Sensor and Simulation Notes

Note 119

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Waveforms on a Nonuniform Surface Transmission Line With an RC-Load

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Abstract

The current and voltage on a nonuniform surface transmission line are calculated by using the conventional transmission-line theory when the line is excited by a step-function voltage generator at one end and terminated by an RC-load at the other. To show how the optimal value of the capacitance in the RC-load can be chosen the current and voltage waveforms are calculated in detail for some typical dimensions of the transmission line. These calculations show that an improvement in the behavior of the waveforms can be obtained with an RC-load. Analytical expressions for the low-frequency and high-frequency behavior of the input impedance of the transmission line are derived and for intermediate frequencies the input impedance is calculated numerically.

I. Introduction

One of the common simulators for the nuclear electromagnetic pulse (EMP) over a ground surface (earth) is a surface transmission line. Due to the lossy nature of earth, a simulated EMP, especially its early portion, attenuates as it propagates down the line. By making the transmission line nonuniform[1] instead of uniform[2,3,4] it has been shown that an improvement can be obtained on the performance (especially the high-frequency performance) of the surface transmission line as an EMP simulator.

In a previous note a purely resistive termination of the transmission line has been used to match the high-frequency limit of the characteristic impedance of a nonuniform line. In this note we seek to improve the performance of the lossy nonuniform surface transmission line with an RC-load at one end. The line is assumed to be driven by a step-function voltage generator at the other end.

The analysis employed here is based on the assumption that the behavior of the actual surface transmission line simulator can be described by the conventional transmission-line theory. Comments on this assumption have been made in Sensor and Simulation Notes 60 and 77 and they are also valid in the present consideration. Sections II, III and IV give the underlying mathematical details and numerical methods for the sample of calculations that will be performed in the following and for a parametric study that may be undertaken in the future. In this parametric study we intend to vary the ground permittivity, the ground conductivity, the length of the transmission line, the height and the slope of the perfectly conducting sheet (or wire array) above the ground. A discussion of the parameters that will be used in the parametric study is given in section V. In the sample calculations performed here we use some typical values of these parameters and display current and voltage waveforms in the time domain at selected positions along the line. Also, we plot the input impedance in a complex plane. It is found that, indeed, the performance of the nonuniform surface transmission line can be improved with an RC-load.

II. Transmission Line Model

A schematic representation of the transmission line considered here is shown in figure 1. The top plate of the transmission line is a perfect conductor and slants downward towards the bottom plate with constant slope (ξ_s). The bottom plate of the line is a lossy medium (earth). We assume here that the influence of the lossy medium can be characterized by a surface impedance Z_g. The transmission line is excited by a step voltage of magneitude V_o at $x = 0$ and terminated by an RC-load, Z_L, at $x = d$ (see figure 1). The resistances R_1 and R_2 of the load are determined from the geometry of the transmission line, while the capacitance C is to be varied. The width of the transmission line is W.

Let I(x,t) and V(x,t) be the current and voltage, respectively, due to a step-function voltage generator of strength V_o. Moreover, introduce the normalized quantities

$$\tau = (ct - x)/d \quad , \tag{1}$$

$$h(x,\tau) = y_o W^{-1} V_o^{-1} Z_o I(x,t) \tag{2}$$

and

$$v(x,\tau) = V(x,t)/V_o \quad , \tag{3}$$

where y_o is the height of the transmission line at $x = 0$, Z_o is the free space wave impedance ($Z_o \approx 377\Omega$) and c is the vacuum speed of light.

Using conventional transmission line theory together with Fourier's integral theorem we get the following integral representations for $h(x,\tau)$ and $e(x,\tau)$ (1)

$$h(x,\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{u(x,k) e^{jkd\tau}}{jk\ z(0,k)}\, dk \tag{4}$$

$$v(x,\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{u(x,k) z(x,k) e^{jkd\tau}}{jk\ z(0,k)}\, dk \tag{5}$$

where u(x,k) satisfies the differential equation

$$u'' - (2jk + \xi_s/y)u' + (jk/y)(\xi_s - Z_g/Z_o)u = 0 \tag{6}$$

and

$$y(x) = y_o - \xi_s x \tag{7}$$

is the height of the top plate over the bottom plate. The normalized impedance, z(x,k), is given by

$$z(x,k) = [y(x)/y_o][1 + jk^{-1}u'(x,k)/u(x,k)] \quad . \tag{8}$$

Here, as well as in the preceeding, the primes denote differentiation with respect to x.

Assume that the effect of the lossy medium can be taken into account by a surface impedance defined by

$$Z_g(k) = \sqrt{jkc\mu_o/(\sigma + jkc\varepsilon)} \tag{9}$$

where σ and ε are the conductivity and dielectric constant of the lossy medium respectively. The characteristic impedance, Z(x,k), and the characteristic admittance, Y(x,k), both per unit length of the transmission line are then given by [1]

$$Z(x,k) = jkc\mu_o W^{-1} y(x) + Z_g/W \tag{10}$$

and

$$Y(x,k) = jkc\varepsilon_o W/y(x) \quad . \tag{11}$$

The boundary conditions for u(x,k) are

$$u(0,k) = 1 \tag{12}$$

and

$$u'(d,k)/u(d,k) = jk - Y(d,k)Z_L(k) \tag{13}$$

where

$$Z_L(k) = R_1(1 + jkcR_2C)/[1 + jkc(R_1 + R_2)C] \quad . \tag{14}$$

In appendix A some analytical properties of $u(x,k)$ are given, and in appendices B and C we derive asymptotic expansions of $u(x,k)$ valid when $|kd| \ll 1$ and $|kd| \gg 1$, respectively.

Equation (6) was solved numerically in the following way. Starting at $x = d$ we assume the boundary conditions $u_o(d,k) = 1$ and $u_o'(d,k) = jk - Y(d,k)Z_L(k)$. Making use of the Runge-Kutta method we then compute $u_o(x,k)$ backward along x to $x = 0$. The solution of the differential equation (6) satisfying the boundary conditions (12) and (13) is then given by $u(x,k) = u_o(x,k)/u_o(0,k)$.

III. Input Impedance

The input impedance Z_{in} of the transmission line is given by

$$Z_{in}(\omega) = Z_o y_o W^{-1} z(0,\omega/c) \tag{15}$$

where $z(x,\omega/c)$ is defined in equation (8).

By putting

$$R_1 = Z_o y_o / W \tag{16}$$

it follows from the analysis in appendix B that we have the following asymptotic expansion of $z(x,k)$, valid for $\omega dc^{-1} < 10^{-2}$,

$$z(x,k) = 1 + \xi_\sigma\sqrt{jkd} - jk\{d\xi_C + y_o\xi_s^{-1}\ln[y(x)/y_d] - 1/2[y^2(x) - y_d^2]y_o^{-1}\xi_s^{-1}\}$$

$$- jky_o\xi_\sigma\sqrt{jkd}\{[2\xi_s^{-1} + 1/2\ \varepsilon_r y_o d^{-1}\xi_\sigma^2][1 - xd^{-1}]$$

$$- 2y_d d^{-1}\xi_s^{-1}\ln[y(x)/y_d]\} + O(k^2d^2) \quad . \tag{17}$$

where

$$k = \omega/c \quad ,$$

$$y_d = y_o - \xi_s d \quad ,$$

$$\xi_C = y_o C/dW\varepsilon_o \quad ,$$

$$\xi_\sigma^2 = d/y_o^2\sigma Z_o$$

and ε_r is the relative dielectric constant of the lossy medium.

In order to minimize the reflections from the load at high frequencies we put

$$R_1R_2/(R_1 + R_2) = Z_o y_d/W \tag{18}$$

(see equation (C5) in appendix C). For $\omega > \omega_o$, where

$$\omega_o = 10 \max\{c\xi_s y_o^{-1}\xi_C^{-1}, c/d, \sigma/\varepsilon_o\} \quad ,$$

it follows from the analysis in appendix C that

$$z(x,k) = \{y(x)/y_o\}\{1 - jk^{-1}d^{-1}[\beta\xi_s dy^{-1}(x)$$

$$- a[y_d/y(x)]^{2\beta+1}e^{-2jk(d-x)}]\} + O(k^{-2}d^{-2}) \tag{19}$$

where

$$\beta = 1/2\ (\xi_s^{-1}\varepsilon_r^{-\frac{1}{2}} - 1)$$

and

$$a = \beta\xi_s dy_d^{-1} - y_d y_o^{-1}\xi_C^{-1} \quad .$$

Notice that with the above choice of R_1 and R_2 we have

$$\lim_{\omega\to 0} Z_{in}(\omega) = \lim_{\omega\to\infty} Z_{in}(\omega) = Z_o y_o/W \quad .$$

For intermediate frequencies the input impedance was calculated numerically and the result is displayed in figures 14-19. The choice of the parameters as well as the values assigned to them are discussed in section V. Ideally, we would like the input impedance to be $y_o Z_o/W$ for all frequencies.

IV. Waveforms for a Step Voltage

A. Current Waveforms

The current on the transmission line due to a step voltage generator is described by (see equation (4))

$$h(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{u(x,k)e^{jkd\tau}}{jk\ z(0,k)}\, dk \quad .$$

The integrand is singular at $k = 0$. To evaluate this Fourier inversion integral numerically we put

$$h(x,\tau) = [1 - \xi_\sigma(\pi\tau)^{-\frac{1}{2}}]U(\tau) + (2\pi)^{-1}\int_{-\infty}^{\infty} H(x,k)e^{jkd\tau}dk \qquad (20)$$

where $U(\tau)$ is the unit step function and

$$H(x,k) = -\,jk^{-1}u(x,k)z^{-1}(0,k) + jk^{-1} + \xi_\sigma\sqrt{d/jk} \quad .$$

It follows from appendices A, B and C that $H(x,k)$ is bounded for k real and that as k approaches infinity $H(x,k)$ tends to zero sufficiently rapidly for the integral to converge.

The initial value theorem together with the analysis in appendix C gives

$$\lim_{\tau\to 0+} h(x,\tau) = [y(x)/y_o]^\beta \qquad (21)$$

and, of course, this result agrees with the corresponding result in reference 1.

The result of the numerical computations are shown in figures 2-7. The discontinuities in the slopes of the curves are due to reflections of the wave front at the load and at the generator. Ideally, we would like $h(x,\tau)$ to be unity.

B. Voltage Waveforms

The voltage on the transmission line is described by (see equations (5) and (20))

$$v(x,\tau) = [1 - d\xi_\sigma(\pi\tau y_o^2)^{-\frac{1}{2}}]U(\tau) + (2\pi)^{-1} \int_{-\infty}^{\infty} E(x,k)e^{jkd\tau}dk \tag{22}$$

where

$$E(x,k) = - jk^{-1}u(x,k)z(x,k)z^{-1}(0,k) + jk^{-1} + \xi_\sigma y_o^{-1}\sqrt{d^3/jk}$$

Moreover, we have

$$\lim_{\tau\to 0+} v(x,\tau) = [y(x)/y_o]^{\beta+1} \tag{23}$$

and, of course, $v(0,\tau) = U(\tau)$. The result of the numerical Fourier inversion is shown in figures 8-13.

In the next section, we first introduce a set of parameters suitable for a parametric study of the current and voltage waveforms on the transmission line considered here and then calculate the waveforms for some typical values of these parameters. The complete parametric study will be undertaken in a future note.

V. Numerical Results

Before we proceed with the discussion of the numerical results we list all symbols that will be used.

A. List of Symbols

$h(x,\tau) = h(\xi,\tau|\varepsilon_r,\xi_\sigma,\xi_o,\xi_s,\xi_C) = y_o W^{-1} V_o^{-1} Z_o I(x,t)$

$v(x,t) = v(\xi,\tau|\varepsilon_r,\xi_\sigma,\xi_o,\xi_s,\xi_C) = V(x,t)/V_o$

$z_{in}(\nu|\varepsilon_r,\xi_\sigma,\xi_o,\xi_s,\xi_C) = W y_o^{-1} Z_o^{-1} Z_{in}(\omega)$

$\Delta = \max_\nu\{|z_{in}(\nu|\varepsilon_r,\xi_\sigma,\xi_o,\xi_s,\xi_C) - 1|\}$

$\delta = \min_{\xi_C}\{\Delta(\varepsilon_r,\xi_\sigma,\xi_o,\xi_s,\xi_C)\}$

$\xi_\sigma = (y_o^2 \sigma Z_o/d)^{-\frac{1}{2}}$

$\xi_o = y_o/d$

$\xi_C = y_o C(dW\varepsilon_o)^{-1}$

$\xi = x/d$

$\tau = (ct - x)/d$

$\nu = \omega d/c$

$I(x,t)$ = current on transmission line

$V(x,t)$ = voltage on transmission line

$V_o U(t)$ = generator voltage

$U(t)$ = unit step function

$Z_{in}(\omega)$ = input impedance of transmission line

ε_r = relative dielectric constant of lossy medium

σ = conductivity of lossy medium

C = capacitance of RC-load

ξ_s = slope of transmission line

y_o = height of transmission line at generator

d = length of transmission line

W = width of transmission line

x = distance along transmission line from generator

t = time

ω = angular frequency

c = vacuum speed of light

Z_o = free space wave impedance

ε_o = dielectric constant of free space

B. Current and Voltage Waveforms

From the analysis given in sections II and IV we see that two variables (τ,ξ) and five parameters $(\varepsilon_r,\xi_\sigma,\xi_o,\xi_s,\xi_C)$ are needed to describe h and v, i.e.,

$$h = h(\xi,\tau|\varepsilon_r,\xi_\sigma,\xi_o,\xi_s,\xi_C)$$

$$v = v(\xi,\tau|\varepsilon_r,\xi_\sigma,\xi_o,\xi_s,\xi_C) \quad .$$

The time variations of v and h for ξ = 0, 0.5 and 1 were calculated in the following way. First, the differential equation (6) was solved numerically by using the Runge-Kutta method. Then, the inverse Fourier integrals (20) and (22) were evaluated numerically hereby making use of the high-frequency approximate expressions derived in appendix C.

