Sensor and Simulation Notes
Note 125

23 January 1971

Some Characteristics of Electric and Magnetic Dipole Antennas for Radiating Transient Pulses

Capt Carl E. Baum
Air Force Weapons Laboratory

Abstract

This note considers some of the properties of electric and magnetic dipole antennas for radiating transient electromagnetic pulses. Each type has certain advantages and disadvantages associated with the electrical generators one might use with it. In particular one might use a magnetic dipole radiator with an appropriate pulse generator to maximize the low frequencies radiated in the pulse waveform. However, such a pulse generator might typically have a comparatively slow rise time with an accompanying loss of the high frequencies radiated. One application of electric and magnetic dipole radiators would be to use them in combination and obtain the best features of both. This note considers some general properties of radiating current distributions leading to electric and magnetic dipole radiators for best low frequency radiation. Electric and magnetic dipoles are then combined and some of the properties of the combination are considered.

Foreword

Initially this started out to be a rather brief note expounding some of the features of crossed electric and magnetic dipoles. One thing led to the next and before I knew it I was writing an extensive treatment of the low frequency characteristics of the distant fields and potentials associated with charge and current density distributions confined to finite volumes applying to antennas of finite size. This applies to both electric and magnetic dipole antennas, both singly and in combination, as the dominant antennas for low frequencies and large distances from the source (antenna).

I would like to thank Dr. Kelvin Lee of Northrop (Pasadena) as well as Sgt. Richard Clark of AFWL and Mr. Joe Martinez of Dikewood for reading over the draft to catch some of the errors in such a lengthy note. Thanks in particular goes to Mrs. Joanne Long for typing this note as well as many other recent notes.

"Cheshire Puss," she began, rather timidly, as she did not at all know whether it would like the name; however, it only grinned a little wider. "Come, it's pleased so far," thought Alice, and she went on, "Would you tell me, please, which way I ought to walk from here?"

"That depends a good deal on where you want to get to," said the Cat.

"I don't much care where," said Alice.

"Then it doesn't matter which way you walk," said the Cat.

"--so long as I get somewhere," Alice added as an explanation.

"Oh, you're sure to do that," said the Cat, "if you only walk long enough!"

(Lewis Carroll, Alice in Wonderland)

Contents

List of Illustrations

I. Introduction

One general class of simulators for the nuclear electromagnetic pulse (EMP) consists of pulse radiating antennas. For such simulators one radiates an electromagnetic pulse from the antenna to an observation position far from the radiator in terms of the size of the radiating antenna. Previous notes have considered various features of such pulse-radiating antennas, particularly oriented toward long thin electric dipoles. Another type briefly considered is the pulse-radiating planar array.[1] In this note we generalize some of the previous considerations of electric dipole radiating antennas to include magnetic dipoles.

A significant limitation of an electromagnetic pulse radiator is its poor efficiency in radiating the low-frequency portion of the pulse with corresponding wavelengths significantly larger than the dimensions of the radiating antenna. With limited energy available then the Fourier transform of the radiated pulse goes to zero in the limit of low frequencies; this can be a significant limitation if one is trying to simulate an EMP which has a Fourier transform which does not roll off for low frequencies of interest. One type of pulse radiating antenna which has been used for EMP work is basically a capacitive electric dipole driven by a capacitive high voltage pulse generator; this can give the electric dipole a late time electric dipole moment to maximize the low frequencies in the radiated pulse and also give a fast risetime to the radiated pulse. Given an appropriate pulse generator an inductive magnetic dipole (or basically a loop antenna) can also be used to radiate an electromagnetic pulse with similar low-frequency performance characteristics. From a practical standpoint one difficulty with a radiating magnetic dipole is matching it to an appropriate pulse generator which is basically inductive at low frequencies, yet still have it switch into the antenna fast enough to give a sufficiently fast risetime to the radiated pulse. However certain types of inductive pulse generators can store and deliver quite large energies to an inductive load. One possible use of a radiating magnetic dipole is then to combine it with a radiating electric dipole or other high-frequency radiator. The magnetic dipole would be used to maximize the radiated low frequencies.

In this note we first consider some of the general characteristics of radiating current distributions, leading to the formulation of electric and magnetic dipole terms as the dominant low-frequency radiation terms. This is followed by a discussion of some of the characteristics of a few electric and magnetic dipole antennas together with some of the generators one might use with them. Finally we consider some possible geometries of combined electric and magnetic dipole radiating antennas and some of the electromagnetic characteristics of the combination.

II. Separation of the Current Density into Solenoidal and Irrotational Terms

In calculating the electromagnetic fields associated with a current density distribution in free space one has the vector and scalar potentials given by (all units rationalized MKSA)

$$\vec{A}(\vec{r},t) = \mu_o \int_{V'} \frac{\vec{J}\left(\vec{r}\,',t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{4\pi|\vec{r}-\vec{r}\,'|}\, dV'$$

$$\Phi(\vec{r},t) = \frac{1}{\epsilon_o} \int_{V'} \frac{\rho\left(\vec{r}\,',t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{4\pi|\vec{r}-\vec{r}\,'|}\, dV' \tag{2.1}$$

where the primed coordinates ($\vec{r}\,'$) are used as integration variables for the current density $\vec{J}$ and the charge density ρ which are contained in a volume V' such that $\vec{J}$ and ρ and all their successive derivatives are zero on S', the surface of V'. As shown in figure 1 the position at which the fields and potentials are to be calculated will typically be listed with coordinates $\vec{r}$. The time is t and the retarded time $t - |\vec{r}-\vec{r}\,'|/c$ is used to give the solution subject to the radiation condition at infinity. The speed of light and wave impedance of free space are

$$c \equiv \frac{1}{\sqrt{\mu_o \epsilon_o}}, \qquad Z_o \equiv \sqrt{\frac{\mu_o}{\epsilon_o}} \tag{2.2}$$

where μ_o and ϵ_o are respectively the permeability and permittivity of free space. Using a tilde ~ over a quantity for the two sided Laplace transform the potentials are written as

$$\tilde{\vec{A}}(\vec{r}) = \mu_o \int_{V'} \tilde{\vec{J}}(\vec{r}\,')\, \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}\, dV'$$

$$\tilde{\Phi}(\vec{r}) = \frac{1}{\epsilon_o} \int_{V'} \tilde{\rho}(\vec{r}\,')\, \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}\, dV' \tag{2.3}$$

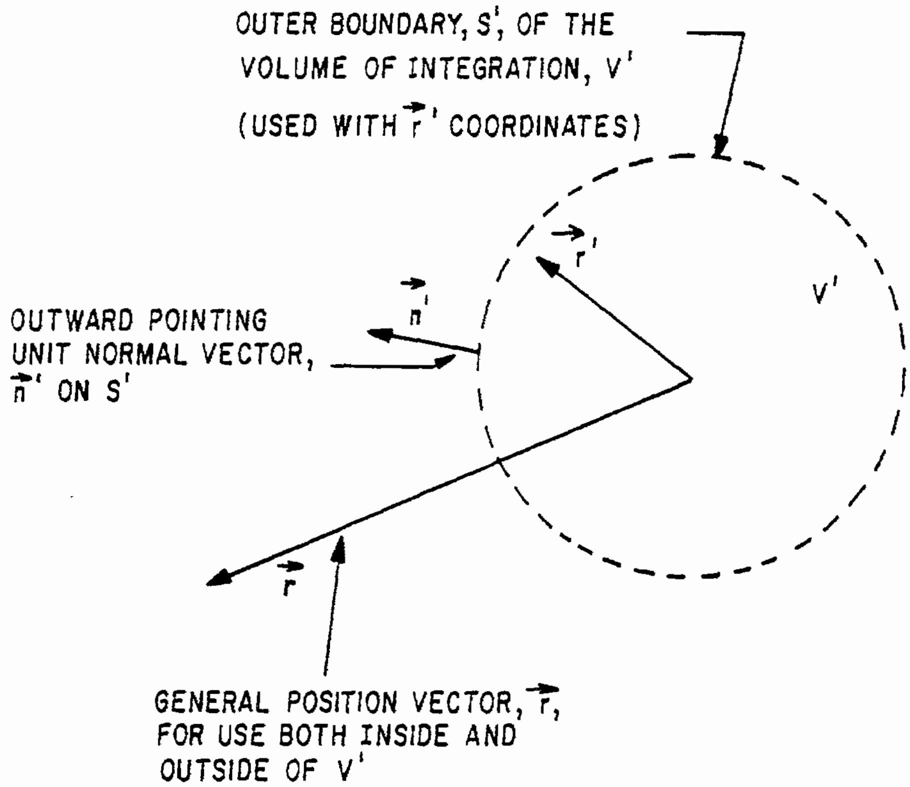


FIGURE 1. COORDINATES FOR ELECTROMAGNETIC QUANTITIES

The current density and charge density are related by the equation of continuity

$$\nabla \cdot \vec{J}(\vec{r},t) + \frac{\partial \rho}{\partial t}(\vec{r},t) = 0 \;, \quad \nabla \cdot \tilde{\vec{J}}(\vec{r}) + s\tilde{\rho}(\vec{r}) = 0 \tag{2.4}$$

Note that ∇ operates on the $\vec{r}$ coordinates, ∇' on the $\vec{r}\,'$ coordinates, etc. The fields are found from the potentials using

$$\vec{B} = \nabla \times \vec{A} \;, \quad \vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} \tag{2.5}$$

The vector and scalar potentials in this formulation are related by the Lorentz gauge as

$$\nabla \cdot \vec{A} + \varepsilon_o \mu_o \frac{\partial \Phi}{\partial t} = 0 \tag{2.6}$$

The Laplace transform variable is s (sometimes not shown) and the propagation constant is

$$\gamma = ik = \frac{s}{c} \tag{2.7}$$

For Fourier transforms s can be replaced by $i\omega$.

Maxwell's equations in free space are

$$\begin{aligned} &\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \;, \quad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \\ &\nabla \cdot \vec{B} = 0 \;, \quad \nabla \cdot \vec{D} = \rho \end{aligned} \tag{2.8}$$

and the constitutive relations are

$$\vec{B} = \mu_o \vec{H} \;, \quad \vec{D} = \varepsilon_o \vec{E} \tag{2.9}$$

Equations 2.1 through 2.7 give the solution of equations 2.8 and 2.9 in terms of $\vec{J}$ and ρ subject to the radiation condition at infinity.

With these preliminaries out of the way let us consider the current density $\vec{J}$ a little more closely in this section. By the Helmholz theorem we can write a vector field as the sum of the gradient of a scalar field and the curl of a vector field which can be chosen to have zero divergence.[2] Thus write the current density as

$$\vec{J} = \vec{J}_e + \vec{J}_h \tag{2.10}$$

where

$$\vec{J}_e = \nabla \Phi_j \ , \quad \vec{J}_h = \nabla \times \vec{A}_j \tag{2.11}$$

The subscripts "e" and "h" are used to associate the two terms with $\vec{E}$ and $\vec{H}$ and the two terms comprising $\vec{J}$ might be roughly thought of as electric and magnetic portions respectively of the current density, as will become clearer later. The subscript "j" is used with $\vec{A}_j$ and Φ_j to denote what might be respectively called the current density vector potential and the current density scalar potential. Note that they are not the same as $\vec{A}$ and Φ as in equations 2.1 and 2.3. Of course there are requirements on the smooth behavior and behavior at infinity of $\vec{J}$ for this splitting to strictly hold, but we will generally deal with $\vec{J}$ confined to some volume of space and if it is discontinuous or singular, then taken as the limit of a well behaved $\vec{J}$.

As an aid in constructing $\vec{J}_e$ and $\vec{J}_h$ from $\vec{J}$ consider another vector function $\vec{\Lambda}_j$ (perhaps called the current density super potential) calculated as

$$\vec{\Lambda}_j(\vec{r}) = \int_{V'} \frac{\vec{J}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} \, dV' \tag{2.12}$$

which is a solution of the vector Poisson equation

$$\nabla^2 \vec{\Lambda}_j(\vec{r}) = -\vec{J}(\vec{r}) \tag{2.13}$$

Note that these functions are all functions of t (suppressed) and that retarded time is not used in the solution (e.g. see equation 2.12). This splitting is also carried over into the Laplace transform domain.

Expand the vector Laplacian in equation 2.13 to give

$$\nabla[\nabla \cdot \vec{\Lambda}_j(\vec{r})] - \nabla \times [\nabla \times \vec{\Lambda}_j(\vec{r})] = -\vec{J}(\vec{r}) \tag{2.14}$$

Comparing this to equations 2.11 we identify

$$\begin{aligned} \Phi_j(\vec{r}) &\equiv -\nabla \cdot \vec{\Lambda}_j(\vec{r}) \\ \vec{A}_j(\vec{r}) &\equiv \nabla \times \vec{\Lambda}_j(\vec{r}) \end{aligned} \tag{2.15}$$

so that

$$\begin{aligned} \vec{J}_e(\vec{r}) &= \nabla\Phi_j(\vec{r}) = -\nabla[\nabla \cdot \vec{\Lambda}_j(\vec{r})] \\ \vec{J}_h(\vec{r}) &= \nabla \times \vec{A}_j(\vec{r}) = \nabla \times [\nabla \times \vec{\Lambda}_j(\vec{r})] \end{aligned} \tag{2.16}$$

Thus $\vec{A}_j$, Φ_j, $\vec{J}_e$, and $\vec{J}_h$ are all known in terms of $\vec{\Lambda}_j$ and thus in terms of $\vec{J}$ from equation 2.12. Note from equations 2.15 that $\vec{A}_j$ is given as the curl of a vector field which makes $\vec{A}_j$ have zero divergence. One could add another term to $\vec{A}_j$ which was the gradient of a scalar field, but since only the curl of $\vec{A}_j$ is used and the curl of a gradient is zero it would give no contribution. Thus equations 2.13 and 2.15 can be considered to define the current density potentials.

With $\vec{J}$ split as in equations 2.16 one sees an immediate advantage in that one term has zero curl and the other has zero divergence as

$$\begin{aligned} \nabla \times \vec{J}_e(\vec{r}) &= \nabla \times [\nabla\Phi_j(\vec{r})] = \vec{0} \\ \nabla \cdot \vec{J}_h(\vec{r}) &= \nabla \cdot [\nabla \times \vec{A}_j(\vec{r})] = 0 \end{aligned} \tag{2.17}$$

and the equation of continuity can be written as

$$\nabla \cdot \vec{J}(\vec{r}) = \nabla \cdot \vec{J}_e(\vec{r}) = -\frac{\partial}{\partial t}\rho(\vec{r}) \tag{2.18}$$

Combining this with equations 2.11 gives a scalar Poisson equation

$$\nabla^2 \Phi_j(\vec{r}) = -\frac{\partial}{\partial t}\rho(\vec{r}) \tag{2.19}$$

which has the solution

$$\Phi_j(\vec{r}) = \frac{\partial}{\partial t}\int_{V'} \frac{\rho(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\,dV' \tag{2.20}$$

From this result we see that $\tilde{\Phi}_j/(s\varepsilon_o)$ is the same as $\tilde{\Phi}$ in the low frequency or static limit by looking at equations 2.1 and 2.3. However the solution for Φ_j is used at arbitrary frequency or with arbitrary time dependence.

One can similarly eliminate $\vec{J}_e$ by writing a curl equation as

$$\nabla \times \vec{J}(\vec{r}) = \nabla \times \vec{J}_h(\vec{r}) \equiv \vec{\lambda}(\vec{r}) \tag{2.21}$$

where we have defined a new vector field $\vec{\lambda}$ which is related only to $\vec{J}_h$ just as ρ is related only to $\vec{J}_e$. Combining with equations 2.11 gives

$$\vec{\lambda}(\vec{r}) = \nabla \times [\nabla \times \vec{A}_j(\vec{r})] \tag{2.22}$$

and since from equations 2.15

$$\nabla \cdot \vec{A}_j(\vec{r}) = 0 \tag{2.23}$$

we have

$$\nabla^2\vec{A}_j(\vec{r}) = \nabla[\nabla \cdot \vec{A}(\vec{r})] - \nabla \times [\nabla \times \vec{A}(\vec{r})] = -\vec{\lambda}(\vec{r}) \tag{2.24}$$

This is another vector Poisson equation with the solution

$$\vec{A}_j(\vec{r}) = \int_{V'} \frac{\vec{\Lambda}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} dV' \tag{2.25}$$

Note that $\mu_o\vec{\Lambda}_j$ is the same as $\vec{A}$ in the static or low frequency limit as is seen by comparing equation 2.12 to equations 2.1 and 2.3. Since from equations 2.15 we have $\vec{A}_j$ in terms of $\vec{\Lambda}_j$ then we can see that $\mu_o\vec{A}_j$ is the same as $\nabla \times \vec{A}$ in the low-frequency or static limit. From equations 2.5 and 2.9 then $\vec{A}_j$ is the same as $\vec{H}$ in the low frequency or static limit, but this solution is applied to arbitrary frequencies or time dependence.

Now that the current density potential functions have been calculated in terms of $\vec{J}$, next consider some explicit representations of $\vec{J}_e$ and $\vec{J}_h$ in terms of $\vec{J}$. Starting with equation 2.12 we have

$$\Phi_j(\vec{r}) = -\nabla \cdot \vec{\Lambda}_j(\vec{r}) = -\nabla \cdot \int_{V'} \frac{\vec{J}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} dV' \tag{2.26}$$

Using various formulas of vector analysis in three dimensions[3] and the relation

$$\nabla\left[\frac{1}{4\pi|\vec{r}-\vec{r}\,'|}\right] = -\nabla'\left[\frac{1}{4\pi|\vec{r}-\vec{r}\,'|}\right] \tag{2.27}$$

where a prime on the ∇ operator means that it operates on the primed coordinates, we can manipulate the integrals into other forms. Equation 2.26 can then be written as

$$\Phi_j(\vec{r}) = \int_{V'} \vec{J}(\vec{r}\,') \cdot \nabla'\left[\frac{1}{4\pi|\vec{r}-\vec{r}\,'|}\right] dV'$$

$$= \int_{V'} \nabla' \cdot \left[\frac{\vec{J}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\right] dV' - \int_{V'} \frac{\nabla'\cdot\vec{J}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} dV' \tag{2.28}$$

From Gauss' theorem we have

$$\int_{V'} \nabla' \cdot \left[\frac{\vec{J}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\right] dV' = \int_{S'} \frac{\vec{n}\,'\cdot\vec{J}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} dS' = 0 \tag{2.29}$$

since $\vec{J} = \vec{0}$ on S' (the boundary of V') by hypothesis where $\vec{n}'$ is the outward pointing unit normal vector on S'. Thus Φ_j can be written as

$$\Phi_j(\vec{r}) = -\int_{V'} \frac{\nabla'\cdot\vec{J}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\, dV' = \frac{\partial}{\partial t}\int_{V'} \frac{\rho(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\, dV' \qquad (2.30)$$

so that the $\nabla\cdot$ operator in equation 2.26 has just moved inside the integral and has become a $\nabla'\cdot$ operator operating on $\vec{J}(\vec{r}\,')$.

Using equations 2.11 we can write $\vec{J}_e$ as

$$\vec{J}_e(\vec{r}) = \nabla\Phi_j(\vec{r}) = -\nabla\left\{\nabla \cdot \int_{V'} \frac{\vec{J}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\, dV'\right\} \qquad (2.31)$$

where ∇ is used outside the integral. Moving the ∇ operators inside we first have

$$\vec{J}_e(\vec{r}) = -\nabla\int_{V'} \frac{\nabla'\cdot\vec{J}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\, dV' = \frac{\partial}{\partial t}\nabla\int_{V'} \frac{\rho(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\, dV'$$

$$= \int_{V'} [\nabla' \cdot \vec{J}(\vec{r}\,')]\nabla'\left[\frac{1}{4\pi|\vec{r}-\vec{r}\,'|}\right]dV'$$

$$= \int_{V'} \nabla'\left[\frac{\nabla'\cdot\vec{J}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\right]dV' - \int_{V'} \frac{\nabla'[\nabla'\cdot\vec{J}(\vec{r}\,')]}{4\pi|\vec{r}-\vec{r}\,'|}\, dV' \qquad (2.32)$$

From another form of Gauss' theorem

$$\int_{V'} \nabla'\left[\frac{\nabla'\cdot\vec{J}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\right]dV' = \int_{S'} \frac{\nabla'\cdot\vec{J}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\,\vec{n}dS' = \vec{0} \qquad (2.33)$$

since $\vec{J}$ and all its derivatives are assumed to be zero on S'.

Thus $\vec{J}_e$ can be written as

$$\vec{J}_e(\vec{r}) = -\int_{V'} \frac{\nabla'[\nabla'\cdot\vec{J}(\vec{r}')]}{4\pi|\vec{r}-\vec{r}'|}\,dV' = \frac{\partial}{\partial t}\int_{V'} \frac{\nabla'\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|}\,dV' \qquad (2.34)$$

Equations 2.31, 2.32, and 2.34 have alternate forms for $\vec{J}_e$ explicitly given in terms of $\vec{J}$.

The current density vector potential can be written from equation 2.12 as

$$\vec{A}_j(\vec{r}) = \nabla\times\vec{\Lambda}_j(\vec{r}) = \nabla\times\int_{V'} \frac{\vec{J}(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|}\,dV' \qquad (2.35)$$

This can be manipulated to the form

$$\vec{A}_j(\vec{r}) = -\int_{V'} \nabla'\left[\frac{1}{4\pi|\vec{r}-\vec{r}'|}\right]\times\vec{J}(\vec{r}')$$

$$= -\int_{V'} \nabla'\times\left[\frac{\vec{J}(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|}\right]dV' + \int_{V'} \frac{\nabla'\times\vec{J}(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|}\,dV' \qquad (2.36)$$

From yet another form of Gauss' theorem

$$\int_{V'} \nabla'\times\left[\frac{\vec{J}(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|}\right]dV' = \int_{S'} \vec{n}\times\left[\frac{\vec{J}(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|}\right]dS' = \vec{0} \qquad (2.37)$$

since $\vec{J} = \vec{0}$ on S'. Thus $\vec{A}_j$ can also be written as

$$\vec{A}_j(\vec{r}) = \int_{V'} \frac{\nabla'\times\vec{J}(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|}\,dV' = \int_{V'} \frac{\vec{\lambda}(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|}\,dV' \qquad (2.38)$$

Now $\vec{J}_h$ can be written as

$$\vec{J}_h(\vec{r}) = \nabla\times\vec{A}_j(\vec{r}) = \nabla\times\left\{\nabla\times\int_{V'} \frac{\vec{J}(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|}\,dV'\right\} \qquad (2.39)$$

with ∇ used outside the integral. Using equation 2.38 one can also write

$$\vec{J}_h(\vec{r}) = \nabla \times \int_{V'} \frac{\nabla' \times \vec{J}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\, dV' = \nabla \times \int_{V'} \frac{\vec{\lambda}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\, dV' \qquad (2.40)$$

This result is just like equation 2.35 except $\vec{J}(\vec{r}\,')$ is replaced by $\vec{\lambda}(\vec{r}\,')$. Thus the steps of equations 2.36 and 2.37 can be repeated noting that $\vec{\lambda} = \vec{0}$ on S'. Thus we can write

$$\vec{J}_h(\vec{r}) = \int_{V'} \frac{\nabla' \times [\nabla' \times \vec{J}(\vec{r}\,')]}{4\pi|\vec{r}-\vec{r}\,'|}\, dV' = \int_{V'} \frac{\nabla' \times \vec{\lambda}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\, dV' \qquad (2.41)$$

By adding $\vec{J}_e$ and $\vec{J}_h$ in equations 2.31 and 2.39 or equations 2.34 and 2.41 and combining the curl curl operator with the gradient divergence operator to form a vector Laplacian, one obtains (as a check) an identity for the vector field $\vec{J}$ (which goes to zero for large $|\vec{r}|$) based on the solution of a vector Poisson equation.

