\} e^{\xi q}\, d\xi$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} g(\xi)\, e^{\xi q} d\xi + \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} r(\xi)\, e^{\xi q}\, d\xi$$

$$= G(q) + R(q) \tag{B8}$$

Now let $\xi = \gamma + i\lambda$, R(q) can be written as

$$R(q) = \frac{1}{2\pi}\int_{-\infty}^{\infty} r(\gamma + i\lambda)\; e^{(\gamma + i\lambda)q}\; d\lambda \tag{B9}$$

For all $\gamma \geq \gamma_o > 0$ where $\gamma_o$ is a real number chosen to the right of all singularities in $r(\xi)$, R(q) can be evaluated. Now let $\gamma = \alpha/q$ where $\alpha > 0$. Then for all q such that $0 < q \leq q_o$ where $q_o = \dfrac{\alpha}{\gamma_o}$

$$\left|R(q)\right| = \left|\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha+i\lambda q)}\, r\left(\frac{\alpha}{q} + i\lambda\right) d\lambda\right|$$

$$< \frac{C}{2\pi}\int_{-\infty}^{\infty} \frac{\left|e^{(\alpha+i\lambda q)}\right|}{\left[\sqrt{\left(\frac{\alpha}{q}\right)^2 + \lambda^2}\right]^{\mu}} d\lambda \qquad \text{(B10)}$$

where condition (B7) has been used. Therefore,[8]

$$\left|R(q)\right| < \frac{C e^{\alpha}}{2\pi}\int_{-\infty}^{\infty} \frac{d\lambda}{\left[\left(\frac{2}{q}\right)^2 + \lambda^2\right]^{\mu}}$$

$$= \frac{C e^{\alpha}}{2\pi}\left(\frac{q}{\alpha}\right)^{\mu-1}\int_{-\infty}^{\infty} \frac{1}{\left[\sqrt{1+\rho^2}\right]^{\mu}} d\rho$$

$$= \frac{C e^{\alpha}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\mu-1}{2}\right)}{2\pi\Gamma(\mu)}\left(\frac{q}{\alpha}\right)^{\mu-1}$$

$$= C_1 q^{\mu-1} \qquad \text{(B11)}$$

where $\rho = q\lambda/\alpha$ and $C_1$ is a constant. Thus

$$\left|R(q)\right| = O\left(q^{\mu-1}\right) \qquad \text{(B12)}$$

This is the required result and the proof is complete. Theorems and derivations similar to this theorem can be found in the following references: Doetsch, G., Theorie und Anwendung der Laplace-Transformation; Kap. 13, (Berlin, 1937); and Carslaw, H. S., and Jaeger, J. C., Operational Methods in Applied Mathematics, Chapter 13, Dover Edition, 1963.

If $f(\xi)$ has the asymptotic power series expansion for $\xi \to \infty$ with $R_e(\xi) > \xi_a$

$$f(\xi) = \sum_{n=1}^{k} \Lambda_n \xi^{-\mu_n} + O\left(\xi^{-\mu_{k+1}}\right) \quad \text{(B13)}$$

where $0 < \mu_1 < \mu_2 < \cdot \cdot \cdot$, then it follows from the theorem, as $q \to 0$,

$$F(q) = \sum_{n=1}^{k} \frac{\Lambda_n}{\Gamma(\mu_n)} q^{(\mu_n - 1)} + O\left(q^{(\mu_{k+1}^{-1})}\right) \quad \text{(B14)}$$

where $q < q_o$. This result, Eqn.(B14), was used to calculate the asymptotic expansion of $F(\zeta)$ for $\zeta \to 0$.

## Appendix C. Series Approximation for $F(\zeta)$

A regular function $F(\zeta)$ can be approximated by a least squares polynomial fit to give a polynomial approximation $P(\zeta)$ given by

$$P(\zeta) = \sum_{m=0}^{n} b_m \zeta^m \tag{C1}$$

The relative error of approximation $\Delta(\zeta)$ is given by

$$\Delta(\zeta) = \frac{P(\zeta) - F(\zeta)}{F(\zeta)} \tag{C2}$$

and

$$P(\zeta) = F(\zeta) \left[1 + \Delta(\zeta)\right] \tag{C3}$$

To measure the accuracy of the approximation, define the maximum relative error in the range of approximation as $\Delta_m$.