The values of the parameters $\varepsilon_r,\xi_\sigma,\xi_o,\xi_s$ and ξ_C were chosen in the following way. For each fixed combination of values of $\varepsilon_r,\xi_\sigma,\xi_o$ and ξ_s we have chosen the value of ξ_C that would give as closely as possible the ideal current and voltage waveforms which are given by

$$h_i(\xi,\tau) = (1 - \xi\xi_s\xi_o^{-1})^{-1}v_i(\xi,\tau) = U(\tau) \quad .$$

As mentioned earlier the wave front is reflected at the load and at the generator. In general, for a value of ξ_C less than the chosen value these reflections cause a drop in the current waveform and an overshooting in the voltage waveform. On the other hand a value of ξ_C greater than the chosen value will make the current waveform overshoot its desired value of unity and the rise time of the voltage waveform will become larger. However, a deviation less than 20% of ξ_C from its desired value has negligable influence on the waveforms.

In our sample calculations the parameters are given the values

$$\varepsilon_r = 10$$
$$\xi_\sigma = .607$$
$$\xi_o = .12$$
$$\xi_s = 0, .01, .02, .04, .06, .10$$
$$\xi_C = .1, .1, .1, .3, .5, .5,$$

respectively. A typical surface transmission line that gives the above parametric values can be described by

$$\sigma = 10^{-2}\Omega^{-1}m^{-1}$$
$$d = 50 \text{ m}$$
$$y_o = 6 \text{ m}$$
$$y_d = 6, 5.5, 5, 4, 3, 1 \text{ m}$$
$$R_1/W = R_2/W = 2262 \ \Omega m^{-1}$$
$$C/W = 7.4, 7.4, 7.4, 22, 37, 37 \text{ pFm}^{-1},$$

respectively (1 pF = 10^{-12} F).

Both the current and the voltage waveforms are graphed for $0 \le \tau \le 7$ and for $\xi = 0, .5, 1$ in figures 2-13. The voltage at $\xi = 0$ is simply a step function.

C. Input Impedance

Introduce the normalized input impedance z_{in} by

$$z_{in} = Wy_o^{-1} Z_o^{-1} Z_{in} \quad . \tag{24}$$

From the analysis in sections II and III we see that one variable (ν) and five parameters ($\varepsilon_r, \xi_\sigma, \xi_o, \xi_s, \xi_C$) are required to describe z_{in}, i.e.,

$$z_{in} = z_{in}(\nu | \varepsilon_r, \xi_\sigma, \xi_o, \xi_s, \xi_C)$$

where the normalized frequency ν is defined by

$$\nu = \omega d/c \quad . \tag{25}$$

For each combination of values of the parameters $\varepsilon_r, \xi_\sigma, \xi_o, \xi_s$ and ξ_C we have graphed z_{in} in a complex diagram with ν as a variable (see figures 14-19). As $\nu \to \infty$ z_{in} spirals towards 1.

For given $\varepsilon_r, \xi_\sigma, \xi_o, \xi_s$ let ξ_C^* be the value of ξ_C that minimizes Δ, the maximum deviation of z_{in} from unity over $0 < \nu < \infty$. Let this minimum value of Δ be denoted by δ, i.e., in mathematical terms

$$\delta = \min_{\xi_C} \{ \max_{\nu} [| z_{in}(\nu | \varepsilon_r, \xi_\sigma, \xi_o, \xi_s, \xi_C) - 1 |] \} \quad . \tag{26}$$

In table 1 we give δ and ξ_C^* for different values of ξ_s with $\varepsilon_r = 10$, $\xi_\sigma = .607$ and $\xi_s = .12$.

Table 1

ξ_s	δ	ξ_C^*
0	.48	.1
.01	.39	.1
.02	.30	.1
.04	.25	.3
.06	.30	.5
.10	.68	.5

This ξ_C^* that makes the input impedance deviate the least from its desired value of unity also gives the most desired current and voltage waveforms.

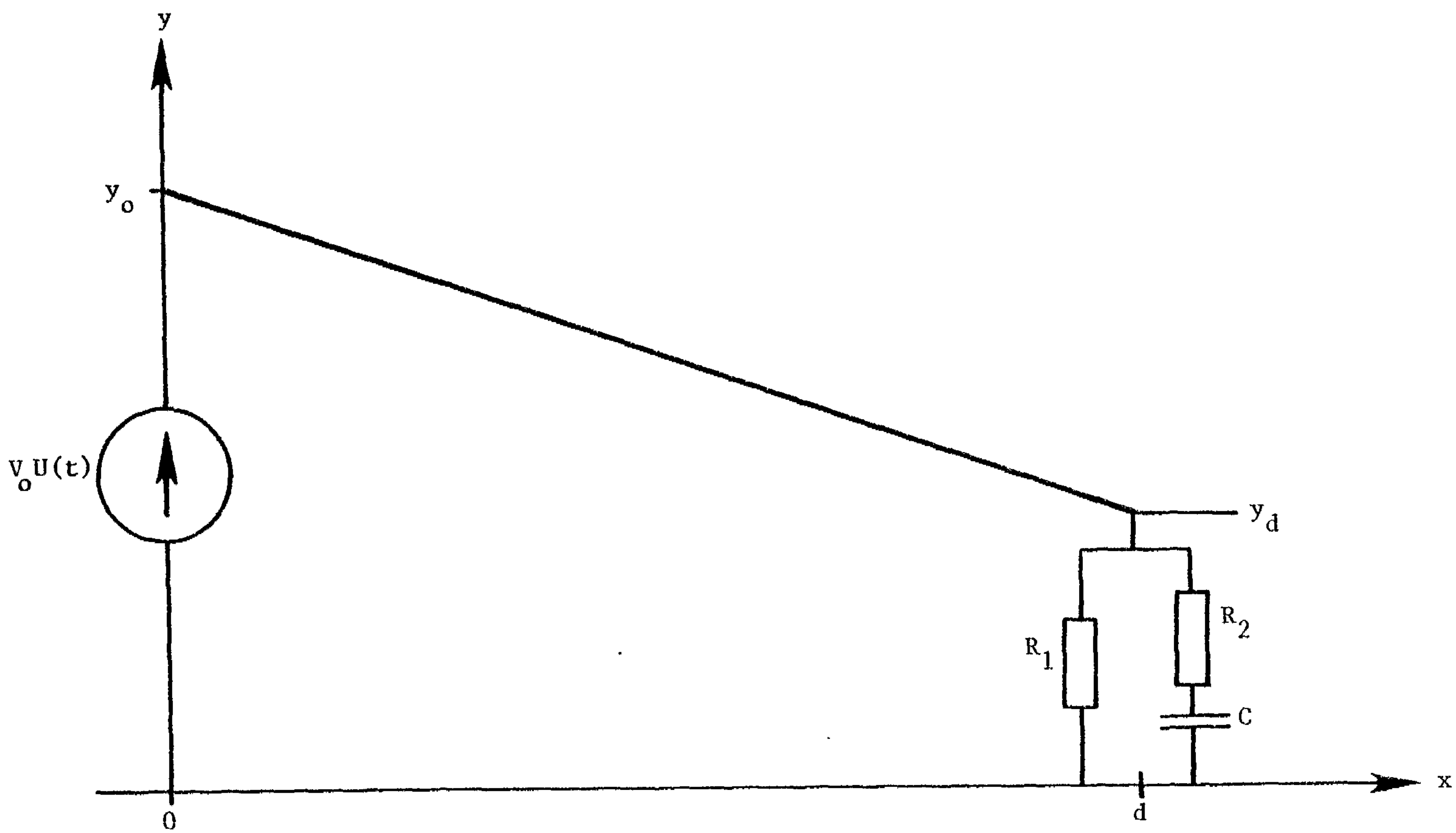

Figure 1. Schematic representation of a nonuniform transmission line with constant slope, terminated by an RC-load and excited by a step voltage.

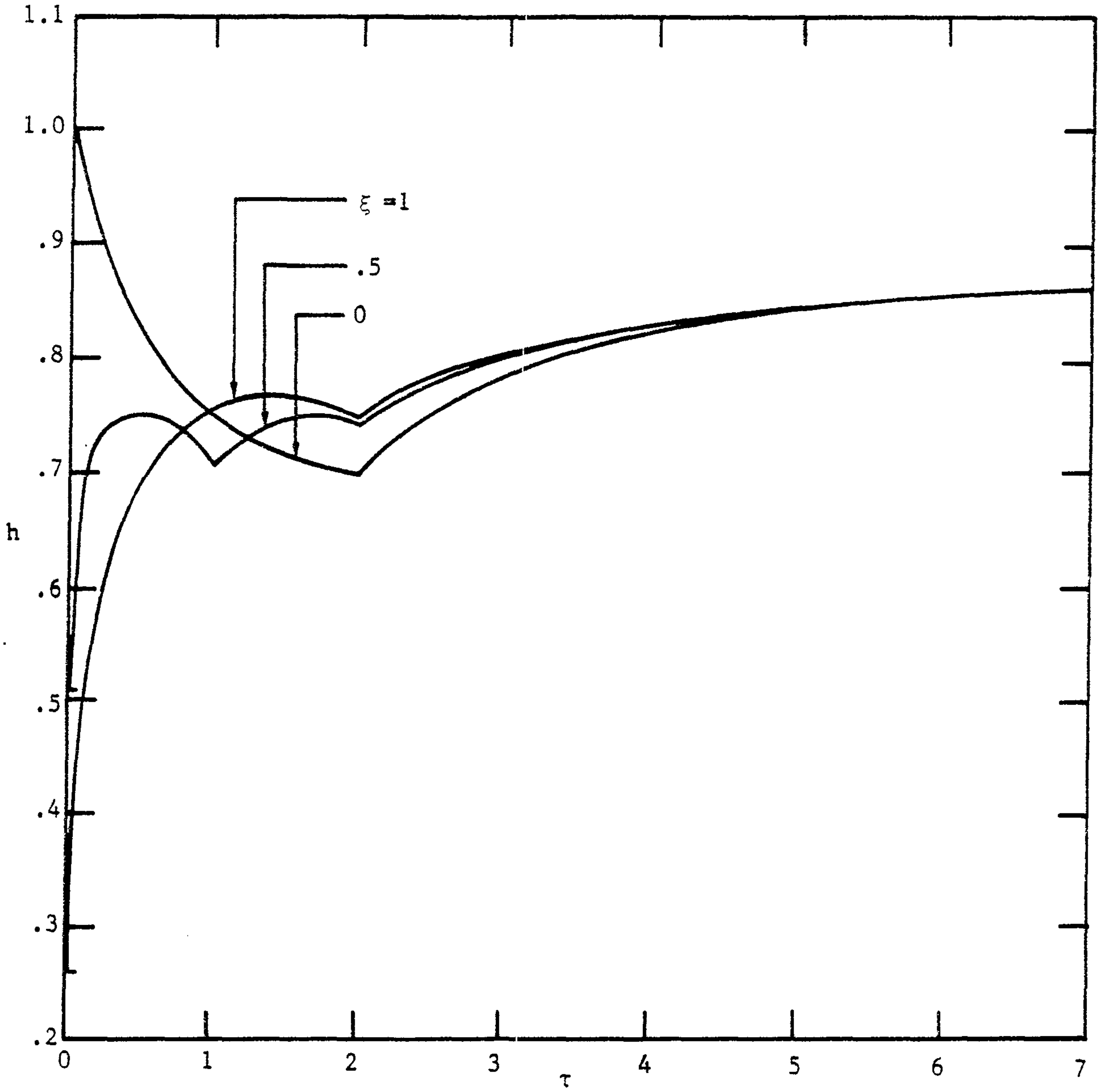

Figure 2. Current wave forms on nonuniform transmission line excited by a step voltage when ε_r=10, ξ_σ=.607, ξ_o=.12, ξ_s=0, ξ_C=.1.