Given a distribution of $\vec{J}$ which is sufficiently well behaved in some finite volume (or more generally goes to zero for large $|\vec{r}|$), then one can split it into irrotational ($\vec{J}_e$) and solenoidal ($\vec{J}_h$) parts. Alternatively one can use this splitting technique to construct solenoidal or irrotational current densities from any well behaved vector field. The application of $\vec{J}_e$ and $\vec{J}_h$ to static field problems is rather clear. Electrostatic problems involve the charge density ρ which is related to the divergence of $\vec{J}_e$. Magnetostatic problems only involve $\vec{J}_h$ because if a significant $\vec{J}_e$ is included there is a time changing ρ which gives a large low-frequency $\tilde{\rho}$ and thus large electric fields at low frequencies. For devices which are used to produce low-frequency or static electric or magnetic fields then one normally considers respectively distributions of ρ or $\vec{J}_h$. One type of such a low-frequency device is a dipole, either an electric or magnetic dipole. An electric dipole separates charge and a magnetic dipole has current flowing around an area. Besides producing local static electric or magnetic fields, dipoles can also be used as low-frequency radiating antennas, but which have limited efficiency as radiators at low frequencies. Electric and magnetic dipoles can also be used as sensors for electric and magnetic fields respectively and splitting into ρ and $\vec{J}_h$ applies here as well.[4]

III. Electric and Magnetic Dipoles

With $\vec{J}$ now explicitly represented in terms of its solenoidal and irrotational parts let us now consider some of the moments of the current distribution and their relation to ρ and $\vec{\lambda}$. Referring back to the equations for the vector and scalar potentials (equations 2.1 and 2.3) note that the scalar potential Φ is written as an integral over ρ. Also from equations 2.5 one of the terms for $\vec{E}$ is $-\nabla\Phi$ and thus depends on ρ. From the equation of continuity (equations 2.4) ρ is related to $\vec{J}$ but not completely determined by $\vec{J}$. Specifically, given $\vec{J}$ one can calculate the time derivative of ρ. This only determines ρ to within a constant (independent of t). By assumption $\vec{J}$ and ρ are constrained to be zero everywhere except within V' which has finite linear dimensions, and $\vec{J}$ and ρ and all their derivatives are assumed zero on S'. Consider then the total charge in V'

$$Q \equiv \int_{V'} \rho(\vec{r}',t)\,dV' \tag{3.1}$$

One might also call Q the electric monopole moment.

Taking the time derivative of Q and using the equation of continuity and Gauss' theorem gives

$$\frac{dQ}{dt} = \int_{V'} \frac{\partial}{\partial t} \rho(\vec{r}',t)\,dV' = -\int_{V'} \nabla' \cdot \vec{J}(\vec{r}',t)\,dV'$$

$$= -\int_{S'} \vec{n}' \cdot \vec{J}(\vec{r}',t)\,dS' = 0 \tag{3.2}$$

Requiring no current density to pass through S' then makes Q time independent. This is because, strictly speaking, only the time derivative of ρ is related to $\vec{J}$ or equivalently $\vec{J}_e$. An arbitrary constant ρ, say $\rho_o(\vec{r}')$, can be added with no change in $\vec{J}$.

If $\rho(\vec{r}',t)$ is assumed zero before some time, say t_o, then Q is zero for $t < t_o$ and the result of equation 3.2 makes $Q = 0$ for all time. However if Q is allowed to be non zero and thus constant for all time this can introduce difficulties into the two-sided Laplace transform since $\tilde{Q}$ does not exist if $Q \neq 0$. If a one-sided Laplace transform is used then one needs initial conditions for not only Q but also ρ and the related fields.

Typically for transient problems Q is taken zero, but if $Q \neq 0$ this this term can be treated separately.

The electric dipole moment is just

$$\vec{p}(t) \equiv \int_{V'} \vec{r}'\rho(\vec{r}',t)dV' \tag{3.3}$$

This term is the charge separation in the volume V'. Provided $\rho = 0$ for $t < t_o$ then we can write

$$\tilde{\vec{p}} = \int_{V'} \vec{r}'\tilde{\rho}(\vec{r}')dV' = -\frac{1}{s} \int_{V'} \vec{r}'\nabla' \cdot \tilde{\vec{J}}(\vec{r}')dV'$$

$$= -\frac{1}{s} \int_{S'} [\vec{n}'\cdot\tilde{\vec{J}}(\vec{r}')]\vec{r}'dS' + \frac{1}{s} \int_{V'} \tilde{\vec{J}}(\vec{r}')dV'$$

$$= \frac{1}{s} \int_{V'} \tilde{\vec{J}}(\vec{r}')dV' \tag{3.4}$$

since $\vec{J}$ is zero on S'. In the time domain we have

$$\frac{d}{dt}\vec{p}(t) = \int_{V'} \vec{J}(\vec{r}',t)dV' = \int_{V'} \vec{r}\frac{\partial}{\partial t}\rho(\vec{r}',t)dV' \tag{3.5}$$

Note by its relation to ρ that $\vec{p}$ is related to $\vec{J}_e$ except for a constant term. Call this constant term in ρ as ρ_o where the total charge is just

$$Q = \int_{V'} \rho_o(\vec{r}')dV' \tag{3.6}$$

and for the charge difference, $\rho - \rho_o$, we have then

$$0 = \int_{V'} [\rho(\vec{r}',t) - \rho_o(\vec{r}')]dV' \tag{3.7}$$

Similarly we can have a constant electric dipole term defined by

$$\vec{p}_o \equiv \int_{V'} \vec{r}\,'\rho_o(\vec{r}\,')dV' \qquad (3.8)$$

Even with a constant term ρ_o in the charge density Q could still be zero. Rewrite equation 3.8 as

$$\vec{p}_o = \int_{V'} (\vec{r}\,'-\vec{r}_c)\rho_o(\vec{r}\,')dV' + \int_{V'} \vec{r}_c\rho_o(\vec{r}\,')dV'$$

$$= \int_{V'} (\vec{r}\,'-\vec{r}_c)\rho_o(\vec{r}\,')dV' + \vec{r}_c Q \qquad (3.9)$$

where $\vec{r}_c$ is some constant position vector. If $Q \neq 0$ then we can find some value of $\vec{r}_c$ such that

$$\vec{p}_o = \vec{r}_c Q$$

$$\int_{V'} (\vec{r}\,'-\vec{r}_c)\rho_o(\vec{r}\,')dV' = \vec{0} \qquad (3.10)$$

This defines $\vec{r}_c$ as the charge center for the constant ρ_o and by a shift of coordinates from $\vec{r}$ to $\vec{r} - \vec{r}_c$ and similarly for $\vec{r}\,'$ the electric dipole term can be made zero. If $Q = 0$ however we have

$$\vec{p}_o = \int_{V'} (\vec{r}\,'-\vec{r}_c)\rho_o(\vec{r}\,')dV' \qquad (3.11)$$

independent of $\vec{r}_c$ so that $\vec{p}_o$ cannot be made zero by a shift in coordinates. If $\rho = 0$ for $t < t_o$ neither Q nor $\vec{p}_o$ is present and $\vec{p}(t)$ is independent of the choice of coordinate origin as indicated by a choice of $\vec{r}_c$.

Having only electric currents in Maxwell's equations there is no magnetic monopole term comparable to the electric monopole term Q. However we do have a magnetic dipole term defined by

$$\vec{m}(t) \equiv \int_{V'} \frac{1}{2}[\vec{r}'\times\vec{J}(\vec{r}',t)]dV' \tag{3.12}$$

This term is basically the current circulation in the volume V' and can be thought of as current flowing around an area where the area has a vector orientation. Noting that $\vec{r}' = \nabla'(\vec{r}'\cdot\vec{r}')/2$ we can write equation 3.12 as

$$\vec{m}(t) = \frac{1}{4}\int_{V'} \nabla' \times [(\vec{r}'\cdot\vec{r}')\vec{J}(\vec{r}',t)]dV' - \frac{1}{4}\int_{V'} (\vec{r}'\cdot\vec{r}')\nabla' \times \vec{J}(\vec{r}',t)dV' \tag{3.13}$$

From a form of Gauss' theorem we have

$$\int_{V'} \nabla' \times [(\vec{r}'\cdot\vec{r}')\vec{J}(\vec{r}',t)]dV' = \int_{S'} \vec{n}' \times [(\vec{r}'\cdot\vec{r}')\vec{J}(\vec{r}',t)]dS' = \vec{0} \tag{3.14}$$

Thus the magnetic dipole moment can be written as

$$\vec{m}(t) = -\frac{1}{4}\int_{V'} (\vec{r}'\cdot\vec{r}')\nabla' \times \vec{J}(\vec{r}',t)dV'$$

$$= -\frac{1}{4}\int_{V'} (\vec{r}'\cdot\vec{r}')\vec{\lambda}(\vec{r}',t)dV' \tag{3.15}$$

Since $\vec{m}$ can be written in terms of $\vec{\lambda}$ it is then a $\vec{J}_h$ type of quantity. However, as will be seen further on, since we have restricted $\vec{J}$ to inside V' then $\vec{J}_e$ and $\vec{J}_h$ are not completely independent.

Note that $\vec{m}$ is not necessarily independent of the choice of coordinate center. Rewrite equation 3.12 with some arbitrary coordinate vector $\vec{r}_c$ as

$$\vec{m}(t) = \int_{V'} \frac{1}{2}[(\vec{r}\,'-\vec{r}_c)\times\vec{J}(\vec{r}\,',t)]dV' + \int_{V'} \frac{1}{2}[\vec{r}_c\times\vec{J}(\vec{r}\,',t)]dV'$$

$$= \int_{V'} \frac{1}{2}[(\vec{r}\,'-\vec{r}_c)\times\vec{J}(\vec{r}\,',t)]dV' + \frac{1}{2}\,\vec{r}_c \times \left[\frac{d}{dt}\,\vec{p}(t)\right] \qquad (3.16)$$

If a coordinate system were centered on $\vec{r}\,' = \vec{r}_c$ and $\vec{m}$ were calculated in that system then the result would depend on the choice of $\vec{r}_c$ except for the component of $\vec{r}_c$ parallel to the time derivative of $\vec{p}$. Using two different values of $\vec{r}_c$ the two different calculations for $\vec{m}$ may not even have the same time dependence. Of course if the time derivative of $\vec{p}$ is zero then $\vec{m}$ is independent of the choice of coordinate center. Thus one must be cautious in considering $\vec{m}$ and thinking of it as a $\vec{J}_h$ quantity. In some cases one might choose the coordinate origin to make $\vec{m} = \vec{0}$ if this is possible for all frequencies or times of interest. This depends on $\vec{m}$ behaving as the time derivative of $\vec{r}_c \times \vec{p}$ for some fixed coordinate vector $\vec{r}_c$. Typically one chooses the coordinate center as some symmetry position for the current density distribution to simplify the calculations; such a choice may automatically make $\vec{m}$ go to zero for appropriate types of electric dipole antennas.

This leads us to consider the behavior of $\vec{J}_e$ and $\vec{J}_h$ outside of V', i.e. $\vec{J}_e$ and $\vec{J}_h$ can be non zero outside of V' as long as their sum is zero. For convenience define

$$r \equiv |\vec{r}| \;, \qquad r' \equiv |\vec{r}\,'| \qquad (3.17)$$

and let $\vec{e}_r$ be the unit vector in the $\vec{r}$ direction (and similarly we have unit vectors for other coordinates). We are going to be concerned with the behavior of $\vec{J}_e$ and $\vec{J}_h$ for large r associated with various characteristics of the current density $\vec{J}$ confined to V'. For this purpose (and later use) we also consider a volume of space V_∞ bounded by a closed surface S_∞. The surface S_∞ is taken as a sphere of radius r_∞ centered on the coordinate origin and we consider the limiting case as $r_\infty \to \infty$. These integration regions V_∞ and S_∞ are appropriate for quantities not confined within V'.

Recall that our expressions for $\vec{J}_e$ and $\vec{J}_h$ involved integrals over $\vec{J}$ with $|\vec{r} - \vec{r}\,'|$ in the denominator and combined with divergence curl, and/or gradient operating on $\vec{r}$ or $\vec{r}\,'$ coordinates. First consider $\vec{J}_e$ which can be written as (from equation 2.32)

$$\vec{J}_e(\vec{r}) = \frac{\partial}{\partial t} \int_{V'} \rho(\vec{r}')\nabla\left[\frac{1}{4\pi|\vec{r}-\vec{r}'|}\right]dV' \tag{3.18}$$

Consider the gradient term. For $\vec{r} \neq \vec{r}'$, which is guaranteed for $\vec{r}$ outside of V', consider $\vec{r} - \vec{r}'$ as the coordinates and let ∇ operate with respect to these coordinates. This is then basically the gradient of a $|\vec{r} - \vec{r}'|^{-1}$ potential which gives a $|\vec{r} - \vec{r}'|^{-2}$ field times a unit vector in the $-\vec{r} + \vec{r}'$ direction. Thus we have

$$\nabla\left[\frac{1}{4\pi|\vec{r}-\vec{r}'|}\right] = -\nabla'\left[\frac{1}{4\pi|\vec{r}-\vec{r}'|}\right]$$

$$= -\frac{\vec{r}-\vec{r}'}{4\pi|\vec{r}-\vec{r}'|^3}$$

$$= \frac{1}{4\pi}\left[-\frac{\vec{e}_r}{r^2} + \frac{\vec{r}'}{r^3}\right]\left|\vec{e}_r - \frac{\vec{r}'}{r}\right|^{-3}$$

$$= \frac{1}{4\pi}\left[-\frac{\vec{e}_r}{r^2} + \frac{\vec{r}'}{r^3}\right]\left[1 - 2\frac{\vec{e}_r\cdot\vec{r}'}{r} + \left(\frac{r'}{r}\right)^2\right]^{-3/2}$$

$$= \frac{1}{4\pi}\left[-\frac{\vec{e}_r}{r^2} + \frac{\vec{r}'}{r^3}\right]\sum_{\ell=0}^{\infty}\binom{-\frac{3}{2}}{\ell}\left[-2\frac{\vec{e}_r\cdot\vec{r}'}{r} + \left(\frac{r'}{r}\right)^2\right]^{\ell} \tag{3.19}$$

which is an absolutely convergent power series representation provided that

$$\left|-2\frac{\vec{e}_r\cdot\vec{r}'}{r} + \left(\frac{r'}{r}\right)^2\right| < 1 \tag{3.20}$$

Let r_o be the maximum value of r' attained on S' and maximize $-\vec{e}_r\cdot\vec{r}'$ as r_o. Then if

$$2\frac{r_o}{r} + \left(\frac{r_o'}{r}\right)^2 = \left[1 + \frac{r_o}{r}\right]^2 - 1 < 1 \tag{3.21}$$

or

$$r > [1 + \sqrt{2}]r_o \tag{3.22}$$

the above series representation is valid. The binomial coefficient is given by

$$\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$$

$$= \frac{(\alpha-\beta+1)_\beta}{\beta!} \tag{3.23}$$

where the Pochhammer symbol is, for $\beta = \ell$ (an integer assumed ≥ 0),

$$(\alpha)_o = 1$$

$$(\alpha)_\ell = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+\ell-1) = \frac{\Gamma(\alpha+\ell)}{\Gamma(\alpha)} \tag{3.24}$$

The expression in equation 3.19 can be converted to a series in reciprocal powers of r with coefficients as collections of terms involving $\vec{e}_r$ and $\vec{r}'$ in the form

$$\nabla\left[\frac{1}{4\pi|\vec{r}-\vec{r}'|}\right] = \sum_{n=2}^{\infty} \vec{g}_n r^{-n} \tag{3.25}$$

Note that r^{-2} is the first non zero term in such a series. Expand the terms in equation 3.19 as

$$\left[-2\,\frac{\vec{e}_r\cdot\vec{r}'}{r} + \left(\frac{r'}{r}\right)^2\right]^\ell = \sum_{\ell'=o}^{\ell} \binom{\ell}{\ell'} \left(-2\,\frac{\vec{e}_r\cdot\vec{r}'}{r}\right)^{\ell-\ell'} \left(\frac{r'}{r}\right)^{2\ell'}$$

$$= \sum_{\ell'=o}^{\ell} \binom{\ell}{\ell'} (-2\vec{e}_r\cdot\vec{r}')^{\ell-\ell'} r'^{2\ell'} r^{-(\ell+\ell')} \tag{3.26}$$

Collecting terms we have

$$\vec{g}_n = -\frac{1}{4\pi}\left\{\sum_{\ell=S\left(\frac{n-2}{2}\right)}^{n-2} \binom{-\frac{3}{2}}{\ell}\binom{\ell}{n-2-\ell}(-2\vec{e}_r\cdot\vec{r}\,')^{2\ell-n+2}r'^{\,2n-4-2\ell}\right\}\vec{e}_r$$

$$+\frac{1}{4\pi}\left\{\sum_{\ell=S\left(\frac{n-3}{2}\right)}^{n-3} \binom{-\frac{3}{2}}{\ell}\binom{\ell}{n-3-\ell}(-2\vec{e}_r\cdot\vec{r}\,')^{2\ell-n+3}r'^{\,2n-6-2\ell}\right\}\vec{r}\,' \qquad (3.27)$$

where S(x) means the smallest integer $\geq x$. The first sum is non zero for $n \geq 2$ and the second for $n \geq 3$. Looking at the form of the coefficients one can see that for r sufficiently large compared to r_o (the maximum r') the series representation in equation 3.25 is convergent. The first few coefficients are

$$4\pi\vec{g}_2 = -\vec{e}_r$$

$$4\pi\vec{g}_3 = -3(\vec{e}_r\cdot\vec{r}\,')\vec{e}_r + \vec{r}\,' \qquad (3.28)$$

$$4\pi\vec{g}_4 = \left[\frac{3}{2}r'^2 - \frac{15}{2}(\vec{e}_r\cdot\vec{r}\,')^2\right]\vec{e}_r + 3(\vec{e}_r\cdot\vec{r}\,')\vec{r}\,'$$

Now return to the expression for $\vec{J}_e$ (equation 3.18) and write it as a series in the form

$$\vec{J}_e(\vec{r}) = \sum_{n=3}^{\infty}\vec{a}_n r^{-n} \qquad (3.29)$$

where n = 2 is not included as discussed below and where the coefficients are functions of $\vec{e}_r$ as

$$\vec{a}_n(\vec{e}_r) = \frac{\partial}{\partial t}\int_{V'}\rho(\vec{r}\,')\vec{g}_n(\vec{e}_r,\vec{r}\,')dV' \qquad (3.30)$$

Consider the first few coefficients. For n = 2 we have

$$\vec{a}_2 = -\frac{\vec{e}_r}{4\pi}\frac{\partial}{\partial t}\int_{V'}\rho(\vec{r}\,')dV' = -\frac{\vec{e}_r}{4\pi}\frac{\partial}{\partial t}Q = 0 \tag{3.31}$$

Thus $\vec{J}_e$ has no r^{-2} term and is therefore $O(r^{-3})$ as $r \to \infty$. For n = 3 we have

$$\vec{a}_3(\vec{e}_r) = -\frac{3\vec{e}_r}{4\pi}\left\{\vec{e}_r\cdot\frac{\partial}{\partial t}\int_{V'}\vec{r}\,'\rho(\vec{r}\,')dV'\right\} + \frac{1}{4\pi}\frac{\partial}{\partial t}\int_{V'}\vec{r}\,'\rho(\vec{r}\,')dV'$$

$$= \frac{1}{4\pi}\frac{\partial}{\partial t}\vec{p} - \frac{3\vec{e}_r}{4\pi}\left[\vec{e}_r\cdot\frac{\partial}{\partial t}\vec{p}\right] \tag{3.32}$$

Thus $\vec{a}_3$ is zero if and only if the time derivative of the electric dipole moment is zero. For n = 4 we start to get electric quadrupole moments and similarly for higher order terms.

Now since $\vec{J} = \vec{J}_e + \vec{J}_h$ and since $\vec{J}$ is zero outside V' then the representation of $\vec{J}_e$ in equation 3.29 also applies to $\vec{J}_h$ with a simple change in sign. Referring to equation 3.18 note that $\vec{J}_e$ is represented as an integral over ρ or $\nabla\cdot\vec{J}$; there is no dependence on $\vec{\lambda}$ or $\nabla\times\vec{J}$. Then outside V' both $\vec{J}_e$ and $\vec{J}_h$ are represented by integrals over only the irrotational part of $\vec{J}$. Likewise both $\vec{J}_e$ and $\vec{J}_h$ outside V' are represented by integrals over only the solenoidal part of $\vec{J}$ (i.e. $\vec{\lambda}$) as in equations 2.39 through 2.41. This leads to the very interesting result that in general there are both $\vec{J}_e$ and $\vec{J}_h$ terms associated with a time changing charge distribution. Even simple electric moments like electric dipoles must produce a $\vec{J}_h$ outside V' and since $\vec{J}_h$ is represented by an integral over $\vec{\lambda}$ then $\vec{\lambda}$ must be non zero in such cases and $\vec{J}$ must have a non zero curl. One cannot set up a purely electric charge distribution with certain time changing electric moments confined to V' without setting up solenoidal currents as well. To have no $\vec{J}_h$ it is necessary to have no $\vec{J}_e$ outside of V'. This would imply that the time derivative of the integral over the charge with the Green's function be zero everywhere outside V' which in turn implies that various electric moments be time independent. One could still have a time changing ρ and thus a $\vec{J}_e$ inside V' without any $\vec{J}_h$ provided these electric moments were all time independent. For example, one could have a current density distribution inside V' which was everywhere parallel to the local $\vec{r}\,'$ giving a spherically symmetric radial current distribution with a spatial dependence only on r' such that no current crossed V'. This distribution would have zero curl and no time changing electric moments while still having non zero ρ. Clearly superposition of an arbitrary number of such spherically symmetric current density distributions, even with different centers inside V', will also give no $\vec{J}_h$. Of course both $\vec{J}_e$ and $\vec{J}_h$ outside V' would be zero for such a case.