To remove the singularity in $F(\zeta)$ at $\zeta = 0$, define a new function $F_1(\zeta)$ as

$$F_1(\zeta) = \frac{F(\zeta)}{f(\zeta)} \tag{C4}$$

where $f(\zeta)$ is the asymptotic form of $F(\zeta)$ for $\zeta \to 0$ given by

$$f(\zeta) = \frac{\sqrt{2}}{\pi\sqrt{\zeta}} \tag{C5}$$

For $0 \leq \zeta \leq \zeta_1$, $F_1(\zeta)$ is regular and can be approximated by a polynomial. The approximation for $F(\zeta)$ can now be written as

$$F(\zeta) \simeq P(\zeta) = \sum_{m=0}^{n} a_m \zeta^{m+k} \qquad \text{(C6)}$$

where $k = \begin{cases} -\frac{1}{2} & \text{if } \zeta \leq \zeta_1 \\ 0 & \text{if } \zeta \geq \zeta_1 \end{cases}$

The series approximation of $F(\zeta)$ was calculated by a least squares polynomial fit with $\zeta_1 = 1.0$ and $n = 10$. The coefficients $a_m$ are tabulated in Table 1C for six ranges of $\zeta$. The values of $F(\zeta)$, $P(\zeta)$, and the asymptotic form for large and small $\zeta$ are tabulated in Table 2C.

Table 1Ca. Values of $b_m$ for $0.0 \leq \zeta \leq 1.0$.

$$\frac{\pi\sqrt{\zeta}}{\sqrt{2}} F(\zeta) = \sum_{m=0}^{10} b_m \zeta^m .$$

$$a_m = \frac{\sqrt{2}}{\pi} b_m$$

| Range of $\zeta$ | $0 \leq \zeta \leq .01$ | $.01 \leq \zeta \leq 1.0$ |
|---|---|---|
| m \ $\Delta_m$ | 5.4312 E - 07 | 6.6312 E - 06 |
| 0 | 1.0000 E + 00 | 1.0001 E + 00 |
| 1 | 3.5554 E - 01 | 2.5745 E - 01 |
| 2 | -9.7015 E + 01 | -1.9527 E - 01 |
| 3 | 7.2158 E + 04 | 1.0017 E + 00 |
| 4 | -3.4541 E + 07 | -4.6672 E + 00 |
| 5 | 1.0601 E + 10 | 1.4199 E + 01 |
| 6 | -2.1050 E + 12 | -2.8027 E + 01 |
| 7 | 2.6856 E + 14 | 3.5595 E + 01 |
| 8 | -2.1220 E + 16 | -2.8028 E + 01 |
| 9 | 9.4404 E + 17 | 1.2437 E + 01 |
| 10 | -1.8066 E + 19 | -2.3750 E + 00 |

The values of $b_m$ above are given in the E-format, i.e., $(Y)\,10^x$ = Y E + 0x.

Table 1Cb. Values of $a_m$ for $1.0 \leq \zeta \leq 10,000.0$.

$$F(\zeta) = \sum_{n=0}^{10} a_m \zeta^m$$

| Range of $\zeta$ | $1.0 \leq \zeta \leq 10.0$ | $10.0 \leq \zeta \leq 100.0$ | $100.0 \leq \zeta \leq 1000.0$ | $1000.0 \leq \zeta \leq 10,000.0$ |
|---|---|---|---|---|
| $\Delta_m$ / m | -3.5485 E-04 | -1.7943 E-04 | -1.0808 E-04 | -7.3248 E-05 |
| 0 | 9.8379 E-01 | 4.1086 E-01 | 2.3447 E-01 | 1.5806 E-01 |
| 1 | -8.7481 E-01 | -2.5827 E-02 | -1.0873 E-03 | -5.6405 E-05 |
| 2 | 6.8449 E-01 | 1.9382 E-03 | 7.8962 E-06 | 3.9949 E-08 |
| 3 | -3.4999 E-01 | -9.7520 E-05 | -3.9193 E-08 | -1.9609 E-11 |
| 4 | 1.1955 E-01 | 3.3041 E-06 | 1.3183 E-10 | 6.5521 E-15 |
| 5 | -2.7728 E-02 | -7.6275 E-08 | -3.0297 E-13 | -1.4992 E-18 |
| 6 | 4.3736 E-03 | 1.1995 E-09 | 4.7504 E-16 | 2.3434 E-22 |
| 7 | -4.6122 E-04 | -1.2623 E-11 | -4.9887 E-19 | -2.4556 E-26 |
| 8 | 3.1082 E-05 | 8.4935 E-14 | 3.3517 E-22 | 1.6472 E-30 |
| 9 | -1.2090 E-06 | -3.3000 E-16 | -1.3007 E-25 | -6.3852 E-35 |
| 10 | 2.0633 E-08 | 5.6267 E-19 | 2.2156 E-29 | 1.0868 E-39 |