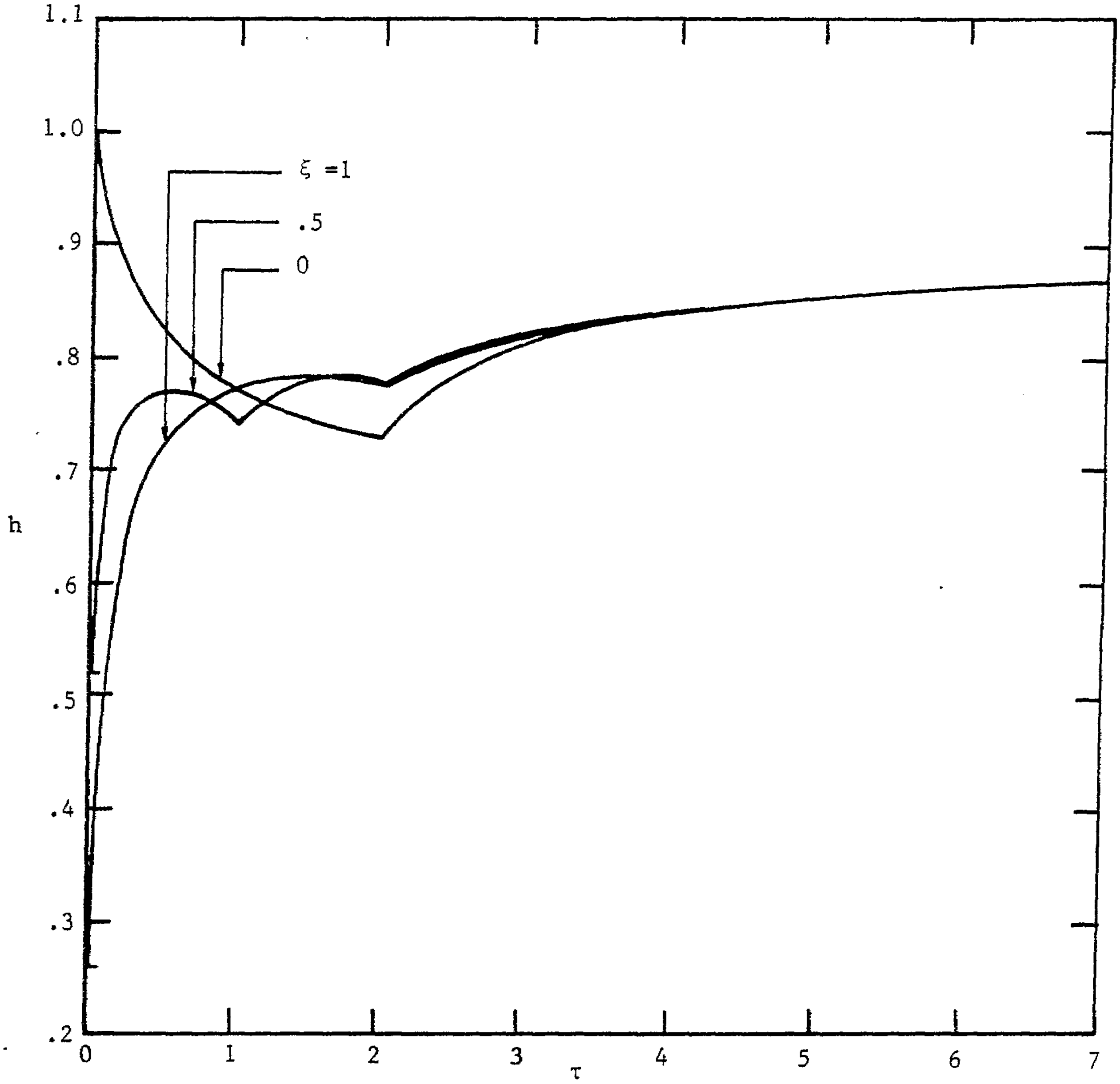

Figure 3. Current wave forms on nonuniform transmission line excited by a step voltage when ε_r=10, ξ_σ=.607, ξ_o=.12, ξ_s=.01, ξ_C=.1.

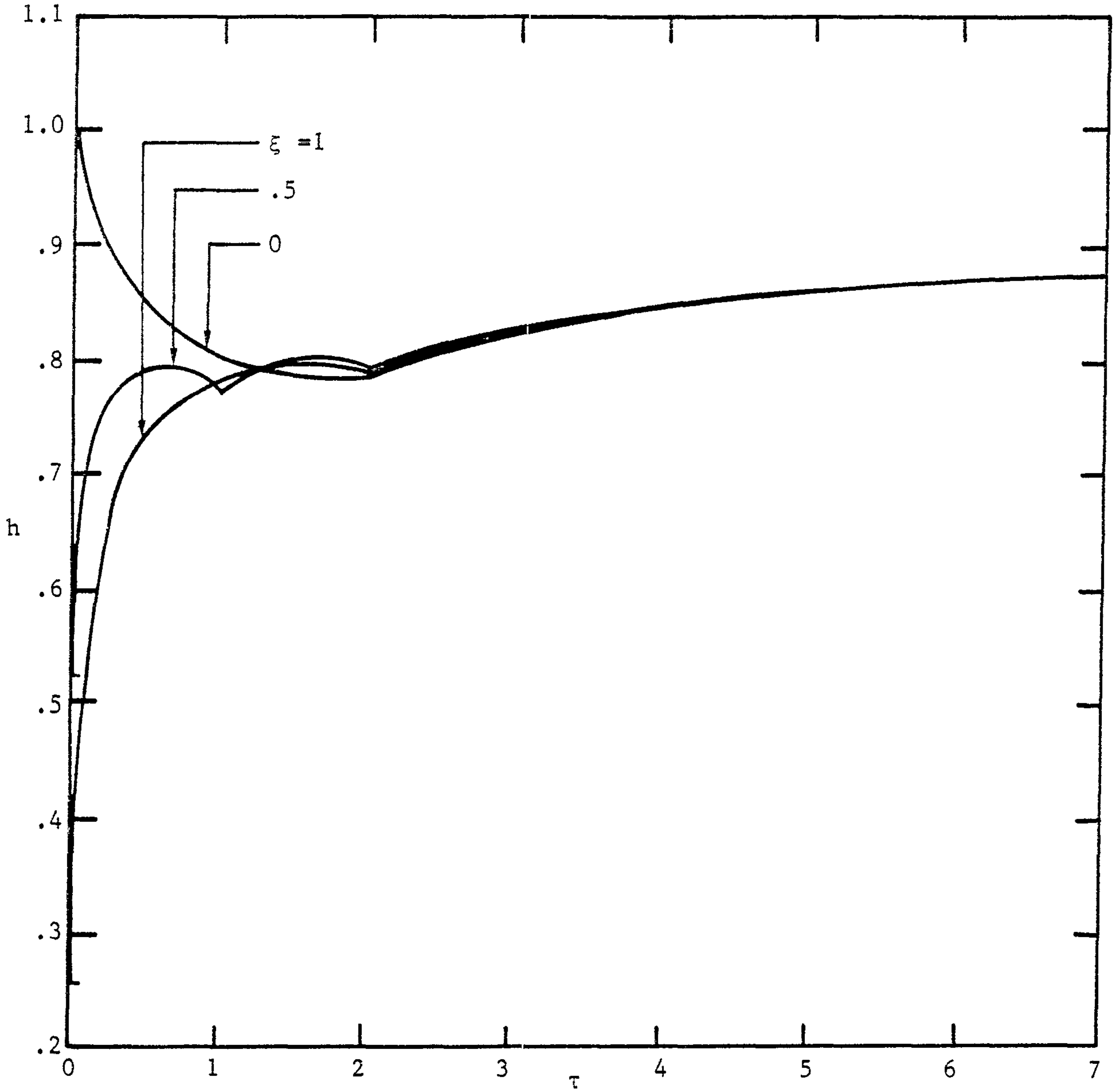

Figure 4. Current wave forms on nonuniform transmission line excited by a step voltage when ε_r=10, ξ_σ=.607, ξ_o=.12, ξ_s=.02, ξ_C=.1.

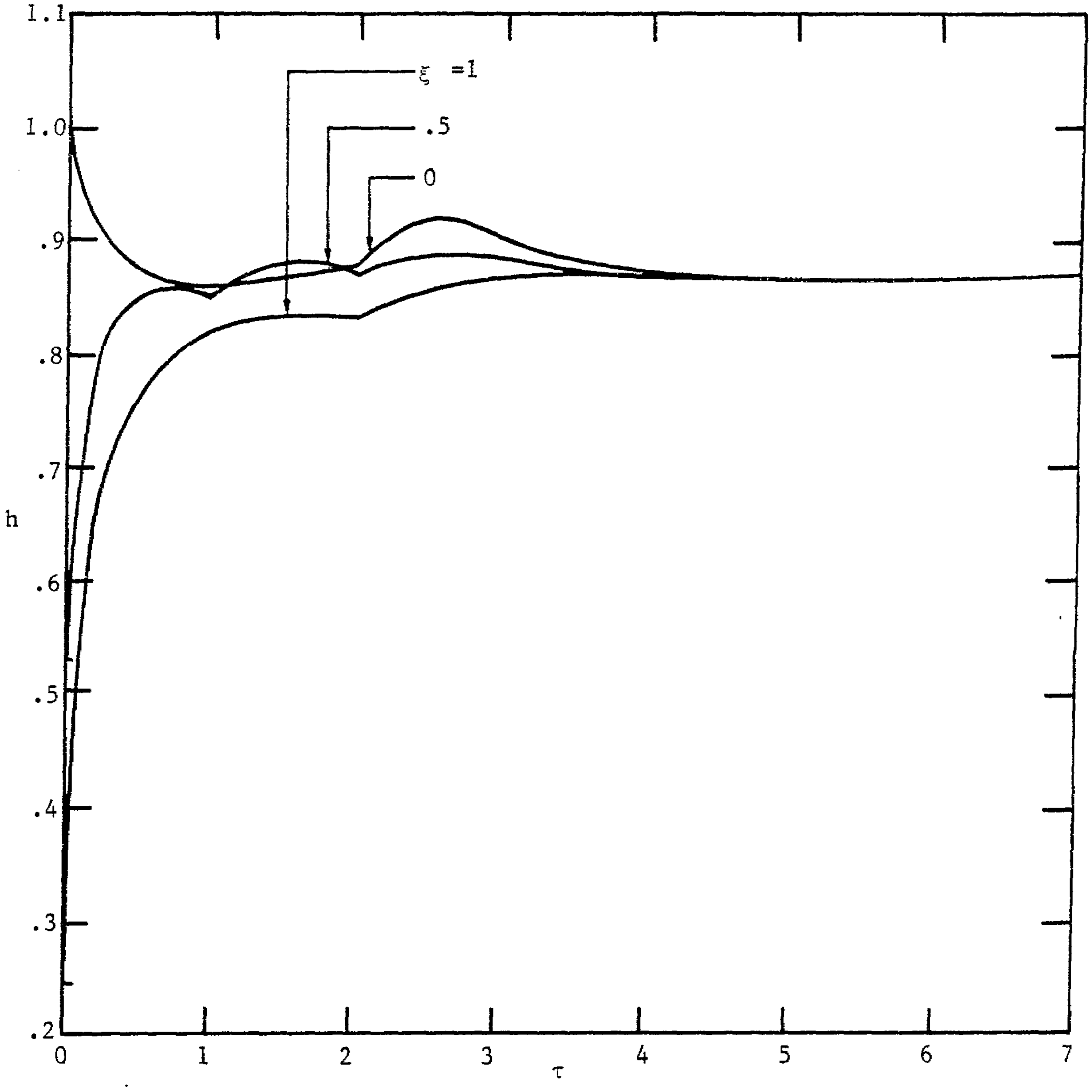

Figure 5. Current wave forms on nonuniform transmission line excited by a step voltage when ε_r=10, ξ_σ=.607, ξ_σ=.12, ξ_s=.06, ξ_C=.3.

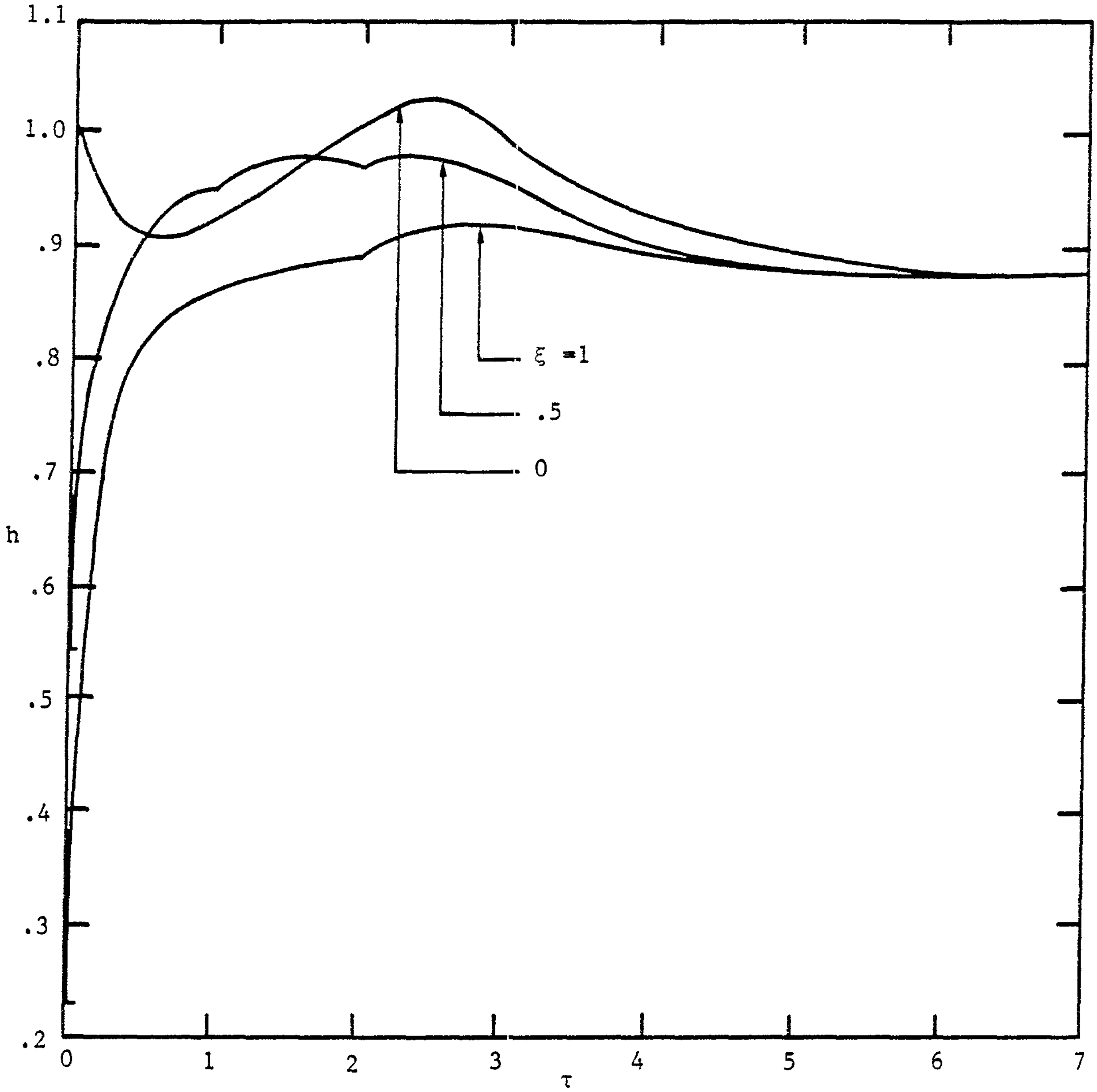

Figure 6. Current wave forms on nonuniform transmission line excited by a step voltage when ε_r=10, ξ_σ=.607, ξ_o=.12, ξ_s=.06, ξ_C=.5.