Now consider the reverse question. Can one have a current distribution inside V' with zero divergence, i.e. only $\vec{J}_h$, with magnetic moments such as the magnetic dipole? This can in fact be done. Consider a closed current path in V' carrying the same current I (which may be changing in time) all along the path. There is no ρ anywhere associated with this time changing current and so $\vec{J}_e$ computed from equation 3.18 is identically zero. The closed current path has a magnetic dipole moment given by I times a vector area associated with the current loop. Clearly one can superimpose an arbitrary number of such current loops with the same result. Thus it is clearly possible to have a $\vec{J}_h$ confined to V' with no $\vec{J}_e$ anywhere; this $\vec{J}_h$ can have magnetic moments which can produce time changing electromagnetic fields outside of V'.

Now that we know something about the behavior of $\vec{J}_e$ and $\vec{J}_h$ at large r we can use this knowledge in splitting electromagnetic quantities that are integrals over $\vec{J}$ into two parts, based on integrals over $\vec{J}_e$ and $\vec{J}_h$ separately where the volume of integration is V_∞, all space. This splitting then requires that these integrals exist. Taking S_∞ as a sphere of radius r_∞ with $r_\infty \to \infty$ one can take the dependence of $\vec{J}_e(\vec{r}\,')$ and $\vec{J}_h(\vec{r}\,')$ for large r' and determine the convergence of the integral in question as $r_\infty \to \infty$. The leading term in $\vec{J}_e$ and $\vec{J}_h$ is an r'^{-3} term involving the electric dipole moment. This can be combined with other terms in the integrand. For the potentials and fields the integrand involves an r'^{-1} term in addition to the current density so that the integral over V_∞ converges. However for various of the moments of the current distribution the integrand involves positive powers of r' so that the integral over V_∞ may not converge unless special restrictions are made on how S_∞ expands toward infinity, say as a sphere of radius r_∞. Thus the splitting of the current density into $\vec{J}_e$ and $\vec{J}_h$ may not be applicable for the various moments of $\vec{J}$ except in special cases. This restriction applies to the asymptotic expansion of the potentials and fields for large r because the various terms involve these moments in the coefficients.

In this section we have considered the first few moments of the current distribution. This can be extended to the general electric and magnetic multipoles and related to spherical harmonics based on some chosen spherical coordinate system. In the present note we are concerned primarily with electric and magnetic dipoles. However more general multipoles may have application for EMP simulators, sensors, etc.; perhaps some multipole topics can be included in future notes.

IV. Relation of the Electromagnetic Potentials and Fields to the Current Density

Now we go on from the splitting of $\vec{J}$ to consider the potentials and fields. Since $\vec{J}$ is split into $\vec{J}_e$ and $\vec{J}_h$, all the potentials and fields can be similarly split since they can all be related via integrals to $\vec{J}$. Thus for the scalar potential we have

$$\tilde{\Phi}(\vec{r}) = \tilde{\Phi}_e(\vec{r}) + \tilde{\Phi}_h(\vec{r}) = \frac{1}{\varepsilon_o} \int_{V'} \tilde{\rho}(\vec{r}\,') \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} \, dV'$$

$$= -\frac{1}{\varepsilon_o s} \int_{V'} [\nabla' \cdot \tilde{\vec{J}}(\vec{r}\,')] \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} \, dV' \tag{4.1}$$

Since the divergence of $\vec{J}_h$ is zero then we can write

$$\tilde{\Phi}_e(\vec{r}) = \tilde{\Phi}(\vec{r}) = -\frac{1}{\varepsilon_o s} \int_{V'} [\nabla' \cdot \tilde{\vec{J}}_e(\vec{r}\,')] \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} \, dV'$$

$$\tilde{\Phi}_h(\vec{r}) = 0 \tag{4.2}$$

In the time domain this is

$$\Phi_e(\vec{r},t) = \Phi(\vec{r},t) = \frac{1}{\varepsilon_o} \int_{V'} \frac{\rho\left(\vec{r}\,', t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{4\pi|\vec{r}-\vec{r}\,'|} \, dV'$$

$$\Phi_h(\vec{r},t) = 0 \tag{4.3}$$

The vector potential is

$$\tilde{\vec{A}}(\vec{r}) = \tilde{\vec{A}}_e(\vec{r}) + \tilde{\vec{A}}_h(\vec{r}) = \mu_o \int_{V'} \tilde{\vec{J}}(\vec{r}\,') \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} \, dV'$$

$$\vec{A}(\vec{r},t) = \vec{A}_e(\vec{r},t) + \vec{A}_h(\vec{r},t) = \mu_o \int_{V'} \frac{\vec{J}\left(\vec{r}\,', t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{4\pi|\vec{r}-\vec{r}\,'|} \, dV' \tag{4.4}$$

where

$$\tilde{\vec{A}}_e(\vec{r}) = \mu_o \int_{V_\infty} \tilde{\vec{J}}_e(\vec{r}\,') \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} \, dV'$$

$$\tilde{\vec{A}}_h(\vec{r}) = \mu_o \int_{V_\infty} \tilde{\vec{J}}_h(\vec{r}\,') \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} \, dV' \tag{4.5}$$

which in the time domain is

$$\vec{A}_e(\vec{r},t) = \mu_o \int_{V_\infty} \frac{\vec{J}_e\left(\vec{r}\,', t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{4\pi|\vec{r}-\vec{r}\,'|} \, dV'$$

$$\vec{A}_h(\vec{r},t) = \mu_o \int_{V_\infty} \frac{\vec{J}_h\left(\vec{r}\,', t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{4\pi|\vec{r}-\vec{r}\,'|} \, dV' \tag{4.6}$$

Note the use of V_∞ because in general $\vec{J}_e$ and $\vec{J}_h$ can extend outside of V'. As discussed in the previous section $\vec{J}_e$ and $\vec{J}_h$ are both $O(r'^{-3})$ as $r' \to \infty$. With the additional factor of r'^{-1} in the integrand then these integrals clearly converge as $r' \to \infty$.

Having split the vector potential as above recall the Lorentz gauge from equation 2.6 as

$$\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0 \tag{4.7}$$

Using the results of equations 4.2 and 4.3 the divergence of $\vec{A}_e$ and $\vec{A}_h$ can be written as

$$\nabla \cdot \vec{A}_e(\vec{r},t) = -\frac{1}{c^2}\frac{\partial \Phi_e(\vec{r},t)}{\partial t} = -\frac{1}{c^2}\frac{\partial \Phi(\vec{r},t)}{\partial t}$$

$$\nabla \cdot \tilde{\vec{A}}_e(\vec{r}) = -\frac{s}{c^2}\tilde{\Phi}_e(\vec{r}) = -\frac{s}{c^2}\tilde{\Phi}(\vec{r}) \tag{4.8}$$

$$\nabla \cdot \vec{A}_h(\vec{r},t) = 0 \ , \quad \nabla \cdot \tilde{\vec{A}}_h(\vec{r}) = 0$$

These results can also be obtained by use of the divergence operator on equations 4.5 and moving ∇ inside the integrals, converting to ∇', manipulating the ∇' onto the $\vec{J}$ terms and using the results for the divergence of $\vec{J}_e$ and $\vec{J}_h$. Similarly consider the curl of $\vec{A}_e$ and $\vec{A}_h$, giving

$$\nabla \times \tilde{\vec{A}}_{e,h}(\vec{r}) = \mu_o \nabla \times \int_{V_\infty} \tilde{\vec{J}}_{e,h}(\vec{r}')\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\,dV' \tag{4.9}$$

where the subscript e,h simply indicates that either e or h can be used. Equation 4.9 can be manipulated to

$$\nabla \times \tilde{\vec{A}}_{e,h}(\vec{r}) = -\mu_o \int_{V_\infty} \nabla'\left[\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\right] \times \tilde{\vec{J}}_{e,h}(\vec{r}')\,dV'$$

$$= -\mu_o \int_{V_\infty} \nabla' \times \left[\tilde{\vec{J}}_{e,h}(\vec{r}')\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\right]dV'$$

$$+ \mu_o \int_{V_\infty} [\nabla' \times \tilde{\vec{J}}_{e,h}(\vec{r}')]\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\,dV' \tag{4.10}$$

From a form of Gauss' theorem we have

$$\int_{V_\infty} \nabla' \times \left[\tilde{\vec{J}}_{e,h}(\vec{r}')\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\right] dV'$$

$$= \int_{S_\infty} \vec{n}_\infty \times \left[\tilde{\vec{J}}_{e,h}(\vec{r}')\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\right] dS' = \vec{0} \tag{4.11}$$

where we use the fact that $\tilde{\vec{J}}_{e,h}$ is $O(r'^{-3})$ as $r' \to \infty$ so that the integral over S_∞ (area $O(r'^2)$ as $r' \to \infty$) is zero; $\vec{n}_\infty$ is the outward pointing normal on S_∞ and for convenience is typically just $\vec{e}_r'$. Thus from equations 4.10 and 4.11 with the results for the curl of $\vec{J}_e$ and $\vec{J}_h$ we have

$$\nabla \times \tilde{\vec{A}}_e(\vec{r}) = \vec{0} \;, \quad \nabla \times \vec{A}_e(\vec{r},t) = \vec{0}$$

$$\nabla \times \tilde{\vec{A}}_h(\vec{r}) = \mu_o \int_{V'} \tilde{\vec{\lambda}}(\vec{r}')\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} dV' \tag{4.12}$$

$$\nabla \times \vec{A}_h(\vec{r},t) = \mu_o \int_{V'} \frac{\vec{\lambda}\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{4\pi|\vec{r}-\vec{r}'|} dV'$$

where we only integrate over the finite volume V' since $\vec{\lambda}$ is zero outside it. From equations 4.8 and 4.12 we observe that $\vec{A}$ is split into a solenoidal part $\vec{A}_h$ and an irrotational part $\vec{A}_e$, corresponding directly to the same split in $\vec{J}$. Note that the result for $\nabla \times \vec{A}_h$ also applies to $\nabla \times \vec{A}$ and the result for $\nabla \cdot \vec{A}_e$ also applies to $\nabla \cdot \vec{A}$.

In making the expansions for large r the frequency-domain Green's function, common to both scalar and vector potentials, is very significant. For convenience, it can be rewritten as

$$\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} = \frac{e^{-\gamma r}}{4\pi r}\left|\vec{e}_r - \frac{\vec{r}'}{r}\right|^{-1} e^{-\gamma r\left[\left|\vec{e}_r-\frac{\vec{r}'}{r}\right|-1\right]}$$

$$= \frac{e^{-\gamma r}}{4\pi r}\, \alpha^{-1/2} e^{-\gamma r[\alpha^{1/2}-1]} \tag{4.13}$$

where we have the dimensionless term

$$\alpha \equiv \left| \vec{e}_r - \frac{\vec{r}\,'}{r} \right|^2 = 1 - 2\,\frac{\vec{e}_r \cdot \vec{r}\,'}{r} + \left(\frac{r'}{r}\right)^2 \tag{4.14}$$

Note that $e^{-\gamma r}$ is factored out; for $\gamma = s/c$ this is just the delay term and is removed before considering the asymptotic form of the rest of the expression for large r.

The magnetic field is given by

$$\tilde{\vec{H}}(\vec{r}) = \frac{1}{\mu_o} \nabla \times \tilde{\vec{A}}(\vec{r}) = \nabla \times \int_{V'} \tilde{\vec{J}}(\vec{r}\,') \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} dV'$$

$$= \int_{V'} \nabla \left[\frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} \right] \times \tilde{\vec{J}}(\vec{r}\,') dV' \tag{4.15}$$

From equations 4.12 we have the result that only $\vec{A}_h$ has a non zero curl. Thus considering $\vec{J}_e$ and $\vec{J}_h$ separately in equation 4.15 in the integral for $\vec{H}$ we have

$$\tilde{\vec{H}}_e(\vec{r}) = \vec{0}\ , \quad \vec{H}_e(\vec{r},t) = \vec{0}$$

$$\tilde{\vec{H}}_h(\vec{r}) = \int_{V_\infty} \nabla \left[\frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} \right] \times \tilde{\vec{J}}(\vec{r}\,') dV'$$

$$= \int_{V'} \tilde{\vec{\lambda}}(\vec{r}\,') \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} dV' \tag{4.16}$$

$$\vec{H}_h(\vec{r},t) = \int_{V'} \frac{\vec{\lambda}\left(\vec{r}\,', t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{4\pi|\vec{r}-\vec{r}\,'|} dV'$$

Thus $\vec{H}$ is only dependent on $\vec{J}_h$. A pure $\vec{J}_e$ current density distribution has no magnetic field and as such cannot radiate a spherical TEM wave which has both electric and magnetic fields. As was observed, however, in a previous section (section III) a pure $\vec{J}_e$ current distribution confined to the finite volume V' can have no electric dipole moment or higher order electric multipole which would require $\vec{J}_e$ and thus $\vec{J}_h$ to be non zero outside V'. In normal applications we have a $\vec{J}_h$ associated with $\vec{J}_e$, as in the case of a time changing electric dipole.

The gradient of the Green's function is

$$\nabla\left[\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\right] = -\nabla'\left[\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\right]$$

$$= -\frac{\vec{r}-\vec{r}'}{4\pi|\vec{r}-\vec{r}'|^3}[1+\gamma|\vec{r}-\vec{r}'|]e^{-\gamma|\vec{r}-\vec{r}'|}$$

$$= -\left[\vec{e}_r - \frac{\vec{r}'}{r}\right]\frac{e^{-\gamma r}}{4\pi r^2}\,\beta \qquad (4.17)$$

where we have defined a dimensionless factor

$$\beta \equiv \left|\vec{e}_r - \frac{\vec{r}'}{r}\right|^{-3}\left[1+\gamma r\left|\vec{e}_r - \frac{\vec{r}'}{r}\right|\right]e^{-\gamma r\left[|\vec{e}_r-\frac{\vec{r}'}{r}|-1\right]}$$

$$= \left\{\left|\vec{e}_r - \frac{\vec{r}'}{r}\right|^{-3} + \gamma r\left|\vec{e}_r - \frac{\vec{r}'}{r}\right|^{-2}\right\}e^{-\gamma r\left[|\vec{e}_r-\frac{\vec{r}'}{r}|-1\right]}$$

$$= \{\alpha^{-3/2}+\gamma r\alpha^{-1}\}e^{-\gamma r[\alpha^{1/2}-1]} \qquad (4.18)$$

This is a generalization of the result in equation 3.19 from a static to a frequency dependent Green's function. Note that it is a vector in the $-\vec{r}+\vec{r}'$ direction times a complex factor and is only a function of γ and $\vec{r}-\vec{r}'$.

Writing the magnetic field with the gradient term written out gives

$$\tilde{\vec{H}}(\vec{r}) = \int_{V'} [\tilde{\vec{J}}(\vec{r}')\times[\vec{r}-\vec{r}']]\frac{1}{4\pi|\vec{r}-\vec{r}'|^3}[1+\gamma|\vec{r}-\vec{r}'|]e^{-\gamma|\vec{r}-\vec{r}'|}dV' \qquad (4.19)$$

Noting for $\gamma = s/c$ that the exponential factor is simply a delay while s represents a time derivative we can write in the time domain

$$\vec{H}(\vec{r},t) = \int_{V'} \left\{ \frac{1}{4\pi|\vec{r}-\vec{r}'|^3} \vec{J}\left(\vec{r}',t - \frac{|\vec{r}-\vec{r}'|}{c}\right) \times [\vec{r}-\vec{r}'] \right.$$

$$\left. + \frac{1}{4\pi|\vec{r}-\vec{r}'|^2} \frac{1}{c} \frac{\partial}{\partial t} \vec{J}\left(\vec{r}',t - \frac{|\vec{r}-\vec{r}'|}{c}\right) \times [\vec{r}-\vec{r}'] \right\} dV' \quad (4.20)$$

As discussed before this is a $\vec{J}_h$ type quantity and could thus be written as an integral over all space involving only $\vec{J}_h$ in the integrand.

The electric field is given by

$$\tilde{\vec{E}}(\vec{r}) = -\nabla\tilde{\Phi}(\vec{r}) - s\tilde{\vec{A}}(\vec{r})$$

$$= -\frac{1}{\varepsilon_o} \int_{V'} \tilde{\rho}(\vec{r}') \nabla\left[\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\right] dV' - \mu_o s \int_{V'} \tilde{\vec{J}}(\vec{r}') \frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} dV'$$

(4.21)

Here we have the Green's function and its gradient as in equations 4.13 and 4.17; the dimensionless factors α and β then apply for the electric field as well as the magnetic field. The electric field can be rewritten as

$$\tilde{\vec{E}}(\vec{r}) = \frac{1}{\varepsilon_o} \int_{V'} \tilde{\rho}(\vec{r}') \frac{\vec{r}-\vec{r}'}{4\pi|\vec{r}-\vec{r}'|^3} [1+\gamma|\vec{r}-\vec{r}'|] e^{-\gamma|\vec{r}-\vec{r}'|} dV'$$

$$- \mu_o s \int_{V'} \tilde{\vec{J}}(\vec{r}') \frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} dV' \quad (4.22)$$

In the time domain the electric field can be written as

$$\vec{E}(\vec{r},t) = \frac{1}{\varepsilon_o} \int_{V'} \left\{ \frac{\vec{r}-\vec{r}'}{4\pi|\vec{r}-\vec{r}'|^3} \rho\left(\vec{r}',t - \frac{|\vec{r}-\vec{r}'|}{c}\right) \right.$$

$$\left. + \frac{\vec{r}-\vec{r}'}{4\pi|\vec{r}-\vec{r}'|^2} \frac{1}{c} \frac{\partial}{\partial t} \rho\left(\vec{r}',t - \frac{|\vec{r}-\vec{r}'|}{c}\right) \right\} dV'$$

$$- \mu_o \frac{\partial}{\partial t} \int_{V'} \frac{\vec{J}\left(\vec{r}',t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{4\pi|\vec{r}-\vec{r}'|} dV' \qquad (4.23)$$

Again the electric field can be split into terms associated with $\vec{J}_e$ and $\vec{J}_h$ with appropriate care near $\vec{r}' = \vec{r}$. The terms involving ρ go with $\vec{E}_e$ while the one involving $\vec{J}$ splits to both $\vec{E}_e$ and $\vec{E}_h$ giving

$$\tilde{\vec{E}}_e(\vec{r}) = -\nabla\tilde{\Phi}_e(\vec{r}) - s\tilde{\vec{A}}_e(\vec{r}) \ , \quad \tilde{\vec{E}}_h(\vec{r}) = -s\tilde{\vec{A}}_h(\vec{r})$$

$$\vec{E}_e(\vec{r},t) = -\nabla\Phi_e(\vec{r},t) - \frac{\partial}{\partial t}\vec{A}_e(\vec{r},t) \ , \quad \vec{E}_h(\vec{r},t) = -\frac{\partial}{\partial t}\vec{A}_h(\vec{r},t)$$

$$\nabla \cdot \tilde{\vec{E}}_h(\vec{r}) = 0 \ , \quad \nabla \cdot \vec{E}_h(\vec{r},t) = 0 \qquad (4.24)$$

$$\nabla \times \tilde{\vec{E}}_e(\vec{r}) = \vec{0} \ , \quad \nabla \times \vec{E}_e(\vec{r},t) = \vec{0}$$

where the integrals for the potentials based on $\vec{J}_e$ and $\vec{J}_h$ are given previously. Note that a zero curl for $\vec{E}_e$ is consistent with the result from equations 4.16 that $\vec{H}_e$ is identically zero.

With the results of equations 4.16 and 4.24 we can make a quite general observation. Associated with $\vec{J}_e$ are only what one might call quasi static electric fields. $\vec{H}_e$ is zero and so no distant electromagnetic fields are radiated as a TEM wave in the limit of large r from a pure $\vec{J}_e$ distribution. For radiated fields it is essential to have a $\vec{J}_h$. This does not imply that radiated field results are not expressible in terms of electric moments because as we have seen electric moments such as the electric dipole necessarily have a $\vec{J}_h$ associated with it if $\vec{J}$ is confined to a finite volume V'.

Comparing equations 4.22 and 4.23 for the electric field with equations 4.19 and 4.20 for the magnetic field one can observe various similarities and differences between the terms involving ρ and $\vec{J}$. These are related to the contributions that various multipole moments, such as electric and magnetic dipoles, give to the different fields. The contributions of the various moments to the fields are seen more explicitly in the terms of the asymptotic expansion for large r as these terms are taken to their low frequency representations.

V. Asymptotic Forms for the Potentials and Fields for Large r

Having considered the behavior of $\vec{J}_e$ and $\vec{J}_h$ and the first few moments of the current distribution, and having considered the scalar and vector potentials and electric and magnetic fields as they relate to $\vec{J}_e$ and $\vec{J}_h$, now consider the behavior of the potentials and fields for large r. In particular we look at what happens for large r by considering the first few terms of the asymptotic expansion as $r \to \infty$. This expansion is performed in the Laplace transform domain. By using the inverse Laplace transform on the terms of the asymptotic expansion we obtain time domain forms. The 1/r terms in such expansions are what are usually termed the far fields or far potentials. However the frequency dependence of these various terms is quite different. In particular, for small $|s|$ for large but fixed r the 1/r term may not be dominant if $|s|$ is sufficiently small. Then for sufficiently low frequencies other terms besides the far fields become of interest, even for reasonably large r compared to the linear dimensions of V'.

This asymptotic expansion as $r \to \infty$ for fixed s does not strictly apply for $|s| \to \infty$, nor for those time domain features which rely on the limit of large $|s|$. Nevertheless this expansion can be used to define the far fields and other terms in both frequency and time domains. However, as $|s| \to \infty$ this expansion may not apply because the value of r required for a given accuracy of the first so many terms as compared to the exact result may increase without bound as $|s| \to \infty$. This is a question of how far to the far field or the first so many terms as an accurate representation for large $|s|$. The answer depends on the form of $\vec{J}$ for large $|s|$ and is not considered in this note.