Table 2C. Values of F(ζ), P(ζ), and Asymptotic Forms

| ζ | F(ζ) | P(ζ) | Asymptotic Form Small ζ | Asymptotic Form Large ζ |
|---|---|---|---|---|
| .00010 | 45.01791 | 45.01794 | 45.01582 | |
| .00015 | 36.75761 | 36.75761 | 36.75526 | |
| .00020 | 31.83355 | 31.83354 | 31.83099 | |
| .00030 | 25.99281 | 25.99281 | 25.98989 | |
| .00050 | 20.13517 | 20.13517 | 20.13168 | |
| .00070 | 17.01832 | 17.01832 | 17.01438 | |
| .00100 | 14.23977 | 14.23977 | 14.23525 | |
| .00150 | 11.62834 | 11.62834 | 11.62303 | |
| .00200 | 10.07182 | 10.07182 | 10.06584 | |
| .00300 | 8.22582 | 8.22582 | 8.21873 | |
| .00500 | 6.37506 | 6.37506 | 6.36620 | |
| .00700 | 5.39071 | 5.39071 | 5.38042 | |
| .01000 | 4.51367 | 4.51367 | 4.50158 | |
| .01500 | 3.69007 | 3.69007 | 3.67553 | |
| .02000 | 3.19971 | 3.19970 | 3.18310 | |
| .03000 | 2.61903 | 2.61902 | 2.59899 | |
| .05000 | 2.03857 | 2.03858 | 2.01317 | |
| .07000 | 1.73113 | 1.73113 | 1.70144 | |
| .10000 | 1.45852 | 1.45852 | 1.42353 | |
| .15000 | 1.20437 | 1.20437 | 1.16230 | |
| .20000 | 1.05440 | 1.05440 | 1.00658 | |
| .30000 | .87882 | .87882 | .82187 | |
| .50000 | .70661 | .70662 | .63662 | |
| .70000 | .61729 | .61729 | .53804 | |
| 1.0000 | .53941 | .53941 | .45016 | |
| 1.5000 | .46785 | .46779 | | |
| 2.0000 | .42589 | .42590 | | |
| 3.0000 | .37653 | .37650 | | |
| 5.0000 | .32689 | .32690 | | |
| 7.0000 | .30000 | .30006 | | |
| 10.000 | .27543 | .27548 | | |
| 15.000 | .25149 | .25148 | | |
| 20.000 | .23663 | .23664 | | |
| 30.000 | .21816 | .21816 | | |
| 50.000 | .19831 | .19831 | | |
| 70.000 | .18693 | .18695 | | |
| 100.00 | .17612 | .17613 | | .21181 |
| 150.00 | .16515 | .16514 | | .19506 |
| 200.00 | .15811 | .15811 | | .18470 |
| 300.00 | .14909 | .14909 | | .17183 |
| 500.00 | .13902 | .13903 | | .15796 |
| 700.00 | .13307 | .13308 | | .14999 |

Table 2C. Values of $F(\zeta)$, $P(\zeta)$, and Asymptotic Forms (Continued)

| $\zeta$ | $F(\zeta)$ | $P(\zeta)$ | Asymptotic Form | |
|---|---|---|---|---|
| | | | Small $\zeta$ | Large $\zeta$ |
| 1000.0 | .12727 | .12727 | | .14238 |
| 1500.0 | .12123 | .12123 | | .13460 |
| 2000.0 | .11728 | .11728 | | .12959 |
| 3000.0 | .11210 | .11211 | | .12312 |
| 5000.0 | .10618 | .10619 | | .11583 |
| 7000.0 | .10260 | .10261 | | .11149 |
| 10000. | .09905 | .09907 | | .10722 |

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