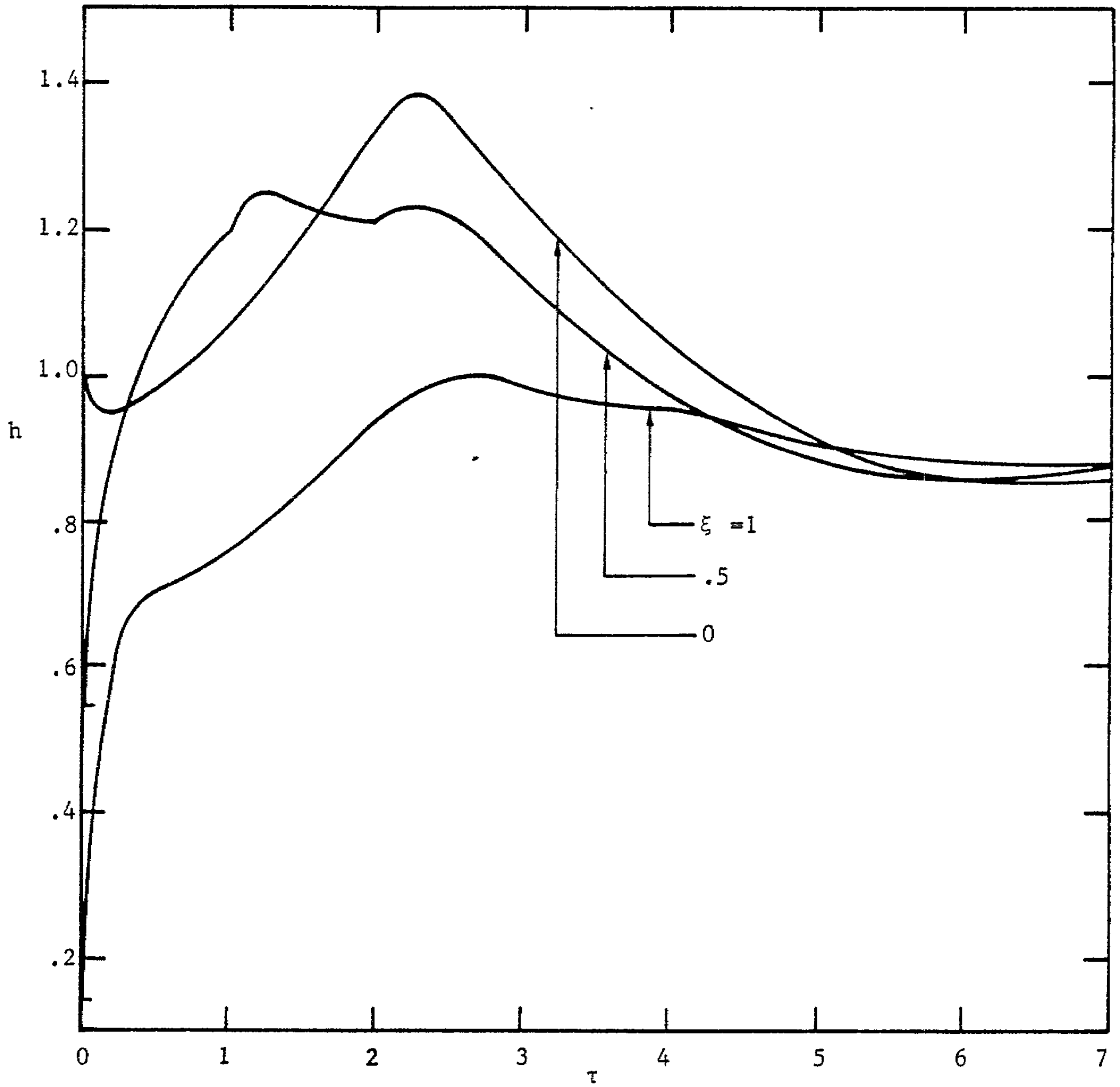

Figure 7. Current wave forms on nonuniform transmission line excited by a step voltage when ε_r=10, ξ_σ=.607, ξ_o=.12, ξ_s=.10, ξ_C=.5.

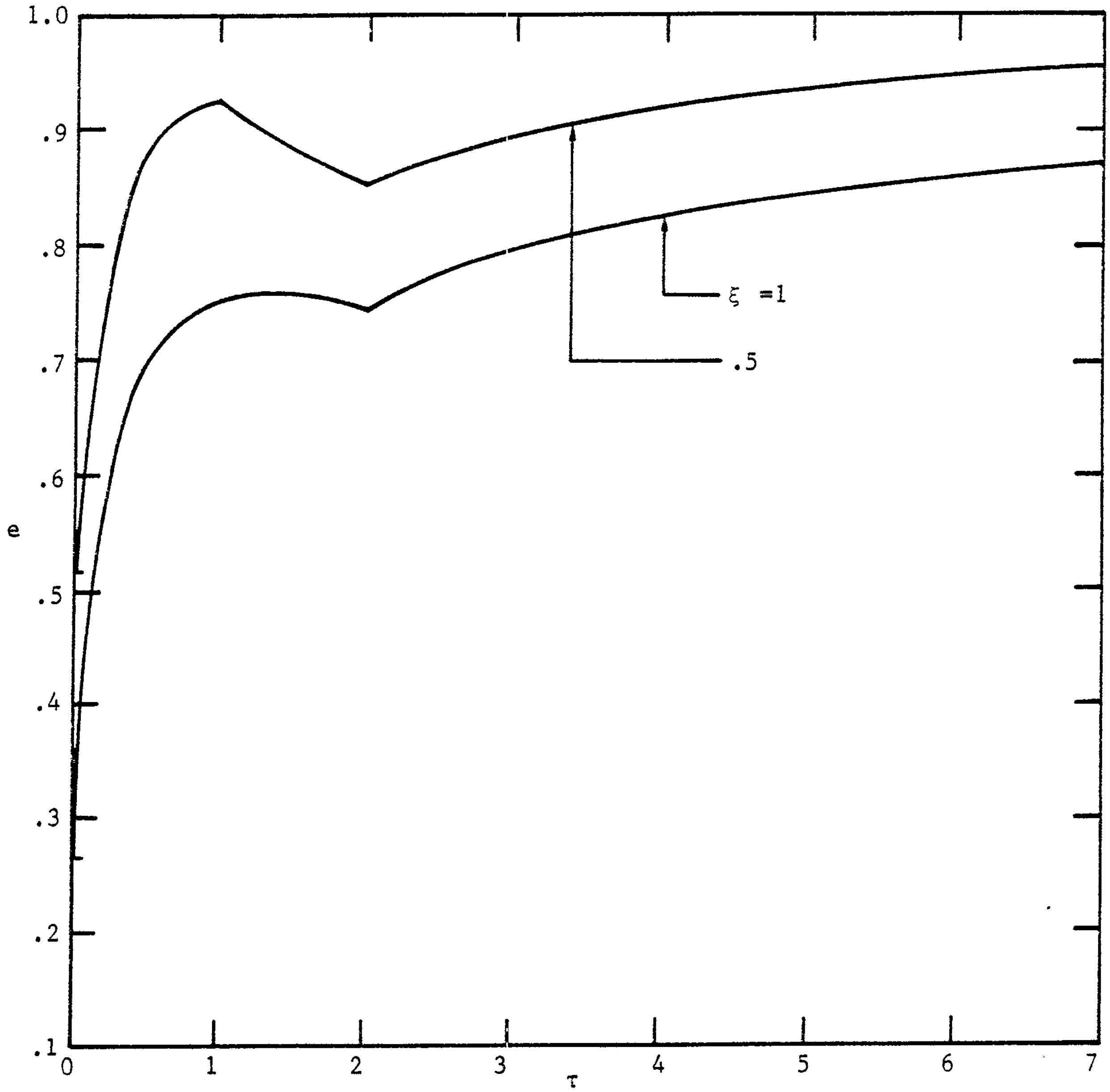

Figure 8. Voltage wave forms on nonuniform transmission line excited by a step voltage when ε_r=10, ξ_σ=.607, ξ_o=.12, ξ_s=0, ξ_C=.1.

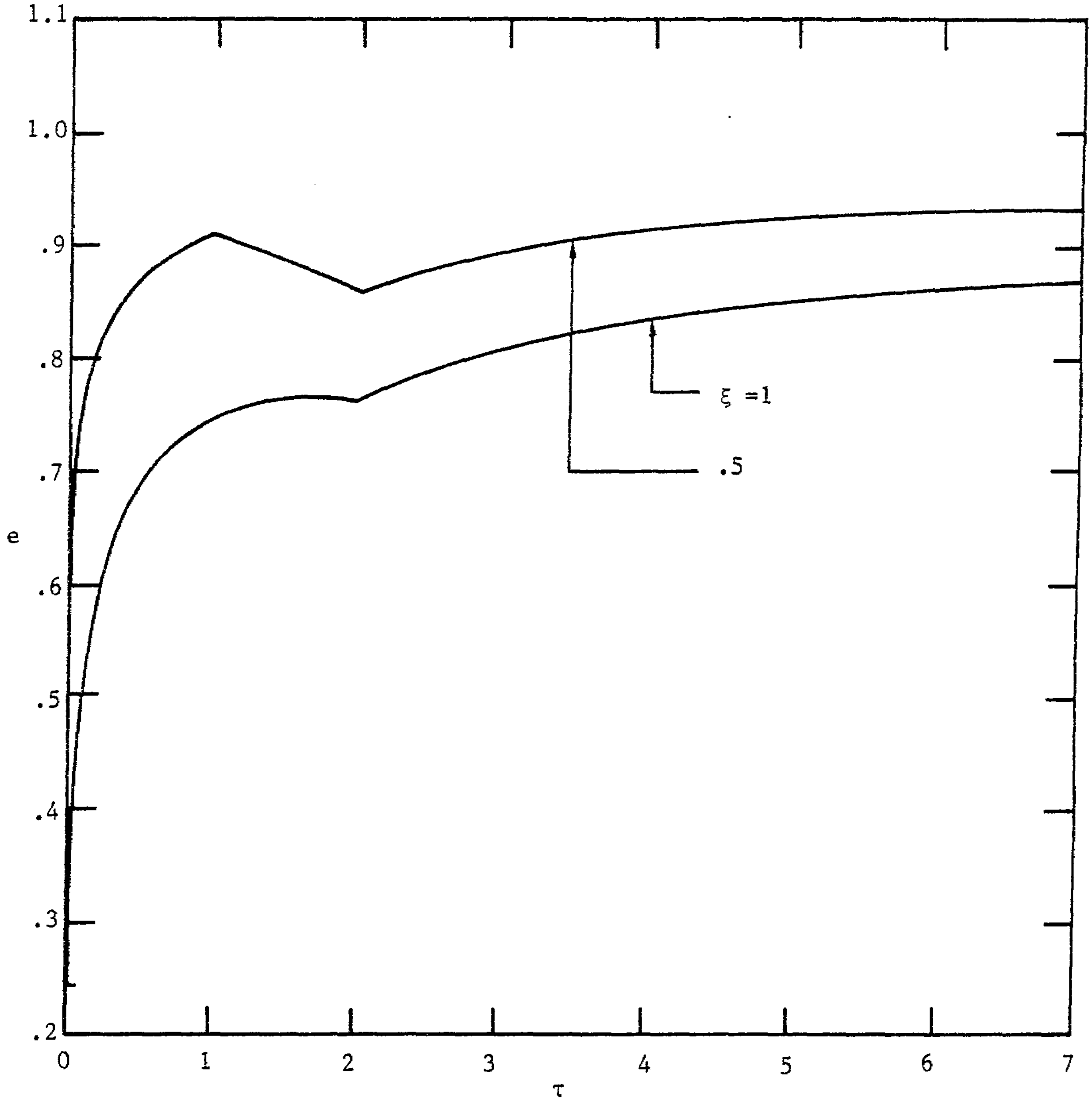

Figure 9. Voltage wave forms on nonuniform transmission line excited by a step voltage when ε_r=10, ξ_σ=.607, ξ_o=.12, ξ_s=.01, ξ_C=.1.