The expressions for the potentials and fields in terms of $\vec{J}$ and ρ can be asymptotically expanded for large r. The Green's function and its gradient can be expanded for large r so that the various terms for the potentials and fields can be written as integrals involving $\vec{J}$, ρ, $\vec{e}_r$, and $\vec{r}\,'$ times a reciprocal power of r times a delay factor $e^{-\gamma r}$. Thus we write the asymptotic expansions as $r \to \infty$ in the forms

$$\tilde{\vec{A}}(\vec{r}) = \sum_{n=1}^{N} \tilde{\vec{A}}_n + O(e^{-\gamma r} r^{-N-1})$$

$$\tilde{\Phi}(\vec{r}) = \sum_{n=1}^{N} \tilde{\Phi}_n + O(e^{-\gamma r} r^{-N-1})$$

$$\tilde{\vec{E}}(\vec{r}) = \sum_{n=1}^{N} \tilde{\vec{E}}_n + O(e^{-\gamma r} r^{-N-1}) \tag{5.1}$$

$$\tilde{\vec{H}}(\vec{r}) = \sum_{n=1}^{N} \tilde{\vec{H}}_n + O(e^{-\gamma r} r^{-N-1})$$

where $N \geq 0$ is a positive integer chosen on the basis of the number of terms one wishes to consider. Each of the terms has the form of $e^{-\gamma r}/r^n$ times a scalar or vector coefficient which is independent of r. A time domain form for each of the terms in the expansions in equations 5.1 is defined by the inverse Laplace transform of the corresponding Laplace or frequency domain form. The time domain forms are then represented as scalar or vector functions of $t - r/c$ times r^{-n} for each of the terms in the expansion for the case of free space with $\gamma = s/c$. The first terms ($n = 1$) are called the far potentials and far fields. The successive terms represent corrections which can be important in some cases, such as at lower frequencies if r is not too large. Various features of these terms might be associated with static field distributions. Together these terms for $n \geq 2$ might be called near potentials and near fields. Note for this asymptotic expansion that we have $r >> r_o$ (the maximum value of r' on S'). If we split $\vec{J}$ into $\vec{J}_e$ and $\vec{J}_h$ and consider cases where these latter are non zero outside V' then we do not have $r >> r'$ for all r' in the volume of integration, V_∞. Thus this asymptotic expansion for $r \to \infty$ is not necessarily directly applicable to the potentials and fields which are split into separate terms based on $\vec{J}_e$ and $\vec{J}_h$ which are not confined inside a finite volume like V'.

In this note we consider the individual terms of these expansions up through $n = 3$. To calculate these terms we need the asymptotic representation of the Green's function and its gradient up through terms of order $e^{-\gamma r} r^{-3}$. The Green's function from equation 4.13 is

$$\frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} = \frac{e^{-\gamma r}}{4\pi r}\, \alpha^{-1/2} e^{-\gamma r[\alpha^{1/2}-1]} \tag{5.2}$$

and its gradient from equation 4.17 is

$$\nabla\left[\frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}\right] = -\frac{\vec{r}-\vec{r}\,'}{4\pi|\vec{r}-\vec{r}\,'|^3}[1+\gamma|\vec{r}-\vec{r}\,'|]e^{-\gamma|\vec{r}-\vec{r}\,'|}$$

$$= -\left[\vec{e}_r - \frac{\vec{r}\,'}{r}\right]\frac{e^{-\gamma r}}{4\pi r^2}\,\beta \tag{5.3}$$

where we have the dimensionless factors

$$\alpha = 1 - 2\,\frac{\vec{e}_r\cdot\vec{r}\,'}{r} + \left(\frac{r'}{r}\right)^2$$

$$\beta = \{\alpha^{-3/2}+\gamma r\alpha^{-1}\}e^{-\gamma r[\alpha^{1/2}-1]} \tag{5.4}$$

Consider the exponential factor containing α. First we have for large r

$$\alpha^{1/2} = \sum_{\ell=0}^{\infty}\binom{\frac{1}{2}}{\ell}\left[-2\,\frac{\vec{e}_r\cdot\vec{r}\,'}{r} + \left(\frac{r'}{r}\right)^2\right]^{\ell}$$

$$= 1 + \frac{1}{2}\left[-2\,\frac{\vec{e}_r\cdot\vec{r}\,'}{r} + \left(\frac{r'}{r}\right)^2\right] - \frac{1}{8}\left[-2\,\frac{\vec{e}_r\cdot\vec{r}\,'}{r} + \left(\frac{r'}{r}\right)^2\right]^2$$

$$+ \frac{1}{16}\left[-2\,\frac{\vec{e}_r\cdot\vec{r}\,'}{r} + \left(\frac{r'}{r}\right)^2\right]^3 + O(r^{-4})$$

$$= 1 - \frac{\vec{e}_r\cdot\vec{r}\,'}{r} + \frac{1}{2r^2}\left[r'^2-(\vec{e}_r\cdot\vec{r}\,')^2\right]$$

$$+ \frac{1}{2r^3}(\vec{e}_r\cdot\vec{r}\,')\left[r'^2-(\vec{e}_r\cdot\vec{r}\,')^2\right] + O(r^{-4}) \tag{5.5}$$

The exponential factor then becomes

$$e^{-\gamma r[\alpha^{1/2}-1]} = \exp\left\{\gamma\vec{e}_r\cdot\vec{r}' - \frac{\gamma}{2r}\left[r'^2-(\vec{e}_r\cdot\vec{r}')^2\right]\right.$$

$$\left. - \frac{\gamma}{2r^2}(\vec{e}_r\cdot\vec{r}')\left[r'^2-(\vec{e}_r\cdot\vec{r}')^2\right] + O(r^{-3})\right\}$$

$$= e^{\gamma\vec{e}_r\cdot\vec{r}'}\left\{1+\left[-\frac{\gamma}{2r}\left[r'^2-(\vec{e}_r\cdot\vec{r}')^2\right] - \frac{\gamma}{2r^2}(\vec{e}_r\cdot\vec{r}')\left[r'^2-(\vec{e}_r\cdot\vec{r}')^2\right]\right]\right.$$

$$\left. + \frac{\gamma^2}{8r^2}\left[r'^2-(\vec{e}_r\cdot\vec{r}')^2\right]^2 + O(r^{-3})\right\}$$

$$= e^{\gamma\vec{e}_r\cdot\vec{r}'}\left\{1 - \frac{1}{r}\frac{\gamma}{2}\left[r'^2-(\vec{e}_r\cdot\vec{r}')^2\right]\right.$$

$$+ \frac{1}{r^2}\left[-\frac{\gamma}{2}(\vec{e}_r\cdot\vec{r}')\left[r'^2-(\vec{e}_r\cdot\vec{r}')^2\right] + \frac{\gamma^2}{8}\left[r'^2-(\vec{e}_r\cdot\vec{r}')^2\right]^2\right]$$

$$\left. + O(r^{-3})\right\} \tag{5.6}$$

where we have used the power series expansion for the exponential to expand the terms of order r^{-1} and higher order.

For the Green's function we first need the negative square root expansion as

$$\alpha^{-1/2} = \sum_{\ell=0}^{\infty}\binom{-\frac{1}{2}}{\ell}\left[-2\,\frac{\vec{e}_r\cdot\vec{r}'}{r} + \left(\frac{r'}{r}\right)^2\right]^{\ell}$$

$$= 1 - \frac{1}{2}\left[-2\,\frac{\vec{e}_r\cdot\vec{r}'}{r} + \left(\frac{r'}{r}\right)^2\right] + \frac{3}{8}\left[-2\,\frac{\vec{e}_r\cdot\vec{r}'}{r} + \left(\frac{r'}{r}\right)^2\right]^2 + O(r^{-3})$$

$$= 1 + \frac{\vec{e}_r\cdot\vec{r}'}{r} + \frac{1}{r^2}\left[-\frac{r'^2}{2} + \frac{3}{2}(\vec{e}_r\cdot\vec{r}')^2\right] + O(r^{-3}) \tag{5.7}$$

Combining equations 5.2, 5.6, and 5.7 gives the Green's function for $r \to \infty$ as

$$\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} = \frac{e^{-\gamma r+\gamma \vec{e}_r\cdot\vec{r}'}}{4\pi}\left\{\frac{1}{r} + \frac{1}{r^2}\left[\vec{e}_r\cdot\vec{r}' - \frac{\gamma}{2}\left[r'^2-(\vec{e}_r\cdot\vec{r}')^2\right]\right]\right.$$

$$+ \frac{1}{r^3}\left[-\frac{r'^2}{2} + \frac{3}{2}(\vec{e}_r\cdot\vec{r}')^2-\gamma(\vec{e}_r\cdot\vec{r}')\left[r'^2-(\vec{e}_r\cdot\vec{r}')^2\right]\right.$$

$$\left.\left. + \frac{\gamma^2}{8}\left[r'^2-(\vec{e}_r\cdot\vec{r}')^2\right]^2\right] + O(r^{-4})\right\} \qquad (5.8)$$

For the gradient of the Green's function we need β, for which we need two more powers of α for $r \to \infty$ as

$$\alpha^{-1} = \sum_{\ell=0}^{\infty}\binom{-1}{\ell}\left[-2\,\frac{\vec{e}_r\cdot\vec{r}'}{r} + \left(\frac{r'}{r}\right)^2\right]^{\ell}$$

$$= 1 - \left[-2\,\frac{\vec{e}_r\cdot\vec{r}'}{r} + \left(\frac{r'}{r}\right)^2\right] + \left[-2\,\frac{\vec{e}_r\cdot\vec{r}'}{r} + \left(\frac{r'}{r}\right)^2\right]^2 + O(r^{-3})$$

$$= 1 + \frac{2}{r}\,\vec{e}_r\cdot\vec{r}' - \frac{1}{r^2}\left[r'^2-4(\vec{e}_r\cdot\vec{r}')^2\right] + O(r^{-3}) \qquad (5.9)$$

and

$$\alpha^{-3/2} = \sum_{\ell=0}^{\infty}\binom{-\frac{3}{2}}{\ell}\left[-2\,\frac{\vec{e}_r\cdot\vec{r}'}{r} + \left(\frac{r'}{r}\right)^2\right]^{\ell}$$

$$= 1 - \frac{3}{2}\left[-2\,\frac{\vec{e}_r\cdot\vec{r}'}{r} + \left(\frac{r'}{r}\right)^2\right] + \frac{15}{8}\left[-2\,\frac{\vec{e}_r\cdot\vec{r}'}{r}\right]^2 + O(r^{-3})$$

$$= 1 + \frac{3}{r}\,\vec{e}_r\cdot\vec{r}' + \frac{1}{r^2}\left[-\frac{3}{2}\,r'^2 + \frac{15}{2}(\vec{e}_r\cdot\vec{r}')^2\right] + O(r^{-3}) \qquad (5.10)$$

Then for $r \to \infty$ we have one factor of β as

$$\alpha^{-3/2} + \gamma r \alpha^{-1} = \gamma r + [1+2\gamma \vec{e}_r \cdot \vec{r}\,']$$

$$+ \frac{1}{r}\left[3e_r \cdot r' - \gamma\left[r'^2 - 4(\vec{e}_r \cdot \vec{r}\,')^2\right]\right] + O(r^{-2}) \qquad (5.11)$$

Combining this factor with equations 5.3, 5.4, and 5.6 gives the gradient of the Green's function for $r \to \infty$ as

$$\nabla\left[\frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}\right] = -\left[\vec{e}_r - \frac{\vec{r}\,'}{r}\right]\frac{e^{-\gamma r+\gamma \vec{e}_r \cdot \vec{r}\,'}}{4\pi}\left\{\frac{\gamma}{r}\right.$$

$$+ \frac{1}{r^2}\left[1+2\gamma \vec{e}_r \cdot \vec{r}\,' - \frac{\gamma^2}{2}\left[r'^2 - (\vec{e}_r \cdot \vec{r}\,')^2\right]\right]$$

$$+ \frac{1}{r^3}\left[3\vec{e}_r \cdot \vec{r}\,' - \frac{3}{2}\gamma\left[r'^2 - 3(\vec{e}_r \cdot \vec{r}\,')^2\right]\right.$$

$$\left. - \frac{3}{2}\gamma^2(\vec{e}_r \cdot \vec{r}\,')\left[r'^2 - (\vec{e}_r \cdot \vec{r}\,')^2\right] + \frac{\gamma^3}{8}\left[r'^2 - (\vec{e}_r \cdot \vec{r}\,')^2\right]^2\right]$$

$$\left. + O(r^{-4})\right\}$$

$$= \frac{e^{-\gamma r+\gamma \vec{e}_r \cdot \vec{r}\,'}}{4\pi}\left\{-\vec{e}_r \frac{1}{r}\gamma\right.$$

$$+ \frac{1}{r^2}\left[-\vec{e}_r\left[1+2\gamma \vec{e}_r \cdot \vec{r}\,' - \frac{\gamma^2}{2}\left[r'^2 - (\vec{e}_r \cdot \vec{r}\,')^2\right]\right] + \vec{r}\,'\gamma\right]$$

$$+ \frac{1}{r^3}\left[-\vec{e}_r\left[3\vec{e}_r \cdot \vec{r}\,' - \frac{3}{2}\gamma\left[r'^2 - 3(\vec{e}_r \cdot \vec{r}\,')^2\right]\right.\right.$$

$$- \frac{3}{2}\gamma^2(\vec{e}_r\cdot\vec{r}')\left[r'^2-(\vec{e}_r\cdot\vec{r}')^2\right] + \frac{\gamma^3}{8}\left[r'^2-(\vec{e}_r\cdot\vec{r}')^2\right]^2\Bigg]$$

$$+ r'\left[1+2\gamma\vec{e}_r\cdot\vec{r}' - \frac{\gamma^2}{2}\left[r'^2-(\vec{e}_r\cdot\vec{r}')^2\right]\right]\Bigg]$$

$$+ O(r^{-4})\Bigg\} \qquad (5.12)$$

Both the Green's function and its gradient are now represented as a sum of terms of the form $e^{-\gamma r}r^{-n}$ for their r dependence at large r up through $n = 3$. The coefficients of each of these terms are functions of $\vec{e}_r$, $\vec{r}'$, and γ. Substituting these terms back into the expansions for large r of the potentials and fields the individual terms of the asymptotic expansion are given by integrals of $\tilde{\rho}$ and/or $\vec{J}$ over V'. We now go on to consider several of these terms, particularly for low frequencies.

VI. The Far Potentials and Far Fields: Order r^{-1}

Consider now the far potentials and far fields, expanding on some of the considerations in a previous note.[5] The far potentials and far fields are by definition the first terms in the asymptotic expansions in equations 5.1; their dependence on r is in the form $e^{-\gamma r}r^{-1}$ and they can be calculated by using the terms in the Green's function and/or its gradient with this dependence on r since $\vec{J}$ and ρ are not functions of r. The far potentials and far fields are denoted by a subscript 1.

For convenience the radiation vector is defined by

$$\tilde{\vec{N}}(\vec{e}_r) \equiv \frac{\mu_o}{4\pi} \int_{V'} e^{\gamma \vec{e}_r \cdot \vec{r}\,'} \tilde{\vec{J}}(\vec{r}\,')\, dV' \tag{6.1}$$

which in the time domain (for $\gamma = s/c$) is

$$\vec{N}(\vec{e}_r, t) = \frac{\mu_o}{4\pi} \int_{V'} \vec{J}\left(\vec{r}\,', t + \frac{\vec{e}_r \cdot \vec{r}\,'}{c}\right) dV' \tag{6.2}$$

Note that the radiation vector is a function of the direction, $\vec{e}_r$, from the coordinate center to the observer, but not a function of the distance, r, from the coordinate center to the observer.

Using equation 5.8 for the Green's function we have the far potentials

$$\tilde{\vec{A}}_1(\vec{r}) = \mu_o \frac{e^{-\gamma r}}{4\pi r} \int_{V'} e^{\gamma \vec{e}_r \cdot \vec{r}\,'} \tilde{\vec{J}}(\vec{r}\,')\, dV'$$

$$= \frac{e^{-\gamma r}}{r} \tilde{\vec{N}}(\vec{e}_r) \tag{6.3}$$

$$\tilde{\Phi}_1(\vec{r}) = \frac{1}{\epsilon_o} \frac{e^{-\gamma r}}{4\pi r} \int_{V'} e^{\gamma \vec{e}_r \cdot \vec{r}\,'} \tilde{\rho}(\vec{r}\,')\, dV'$$

which in the time domain are

$$\vec{A}_1(\vec{r},t) = \frac{\mu_o}{4\pi r} \int_{V'} \vec{J}\left(\vec{r}', t-\frac{r}{c}+\frac{\vec{e}_r\cdot\vec{r}'}{c}\right) dV'$$

$$= \frac{1}{r}\,\vec{N}\left(\vec{e}_r, t-\frac{r}{c}\right) \tag{6.4}$$

$$\Phi_1(\vec{r},t) = \frac{1}{\varepsilon_o}\,\frac{1}{4\pi r} \int_{V'} \rho\left(\vec{r}', t-\frac{r}{c}+\frac{\vec{e}_r\cdot\vec{r}'}{c}\right) dV'$$

The far scalar potential can be written in terms of the radiation vector by manipulating $\tilde{\Phi}_1$ as

$$\tilde{\Phi}_1(\vec{r}) = \frac{1}{\varepsilon_o}\,\frac{e^{-\gamma r}}{4\pi r} \int_{V'} e^{\gamma \vec{e}_r\cdot\vec{r}'}\,\tilde{\rho}(\vec{r}')\,dV'$$

$$= -\frac{1}{\varepsilon_o}\,\frac{e^{-\gamma r}}{4\pi r}\,\frac{1}{s} \int_{V'} e^{\gamma \vec{e}_r\cdot\vec{r}'}\,\nabla'\cdot\tilde{\vec{J}}(\vec{r}')\,dV'$$

$$= -\frac{1}{\varepsilon_o}\,\frac{e^{-\gamma r}}{4\pi r}\,\frac{1}{s} \int_{V'} \nabla'\cdot\left[e^{\gamma \vec{e}_r\cdot\vec{r}'}\,\tilde{\vec{J}}(\vec{r}')\right] dV'$$

$$+ \frac{1}{\varepsilon_o}\,\frac{e^{-\gamma r}}{4\pi r}\,\frac{1}{s} \int_{V'} \nabla'\left[e^{\gamma \vec{e}_r\cdot\vec{r}'}\right]\cdot\tilde{\vec{J}}(\vec{r}')\,dV' \tag{6.5}$$

From Gauss' theorem we have

$$\int_{V'} \nabla'\cdot\left[e^{\gamma \vec{e}_r\cdot\vec{r}'}\,\tilde{\vec{J}}(\vec{r}')\right] dV' = \int_{S'} e^{\gamma \vec{e}_r\cdot\vec{r}'}\,\vec{n}'\cdot\tilde{\vec{J}}(\vec{r}')\,dS' = 0 \tag{6.6}$$

since $\vec{J} = \vec{0}$ on S'. Now we also have

$$\nabla'\left[e^{\gamma \vec{e}_r\cdot\vec{r}'}\right] = \gamma e^{\gamma \vec{e}_r\cdot\vec{r}'}\,\nabla'[\vec{e}_r\cdot\vec{r}'] = \gamma e^{\gamma \vec{e}_r\cdot\vec{r}'}\,\vec{e}_r \tag{6.7}$$

The far scalar potential can then be written as

$$\tilde{\Phi}_1(\vec{r}) = \frac{1}{\varepsilon_o}\,\frac{e^{-\gamma r}}{4\pi r}\,\frac{\gamma}{s}\,\vec{e}_r\cdot\int_{V'} e^{\gamma\vec{e}_r\cdot\vec{r}\,'}\,\tilde{\vec{J}}(\vec{r}\,')\,dV'$$

$$= c\,\frac{e^{-\gamma r}}{r}\,\vec{e}_r\cdot\tilde{\vec{N}}(\vec{e}_r) \tag{6.8}$$

which in the time domain is

$$\Phi_1(\vec{r},t) = \frac{c}{r}\,\vec{e}_r\cdot\vec{N}\left(\vec{e}_r, t-\frac{r}{c}\right) \tag{6.9}$$

Thus both the scalar and vector far potentials can be expressed in terms of the radiation vector.

Comparing equations 6.1, 6.5, and 6.8 the r component of the radiation vector can be written as

$$\vec{e}_r\cdot\tilde{\vec{N}}(\vec{e}_r) = \frac{\mu_o}{4\pi}\,\vec{e}_r\cdot\int_{V'} e^{\gamma\vec{e}_r\cdot\vec{r}\,'}\,\tilde{\vec{J}}(\vec{r}\,')\,dV'$$

$$= -\frac{Z_o}{4\pi}\,\frac{1}{s}\int_{V'} e^{\gamma\vec{e}_r\cdot\vec{r}\,'}\,\nabla'\cdot\tilde{\vec{J}}(\vec{r}\,')\,dV'$$

$$= \frac{Z_o}{4\pi}\int_{V'} e^{\gamma\vec{e}_r\cdot\vec{r}\,'}\,\tilde{\rho}(\vec{r}\,')\,dV' \tag{6.10}$$

Thus the r component of the radiation vector only depends on ρ, and might then be considered a $\vec{J}_e$ type of term. Both Φ_1 and the r component of $\vec{A}_1$ might then be considered as $\vec{J}_e$ types of terms.

Consider the part of the radiation vector transverse to $\vec{e}_r$ which we can manipulate as

$$\vec{e}_r \times \tilde{\vec{N}}(\vec{e}_r) = \frac{\mu_o}{4\pi} \vec{e}_r \times \int_{V'} e^{\gamma \vec{e}_r \cdot \vec{r}\,'} \tilde{\vec{J}}(\vec{r}\,')\, dV'$$

$$= \frac{\mu_o}{4\pi} \frac{1}{\gamma} \int_{V'} \nabla' \left[e^{\gamma \vec{e}_r \cdot \vec{r}\,'} \right] \times \tilde{\vec{J}}(\vec{r}\,')\, dV'$$

$$= \frac{\mu_o}{4\pi} \frac{1}{\gamma} \int_{V'} \nabla' \times \left[e^{\gamma \vec{e}_r \cdot \vec{r}\,'} \tilde{\vec{J}}(\vec{r}\,') \right] dV'$$

$$- \frac{\mu_o}{4\pi} \frac{1}{\gamma} \int_{V'} e^{\gamma \vec{e}_r \cdot \vec{r}\,'} \nabla' \times \tilde{\vec{J}}(\vec{r}\,')\, dV' \qquad (6.11)$$

From a form of Gauss' theorem we have

$$\int_{V'} \nabla' \times \left[e^{\gamma \vec{e}_r \cdot \vec{r}\,'} \tilde{\vec{J}}(\vec{r}\,') \right] dV' = \int_{S'} e^{\gamma \vec{e}_r \cdot \vec{r}\,'} \vec{n}\,' \times \tilde{\vec{J}}(\vec{r}\,')\, dS' = \vec{0} \qquad (6.12)$$

since $\vec{J}$ is zero on S'. The transverse part of the radiation vector can then be written (with $\gamma = s/c$) as

$$\vec{e}_r \times \tilde{\vec{N}}(\vec{e}_r) = \frac{\mu_o}{4\pi} \vec{e}_r \times \int_{V'} e^{\gamma \vec{e}_r \cdot \vec{r}\,'} \tilde{\vec{J}}(\vec{r})\, dV'$$

$$= -\frac{Z_o}{4\pi} \frac{1}{s} \int_{V'} e^{\gamma \vec{e}_r \cdot \vec{r}\,'} \nabla' \times \tilde{\vec{J}}(\vec{r}\,')\, dV'$$

$$= -\frac{Z_o}{4\pi} \frac{1}{s} \int_{V'} e^{\gamma \vec{e}_r \cdot \vec{r}\,'} \tilde{\vec{\lambda}}(\vec{r}\,')\, dV' \qquad (6.13)$$

Thus the transverse part (a vector) of the radiation vector (with respect to $\vec{e}_r$) depends only on $\vec{\lambda}$ and might then be considered as a $\vec{J}_h$ type of term. Thus the transverse part of $\vec{A}_1$ might be considered as a $\vec{J}_h$ type of term.