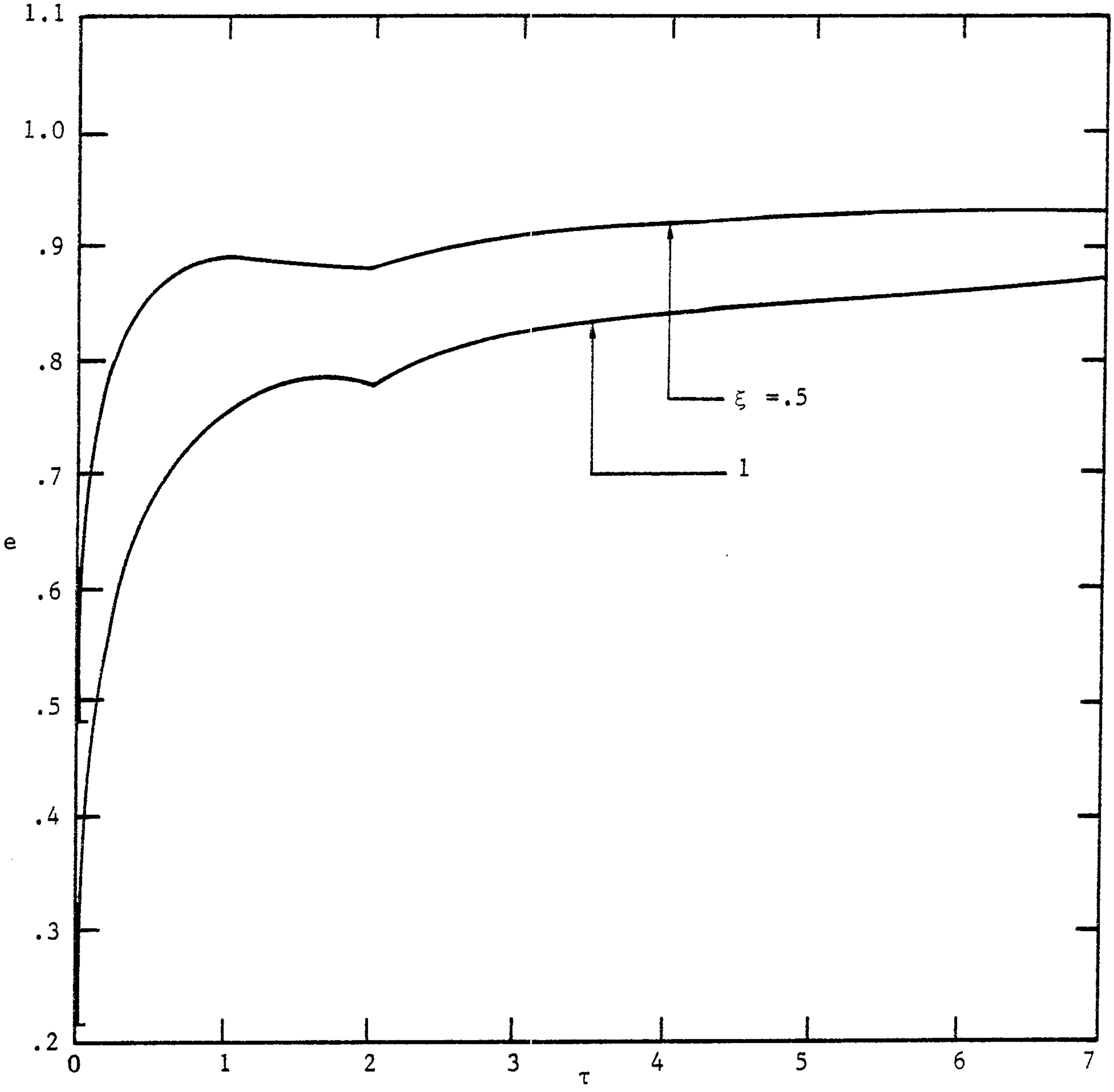

Figure 10. Voltage wave forms on nonuniform transmission line excited by a step voltage when ε_r=10, ξ_σ=.607, ξ_o=.12, ξ_s=.02, ξ_C=.1.

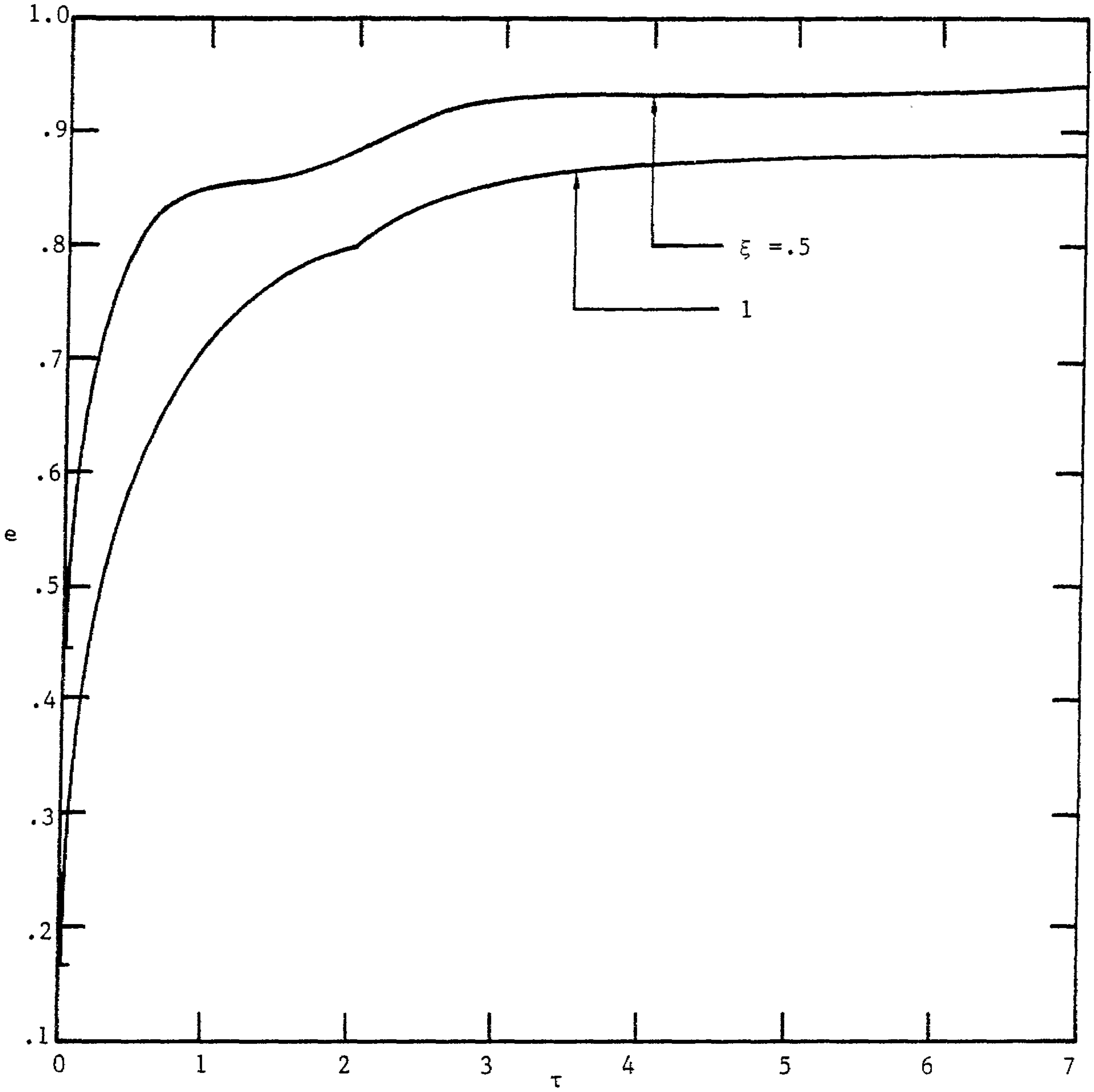

Figure 11. Voltage wave forms on nonuniform transmission line excited by a step voltage when ε_r=10, ξ_σ=.607, ξ_o=.12, ξ_s=.04, ξ_C=.3.

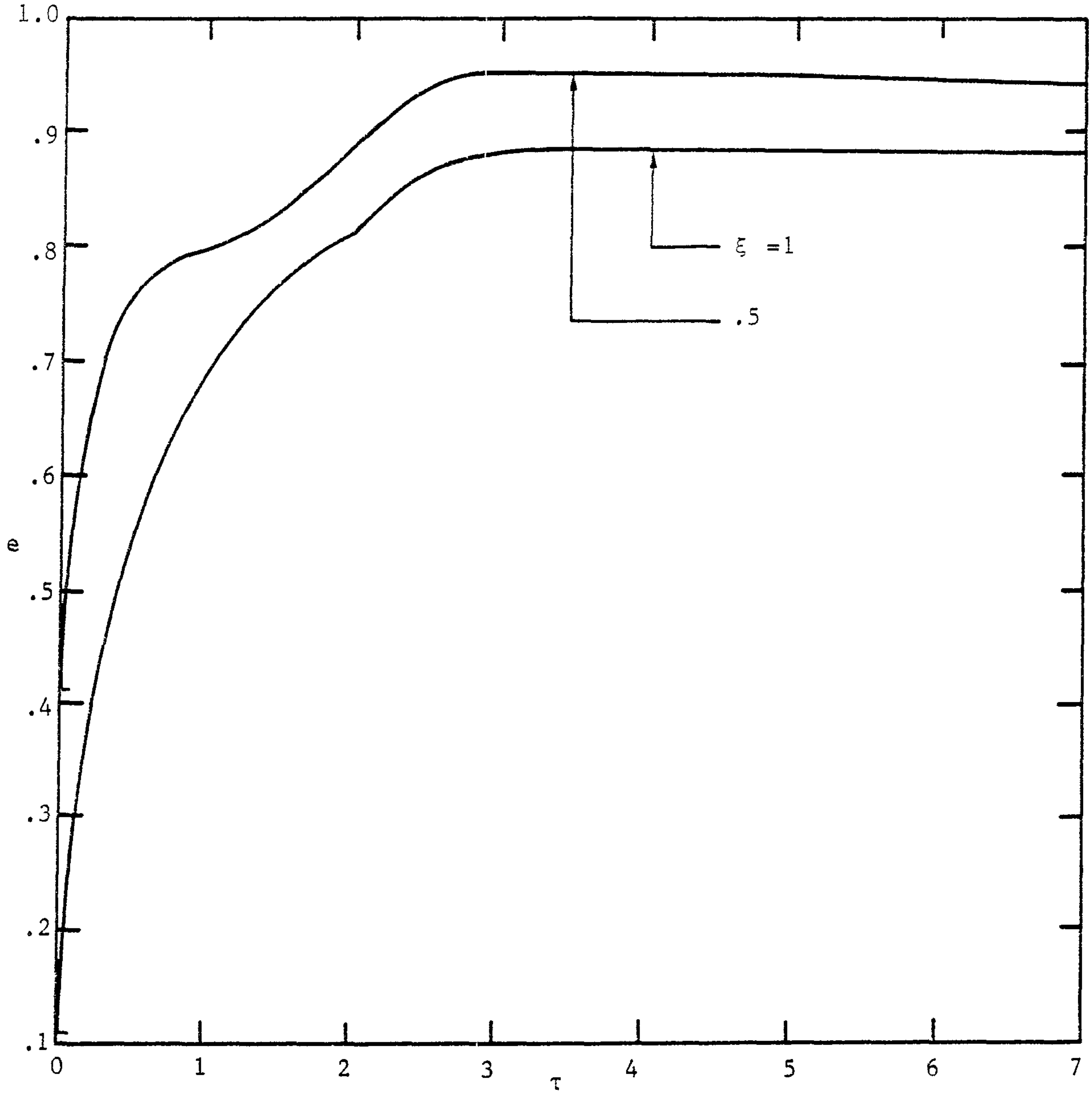

Figure 12. Voltage wave forms on nonuniform transmission line excited by a step voltage when ε_r=10, ξ_σ=.607, ξ_o=.12, ξ_s=.06, ξ_C=.5.