In the time domain we can express longitudinal and transverse parts of the radiation vector (with respect to $\vec{e}_r$) as

$$\vec{e}_r\cdot\vec{N}(\vec{e}_r,t) = \frac{\mu_o}{4\pi}\,\vec{e}_r\cdot\int_{V'}\vec{J}\left(\vec{r}\,',t+\frac{\vec{e}_r\cdot\vec{r}\,'}{c}\right)dV'$$

$$= \frac{Z_o}{4\pi}\int_{V'}\rho\left(\vec{r}\,',t+\frac{\vec{e}_r\cdot\vec{r}\,'}{c}\right)dV'$$

$$\vec{e}_r\times\vec{N}(\vec{e}_r,t) = \frac{\mu_o}{4\pi}\,\vec{e}_r\times\int_{V'}\vec{J}\left(\vec{r}\,',t+\frac{\vec{e}_r\cdot\vec{r}\,'}{c}\right)dV' \tag{6.14}$$

$$= -\frac{Z_o}{4\pi}\,\frac{1}{s}\int_{V'}\vec{\lambda}\left(\vec{r}\,',t+\frac{\vec{e}_r\cdot\vec{r}\,'}{c}\right)dV'$$

With these results for the longitudinal and transverse parts of the radiation vector we can write the far potentials in terms of ρ and $\vec{\lambda}$ as separate terms corresponding to $\vec{J}_e$ and $\vec{J}_h$. The far fields will also use these results.

Consider now the far fields. Using equation 5.8 for the Green's function and equation 5.12 for its gradient and equations 4.21 and 4.15 for the electric and magnetic fields respectively we have the far fields as

$$\tilde{\vec{E}}_1(\vec{r}) = Z_o\,\frac{e^{-\gamma r}}{4\pi r}\,s\vec{e}_r\int_{V'}e^{\gamma\vec{e}_r\cdot\vec{r}\,'}\,\tilde{\rho}(\vec{r}\,')\,dV'$$

$$-\,\mu_o\,\frac{e^{-\gamma r}}{4\pi r}\,s\int_{V'}e^{\gamma\vec{e}_r\cdot\vec{r}\,'}\,\tilde{\vec{J}}(\vec{r}\,')\,dV' \tag{6.15}$$

$$\tilde{\vec{H}}_1(\vec{r}) = -\frac{1}{c}\,\frac{e^{-\gamma r}}{4\pi r}\,s\vec{e}_r\times\int_{V'}e^{\gamma\vec{e}_r\cdot\vec{r}\,'}\,\tilde{\vec{J}}(\vec{r}\,')\,dV'$$

Using the results of equations 6.1, 6.10, and 6.13 for the radiation vector, the far fields can be written as

$$\tilde{\vec{E}}_1(\vec{r}) = \frac{e^{-\gamma r}}{r} s\{\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{N}}(\vec{e}_r)]-\tilde{\vec{N}}(\vec{e}_r)\}$$

$$= \frac{e^{-\gamma r}}{r} s\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{N}}(\vec{e}_r)] \qquad (6.16)$$

$$\tilde{\vec{H}}_1(\vec{r}) = -\frac{1}{Z_o}\frac{e^{-\gamma r}}{r} s\vec{e}_r \times \tilde{\vec{N}}(\vec{e}_r)$$

Thus the far fields depend only on the transverse part of the radiation vector with respect to $\vec{e}_r$. From equation 6.13 the far fields then depend only on $\vec{\lambda}$ and can thus be considered as a $\vec{J}_h$ type of term. As we will see later some terms in the far fields will involve the charge density, as in the case of the electric dipole moment. However as was mentioned previously these electric moments which give a $\vec{J}_e$ outside V' also are associated with a $\vec{J}_h$ and thus can be related to $\vec{\lambda}$. In the time domain the far fields can be written as

$$\vec{E}_1(\vec{r}) = \frac{1}{r}\vec{e}_r \times \left[\vec{e}_r\times\frac{\partial}{\partial t}\vec{N}\left(\vec{e}_r, t-\frac{r}{c}\right)\right] \qquad (6.17)$$

$$\vec{H}_1(\vec{r}) = -\frac{1}{Z_o}\frac{1}{r}\vec{e}_r \times \frac{\partial}{\partial t}\vec{N}\left(\vec{e}_r, t-\frac{r}{c}\right)$$

Now that the far potentials and far fields are all expressed in terms of the one radiation vector let us consider the behavior of the far potentials and far fields for low frequencies. Thus we consider the asymptotic form of the radiation vector as $s \to 0$. Expanding the exponential in a power series equation 6.1 becomes

$$\tilde{\vec{N}}(\vec{e}_r) = \sum_{\ell=0}^{\infty}\tilde{\vec{N}}_\ell(\vec{e}_r) \qquad (6.18)$$

where we have defined

$$\tilde{\vec{N}}_\ell(\vec{e}_r) \equiv \frac{\mu_o}{4\pi}\frac{\gamma^\ell}{\ell!}\int_{V'}(\vec{e}_r\cdot\vec{r}\,')^\ell\tilde{\vec{J}}(\vec{r}\,')dV' \qquad (6.19)$$

Noting that

$$\nabla'\left[\frac{(\gamma\vec{e}_r\cdot\vec{r}')^{\ell+1}}{(\ell+1)!}\right] = \frac{\gamma^{\ell+1}(\vec{e}_r\cdot\vec{r}')^{\ell}}{\ell!}\,\vec{e}_r \tag{6.20}$$

the terms in the expansion of the radiation vector can be split into longitudinal and transverse parts to obtain alternate representations. The longitudinal part can be manipulated as

$$\vec{e}_r\cdot\tilde{\vec{N}}_\ell(\vec{e}_r) = \frac{\mu_o}{4\pi}\frac{\gamma^\ell}{\ell!}\,\vec{e}_r\cdot\int_{V'}(\vec{e}_r\cdot\vec{r}')^{\ell}\tilde{\vec{J}}(\vec{r}')\,dV'$$

$$= \frac{\mu_o}{4\pi}\frac{1}{\gamma}\int_{V'}\nabla'\left[\frac{(\gamma\vec{e}_r\cdot\vec{r}')^{\ell+1}}{(\ell+1)!}\right]\cdot\tilde{\vec{J}}(\vec{r}')\,dV'$$

$$= \frac{\mu_o}{4\pi}\frac{1}{\gamma}\int_{V'}\nabla'\cdot\left[\frac{(\gamma\vec{e}_r\cdot\vec{r}')^{\ell+1}}{(\ell+1)!}\,\tilde{\vec{J}}(\vec{r}')\right]dV'$$

$$- \frac{\mu_o}{4\pi}\frac{\gamma^\ell}{(\ell+1)!}\int_{V'}(\vec{e}_r\cdot\vec{r}')^{\ell+1}\nabla'\cdot\tilde{\vec{J}}(\vec{r}')\,dV' \tag{6.21}$$

From Gauss' theorem we have

$$\int_{V'}\nabla'\cdot\left[\frac{(\gamma\vec{e}_r\cdot\vec{r}')^{\ell+1}}{(\ell+1)!}\,\tilde{\vec{J}}(\vec{r}')\right]dV' = \frac{\gamma^{\ell+1}}{(\ell+1)!}\int_{S'}(\vec{e}_r\cdot\vec{r}')^{\ell+1}\vec{n}'\cdot\tilde{\vec{J}}(\vec{r}')\,dS' = 0 \tag{6.22}$$

since $\vec{J}$ is zero on S'. Thus we have

$$\vec{e}_r\cdot\tilde{\vec{N}}_\ell(\vec{e}_r) = \frac{Z_o}{4\pi}\frac{\gamma^{\ell+1}}{(\ell+1)!}\int_{V'}(\vec{e}_r\cdot\vec{r}')^{\ell+1}\tilde{\rho}(\vec{r}')\,dV' \tag{6.23}$$

Similarly the transverse part can be manipulated as

$$\vec{e}_r \times \tilde{\vec{N}}_\ell(\vec{e}_r) = \frac{\mu_o}{4\pi}\frac{\gamma^\ell}{\ell!}\vec{e}_r \times \int_{V'} (\vec{e}_r \cdot \vec{r}\,')^\ell \tilde{\vec{J}}(\vec{r}\,')\,dV'$$

$$= \frac{\mu_o}{4\pi}\frac{1}{\gamma}\int_{V'} \nabla'\left[\frac{(\gamma\vec{e}_r \cdot \vec{r}\,')^{\ell+1}}{(\ell+1)!}\right]\times\tilde{\vec{J}}(\vec{r}\,')\,dV'$$

$$= \frac{\mu_o}{4\pi}\frac{1}{\gamma}\int_{V'} \nabla'\times\left[\frac{(\gamma\vec{e}_r \cdot \vec{r}\,')^{\ell+1}}{(\ell+1)!}\right]\tilde{\vec{J}}(\vec{r}\,')\times dV'$$

$$- \frac{\mu_o}{4\pi}\frac{\gamma^\ell}{(\ell+1)!}\int_{V'} (\vec{e}_r \cdot \vec{r}\,')^{\ell+1}\nabla'\times\tilde{\vec{J}}(\vec{r}\,')\,dV' \qquad (6.24)$$

From a form of Gauss' theorem we have

$$\int_{V'} \nabla'\times\left[\frac{(\gamma\vec{e}_r \cdot \vec{r}\,')^{\ell+1}}{(\ell+1)!}\tilde{\vec{J}}(\vec{r}\,')\right]dV' = \frac{\gamma^{\ell+1}}{(\ell+1)!}\int_{S'} (\vec{e}_r \cdot \vec{r}\,')^{\ell+1}\vec{n}\,'\times\tilde{\vec{J}}(\vec{r}\,')\,dS' = \vec{0} \qquad (6.25)$$

since $\vec{J}$ is zero on S'. Thus we have

$$\vec{e}_r \times \tilde{\vec{N}}_\ell(\vec{e}_r) = -\frac{Z_o}{4\pi}\frac{1}{s}\frac{\gamma^{\ell+1}}{(\ell+1)!}\int_{V'} (\vec{e}_r \cdot \vec{r}\,')^{\ell+1}\tilde{\vec{\lambda}}(\vec{r}\,')\,dV' \qquad (6.26)$$

Note in converting from $\vec{J}$ to ρ and $\vec{\lambda}$ that the exponents and factorial have been shifted from ℓ to $\ell + 1$. Now we can consider the first few terms for the low frequency behavior of the radiation vector.

Consider the $\ell = 0$ term for which we have

$$\tilde{\vec{N}}_o(\vec{e}_r) = \frac{\mu_o}{4\pi}\int_{V'} \tilde{\vec{J}}(\vec{r}\,')\,dV' \qquad (6.27)$$

The electric dipole moment is just (equation 3.4)

$$\tilde{\vec{p}} = \int_{V'} \vec{r}'\tilde{\rho}(\vec{r}')dV' = \frac{1}{s}\int_{V'} \tilde{\vec{J}}(\vec{r}')dV' \tag{6.28}$$

so that we have

$$\tilde{\vec{N}}_o = \frac{\mu_o}{4\pi} s\tilde{\vec{p}} \tag{6.29}$$

which is independent of $\vec{e}_r$. This can also be considered with respect to its longitudinal and transverse parts giving

$$\begin{aligned} \vec{e}_r\cdot\tilde{\vec{N}}_o &= \frac{\mu_o}{4\pi} s\vec{e}_r\cdot\tilde{\vec{p}} \\ \vec{e}_r\times\tilde{\vec{N}}_o &= \frac{\mu_o}{4\pi} s\vec{e}_r\times\tilde{\vec{p}} \\ &= -\frac{\mu_o}{4\pi}\int_{V'}(\vec{e}_r\cdot\vec{r}')\tilde{\vec{\lambda}}(\vec{r}')dV' \end{aligned} \tag{6.30}$$

Thus the transverse part of $\tilde{\vec{N}}_o$ can be related to both ρ (though $\vec{p}$) and $\vec{\lambda}$. Note from equations 6.16 that only the transverse part of $\tilde{\vec{N}}$ contributes to the far fields. From equations 6.30 we have the result which shows that the particular moment of $\tilde{\vec{\lambda}}$ is a vector which must be perpendicular to $\tilde{\vec{p}}$ and must go to zero as $s \to 0$ if $\tilde{\vec{p}} = O(1/s)$ as $s \to 0$.

Consider the $\ell = 1$ term for which we have

$$\tilde{\vec{N}}_1(\vec{e}_r) = \frac{\mu_o}{4\pi}\gamma\int_{V'}(\vec{e}_r\cdot\vec{r}')\tilde{\vec{J}}(\vec{r}')dV' \tag{6.31}$$

Note the presence of two vector functions of $\vec{r}'$ in the integrand (i.e. both $\vec{r}'$ and $\tilde{\vec{J}}$). This makes the $\ell = 1$ term basically a quadrupole type of term. Split the $\tilde{\vec{N}}_1$ term into two parts as

$$\tilde{\vec{N}}_1'(\vec{e}_r) = \frac{\mu_o}{4\pi}\gamma\int_{V'}\frac{1}{2}[(\vec{e}_r\cdot\tilde{\vec{J}}(\vec{r}'))\vec{r}'+(\vec{e}_r\cdot\vec{r}')\tilde{\vec{J}}(\vec{r}')]dV'$$

$$\tilde{\vec{N}}_1''(\vec{e}_r) = -\frac{\mu_o}{4\pi}\gamma\int_{V'}\frac{1}{2}[(\vec{e}_r\cdot\tilde{\vec{J}}(\vec{r}'))\vec{r}'-(\vec{e}_r\cdot\vec{r}')\tilde{\vec{J}}(\vec{r}')]dV' \tag{6.32}$$

with

$$\tilde{\vec{N}}_1(\vec{e}_r) = \tilde{\vec{N}}_1'(\vec{e}_r) + \tilde{\vec{N}}_1''(\vec{e}_r) \tag{6.33}$$

The prime and double prime terms represent the electric quadrupole and magnetic dipole terms respectively. For the electric quadrupole we follow Papas' convention because it fits our present purposes in the expansion of the fields.[6] The reader should be aware, however, that other conventions are also used.[7]

Consider first the prime term. For this we define the electric quadrupole moment as

$$\vec{\vec{Q}}(t) \equiv \left(Q_{\alpha_1,\alpha_2}(t)\right) \equiv \int_{V'}\vec{r}'\vec{r}'\rho(\vec{r}',t)dV' \tag{6.34}$$

This is a dyadic or tensor of rank 2 which can be written in matrix form as a symmetric matrix. Its components are designated by the indices α_1 and α_2 which can be coordinates (x, y, and z) or numbers depending on the application; they can be written out in cartesian coordinates as

$$Q_{\alpha_1,\alpha_2} = Q_{\alpha_2,\alpha_1} = \int_{V'}r'_{\alpha_1}r'_{\alpha_2}\rho(\vec{r}')dV' \tag{6.35}$$

where r'_{α_1} and r'_{α_2} can be taken as x', y', and z'. This integral can be changed to one in terms of $\vec{J}$ as

$$\tilde{Q}_{\alpha_1,\alpha_2} = -\frac{1}{s}\int_{V'}r'_{\alpha_1}r'_{\alpha_2}\nabla'\cdot\tilde{\vec{J}}(\vec{r}')dV'$$

$$= -\frac{1}{s}\int_{V'}\nabla'\cdot\left[r'_{\alpha_1}r'_{\alpha_2}\tilde{\vec{J}}(\vec{r}')\right]dV' + \frac{1}{s}\int_{V'}\nabla'\left[r'_{\alpha_1}r'_{\alpha_2}\right]\cdot\tilde{\vec{J}}(\vec{r}')dV' \tag{6.36}$$

From Gauss' theorem we have

$$\int_{V'} \nabla' \cdot \left[r'_{\alpha_1} r'_{\alpha_2} \tilde{\vec{J}}(\vec{r}\,') \right] dV' = \int_{S'} r'_{\alpha_1} r'_{\alpha_2} \vec{n}\,' \cdot \tilde{\vec{J}}(\vec{r}\,') dV' = 0 \tag{6.37}$$

since $\vec{J} = \vec{0}$ on S'. Thus we have

$$\tilde{Q}_{\alpha_1,\alpha_2} = \frac{1}{s} \int_{V'} \left[r'_{\alpha_2} \tilde{J}_{\alpha_1}(\vec{r}\,') + r'_{\alpha_1} \tilde{J}_{\alpha_2}(\vec{r}\,') \right] dV' \tag{6.38}$$

which in dyadic form is

$$\tilde{\overleftrightarrow{Q}} = \frac{1}{s} \int_{V'} [\tilde{\vec{J}}(\vec{r}\,')\vec{r}\,' + \vec{r}\,'\tilde{\vec{J}}(\vec{r}\,')] dV' \tag{6.39}$$

In the time domain this can be written

$$\frac{\partial}{\partial t} \overleftrightarrow{Q}(t) = \int_{V'} [\vec{J}(\vec{r}\,',t)\vec{r}\,' + \vec{r}\,'\vec{J}(\vec{r}\,',t)] dV' \tag{6.40}$$

This gives us the result

$$\tilde{\vec{N}}'_1(\vec{e}_r) = \frac{\mu_o}{c} \frac{1}{8\pi} s^2 \vec{e}_r \cdot \tilde{\overleftrightarrow{Q}}$$

$$= \frac{\mu_o}{c} \frac{1}{8\pi} s^2 \int_{V'} \vec{r}\,'(\vec{e}_r \cdot \vec{r}\,') \tilde{\rho}(\vec{r}\,') dV' \tag{6.41}$$

Note that due to the symmetry of $\overleftrightarrow{Q}(t)$ we have

$$\vec{e}_r \cdot \overleftrightarrow{Q}(t) = \overleftrightarrow{Q}(t) \cdot \vec{e}_r \tag{6.42}$$

The use of the dot product with a dyadic means a contraction summation over the first index if the dot is before the dyadic, and over the second index if the dot is after the dyadic; the dot appears between a vector and a dyadic (in either order) or between two dyadics; the contraction summation applies to both terms and means the product of the elements involved and then summation over the index. With a dyadic written before a dot and then a vector or another dyadic the dot product is the same as the standard matrix-vector or matrix-matrix product.

Next consider the double prime term. For this we can define a magnetic dipole dyadic as

$$\overset{\rightrightarrows}{M}(t) \equiv \left(M_{\alpha_1,\alpha_2}(t)\right) \equiv \int_{V'} \frac{1}{2}[\vec{J}(\vec{r}\,',t)\vec{r}\,'-\vec{r}\,'\vec{J}(\vec{r}\,',t)]dV' \qquad (6.43)$$

so that we have

$$\tilde{\vec{N}}_1''(\vec{e}_r) = -\frac{\mu_o}{c}\frac{1}{4\pi}s\vec{e}_r\cdot\tilde{\overset{\rightrightarrows}{M}} \qquad (6.44)$$

The magnetic dipole dyadic is antisymmetric so that

$$\vec{e}_r\cdot\overset{\rightrightarrows}{M}(t) = -\overset{\rightrightarrows}{M}(t)\cdot\vec{e}_r \qquad (6.45)$$

Note the similarity to the electric quadrupole by comparing equation 6.43 to equation 6.40. This term can be written as a magnetic dipole term through the identity

$$\begin{aligned}\vec{e}_r\cdot\overset{\rightrightarrows}{M}(t) &= \int_{V'} \frac{1}{2}[(\vec{e}_r\cdot\vec{J}(\vec{r}\,',t))\vec{r}\,'-(\vec{e}_r\cdot\vec{r}\,')\vec{J}(\vec{r}\,',t)]dV' \\ &= \int_{V'} \frac{1}{2}\,\vec{e}_r\times[\vec{r}\,'\times\vec{J}(\vec{r}\,',t)]dV' \\ &= \vec{e}_r\times\vec{m}(t) \\ &= -\vec{m}(t)\times\vec{e}_r \qquad (6.46)\end{aligned}$$

where the magnetic dipole moment is defined by

$$\vec{m}(t) \equiv \frac{1}{2}\int_{V'} \vec{r}'\times\vec{J}(\vec{r}',t)\,dV' \qquad (6.47)$$

The magnetic dipole moment can be written in other forms; from equation 3.15 it is representable as an integral over $\vec{\lambda}$ in the form

$$\vec{m}(t) = -\frac{1}{4}\int_{V'} r'^2\vec{\lambda}(\vec{r}',t)\,dV' \qquad (6.48)$$

The double prime term can then also be written as

$$\tilde{\vec{N}}_1''(\vec{e}_r) = -\frac{\mu_o}{c}\,\frac{1}{4\pi}\,s\vec{e}_r\times\tilde{\vec{m}} \qquad (6.49)$$

Consider now longitudinal and transverse parts of $\vec{N}_1$. From equation 6.23 the longitudinal part can be written as

$$\vec{e}_r\cdot\tilde{\vec{N}}_1(\vec{e}_r) = \frac{\mu_o}{c}\,\frac{1}{8\pi}\,s^2\int_{V'} (\vec{e}_r\cdot\vec{r}')^2\tilde{\rho}(\vec{r}')\,dV'$$

$$= \frac{\mu_o}{c}\,\frac{1}{8\pi}\,s^2\vec{e}_r\cdot[\vec{e}_r\cdot\tilde{\vec{\vec{Q}}}]$$

$$= \vec{e}_r\cdot\tilde{\vec{N}}_1'(\vec{e}_r) \qquad (6.50)$$

Thus the longitudinal part of $\tilde{\vec{N}}_1''$ is zero. The transverse part is

$$\vec{e}_r\times\tilde{\vec{N}}_1(\vec{e}_r) = \frac{\mu_o}{c}\left\{\frac{s^2}{8\pi}\,\vec{e}_r\times[\vec{e}_r\cdot\tilde{\vec{\vec{Q}}}]-\frac{s}{4\pi}\,\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{m}}]\right\} \qquad (6.51)$$

Combining equations 6.41 and 6.49 we have the result for $\vec{N}_1$ as

$$\tilde{\vec{N}}_1(\vec{e}_r) = \frac{\mu_o}{c}\left\{\frac{s^2}{8\pi}\,\vec{e}_r\cdot\tilde{\vec{\vec{Q}}}-\frac{s}{4\pi}\,\vec{e}_r\times\tilde{\vec{m}}\right\} \qquad (6.52)$$

Now that the first few terms in the low-frequency expansion of the radiation vector are written out we can relate these terms to the low-frequency behavior of the charge and current densities. For $s \to 0$ we assume the charge and current densities have the following asymptotic forms

$$\tilde{\rho}(\vec{r}\,') = f_\rho(s)\rho_\infty(\vec{r}\,') + o(f_\rho(s))$$

$$\tilde{\vec{J}}(\vec{r}\,') = f_J(s)\vec{J}_\infty(\vec{r}\,') + o(f_J(s)) \tag{6.53}$$

where ρ_∞ and J_∞ are spatial distribution functions and not functions of s, while f_ρ and f_J are only functions of s (which we will later take as 1/s). Since the charge and current densities are related by the equation of continuity

$$\nabla'\cdot\tilde{\vec{J}}(\vec{r}\,') = -s\tilde{\rho}(\vec{r}\,') \tag{6.54}$$

then the low-frequency asymptotic forms in equations 6.53 are not completely unrelated. As one possibility ρ could be identically zero and one could still have a non zero $\vec{J}$ if the current density were of the $\vec{J}_h$ type and confined to V'. Such a case would be provided by a closed loop carrying a uniform current; this would give a magnetic dipole moment but no electric moments. Considering the converse, however, if we have a low-frequency ρ we must have some low-frequency $\vec{J}$ along with it. To relate the low-frequency form of ρ to the low-frequency form of $\vec{J}$ we can write equation 6.54 in integral form using Gauss' theorem as

$$\int_{V_1} \tilde{\rho}(\vec{r}\,')dV' = -\frac{1}{s}\int_{S_1} \vec{n}_1\cdot\tilde{\vec{J}}(\vec{r}\,')dS' \tag{6.55}$$

where V_1 is some non zero closed volume contained in V' and S_1 is the surface of V_1 with finite area; $\vec{n}_1$ is the unit outward pointing normal on S_1. For $s \to 0$ use the formulation in equations 6.53 to give

$$\int_{V_1} \tilde{\rho}(\vec{r}\,')dV' = -\frac{f_J(s)}{s}\int_{S_1} \vec{n}_1\cdot\vec{J}_\infty(\vec{r}\,')dS' + o\left(\frac{f_J(s)}{s}\right) = O\left(\frac{f_J(s)}{s}\right) \tag{6.56}$$

Substituting for ρ gives

$$f_\rho(s)\int_{V_1} \rho_\infty(\vec{r}\,')dV' + o(f_\rho(s)) = O\left(\frac{f_J(s)}{s}\right) \tag{6.57}$$

Constrain ρ_∞ such that at low frequencies there is some V_1 for which the integral of ρ over V_1 is non zero. Then we can write

$$\begin{gathered} f_\rho(s) + o(f_\rho(s)) = O\left(\frac{f_J(s)}{s}\right) \\ 1 + \frac{1}{f_\rho(s)}\, o(f_\rho(s)) = \frac{1}{f_\rho(s)}\, O\left(\frac{f_J(s)}{s}\right) \end{gathered} \tag{6.58}$$

With $s \to 0$ we then have

$$f_\rho(s) = O\left(\frac{f_J(s)}{s}\right) \tag{6.59}$$

Thus we have shown that the low-frequency form of the charge density is constrained to be less than 1/s times the low frequency content of the current density times some positive constant. Now $\vec{J}_\infty$ might have zero divergence, in which case ρ would have even less low-frequency content depending on the most significant term with a non zero divergence in the low-frequency asymptotic expansion of $\vec{J}$.

Let us now apply the low frequency behavior of ρ and $\vec{J}$ to the moments, far potentials, and far fields. For $s \to 0$ we have the electric dipole moment

$$\begin{gathered} \tilde{\vec{p}} = \int_{V'} \vec{r}\,'\tilde{\rho}(\vec{r}\,')dV' = f_\rho(s)\vec{p}_\infty + o(f_\rho(s)) \\ \vec{p}_\infty \equiv \int_{V'} \vec{r}\,'\rho_\infty(\vec{r}\,')dV' \end{gathered} \tag{6.60}$$

the electric quadrupole moment dyadic

$$\tilde{\vec{Q}} = \int_{V'} \vec{r}'\vec{r}'\tilde{\rho}(\vec{r}')dV' = f_\rho(s)\vec{Q}_\infty + o(f_\rho(s))$$

$$\vec{Q}_\infty \equiv \int_{V'} \vec{r}'\vec{r}'\rho_\infty(\vec{r}')dV' \tag{6.61}$$

and the magnetic dipole moment

$$\tilde{\vec{m}} = \frac{1}{2}\int_{V'} \vec{r}'\times\tilde{\vec{J}}(\vec{r}')dV' = f_J(s)\vec{m}_\infty + o(f_\rho(s))$$

$$\vec{m}_\infty \equiv \frac{1}{2}\int_{V'} \vec{r}'\times\vec{J}_\infty(\vec{r}')dV' \tag{6.62}$$

Now consider the radiation vector for low frequencies. For $s \to 0$ we have

$$\tilde{\vec{N}}(\vec{e}_r) = \tilde{\vec{N}}_o + \tilde{\vec{N}}_1(\vec{e}_r) + O\left(s^2 f_J(s)\right) \tag{6.63}$$

where from equations 6.1, 6.18, and 6.19 all terms for $\ell \geq 2$ can be included in the order symbol in equation 6.63. For the $\ell = 0$ term we have for $s \to 0$

$$\tilde{\vec{N}}_o = \frac{\mu_o}{4\pi} s\tilde{\vec{p}} = \frac{\mu_o}{4\pi} sf_\rho(s)\vec{p}_\infty + o(sf_\rho(s)) \tag{6.64}$$

and for the $\ell = 1$ terms we have

$$\tilde{\vec{N}}_1'(\vec{e}_r) = \frac{\mu_o}{c}\frac{1}{8\pi} s^2\vec{e}_r\cdot\tilde{\vec{Q}} = \frac{\mu_o}{c}\frac{1}{8\pi} s^2 f_\rho(s)\vec{e}_r\cdot\vec{Q}_\infty + o\left(s^2 f_\rho(s)\right)$$

$$\tilde{\vec{N}}_1''(\vec{e}_r) = -\frac{\mu_o}{c}\frac{1}{4\pi} s\vec{e}_r\times\tilde{\vec{m}} = -\frac{\mu_o}{c}\frac{1}{4\pi} sf_J(s)\vec{e}_r\times\vec{m}_\infty + o(sf_J(s)) \tag{6.65}$$

Collecting these together gives for $s \to 0$

$$\tilde{\vec{N}}(\vec{e}_r) = \frac{\mu_o}{4\pi} s\tilde{\vec{p}} + \frac{\mu_o}{c}\frac{1}{8\pi} s^2\vec{e}_r\cdot\tilde{\vec{\vec{Q}}} - \frac{\mu_o}{c}\frac{1}{4\pi} s\vec{e}_r\times\tilde{\vec{m}} + O\left(s^2 f_J(s)\right)$$

$$= \frac{\mu_o}{4\pi} sf_\rho(s)\vec{p}_\infty - \frac{\mu_o}{c}\frac{1}{4\pi} sf_J(s)\vec{e}_r\times\vec{m}_\infty + o(sf_\rho(s)) + o(sf_J(s)) \qquad (6.66)$$

Thus keeping only the leading terms in ρ and $\vec{J}$ leaves us with only the electric and magnetic dipole terms at low frequencies, provided they are both non zero. Note that the electric quadrupole term is unimportant at low frequencies compared to the electric dipole term. Note that $f_\rho(s)$ and $f_J(s)$ can be of the same order, consistent with the requirement of equation 6.59. Here we begin to see the dominance of the electric and magnetic dipole terms at low frequencies. These two terms are to some extent independent; they are considered in later sections, both singly and in combination.

Having the low-frequency behavior of the radiation vector consider the far potentials. From equations 6.8, 6.10, 6.30, and 6.50 the far potential for $s \to 0$ is

$$\tilde{\Phi}_1(\vec{r}) = c\,\frac{e^{-\gamma r}}{r}\,\vec{e}_r\cdot\tilde{\vec{N}}(\vec{e}_r)$$

$$= \frac{e^{-\gamma r}}{4\pi r}\left\{Z_o s\vec{e}_r\cdot\tilde{\vec{p}}+O\left(s^2 f_\rho(s)\right)\right\}$$

$$= \frac{e^{-\gamma r}}{4\pi r}\{Z_o sf_\rho(s)\vec{e}_r\cdot\vec{p}_\infty+o(sf_\rho(s))\} \qquad (6.67)$$

Note that the far scalar potential only depends on ρ so no f_J is used; the electric dipole, if non zero, is the dominant term for low frequencies. The far vector potential for $s \to 0$ is

$$\tilde{\vec{A}}_1(\vec{r}) = \frac{e^{-\gamma r}}{r}\,\tilde{\vec{N}}(\vec{e}_r)$$

$$= \frac{e^{-\gamma r}}{4\pi r}\left\{\mu_o s\tilde{\vec{p}}-\frac{\mu_o}{c}\, s\vec{e}_r\times\tilde{\vec{m}}+O\left(s^2 f_\rho(s)\right)+O\left(s^2 f_J(s)\right)\right\}$$

$$= \frac{e^{-\gamma r}}{4\pi r}\left\{\mu_o s f_\rho(s)\vec{p}_\infty - \frac{\mu_o}{c}\, s f_J(s)\vec{e}_r\times\vec{m}_\infty + o(s f_\rho(s)) + o(s f_J(s))\right\} \qquad (6.68)$$

Both electric and magnetic dipoles contribute to the far vector potential at low frequencies. The longitudinal part of the far vector potential for $s \to 0$ is

$$\vec{e}_r\cdot\tilde{\vec{A}}_1(\vec{r}) = \frac{e^{-\gamma r}}{r}\,\vec{e}_r\cdot\tilde{\vec{N}}(\vec{e}_r)$$

$$= \frac{e^{-\gamma r}}{4\pi r}\left\{\mu_o s\vec{e}_r\cdot\tilde{\vec{p}} + O\left(s^2 f_\rho(s)\right)\right\}$$

$$= \frac{e^{-\gamma r}}{4\pi r}\{\mu_o s f_\rho(s)\vec{e}_r\cdot\vec{p}_\infty + o(s f_\rho(s))\} \qquad (6.69)$$

which only depends on ρ; the dominant low frequency term is the electric dipole. The transverse part of the far vector potential depends on both electric and magnetic terms as can be seen from equation 6.68.

Next consider the far fields for low frequency. From equations 6.16 these depend only on the transverse part of the radiation vector. The far electric field for $s \to 0$ is given by

$$\tilde{\vec{E}}_1(\vec{r}) = \frac{e^{-\gamma r}}{r}\, s\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{N}}(\vec{e}_r)]$$

$$= \frac{e^{-\gamma r}}{4\pi r}\left\{\mu_o s^2\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{p}}] + \frac{\mu_o}{c}\, s^2\vec{e}_r\times\tilde{\vec{m}} + O\left(s^3 f_\rho(s)\right) + O\left(s^3 f_J(s)\right)\right\}$$

$$= \frac{e^{-\gamma r}}{4\pi r}\left\{\mu_o s^2 f_\rho(s)\vec{e}_r\times[\vec{e}_r\times\vec{p}_\infty] + \frac{\mu_o}{c}\, s^2 f_J(s)\vec{e}_r\times\vec{m}_\infty + o\left(s^2 f_\rho(s)\right)\right.$$

$$\left. + o\left(s^2 f_J(s)\right)\right\} \qquad (6.70)$$

and the far magnetic field by

$$\tilde{\vec{H}}_1(\vec{r}) = -\frac{1}{Z_o} \frac{e^{-\gamma r}}{r} s\vec{e}_r \times \tilde{\vec{N}}(\vec{e}_r)$$

$$= \frac{e^{-\gamma r}}{4\pi r}\left\{-\frac{1}{c} s^2 \vec{e}_r \times \vec{p} + \frac{1}{c^2} s^2 \vec{e}_r \times [\vec{e}_r \times \tilde{\vec{m}}] + O\left(s^3 f_\rho(s)\right) + O\left(s^3 f_J(s)\right)\right\}$$

$$= \frac{e^{-\gamma r}}{4\pi r}\left\{-\frac{1}{c} s^2 f_\rho(s) \vec{e}_r \times \vec{p}_\infty + \frac{1}{c^2} s^2 f_J(s) \vec{e}_r \times [\vec{e}_r \times \vec{m}_\infty] + o\left(s^2 f_\rho(s)\right)\right.$$

$$\left. + o\left(s^2 f_J(s)\right)\right\} \qquad (6.71)$$

The dominant low-frequency terms for the far fields are the electric and magnetic dipoles, if not made small; again this relies on the fact that ρ and $\vec{J}$ can be specified somewhat independently.

A case of particular interest has ρ and $\vec{J}$ behaving as step functions for their late time and low frequency behavior so that we can choose for $s \to 0$ the asymptotic forms

$$f_\rho(s) = \frac{1}{s} + o\left(\frac{1}{s}\right)$$

$$f_J(s) = \frac{1}{s} + o\left(\frac{1}{s}\right) \qquad (6.72)$$

Note that this is consistent with the relation between ρ and $\vec{J}$ as $s \to 0$ required by equation 6.59. Furthermore the 1/s part of f_J is associated with a $\vec{J}_h$ type of current density which has no charge density (and thus no scalar potential) associated with it. This is important for energy considerations for transient magnetic dipoles which we discuss later. The choices in equations 6.72 give charge and current densities for $s \to 0$ as

$$\tilde{\rho}(\vec{r}\,') = \frac{1}{s} \rho_\infty(\vec{r}\,') + o\left(\frac{1}{s}\right)$$

$$\tilde{\vec{J}}(\vec{r}\,') = \frac{1}{s} \vec{J}_\infty(\vec{r}\,') + o\left(\frac{1}{s}\right) \qquad (6.73)$$

and some of the moments as

$$\tilde{\vec{p}} = \frac{1}{s}\,\vec{p}_\infty + o\left(\frac{1}{s}\right)$$

$$\tilde{\overleftrightarrow{Q}} = \frac{1}{s}\,\overleftrightarrow{Q}_\infty + o\left(\frac{1}{s}\right) \qquad (6.74)$$

$$\tilde{\vec{m}} = \frac{1}{s}\,\vec{m}_\infty + o\left(\frac{1}{s}\right)$$

Using the final value theorem of the Laplace transform (assuming ρ and $\vec{J}$ are zero for $t < t_o$ for some finite t_o) we have the late time charge and current densities

$$\rho(\vec{r}\,',\infty) \equiv \lim_{t\to\infty} \rho(\vec{r}\,',t) = \rho_\infty(\vec{r}\,')$$

$$(6.75)$$

$$\vec{J}(\vec{r}\,',\infty) \quad \lim_{t\to\infty} \vec{J}(\vec{r}\,',t) = \vec{J}_\infty(\vec{r}\,')$$

Similarly some of the late time moments are

$$\vec{p}(\infty) = \vec{p}_\infty$$

$$\overleftrightarrow{Q}(\infty) = \overleftrightarrow{Q}_\infty \qquad (6.76)$$

$$\vec{m}(\infty) = \vec{m}_\infty$$

With this special choice for the low-frequency and late-time charge and current densities (equations 6.72) the scalar and vector potentials (equations 6.67 and 6.68) for low frequencies tend to constant values with $s \to 0$ as

$$\tilde{\Phi}_1(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r}\{Z_o \vec{e}_r\cdot\vec{p}(\infty) + o(1)\}$$

$$(6.77)$$

$$\tilde{\vec{A}}_1(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r}\left\{\mu_o \vec{p}(\infty) - \frac{\mu_o}{c}\,\vec{e}_r\times\vec{m}(\infty) + o(1)\right\}$$

and the far fields for $s \to 0$ behave as

$$\tilde{\vec{E}}_1(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r}\left\{\mu_o s\vec{e}_r\times[\vec{e}_r\times\vec{p}(\infty)]+\frac{\mu_o}{c}\, s\vec{e}_r\times\vec{m}(\infty)+o(s)\right\}$$

$$\tilde{\vec{H}}_1(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r}\left\{-\frac{1}{c}\, s\vec{e}_r\times\vec{p}(\infty)+\frac{1}{c^2}\, s\vec{e}_r\times[\vec{e}_r\times\vec{m}(\infty)]+o(s)\right\} \tag{6.78}$$

This generalizes the result of reference 5 to include both electric and magnetic dipole terms for the low-frequency far fields. Note that a late-time magnetic dipole moment gives the same low-frequency dependence to the far fields as does the late-time electric dipole moment. This requires, however, that the late-time current not go to zero but continue to circulate as a $\vec{J}_h$ type of current such as in a closed conducting current path (i.e. some kind of loop).

Since the late-time magnetic dipole moment gives the same kind of low-frequency characteristics to the far fields as does the late-time electric dipole moment, then it can also give a time-domain waveform for the far fields with only one zero crossing. The arguments in reference 5 apply equally well to late-time electric and magnetic dipoles.

The analysis in this note considers the case that ρ and $\vec{J}$ are zero for $t < t_o$. However these results can be readily generalized to the case that ρ and $\vec{J}$ are non zero, say $\rho_o(\vec{r}\,')$ and $\vec{J}_o(\vec{r}\,')$, for $t < t_o$ by treating these initial conditions as a static problem and only using the Laplace transform (two sided) on the difference of the time-domain quantities from their initial values to assure convergence. One can define various difference quantities such as

$$\Delta\rho(\vec{r}\,',t) \equiv \rho(\vec{r}\,',t) - \rho_o(\vec{r}\,')$$

$$\Delta\vec{J}(\vec{r}\,',t) \equiv \vec{J}(\vec{r}\,',t) - \vec{J}_o(\vec{r}\,')$$

$$\Delta\vec{p}(t) \equiv \vec{p}(t) - \vec{p}_o$$

$$\Delta\vec{m}(t) \equiv \vec{m}(t) - \vec{m}_o \tag{6.79}$$

$$\Delta\vec{p}(\infty) \equiv \vec{p}(\infty) - \vec{p}_o$$

$$\Delta\vec{m}(\infty) \equiv \vec{m}(\infty) - \vec{m}_o$$

where

$$\vec{p}_o \equiv \int_{V'} \vec{r}'\rho_o(\vec{r}')dV'$$

$$\vec{m}_o \equiv \frac{1}{2}\int_{V'} \vec{r}'\times\vec{J}_o(\vec{r}')dV' \tag{6.80}$$

Then, for example, one could replace $\vec{p}(\infty)$ and $\vec{m}(\infty)$ by $\Delta\vec{p}(\infty)$ and $\Delta\vec{m}(\infty)$ in equations 6.78 for the low-frequency far fields. The far fields are still zero for $t < t_o$ since the static fields do not have a $1/r$ term in their expansion for large r. However, some of the other terms in the field expansion for large r would be non zero for $t < t_o$. Note that $\vec{J}_o$ must have zero divergence to be consistent with a constant ρ_o so that $\vec{J}_o$ is a $\vec{J}_h$ type of term.

Actually one does not even necessarily need constant ρ and $\vec{J}$ for $t < t_o$. One merely needs to subtract off enough of ρ and $\vec{J}$ in the time domain such that what is left can be treated with the two sided Laplace transform, i.e. so that the two sided Laplace transform integral converges for an appropriate part of the complex s plane which allows the calculation of the inverse transform integral. Corresponding to that part of ρ and/or $\vec{J}$ subtracted off one might find the potentials and fields by some special solution technique; these results could then be added to the appropriate difference quantities.

VII. The Second Terms in the Asymptotic Expansion of the Potentials and Fields for Large r: Order r^{-2}

Having considered the far potentials and far fields we go on to consider the next terms in the asymptotic expansions for large r, i.e. the terms proportional to $e^{-\gamma r}r^{-2}$. For convenience we call these terms the second order potentials and second order fields. This label can be used to distinguish these terms from the far fields and far potentials which might also be called the first order fields and first order potentials. The second order potentials and second order fields are denoted by a subscript 2 and included as such in equations 5.1 which write out the asymptotic expansions for large r.

To write out the second order potentials and fields we use the terms proportional to $e^{-\gamma r}r^{-2}$ in the asymptotic expansion as $r \to \infty$ of the Green's function (equation 5.8) and its gradient (equation 5.12). The second order potentials can then be written as

$$\tilde{\vec{A}}_2(\vec{r}) = \mu_o \frac{e^{-\gamma r}}{4\pi r^2}\int_{V'} e^{\gamma \vec{e}_r\cdot\vec{r}\,'}\left\{\vec{e}_r\cdot\vec{r}\,'-\frac{\gamma}{2}\left[r'^2-(\vec{e}_r\cdot\vec{r}\,')^2\right]\right\}\tilde{\vec{J}}(\vec{r}\,')dV'$$

$$\tilde{\Phi}_2(\vec{r}) = \frac{1}{\varepsilon_o}\frac{e^{-\gamma r}}{4\pi r^2}\int_{V'} e^{\gamma \vec{e}_r\cdot\vec{r}\,'}\left\{\vec{e}_r\cdot\vec{r}\,'-\frac{\gamma}{2}\left[r'^2-(\vec{e}_r\cdot\vec{r}\,')^2\right]\right\}\tilde{\rho}(\vec{r}\,')dV' \tag{7.1}$$

since the potentials use the Green's function as in equations 4.4 and 4.1. The second order fields can be written as

$$\tilde{\vec{E}}_2(\vec{r}) = -\frac{1}{\varepsilon_o}\frac{e^{-\gamma r}}{4\pi r^2}\int_{V'} e^{\gamma \vec{e}_r\cdot\vec{r}\,'}\left\{-\vec{e}_r\left[1+2\gamma\vec{e}_r\cdot\vec{r}\,'-\frac{\gamma^2}{2}\left[r'^2-(\vec{e}_r\cdot\vec{r}\,')^2\right]\right]\right.$$
$$\left.+\vec{r}\,'\gamma\right\}\tilde{\rho}(\vec{r}\,')dV'$$
$$-\mu_o s\frac{e^{-\gamma r}}{4\pi r^2}\int_{V'} e^{\gamma \vec{e}_r\cdot\vec{r}\,'}\left\{\vec{e}_r\cdot\vec{r}\,'-\frac{\gamma}{2}\left[r'^2-(\vec{e}_r\cdot\vec{r}\,')^2\right]\right\}\tilde{\vec{J}}(\vec{r}\,')dV' \tag{7.2}$$

$$\tilde{\vec{H}}_2(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^2}\int_{V'} e^{\gamma \vec{e}_r\cdot\vec{r}\,'}\left\{-\vec{e}_r\left[1+2\gamma\vec{e}_r\cdot\vec{r}\,'-\frac{\gamma^2}{2}\left[r'^2-(\vec{e}_r\cdot\vec{r}\,')^2\right]\right]+\vec{r}\,'\gamma\right\}$$
$$\times\tilde{\vec{J}}(\vec{r}\,')dV'$$

where the electric field uses both the Green's function and its gradient as in equation 4.21 while the magnetic field uses the gradient as in equation 4.15.