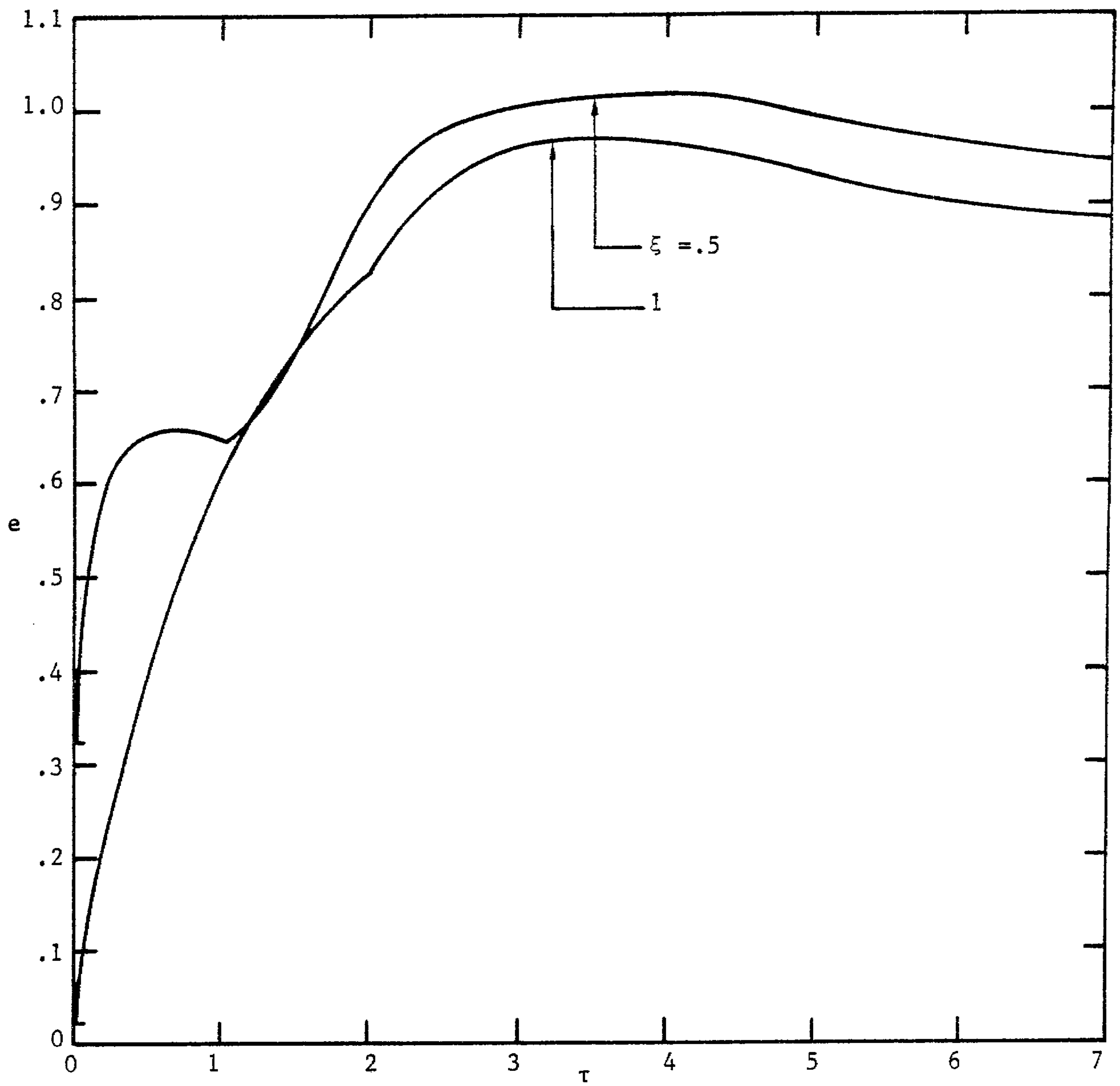

Figure 13. Voltage wave forms on nonuniform transmission line excited by a step voltage when ε_r=10, ξ_σ=.607, ξ_o=.12, ξ_s=.10, ξ_C=.5.

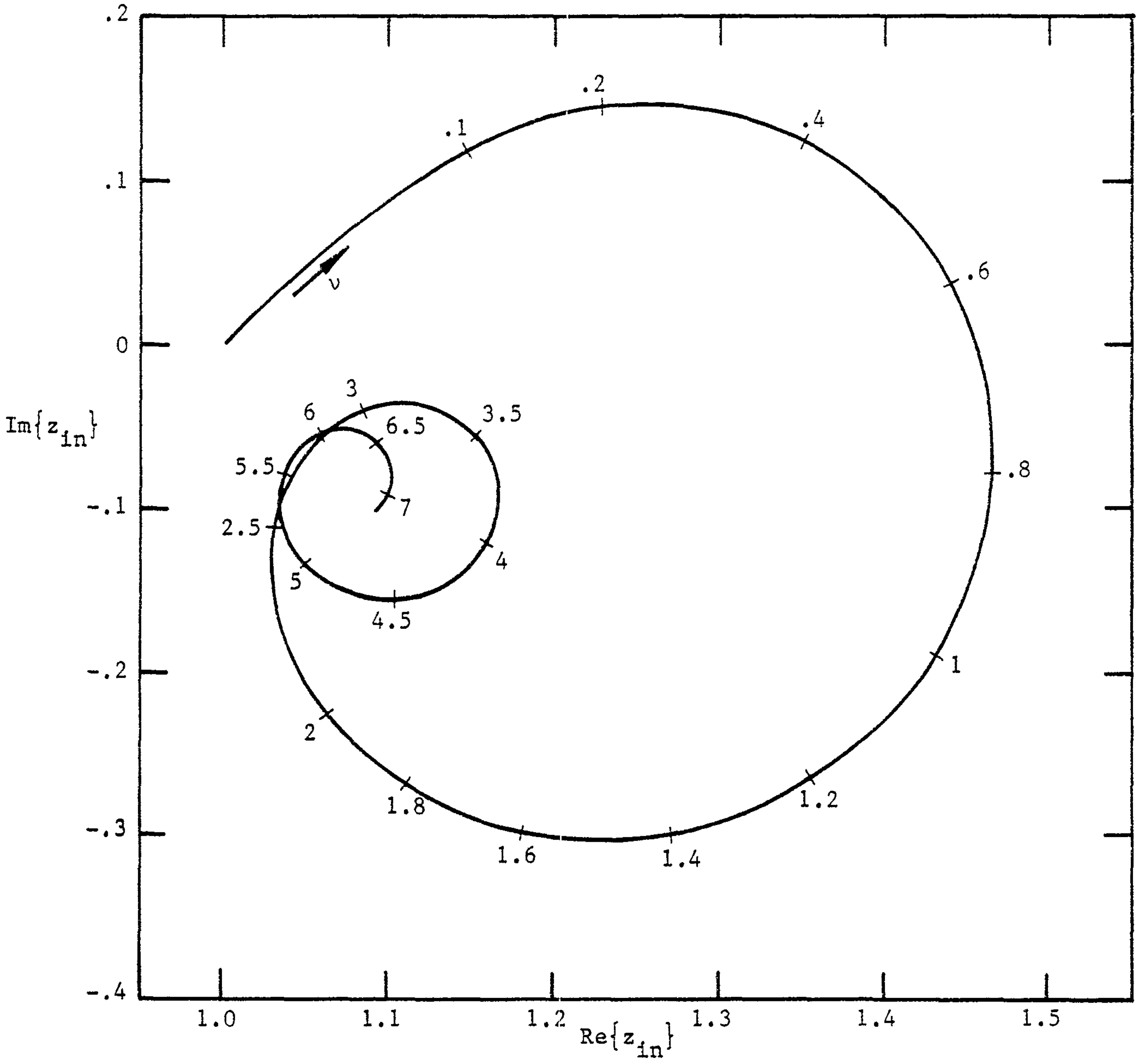

Figure 14. Input impedance of nonuniform transmission line when $\varepsilon_r=10$, $\xi_\sigma=.607$, $\xi_o=.12$, $\xi_s=0$, $\xi_C=.1$.

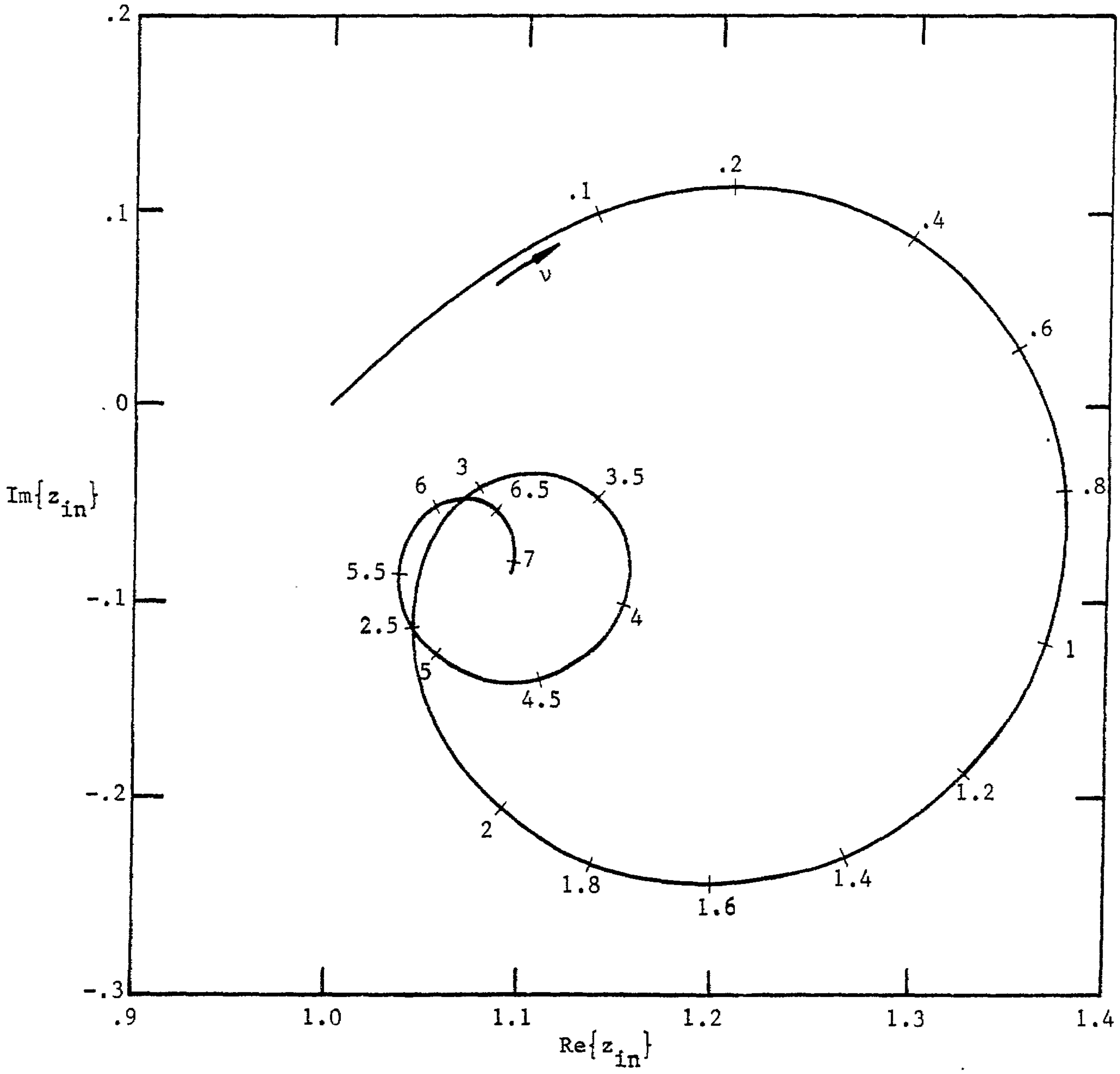

Figure 15. Input impedance of nonuniform transmission line when $\varepsilon_r=10$, $\xi_\sigma=.607$, $\xi_o=.12$, $\xi_s=.01$, $\xi_C=.1$.

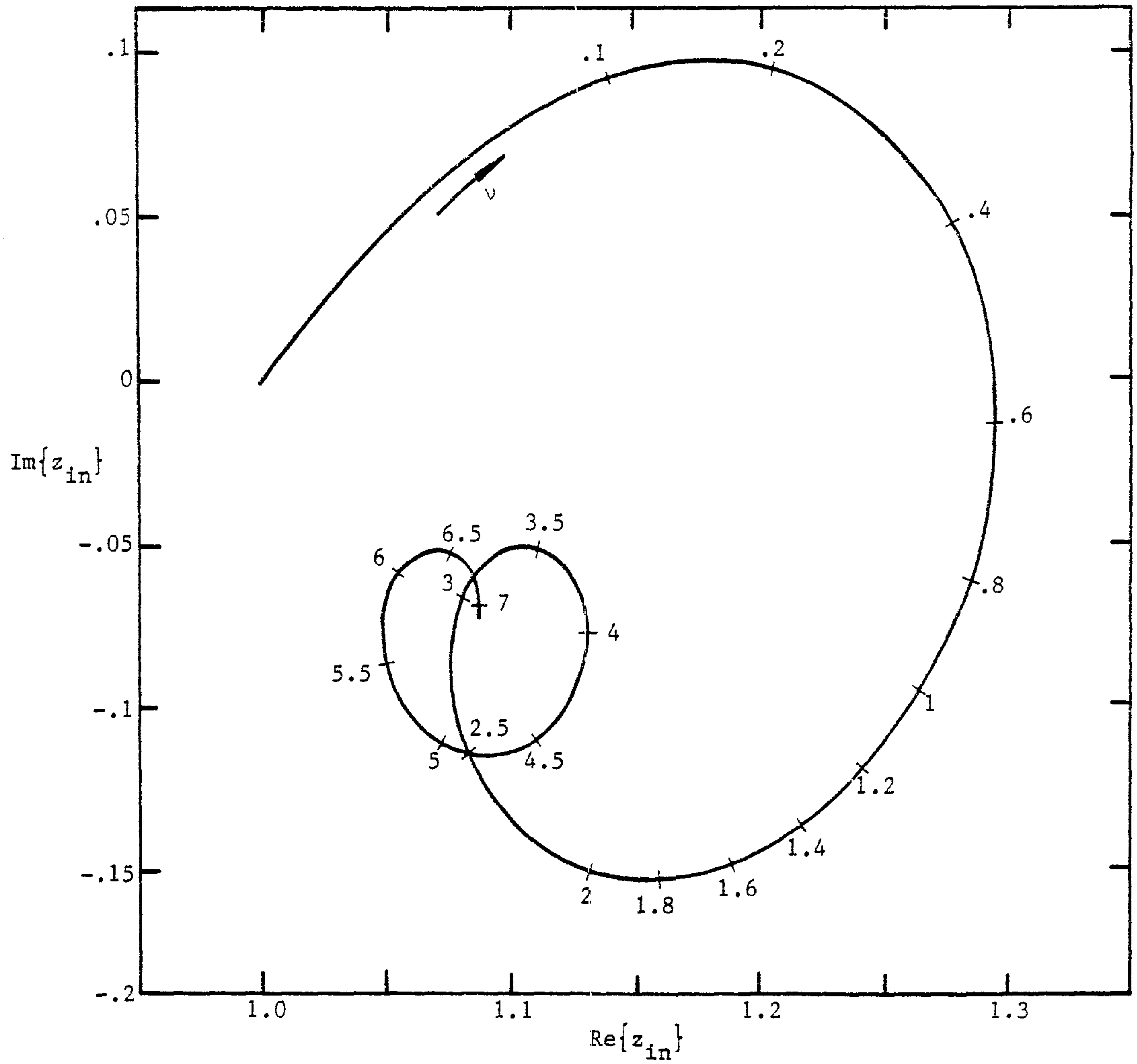

Figure 16. Input impedance of nonuniform transmission line when ε_r=10, ξ_σ=.607, ξ_o=.12, ξ_s=.02, ξ_C=.1.

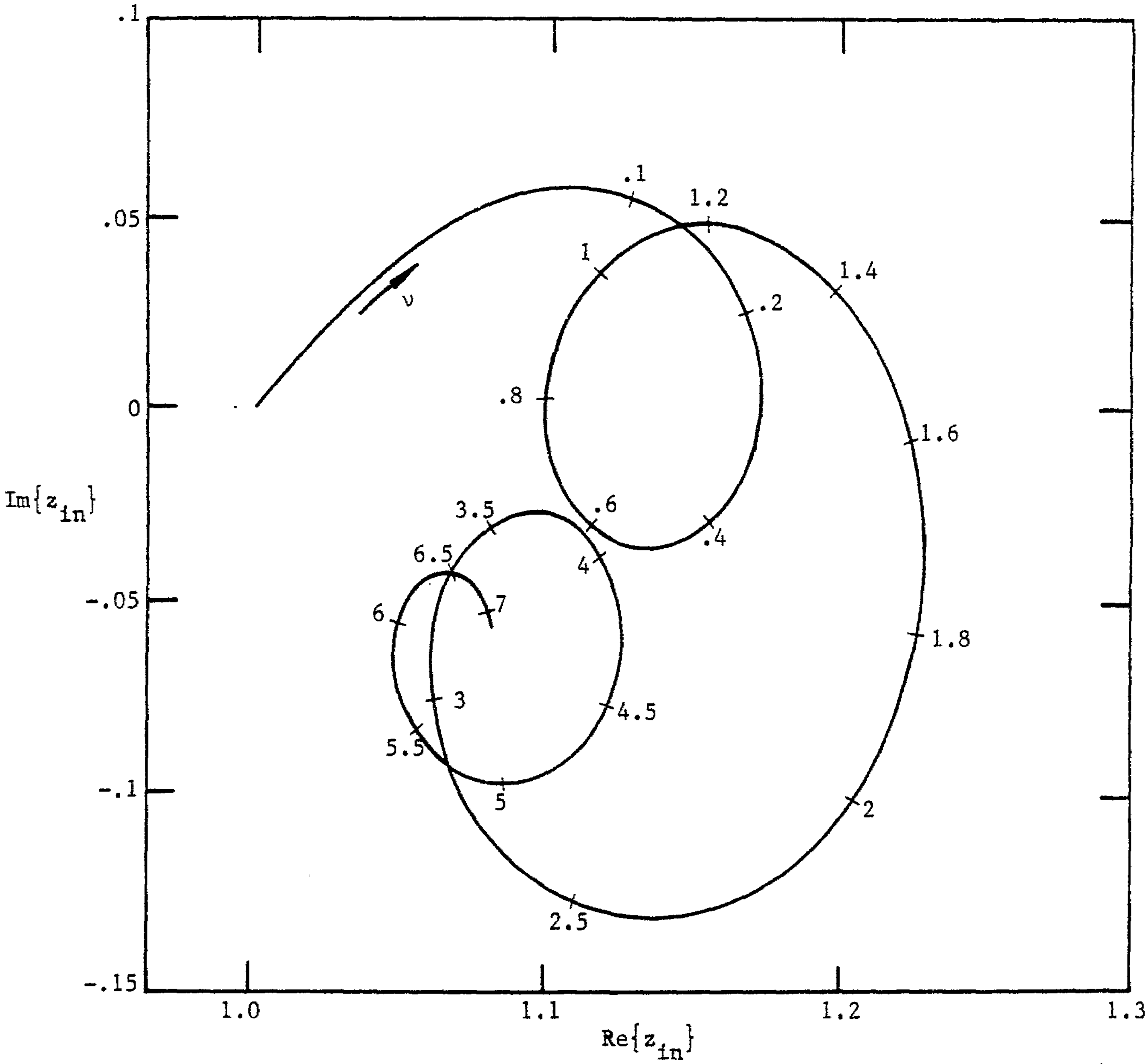

Figure 17. Input impedance of nonuniform transmission line when ε_r=10, ξ_σ=.607, ξ_o=.12, ξ_s=.04, ξ_C=.3.

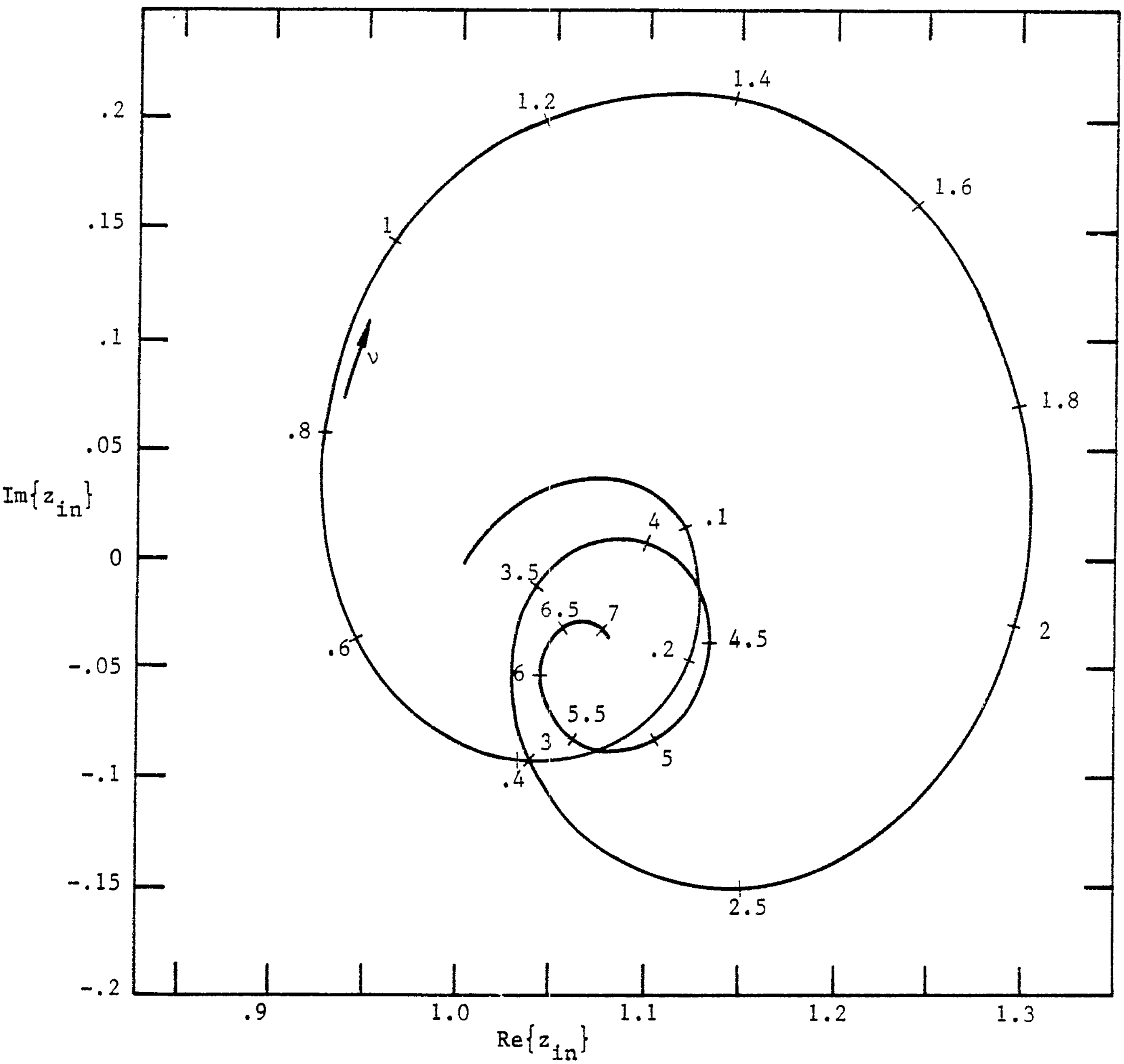

Figure 18. Input impedance of nonuniform transmission line when $\varepsilon_r=10$, $\xi_\sigma=.607$, $\xi_o=.12$, $\xi_s=.06$, $\xi_C=.5$.

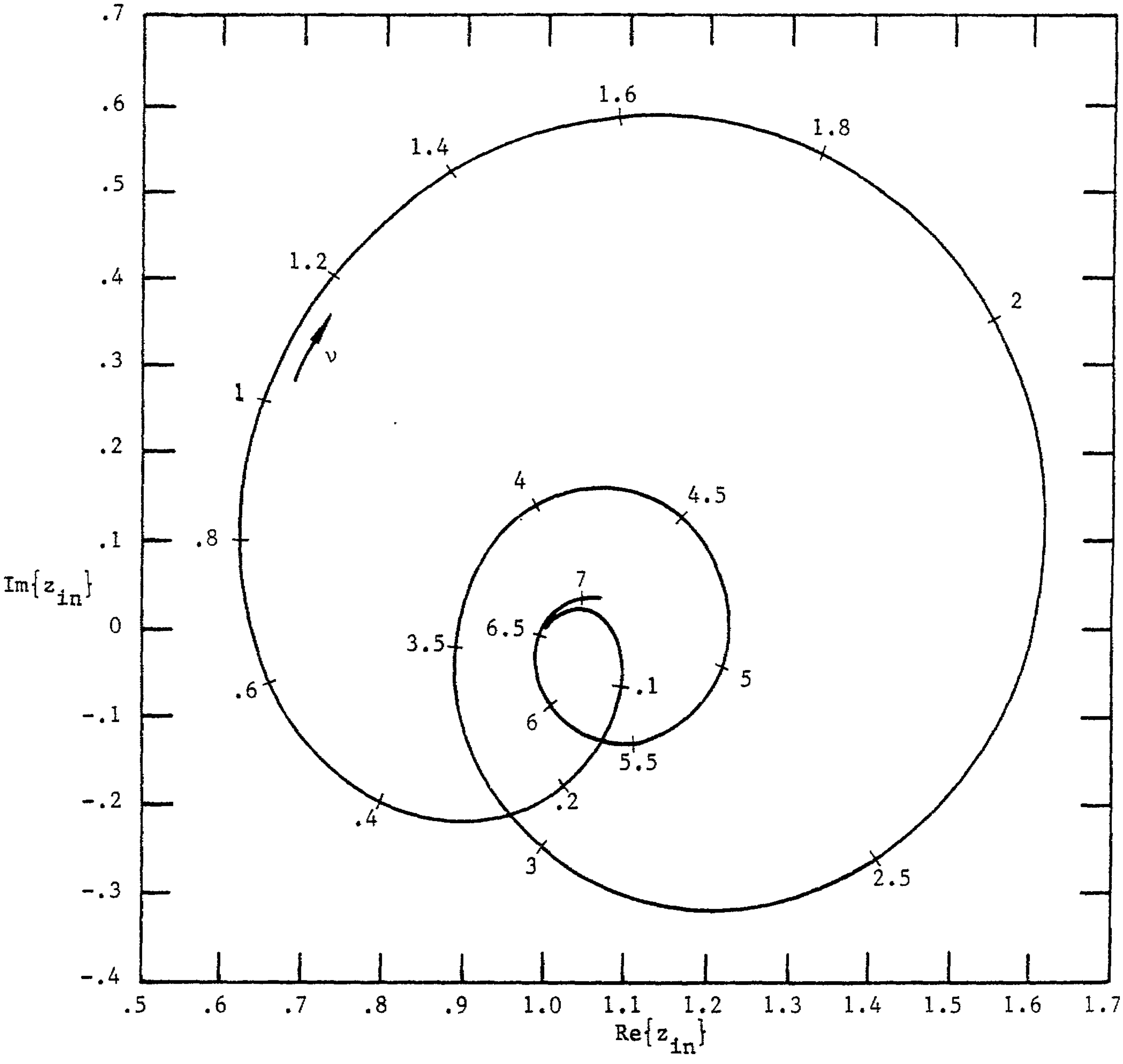

Figure 19. Input impedance of nonuniform transmission line when ε_r=10, ξ_σ=.607, ξ_o=.12, ξ_s=.10, ξ_C=.5.