Comparing these expressions for the second order potentials and fields to equations 6.3 and 6.15 for the far (or first order) potentials and fields, the expressions for the second order potentials and fields are significantly more complex. Equations 7.1 and 7.2 can be expressed in time-domain form as done in the previous section with the far potentials and fields by using $\gamma = s/c$, including the exponentials as time delay (or advance) terms in the time argument for $\vec{J}$ and ρ, and replacing the s terms in the coefficients by time derivatives. For the second order potentials and fields, however, we are interested primarily in their low frequency content as compared to the far potentials and fields. In particular we are interested in seeing at what r (if any) the second order terms become comparable to the first order terms where this r is still large compared to the antenna dimensions. This gives us an estimate of how far down in frequency at some large r one can use the far potentials and fields before the appropriate second or third order terms become significant.

Consider first the second order scalar potential. As in the previous section (equations 6.53) the charge density and current density for $s \to 0$ are assumed to have the asymptotic forms

$$\tilde{\rho}(\vec{r}\,') = f_\rho(s)\rho_\infty(\vec{r}\,') + o(f_\rho(s))$$

$$\tilde{\vec{J}}(\vec{r}\,') = f_J(s)\vec{J}_\infty(\vec{r}\,') + o(f_J(s)) \tag{7.3}$$

where ρ_∞ and $\vec{J}_\infty$ are only functions of $\vec{r}\,'$. Write the exponential term in the integrand for $s \to 0$ as

$$e^{\gamma \vec{e}_r \cdot \vec{r}\,'} = \sum_{\ell=0}^{\infty} \frac{(\gamma \vec{e}_r \cdot \vec{r}\,')^\ell}{\ell!}$$

$$= 1 + \gamma \vec{e}_r \cdot \vec{r}\,' + \frac{1}{2}(\gamma \vec{e}_r \cdot \vec{r}\,')^2 + O(s^3) \tag{7.4}$$

For $s \to 0$ the second order scalar potential can then be written as

$$\tilde{\Phi}_2(\vec{r}) = \frac{1}{\varepsilon_o} \frac{e^{-\gamma r}}{4\pi r^2} \int_{V'} \left\{ \vec{e}_r \cdot \vec{r}' + \frac{3}{2} \gamma (\vec{e}_r \cdot \vec{r}')^2 - \frac{\gamma}{2} r'^2 + O(s^2) \right\} \tilde{\rho}(\vec{r}') dV'$$

$$= \frac{1}{\varepsilon_o} \frac{e^{-\gamma r}}{4\pi r^2} \left\{ \vec{e}_r \cdot \int_{V'} \vec{r}' \tilde{\rho}(\vec{r}') dV' + \frac{3}{2} \gamma \vec{e}_r \left[\vec{e}_r \cdot \int_{V'} \vec{r}'\vec{r}' \tilde{\rho}(\vec{r}') dV' \right] \right.$$

$$\left. - \frac{\gamma}{2} \int_{V'} r'^2 \tilde{\rho}(\vec{r}') dV' + O\left(s^2 f_\rho(s)\right) \right\} \tag{7.5}$$

The first two terms include the electric dipole moment and quadrupole moment dyadic as seen from equations 6.60 and 6.61. The third term is sometimes included in the definition of the electric quadrupole moment dyadic. Here we treat it separately and define it either as a scalar

$$q'(t) \equiv \int_{V'} r'^2 \rho(\vec{r}', t) dV' \tag{7.6}$$

or as a symmetric dyadic

$$\vec{\vec{Q}}'(t) \equiv \left(\delta_{\alpha_1, \alpha_2}\right) q'(t) \tag{7.7}$$

where $(\delta_{\alpha_1,\alpha_2})$ is the unit or identity dyadic. The scalar and dyadic forms are related as

$$q'(t) = \vec{e}' \cdot [\vec{e}' \cdot \vec{\vec{Q}}'(t)] \tag{7.8}$$

where $\vec{e}'$ is an arbitrary unit vector. The second and third terms can be combined as another dyadic of the form

$$\vec{\vec{Q}}''(t) = 3\vec{\vec{Q}}(t) - \vec{\vec{Q}}'(t) = \int_{V'} \left[3\vec{r}'\vec{r}' - r'^2 \left(\delta_{\alpha_1, \alpha_2}\right) \right] \rho(\vec{r}', t) dV' \tag{7.9}$$

As $s \to 0$ we have

$$\tilde{q}' = \int_{V'} r'^2 \tilde{\rho}(\vec{r}')dV' = f_\rho(s)q'_\infty + o(f_\rho(s))$$

$$q'_\infty \equiv \int_{V'} r'^2 \rho_\infty(\vec{r}')dV'$$

$$\tilde{\vec{\vec{Q}}}' = \left(\delta_{\alpha_1,\alpha_2}\right)\tilde{q}' = f_\rho(s)\vec{\vec{Q}}'_\infty + o(f_\rho(s)) \tag{7.10}$$

$$\vec{\vec{Q}}'_\infty \equiv \left(\delta_{\alpha_1,\alpha_2}\right)q'_\infty$$

For $s \to 0$ the second order scalar potential can then be written as

$$\tilde{\Phi}_2(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^2}\left\{\frac{1}{\varepsilon_o}\,\vec{e}_r\cdot\tilde{\vec{p}}+Z_o\frac{s}{2}\,\vec{e}_r\cdot[\vec{e}_r\cdot\tilde{\vec{\vec{Q}}}'']+O\left(s^2 f_\rho(s)\right)\right\}$$

$$= \frac{e^{-\gamma r}}{4\pi r^2}\left\{\frac{1}{\varepsilon_o}\,\vec{e}_r\cdot\tilde{\vec{p}}+\frac{3}{2}\,Z_o s\vec{e}_r\cdot[\vec{e}_r\cdot\tilde{\vec{\vec{Q}}}]-\frac{Z_o}{2}\,s\tilde{q}'+O\left(s^2 f_\rho(s)\right)\right\}$$

$$= \frac{e^{-\gamma r}}{4\pi r^2}\left\{\frac{1}{\varepsilon_o}\,f_\rho(s)\vec{e}_r\cdot\vec{p}_\infty+\frac{3}{2}\,Z_o s f_\rho(s)\vec{e}_r\cdot[\vec{e}_r\cdot\vec{\vec{Q}}\,]-\frac{Z_o}{2}\,s f_\rho(s)q'_\infty + o(sf_\rho(s))\right\} \tag{7.11}$$

or, keeping only the most significant term, as

$$\tilde{\Phi}_2(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^2}\left\{\frac{1}{\varepsilon_o}\,\vec{e}_r\cdot\tilde{\vec{p}}+O(sf_\rho(s))\right\}$$

$$= \frac{e^{-\gamma r}}{4\pi r^2}\left\{\frac{1}{\varepsilon_o}\,f_\rho(s)\vec{e}_r\cdot\vec{p}_\infty+o(f_\rho(s))\right\} \tag{7.12}$$

The leading term is basically an electrostatic potential, except for the delay factor $e^{-\gamma r}$ which goes to unity as $s \to 0$. Comparing this result to the first order potential for $s \to 0$ in equation 6.67 note that the dominant term in both cases is based on the electric dipole moment; as $s \to 0$ we have the result

$$[4\pi r^2 e^{\gamma r}]\tilde{\Phi}_2(\vec{r}) = \frac{c}{s}[4\pi r e^{\gamma r}]\tilde{\Phi}_1(\vec{r}) + O(sf_\rho(s)) \tag{7.13}$$

With

$$s = i\omega \tag{7.14}$$

where ω is the radian frequency then we can define a transition radian frequency

$$\omega_1 \equiv \frac{c}{r} \tag{7.15}$$

which is a function of r. Provided the electric dipole moment in non zero for low frequencies and we neglect higher order terms, then for $\omega > \omega_1$ the far scalar potential dominates the second order potential, and for $\omega < \omega_1$ the second order scalar potential dominates the far (or first order) scalar potential. This result comes from equation 7.13 and gives a fairly simple criterion for when the static scalar potential should be considered along with the far scalar potential. Similarly, one can define a transition radius as

$$r_1 \equiv \frac{c}{\omega} \tag{7.16}$$

which is a function of ω. Given some low radian frequency ω of interest then for $r > r_1$ we have basically the far scalar potential, while for $r < r_1$ the second order scalar potential (and perhaps higher orders) become significant. Note that we still restrict $r >> r_o$ where r_o is the maximum r' on S' because of the asymptotic expansion for large r. Thus r_1 and ω_1 are used with the restriction that $r_1 >> r_o$ and $\omega_1 << c/r_o$.

Next consider the second order vector potential. For $s \to 0$ it can be written as

$$\tilde{\vec{A}}_2(\vec{r}) = \mu_o \frac{e^{-\gamma r}}{4\pi r^2} \int_{V'} \left\{ \vec{e}_r \cdot \vec{r}' + \frac{3}{2} \gamma (\vec{e}_r \cdot \vec{r}')^2 - \frac{\gamma}{2} r'^2 + O(s^2) \right\} \tilde{\vec{J}}(\vec{r}')\, dV'$$

$$= \mu_o \frac{e^{-\gamma r}}{4\pi r^2} \int_{V'} \{ \vec{e}_r \cdot \vec{r}' + O(s) \} \tilde{\vec{J}}(\vec{r}')\, dV'$$

$$= \mu_o \frac{e^{-\gamma r}}{4\pi r^2} \left\{ \vec{e}_r \cdot \int_{V'} \vec{r}' \tilde{\vec{J}}(\vec{r}')\, dV' + O(sf_J(s)) \right\} \qquad (7.17)$$

With the electric quadrupole moment dyadic (equation 6.39) and magnetic dipole dyadic (equation 6.43) we can write for $s \to 0$

$$\tilde{\vec{A}}_2(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^2} \left\{ \frac{\mu_o}{2} s\vec{e}_r \cdot \tilde{\vec{\vec{Q}}} - \mu_o \vec{e}_r \cdot \tilde{\vec{\vec{M}}} + O(sf_J(s)) \right\}$$

$$= \frac{e^{-\gamma r}}{4\pi r^2} \left\{ \frac{\mu_o}{2} s\vec{e}_r \cdot \tilde{\vec{\vec{Q}}} - \mu_o \vec{e}_r \times \tilde{\vec{m}} + O(sf_J(s)) \right\} \qquad (7.18)$$

where the conversion to the magnetic dipole vector uses the result of equation 6.46. As with the far vector potential at low frequencies (equation 6.68), the second order vector potential has both electric and magnetic type terms, thus depending on both $\vec{J}_e$ (or ρ) and $\vec{J}_h$. In terms of the low-frequency forms of the moments equation 7.18 becomes

$$\tilde{\vec{A}}_2(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^2} \left\{ \frac{\mu_o}{2} sf_\rho(s) \vec{\vec{Q}}_\infty - \mu_o f_J(s) \vec{e}_r \times \vec{m}_\infty + o(sf_\rho(s)) + o(f_J(s)) \right\} \qquad (7.19)$$

Note that the longitudinal part of $\tilde{\vec{A}}_2$ with respect to $\vec{e}_r$ for $s \to 0$ is simply

$$\vec{e}_r \cdot \tilde{\vec{A}}_2(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^2} \left\{ \frac{\mu_o}{2} s\vec{e}_r \cdot [\vec{e}_r \cdot \tilde{\vec{\vec{Q}}}] + O(sf_J(s)) \right\}$$

$$= \frac{e^{-\gamma r}}{4\pi r^2} \left\{ \frac{\mu_o}{2} sf_\rho(s) \vec{e}_r \cdot [\vec{e}_r \cdot \vec{\vec{Q}}_\infty] + o(sf_\rho(s)) + O(sf_J(s)) \right\} \qquad (7.20)$$

which does not include the static magnetic dipole term.

The second order scalar potential has an electrostatic term based on the electric dipole moment. As seen in equation 7.12 at low frequencies the second order scalar potential has the same frequency dependence as the electric dipole moment. Considering the second order vector potential the electric quadrupole moment has the same frequency dependence as ρ so equation 7.18 can be written for $s \to 0$ as

$$\tilde{\vec{A}}_2(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^2}\{-\mu_o \vec{e}_r \times \tilde{\vec{m}} + O(sf_\rho(s)) + O(sf_J(s))\}$$

$$= \frac{e^{-\gamma r}}{4\pi r^2}\{-\mu_o f_J(s)\vec{e}_r \times \vec{m}_\infty + O(sf_\rho(s)) + o(f_J(s))\} \qquad (7.21)$$

Comparing the second order vector potential to the far vector potential for $s \to 0$ as in equation 6.68 note the presence of the magnetic dipole term in both cases, but with different coefficients. Analogous to equation 7.13 for the scalar potential we can write one for the vector potential as $s \to 0$, emphasizing the magnetic dipole term, as

$$[4\pi r^2 e^{\gamma r}]\tilde{\vec{A}}_2(\vec{r}) = \frac{c}{s}[4\pi r e^{\gamma r}]\tilde{\vec{A}}_1(\vec{r}) - Z_o\vec{p} + O(sf_\rho(s)) + O(sf_J(s)) \qquad (7.22)$$

If we neglect the electric dipole term and the higher order terms in ρ and $\vec{J}$ and just look at the magnetic dipole contribution we see that ω_1 and r_1 can be used with the vector potential just as with the scalar potential to define the range of r and ω for which the far vector potential is dominant or for which the second order vector potential must be included. If $\vec{J}$ in V' were purely solenoidal (i.e. no ρ) then the magnetic dipole term (if non zero) would be dominant at low frequencies and the split between $\tilde{A}_1$ and $\tilde{A}_2$ dominance at $r = r_1$ or $\omega = \omega_1$ would necessarily follow. However the presence of the electric dipole moment in the far vector potential and the electric quadrupole moment in the second order vector potential can complicate this result somewhat depending on the relative magnitudes of these terms. Note that in the second order vector potential the magnetic dipole term is basically a magnetostatic term for $s \to 0$ since the magnetic dipole term does not have a function of s for a coefficient.

Moving on to the second order electric field, for $s \to 0$ it can be written as

$$\tilde{\vec{E}}_2(\vec{r}) = -\frac{1}{\varepsilon_o}\frac{e^{-\gamma r}}{4\pi r^2}\int_{V'}\left\{-\vec{e}_r\left[1+3\gamma\vec{e}_r\cdot\vec{r}'+\gamma^2\left[3(\vec{e}_r\cdot\vec{r}')^2-\frac{r'^2}{2}\right]\right]\right.$$

$$\left.+r'\left[\gamma+\gamma^2\vec{e}_r\cdot\vec{r}'\right]+O(s^3)\right\}\tilde{\rho}(\vec{r}')\,dV'$$

$$-\mu_o s\frac{e^{-\gamma r}}{4\pi r^2}\int_{V'}\left\{\vec{e}_r\cdot\vec{r}'+\frac{3}{2}\gamma(\vec{e}_r\cdot\vec{r}')^2-\frac{\gamma}{2}r'^2+O(s^2)\right\}\tilde{\vec{J}}(\vec{r}')\,dV' \tag{7.23}$$

Identifying the terms with the various moments and noting that the integral of $\tilde{\rho}$ over V' is zero, except for a possible static term in the time domain, we have

$$\tilde{\vec{E}}_2(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^2}\left\{\left\{3Z_o s\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{p}}]+3\mu_o s^2\vec{e}_r[\vec{e}_r\cdot[\vec{e}_r\cdot\tilde{\vec{\vec{Q}}}]]-\frac{\mu_o}{2}s^2\vec{e}_r\tilde{q}'\right.\right.$$

$$\left.-Z_o s\tilde{\vec{p}}-\mu_o s^2\vec{e}_r\cdot\tilde{\vec{\vec{Q}}}+O\left(s^3f_\rho(s)\right)\right\}$$

$$+\left\{-\frac{\mu_o}{2}s^2\vec{e}_r\cdot\tilde{\vec{\vec{Q}}}+\mu_o s\vec{e}_r\cdot\tilde{\vec{\vec{M}}}\right.$$

$$-\frac{3}{2}\frac{\mu_o}{c}s^2\int_{V'}(\vec{e}_r\cdot\vec{r}')^2\tilde{\vec{J}}(\vec{r}')\,dV'+\frac{1}{2}\frac{\mu_o}{c}s^2\int_{V'}r'^2\tilde{\vec{J}}(\vec{r}')\,dV'$$

$$\left.\left.+O\left(s^3f_J(s)\right)\right\}\right\} \tag{7.24}$$

where we have included a few integrals which involve higher order moments of $\tilde{\vec{J}}$. Considering only the leading terms as $s \to 0$ we have

$$\tilde{\vec{E}}_2(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^2}\left\{Z_o s[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{p}}]-\tilde{\vec{p}}]+\mu_o s\vec{e}_r\times\tilde{\vec{m}}+O\left(s^2f_\rho(s)\right)+O\left(s^2f_J(s)\right)\right\}$$

$$= \frac{e^{-\gamma r}}{4\pi r^2}\left\{Z_o s f_\rho(s)\left[3\vec{e}_r[\vec{e}_r\cdot\vec{p}_\infty]-\vec{p}_\infty\right]+\mu_o s f_J(s)\vec{e}_r\times\vec{m}_\infty\right.$$

$$\left.+o(sf_\rho(s))+o(sf_J(s))\right\} \qquad (7.25)$$

Note the presence of both electric and magnetic dipole terms in the leading terms, but with a factor of s with each. The leading terms in the far electric field (equation 6.70) have s^2 factors with the electric and magnetic dipole terms. Rearranging E_1 and E_2 for $s \to 0$ we have

$$\frac{c}{s}[4\pi r e^{\gamma r}]\tilde{\vec{E}}_1(\vec{r}) = Z_o s\left\{\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{p}}]+\frac{1}{c}\,\vec{e}_r\times\tilde{\vec{m}}+O(sf_\rho(s))+O(sf_J(s))\right\}$$

$$(7.26)$$

$$[4\pi r^2 e^{\gamma r}]\tilde{\vec{E}}_2(\vec{r}) = Z_o s\left\{3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{p}}]-\tilde{\vec{p}}+\frac{1}{c}\,\vec{e}_r\times\tilde{\vec{m}}+O(sf_\rho(s))+O(sf_J(s))\right\}$$

In these two expressions note the magnetic dipole moment enters the same way; the electric dipole moment enters in a different way in each case, but with the same frequency dependence and roughly the same magnitude. Then ω_1 and r_1 can be used with the electric field just as with the potentials to show the range of validity of ω and r respectively for the far electric field and when the second order electric field (and perhaps higher order terms) should be included.

Next, the second order magnetic field for $s \to 0$ can be written as

$$\tilde{\vec{H}}_2(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^2}\left\{-\vec{e}_r\times\int_{V'}\left[1+3\gamma\vec{e}_r\cdot\vec{r}\,'-\gamma^2\left[3(\vec{e}_r\cdot\vec{r}\,')^2-\frac{r'^2}{2}\right]+O(s^3)\right]\tilde{\vec{J}}(\vec{r}\,')\,dV'\right.$$

$$\left.+\int_{V'}\left[\gamma+\gamma^2\vec{e}_r\cdot\vec{r}\,'+O(s^3)\right]\vec{r}\,'\times\tilde{\vec{J}}(\vec{r}\,')\,dV'\right\} \qquad (7.27)$$

Writing this out in terms of the various moments gives

$$\tilde{\vec{H}}_2(\vec{r}\,') = \frac{e^{-\gamma r}}{4\pi r^2}\left\{-s\vec{e}_r\times\tilde{\vec{p}}-\frac{3}{2}\,\frac{1}{c}\,s^2\vec{e}_r\times[\vec{e}_r\cdot\tilde{\vec{\vec{Q}}}]+\frac{3}{c}\,s\vec{e}_r\times[\vec{e}_r\cdot\tilde{\vec{\vec{M}}}]\right.$$

$$+\frac{3}{c^2}\,s^2\vec{e}_r\times\int_{V'}(\vec{e}_r\cdot\vec{r}\,')^2\tilde{\vec{J}}(\vec{r}\,')\,dV'-\frac{1}{2}\,\frac{1}{c^2}\,s^2\vec{e}_r\times\int_{V'}r'^2\tilde{\vec{J}}(\vec{r}\,')\,dV'$$

$$\left.+\frac{2}{c}\,s\tilde{\vec{m}}+\frac{1}{c^2}\,s^2\int_{V'}(\vec{e}_r\cdot\vec{r}\,')\vec{r}\,'\times\tilde{\vec{J}}(\vec{r}\,')\,dV'+O\left(s^3 f_J(s)\right)\right\} \qquad (7.28)$$

Considering only the leading terms as $s \to 0$ gives

$$\tilde{\vec{H}}_2(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^2}\left\{-s\vec{e}_r\times\tilde{\vec{p}}+\frac{3}{c}\,s\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{m}}]+\frac{2}{c}\,s\tilde{\vec{m}}+O\left(s^2 f_\rho(s)\right)+O\left(s^2 f_J(s)\right)\right\}$$

$$= \frac{e^{-\gamma r}}{4\pi r^2}\left\{-s\vec{e}_r\times\tilde{\vec{p}}+\frac{s}{c}\left[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{m}}]-\tilde{\vec{m}}\right]+O\left(s^2 f_\rho(s)\right)+O\left(s^2 f_J(s)\right)\right\}$$

$$= \frac{e^{-\gamma r}}{4\pi r^2}\left\{-s f_\rho(s)\vec{e}_r\times\vec{p}_\infty+\frac{s f_J(s)}{c}\left[3\vec{e}_r[\vec{e}_r\cdot\vec{m}_\infty]-\vec{m}_\infty\right]+o(s f_\rho(s))\right.$$

$$\left.+o(s f_J(s))\right\} \qquad (7.29)$$

This is similar to the result for the second order electric field (equation 7.25) with the roles of the electric and magnetic dipole moments interchanged. Compare the second order magnetic field to the far magnetic field (equation 6.71) for $s \to 0$ by rearranging terms as

$$\frac{c}{s}[4\pi r e^{\gamma r}]\tilde{\vec{H}}_1(\vec{r}) = s\left\{-\vec{e}_r\times\tilde{\vec{p}}+\frac{1}{c}\,\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{m}}]+O(s f_\rho(s))+O(s f_J(s))\right\}$$

$$(7.30)$$

$$[4\pi r^2 e^{\gamma r}]\tilde{\vec{H}}_2(\vec{r}) = s\left\{-\vec{e}_r\times\tilde{\vec{p}}+\frac{1}{c}\left[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{m}}]-\tilde{\vec{m}}\right]+O(s f_\rho(s))+O(s f_J(s))\right\}$$

In these two expressions the terms involving the electric dipole moment are the same; the terms involving the magnetic dipole moment have the same frequency dependence and roughly the same magnitude with the only difference being in the longitudinal part in the $\vec{H}_2$ term where the $\vec{H}_1$ term has none. Thus ω_1 and r_1 can be used with the magnetic field as a criterion for the range of validity of ω and r respectively for the far magnetic field to be dominant and when the second order magnetic field should be included.