Appendix A

Solution of Equation (6)

The solution, $u(x,k)$, of the differential equation (6) in section II can be expressed in terms of Whittaker's confluent hypergeometric functions, $W_{\kappa,\mu}$ and $M_{\kappa,\mu}$,[5]

$$u(x,k) = C_1(k)F_1(x,k) + C_2(k)F_2(x,k) \tag{A1}$$

where

$$F_1(x,k) = \frac{\xi_s e^{z/2}}{2jk\sqrt{z}} W_{\kappa,0}(z) \quad ,$$

$$F_2(x,k) = \frac{\xi_s e^{z/2}}{2jk\sqrt{z}} M_{\kappa,0}(z) \quad ,$$

$$z = -2jk\ y(x)/\xi_s \quad ,$$

and

$$\kappa = 0.5\ Z_g Z_o^{-1} \xi_s^{-1} \quad .$$

The constants of integration, $C_1(k)$ and $C_2(k)$, are determined from the boundary conditions at $x = 0$ and $x = d$ given by equations (12) and (13) in section II. Thus,

$$\left.\begin{aligned} C_1(k) &= [F_2'(d,k) - F(k)F_2(d,k)]/N(k) \\ C_2(k) &= [-F_1'(d,k) + F(k)F_1(d,k)]/N(k) \end{aligned}\right\} \tag{A2}$$

where

$$N(k) = F_1(0,k)[F_2'(d,k) - F(k)F_2(d,k)] - F_2(0,k)[F_1'(d,k) - F(k)F_1(d,k)]$$

and

$$F(k) = jk - Y(d,k)Z_L(k) \quad .$$

From Lemma 2 given below it follows that $N(k) \neq 0$ for k real and $k \neq 0$. Thus, $u(x,k)$ is analytic for k real and $k \neq 0$.

In the two lemmas given below we will make use of the input impedance $Z_{in}(k) = Z_o y_o W^{-1} z(0,k)$, where $z(x,k)$ is defined by equation (8) in section II.

Lemma 1. The input impedance $Z_{in}(k) \neq 0$ for k real and $k \neq 0$.

Proof. Suppose the transmission line is driven by a current generator with harmonic time-dependence (e^{jckt}) and r.m.s. unity. The time averaged power, P_{in}, fed into the transmission line from the generator is then

$$P_{in}(k) = \mathrm{Re}\{Z_{in}(k)\} \quad . \tag{A3}$$

The time averaged power loss in the load, P_L, is

$$P_L(k) = |u(d,k)|^2 \mathrm{Re}\{Z_L(k)\} = |M(d,k)/N(k)|^2 \mathrm{Re}\{Z_L(k)\} \tag{A4}$$

where Z_L is given by equation (13) in section II and

$$M(x,k) = \xi_s z\Gamma(1/2 - \kappa)e^{-z} \quad .$$

This expression for $M(x,k)$ is easily obtained from the Wronskian for the Whittaker functions. Because of the passive nature of the transmission line we have

$$P_{in}(k) \geq P_L(k) \quad . \tag{A5}$$

Suppose there exists k_1 real, $k_1 \neq 0$, such that $Z_{in}(k_1) = 0$. Then $P_{in}(k_1) = 0$, while from equation (A4) it follows that $P_L(k_1) > 0$. This contradicts equation (A5). Thus, $Z_{in}(k) \neq 0$ for k real and $k \neq 0$.

Lemma 2. The input admittance $Y_{in}(k) = Z_{in}^{-1}(k) \neq 0$ for k real and $k \neq 0$.

Proof. Suppose the transmission line is driven by a voltage generator with harmonic time dependence (e^{jkct}) and r.m.s. unity. The time averaged power fed into the transmission line is then

$$P_{in}(k) = \mathrm{Re}\{Y_{in}(k)\} \tag{A6}$$

where

$$Y_{in}(k) = Z_{in}^{-1}(k) = N(k)/A(k) \quad .$$

From Lemma 1 it follows that A(k) is analytic and $A(k) \neq 0$ for k real and $k \neq 0$. The time averaged power loss in the load is

$$P_L(k) = |M(d,k)/A(k)|^2 \mathrm{Re}\{Z_L(k)\} \tag{A7}$$

and again we have

$$P_{in}(k) \geq P_L(k) \quad . \tag{A8}$$

Suppose there exists k_2 real, $k_2 \neq 0$, such that $Y_{in}(k_2) = 0$. Then $P_{in}(k_2) = 0$, while from equation (A7) it follows that $P_L(k_2) > 0$. This is in contradiction to equation (A8). Thus, $Y_{in}(k_2) \neq 0$ for k real and $k \neq 0$.

Using the same kind of argument as in the proof of Lemma 2 one can easily show that $u(x,k) \neq 0$ for k real and $k \neq 0$. Thus, $z(x,k)$ is analytic for k real and $k \neq 0$.

Appendix B

Approximate Solution of Equation (6) for Low Frequencies

In this appendix we will find an approximate solution of equation (6) in section II valid for low frequencies. Substituting

$$u'(x,k)/u(x,k) = f(x,k) + jk + 0.5\ \xi_s/y(x) \tag{B1}$$

and

$$h(x,k) = k^2 - jk\ Z_g Z_o^{-1} y^{-1}(x) + 0.25\ \xi_s^2 y^{-2}(x) \tag{B2}$$

into equation (6) in section II we arrive at the Ricatti equation

$$f'(x,k) + f^2(x,k) + h(x,k) = 0 \tag{B3}$$

with the boundary condition

$$f(d,k) = -\ jk\ WZ_L Z_o^{-1} y_d^{-1} - 0.5\ \xi_s y_d^{-1} \quad .$$

Here $Z_L(k)$ is analytic on the complex k-plane except for one simple pole at $k = k_1$,

$$k_1 = j[c(R_1 + R_2)C]^{-1} \quad .$$

In appendix A we have shown that $f(x,k)$ is analytic for k real and $k \neq 0$. The surface impedance Z_g has a branch-point at $k = 0$, and for $|kd| << 1$ we can express $h(x,k)$ in equation (B2) as

$$h(x,k) = (jk)^{3/2} g(x,k) + \ell(x,k) \tag{B4}$$

where $g(x,k)$ and $\ell(x,k)$ are analytic functions of k for $|kd| << 1$. Suppose now the following expansion of $f(x,k)$ is valid for $|kd| << 1$

$$f(x,k) = (jk)^{\alpha} p(x,k) + q(x,k) \tag{B5}$$

where $p(x,k)$ and $q(x,k)$ are analytic functions of k for $|kd| << 1$ and $p(x,k) \neq 0$. Equations (B3)-(B5) then give

$$(jk)^{\alpha} p'(x,k) + q'(x,k) + (jk)^{2\alpha} p^2(x,k) + q^2(x,k)$$
$$+ 2(jk)^{\alpha} p(x,k) q(x,k) + (jk)^{3/2} g(x,k) + \ell(x,k) = 0 \quad . \tag{B6}$$

From the analyticity condition at $k = 0$ it follows that equation (B6) can be split up into two different systems of equations, namely (I) for $\alpha = 3/4$, $p(x,k)$ and $g(x,k)$ must satisfy the system of differential equations

$$\left.\begin{aligned} &p'(x,k) + 2p(x,k)q(x,k) = 0 \\ &p^2(x,k) + g(x,k) = 0 \\ &q'(x,k) + q^2(x,k) + \ell(x,k) = 0 \end{aligned}\right\} \tag{B7}$$

(II) for $\alpha = 3/2$, $p(x,k)$ and $q(x,k)$ must satisfy the system of differential equations

$$\left.\begin{aligned} &p'(x,k) + 2p(x,k)q(x,k) + g(x,k) = 0 \\ &q'(x,k) - jk^3 p^2(x,k) + q^2(x,k) + \ell(x,k) = 0 \quad . \end{aligned}\right\} \tag{B8}$$

However, it is easy to show that the system of differential equations (B7) is overdetermined and hence has no solution. Thus, we have $\alpha = 3/2$, and $p(x,k)$ $q(x,k)$ must satisfy the system of differential equations (B8). By expanding $p(x,k)$, $q(x,k)$, $g(x,k)$, $\ell(x,k)$ and $f(d,k)$ in a power series in k and substituting these expansions into the system of equations (B8) we get the following approximate solution of $f(x,k)$, valid for $|kd| << 1$,

$$f(x,k) = q_o(x) + jk\, q_1(x) + (jk)^2 q_2(x) + (jk)^{3/2} p_o(x)$$

$$+ (jk)^{5/2} p_1(x) + O(k^3 d^3) \qquad \text{(B9)}$$

where

$$q_o(x) = -\,0.5\ \xi_s/y(x) \quad ,$$

$$q_1(x) = -\ y_o/y(x) \quad ,$$

$$q_2(x) = [y_o/y(x)]\{d\xi_c + y_o\xi_s^{-1}\ln[y(x)/y_d] - 0.5[y^2(x) - y_d^2]y_o^{-1}\xi_s^{-1}\} \ ,$$

$$p_o(x) = \xi_\sigma d^{-\frac{1}{2}}(x - d)y_o/y(x) \quad ,$$

$$p_1(x) = [\xi_\sigma d^{-\frac{1}{2}} y_o^2 y_d/y(x)]\{[2\xi_s^{-1} + 0.5\ \varepsilon_r\xi_\sigma^2 y_o/d](d - x)y_d^{-1}$$

$$-\ 2\xi_s^{-2}\ln[y(x)/y_d]\} \quad ,$$

$$\xi_c = y_o C/Wd\varepsilon_o$$

and

$$\xi_\sigma^2 = d/\sigma Z_o y_o^2 \quad .$$

Moreover, we have for $|kd| << 1$

$$u(x,k) = 1 + jk\ u_1(x) + (jk)^{3/2} u_2(x) + O(k^2 d^2) \qquad \text{(B10)}$$

where

$$u_1(x) = x - y_o\xi_s^{-1}\ln[y_o/y(x)]$$

and

$$u_2(x) = y_o \xi_\sigma \xi_s^{-1} d^{-\frac{1}{2}} \{ y_d \xi_s^{-1} \ln[y_o/y(x)] - x \} \quad .$$

The approximate expression for f(x,k) given by equation (B9) was used when deriving an approximate expression of the input impedance valid for low frequencies (equation (17) in section II). Together equations (B9) and (B10) were used to obtain equations (20) and (22) in section III.

Appendix C

Approximate Solution of Equation (6) for High Frequencies

We will here derive an approximate expression for the solution of the differential equation (6), $u(x,k)$, valid for large k. In the frequency domain the current on the transmission line, $I(x,k)$, satisfies the differential equation[1]

$$I'' - (Y'/Y)I' - YZI = 0 \quad . \tag{C1}$$

With

$$I(x,k) = I_1(k)s(x,k)e^{-jkx}$$

$s(x,k)$ satisfies the differential equation

$$s'' - (2jk + \xi_s/y)s' + (jk/y)(\xi_s - Z_g/Z_o)s = 0 \quad , \tag{C2}$$

and with

$$I(x,k) = I_2(k)p(x,k)e^{jkx}$$

$p(x,k)$ satisfies the differential equation

$$p'' + (2jk - \xi_s/y)p' - (jk/y)(\xi_s + Z_g/Z_o)p = 0 \quad . \tag{C3}$$

Thus, (see equation (6) in section II)

$$u(x,k) = s(x,k) + p(x,k)e^{2jkx} \quad . \tag{C4}$$

For k large we make the expansion

$$s(x,k) = k^{\alpha} \sum_{m=0}^{\infty} (jk)^{-m} s_m(x) \tag{C5}$$

$$p(x,k) = k^{\alpha} \sum_{m=0}^{\infty} (jk)^{-m} p_m(x) \quad . \tag{C6}$$

The boundary condition $u(0,k) = 1$ gives

$$\alpha = 0 \tag{C7}$$

and

$$s_m(0) + p_m(0) = \delta_{om} \tag{C8}$$

where δ_{nm} is the Kronecker symbol. The boundary condition at $x = d$ yields (see equation (13) in section II)

$$u'(d,k)/u(d,k) = jk - Y(d,k)Z_L(k) \quad . \tag{C9}$$

Expanding equations (C2), (C3) and (C9) in a power series in k^{-1} and making use of the fact that $WR_1R_2 = Z_o(R_1 + R_2)y_d$ and $s_o(x) \not\equiv 0$ we arrive at the following expression for $u(x,k)$, valid for $|kd| >> 1$,

$$u(x,k) = f(x) - (j/kd)[h(x) + a\ g(x)e^{-2jk(d-x)} - a\ f(x)e^{-2jkd}] + 0(k^{-2}d^{-2}) \tag{C10}$$

where

$$f(x) = [y(x)/y_o]^{\beta} \quad ,$$

$$g(x) = [y(x)/y_o]^{-\beta-1} \quad ,$$

$$h(x) = 0.5\ \beta^2\xi_s d[y^{-1}(x) - y_o^{-1}] + 0.25\ d\sigma Z_o\xi_s^{-1}\varepsilon_r^{-3/2}f(x)\ln[y_o/y(x)] \quad ,$$

$$\beta = 0.5(\xi_s^{-1}\varepsilon_r^{-1/2} - 1)$$

and

$$a = (\beta\xi_s d\ y_d^{-1} - \xi_C^{-1}y_d y_o^{-1})(y_d/y_o)^{\xi_s^{-1}\varepsilon_r^{-1/2}} \quad .$$

The expression for u(x,k) given by equation (C9) was used when deriving equation (19) in section II and in the numerical evaluation of the integrals in equations (20) and (22) in section III.

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