Near the end of the previous section on the far potentials and far fields we considered a special case where both ρ and $\vec{J}$ behaved as step functions for their late-time and low-frequency characteristics. As in equations 6.72 through 6.76 the electric and magnetic moments are chosen to behave like 1/s times their late-time values such as $\vec{p}(\infty)$ and $\vec{m}(\infty)$ (not to be confused with $\vec{p}_\infty$ and $\vec{m}_\infty$ which are related to the low-frequency asymptotic forms). For this special form of low-frequency excitation then for $s \to 0$ the second order potentials are

$$\tilde{\Phi}_2(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^2}\left\{\frac{1}{\varepsilon_o}\,\frac{1}{s}\,\vec{e}_r\cdot\vec{p}(\infty) + o\left(\frac{1}{s}\right)\right\}$$

$$\tilde{\vec{A}}_2(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^2}\left\{-\mu_o\,\frac{1}{s}\,\vec{e}_r\times\vec{m}(\infty) + o\left(\frac{1}{s}\right)\right\} \tag{7.31}$$

Note for step-function-like dipole moments the second order potentials also behave like step functions at low frequencies. For $s \to 0$ the second order fields for this special excitation behave as

$$\tilde{\vec{E}}_2(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^2}\left\{Z_o\left[3\vec{e}_r[\vec{e}_r\cdot\vec{p}(\infty)] - \vec{p}(\infty)\right] + \mu_o\vec{e}_r\times\vec{m}(\infty) + o(1)\right\}$$

$$\tilde{\vec{H}}_2(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^2}\left\{-\vec{e}_r\times\vec{p}(\infty) + \frac{1}{c}\left[3\vec{e}_r[\vec{e}_r\cdot\vec{m}(\infty)] - \vec{m}(\infty)\right] + o(1)\right\} \tag{7.32}$$

For step-function-like electric and/or magnetic dipoles the second order fields go to a constant at low frequencies. Compare this with the far fields for the same kind of excitation (equations 6.78); the far fields go to zero as $s \to 0$ in a manner proportional to s. This result points out the importance of the second order fields (and perhaps higher orders) for the low frequency content of a radiated waveform, even at large r. This result also shows that for step-function electric and/or magnetic dipoles (at least in late-time behavior) the low-frequency content of the sum of the first and second order fields falls off as r^{-2} at large r; this also implies an r^{-2} fall off of the complete time integral of the sum of the first and second order fields. Another important low-frequency term will be added when we consider the third order fields.

As discussed in the previous section one does not need ρ and $\vec{J}$ to be zero for $t < t_o$ for the results to apply as long as they are constants for $t < t_o$. Static terms as in equations 6.79 and 6.80 can be defined and subtracted from ρ and $\vec{J}$ so that the difference quantities can be treated with two-sided Laplace transform for frequency-domain considerations. The static fields and potentials associated with the constant ρ and $\vec{J}$ terms can then be added to the time domain results for the difference quantities.

VIII. The Third Terms in the Asymptotic Expansion of the Potentials and Fields for Large r: Order r^{-3}

In this section we consider the third terms in the asymptotic expansions for large r, i.e. the terms proportional to $e^{-\gamma r} r^{-3}$. We call these terms the third order potentials and third order fields and denote them with a subscript 3.

Using the terms proportional to $e^{-\gamma r} r^{-3}$ in the Green's function (equation 5.8) and its gradient (equation 5.12) for $r \to \infty$ we can find the third order terms. The third order potentials can be written from equations 4.4 and 4.1 as

$$\tilde{\vec{A}}_3(\vec{r}) = \mu_o \frac{e^{-\gamma r}}{4\pi r^3} \int_{V'} e^{\gamma \vec{e}_r \cdot \vec{r}\,'} \left\{ -\frac{r'^2}{2} + \frac{3}{2}(\vec{e}_r \cdot \vec{r}\,')^2 - \gamma(\vec{e}_r \cdot \vec{r}\,')\left[r'^2 - (\vec{e}_r \cdot \vec{r}\,')^2\right] + \frac{\gamma^2}{8}\left[r'^2 - (\vec{e}_r \cdot \vec{r}\,')^2\right]^2 \right\} \tilde{\vec{J}}(\vec{r}\,')\, dV'$$

(8.1)

$$\tilde{\Phi}_3(\vec{r}) = \frac{1}{\varepsilon_o} \frac{e^{-\gamma r}}{4\pi r^3} \int_{V'} e^{\gamma \vec{e}_r \cdot \vec{r}\,'} \left\{ -\frac{r'^2}{2} + \frac{3}{2}(\vec{e}_r \cdot \vec{r}\,')^2 - \gamma(\vec{e}_r \cdot \vec{r}\,')\left[r'^2 - (\vec{e}_r \cdot \vec{r}\,')^2\right] + \frac{\gamma^2}{8}\left[r'^2 - (\vec{e}_r \cdot \vec{r}\,')^2\right]^2 \right\} \tilde{\rho}(\vec{r}\,')\, dV'$$

From equations 4.21 and 4.15 the third order fields can be written as

$$\tilde{\vec{E}}_3(\vec{r}) = -\frac{1}{\varepsilon_o} \frac{e^{-\gamma r}}{4\pi r^3} \int_{V'} e^{\gamma \vec{e}_r \cdot \vec{r}\,'} \left\{ -\vec{e}_r \left[3\vec{e}_r \cdot \vec{r}\,' - \frac{3}{2}\gamma\left[r'^2 - 3(\vec{e}_r \cdot \vec{r}\,')^2\right] - \frac{3}{2}\gamma^2(\vec{e}_r \cdot \vec{r}\,')\left[r'^2 - (\vec{e}_r \cdot \vec{r}\,')^2\right] + \frac{\gamma^3}{8}\left[r'^2 - (\vec{e}_r \cdot \vec{r}\,')^2\right]^2 \right] + \vec{r}\,'\left[1 + 2\gamma \vec{e}_r \cdot \vec{r}\,' - \frac{\gamma^2}{2}\left[r'^2 - (\vec{e}_r \cdot \vec{r}\,')^2\right]\right] \right\} \tilde{\rho}(\vec{r}\,')\, dV'$$

$$-\mu_o s \frac{e^{-\gamma r}}{4\pi r^3} \int_{V'} e^{\gamma \vec{e}_r \cdot \vec{r}\,'} \left\{ -\frac{r'^2}{2} + \frac{3}{2}(\vec{e}_r \cdot \vec{r}\,')^2 - \gamma(\vec{e}_r \cdot \vec{r}\,')\left[r'^2 - (\vec{e}_r \cdot \vec{r}\,')^2\right] \right.$$

$$+\frac{\gamma^2}{8}\left[r'^2-(\vec{e}_r\cdot\vec{r}')^2\right]^2\Big\}\tilde{\vec{J}}(\vec{r}')dV'$$

(8.2)

$$\tilde{\vec{H}}_3(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^3}\int_{V'} e^{\gamma\vec{e}_r\cdot\vec{r}'}\Big\{-\vec{e}_r\Big[3\vec{e}_r\cdot\vec{r}'-\frac{3}{2}\gamma\left[r'^2-3(\vec{e}_r\cdot\vec{r}')^2\right]$$

$$-\frac{3}{2}\gamma^2(\vec{e}_r\cdot\vec{r}')\left[r'^2-(\vec{e}_r\cdot\vec{r}')^2\right]+\frac{\gamma^3}{8}\left[r'^2-(\vec{e}_r\cdot\vec{r}')^2\right]^2\Big]$$

$$+\vec{r}'\left[1+2\gamma\vec{e}_r\cdot\vec{r}'-\frac{\gamma^2}{2}\left[r'^2-(\vec{e}_r\cdot\vec{r}')^2\right]\right]\Big\}\times\tilde{\vec{J}}(\vec{r}')dV'$$

These results for the third order potentials and fields are rather complex and can be readily expressed in time-domain form as done in previous instances. However we are primarily interested in the low-frequency content of these third order terms as compared to the first and second order potentials and fields.

As in previous sections the charge density and current density are assumed to have the asymptotic forms for $s \to 0$ as

$$\tilde{\rho}(\vec{r}') = f_\rho(s)\rho_\infty(\vec{r}') + o(f_\rho(s))$$

(8.3)

$$\tilde{\vec{J}}(\vec{r}') = f_J(s)\vec{J}_\infty(\vec{r}') + o(f_J(s))$$

Considering first the third order scalar potential for $s \to 0$ we have, expanding the exponential in the integrand,

$$\tilde{\Phi}_3(\vec{r}) = \frac{1}{\varepsilon_0}\frac{e^{-\gamma r}}{4\pi r^3}\int_{V'}\Big\{-\frac{r'^2}{2}+\frac{3}{2}(\vec{e}_r\cdot\vec{r}')^2+O(s)\Big\}\tilde{\rho}(\vec{r}')dV'$$

$$= \frac{e^{-\gamma r}}{4\pi r^3}\Big\{\frac{1}{2}\frac{1}{\varepsilon_0}\vec{e}_r\cdot[\vec{e}_r\cdot\tilde{\vec{\vec{Q}}}'']+O(sf_\rho(s))\Big\}$$

$$= \frac{e^{-\gamma r}}{4\pi r^3}\Big\{\frac{3}{2}\frac{1}{\varepsilon_0}\vec{e}_r\cdot[\vec{e}_r\cdot\tilde{\vec{\vec{Q}}}]-\frac{1}{2}\frac{1}{\varepsilon_0}\tilde{q}'+O(sf_\rho(s))\Big\}$$

$$= \frac{e^{-\gamma r}}{4\pi r^3}\left\{\frac{3}{2}\frac{1}{\varepsilon_o} f_\rho(s)\vec{e}_r\cdot[\vec{e}_r\cdot\vec{\vec{Q}}_\infty]-\frac{1}{2}\frac{1}{\varepsilon_o} f_\rho(s)q'_\infty+o(f_\rho(s))\right\} \quad (8.4)$$

The leading terms (neglecting the delay) are electrostatic terms; they are electric quadrupole moment terms and have the same frequency characteristics (static) as the leading electric dipole term in the second order scalar potential. Thus provided that the electric dipole moment does not have a smaller low frequency content than ρ (equations 8.3) so that $\vec{p}_\infty \neq \vec{0}$, then the second order scalar potential dominates the third order scalar potential at large r for low frequencies. This is because the static term has already appeared in the second order scalar potential.

Next consider the third order vector potential which for $s \to 0$ can be written as

$$\tilde{\vec{A}}_3(\vec{r}) = \mu_o \frac{e^{-\gamma r}}{4\pi r^3}\int_{V'}\left\{-\frac{r'^2}{2}+\frac{3}{2}(\vec{e}_r\cdot\vec{r}')^2+O(s)\right\}\tilde{\vec{J}}(\vec{r}')dV'$$

$$= \frac{e^{-\gamma r}}{4\pi r^3}\left\{\mu_o \int_{V'}\left[\frac{3}{2}(\vec{e}_r\cdot\vec{r}')^2-\frac{r'^2}{2}\right]\tilde{\vec{J}}(\vec{r}')dV'+O(sf_J(s))\right\}$$

$$= \frac{e^{-\gamma r}}{4\pi r^3}\left\{\mu_o f_J(s)\int_{V'}\left[\frac{3}{2}(\vec{e}_r\cdot\vec{r}')^2-\frac{r'^2}{2}\right]\vec{J}_\infty(\vec{r}')dV'+o(f_J(s))\right\} \quad (8.5)$$

The leading term at low frequencies is a moment of the current density of higher order than the magnetic dipole moment. The low frequency content of $\vec{A}_3$ is proportional to that of $\vec{J}$ so this can be considered a magnetostatic term. If the magnetic dipole term has $\vec{m}_\infty \neq \vec{0}$ then the second order vector potential dominates the third order vector potential. If $\vec{m}_\infty = \vec{0}$ such that the low frequency $\vec{J}$ does not give a magnetic dipole with the same low frequency content, then one might consider electric moments. Thus in equation 8.5 part of the integral over $\vec{J}$ could be transformed into one over ρ with a factor of s added from the equation of continuity (much as in the derivation of the electric quadrupole as an integral over $\vec{J}$). This would make $\vec{A}_3$ proportional to $s\tilde{\rho}$ at low frequencies. Note that $\vec{A}_2$ in equation 7.18 has the same low frequency dependence on its electric quadrupole term. Thus even in this case $\vec{A}_2$ would still dominate $\vec{A}_3$ at low frequencies. There are other combinations of non zero moments, however, that one might want to consider for the relative low frequency content of $\vec{A}_2$ and $\vec{A}_3$.

The third order electric field for $s \to 0$ can be written as

$$\tilde{\vec{E}}_3(\vec{r}) = -\frac{1}{\varepsilon_o}\frac{e^{-\gamma r}}{4\pi r^3}\int_{V'}\{-\vec{e}_r[3\vec{e}_r\cdot\vec{r}\,']+\vec{r}\,'+O(s)\}\tilde{\rho}(\vec{r}\,')dV'$$

$$- \mu_o \frac{e^{-\gamma r}}{4\pi r^3} O(sf_J(s))$$

$$= \frac{e^{-\gamma r}}{4\pi r^3}\left\{\frac{1}{\varepsilon_o}[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{p}}]-\tilde{\vec{p}}]+O(sf_\rho(s))+O(sf_J(s))\right\}$$

$$= \frac{e^{-\gamma r}}{4\pi r^3}\left\{\frac{f_\rho(s)}{\varepsilon_o}[3\vec{e}_r[\vec{e}_r\cdot\vec{p}_\infty]-\vec{p}_\infty]+o(f_\rho(s))+O(sf_J(s))\right\} \tag{8.6}$$

The leading term, neglecting the delay, is an electrostatic field term. Assuming $\vec{p}_\infty \neq \vec{0}$ then at low frequencies the third order electric field is proportional to $\tilde{\vec{p}}$ neglecting the terms proportional to $sf_J(s)$ or higher order. Comparing this result to the second order electric field which is proportional to $s\tilde{\vec{p}}$ at low frequencies we see that the third order electric field can dominate the second order electric field. For comparison with equations 7.26 we write for $s \to 0$

$$\frac{s}{c}[4\pi r^3 e^{\gamma r}]\tilde{\vec{E}}_3(\vec{r}) = Z_o s\{3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{p}}]-\tilde{\vec{p}}+O(sf_\rho(s))+O(sf_J(s))\} \tag{8.7}$$

If we consider just the electric dipole term and neglect the magnetic dipole term then for $\omega < \omega_1$ at fixed r, or for $r > r_1$ at fixed ω, the third order electric field dominates both the first and second order electric fields. With the third order electric field we have reached the first electrostatic term with the exception of the monopole charge which must be completely time independent and is not included in the frequency-domain expansions. If on the other hand the electric dipole term is made negligible with $\vec{p}_\infty = \vec{0}$ such that the magnetic dipole term is dominant in $\tilde{\vec{E}}_2$ and the first term in $\tilde{\vec{E}}_3$ is absent, then $\tilde{\vec{E}}_3$ will not necessarily dominate $\tilde{\vec{E}}_2$ at low frequencies.

The third order magnetic field is given for $s \to 0$ by

$$\tilde{\vec{H}}_3(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^3}\int_{V'}\{-\vec{e}_r[3\vec{e}_r\cdot\vec{r}\,']+\vec{r}\,'+O(s)\}\times\tilde{\vec{J}}(\vec{r}\,')\,dV'$$

$$= \frac{e^{-\gamma r}}{4\pi r^3}\left\{-\frac{3}{2}\,s\vec{e}_r\times[\vec{e}_r\cdot\tilde{\vec{\vec{Q}}}]+3\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{m}}]+2\tilde{\vec{m}}+O(sf_J(s))\right\}$$

$$= \frac{e^{-\gamma r}}{4\pi r^3}\{3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{m}}]-\tilde{\vec{m}}+O(sf_\rho(s))+O(sf_J(s))\}$$

$$= \frac{e^{-\gamma r}}{4\pi r^3}\left\{f_J(s)\left[3\vec{e}_r[\vec{e}_r\cdot\vec{m}_\infty]-\vec{m}_\infty\right]+O(sf_\rho(s))+o(f_J(s))\right\} \qquad (8.8)$$

The leading term, neglecting the delay, is a magnetostatic field term. Neglecting higher order terms in ρ and $\vec{J}$ the third order magnetic field is proportional to $\tilde{\vec{m}}$ if it is non zero. For comparison with the first and second order magnetic fields as $s \to 0$ as in equations 7.30 we write

$$\frac{s}{c}[4\pi r^3 e^{\gamma r}]\tilde{\vec{H}}_3(\vec{r}) = s\left\{\frac{1}{c}\left[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{m}}]-\tilde{\vec{m}}\right]+O(sf_\rho(s))+O(sf_J(s))\right\} \qquad (8.9)$$

Considering just the magnetic dipole term and neglecting the electric dipole term and other higher order terms then for $\omega < \omega_1$ at fixed r, or for $r > r_1$ at fixed ω the third order magnetic field dominates the first and second order magnetic fields. If on the other hand the magnetic dipole moment were made negligible at low frequencies then $\tilde{\vec{H}}_3$ will not necessarily dominate $\tilde{\vec{H}}_2$ at low frequencies.

For the third order terms we again consider (as we have done for the first and second order terms) the special case where both ρ and $\vec{J}$ behave as step functions for their late-time and low-frequency characteristics. Equations 6.72 through 6.76 list various electromagnetic parameters for this case in terms of the late-time values of the quantities such as the late time dipole moments $\vec{p}(\infty)$ and $\vec{m}(\infty)$. For this special type of low-frequency excitation the third order potentials for $s \to 0$ are

$$\tilde{\Phi}_3(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^3}\left\{\frac{3}{2}\,\frac{1}{\varepsilon_o}\,\frac{1}{s}\,\vec{e}_r[\vec{e}_r\cdot\vec{\vec{Q}}(\infty)] - \frac{1}{2}\,\frac{1}{\varepsilon_o}\,q'(\infty) + o\left(\frac{1}{s}\right)\right\}$$

$$\tilde{\vec{A}}_3(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^3}\left\{\mu_o\,\frac{1}{s}\int_{V'}\left[\frac{3}{2}(\vec{e}_r\cdot\vec{r}\,')^2 - \frac{r'^2}{2}\right]\vec{J}(\vec{r}\,',\infty)\,dV' + o\left(\frac{1}{s}\right)\right\} \tag{8.10}$$

These are step function results for the low-frequency content which involve higher order moments of ρ and $\vec{J}$ than the dipole moments; the dipole moments give the step-function-like results to the second order potentials in equations 7.31 for this case of step excitation. For $s \to 0$ the third order fields for this special excitation behave as

$$\tilde{\vec{E}}_3(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^3}\left\{\frac{1}{s\varepsilon_o}\left[3\vec{e}_r[\vec{e}_r\cdot\vec{p}(\infty)] - \vec{p}(\infty)\right] + o\left(\frac{1}{s}\right)\right\}$$

$$\tilde{\vec{H}}_3(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^3}\left\{\frac{1}{s}\left[3\vec{e}_r[\vec{e}_r\cdot\vec{m}(\infty)] - \vec{m}(\infty)\right] + o\left(\frac{1}{s}\right)\right\} \tag{8.11}$$

Thus for this special case the third order fields have step function behavior at low frequencies and late times with the third order electric field proportional to the electric dipole moment and the third order magnetic field proportional to the magnetic dipole moment. This result differs from that for the second order fields in equations 7.32 where the leading terms are proportional to both $\vec{p}(\infty)$ and $\vec{m}(\infty)$ for both second order electric field and second order magnetic field.

In considering the third order terms we then find that the third order potentials do not add a new dominant term at low frequencies if the electric and magnetic dipole moments are of the same frequency dependence as the charge and current densities. However the third order fields do add important new terms in the low-frequency limit with the third order electric field proportional to the electric dipole moment and the third order magnetic field proportional to the magnetic dipole moment. The static fields first appear in the third order terms while the static potentials first appear in the second order terms.

IX. The Potentials and Fields in the Low-Frequency Limit

So far we have considered the potentials and fields associated with sources confined to a source region V' with r_0 as the maximum r' on S', asymptotically expanded the potentials and fields for $r \to \infty$ including the terms up through $e^{-\gamma r} r^{-3}$, and considered the low-frequency forms of each of the first three terms in this asymptotic expansion for large r. This analysis shows that for sufficiently low frequencies, even at large r, other terms besides the far potentials and fields can be significant, including both second and third order terms for $\omega < \omega_1 \equiv c/r$ if we are considering the fields, and including second order terms if we are considering the potentials.

Now let us consider the low-frequency content of the potentials and fields at large r from a different approach. In this section we take the asymptotic expansions of the potentials and fields as $s \to 0$ after factoring out the delay term $e^{-\gamma r}$; then we look at the first few terms for large r. This approach will show us that the leading terms for low frequencies are all contained in the first and second order potentials and the first through third order fields provided we have electric and/or magnetic dipoles as leading moments of the source distribution. This implies that for the asymptotic expansion for large r we do not need more than the first two terms for the potentials and the first three terms for the fields to be assured of a valid representation as $s \to 0$ as well. Again this requires dominant electric and/or magnetic dipole terms at low frequencies which can be assured if $\vec{p}$ has the same low-frequency dependence as ρ, and $\vec{m}$ has the same low-frequency dependence as $\vec{J}$.

In section IV we have the general integrals which give the potentials and fields in terms of the charge and current densities. To evaluate the low-frequency asymptotic form of the fields we have the charge density and current density as $s \to 0$ from equations 6.53 as

$$\tilde{\rho}(\vec{r}\,') = f_\rho(s)\rho_\infty(\vec{r}\,') + o(f_\rho(s))$$
$$\tilde{\vec{J}}(\vec{r}\,') = f_J(s)\vec{J}_\infty(\vec{r}\,') + o(f_J(s)) \tag{9.1}$$

with similar notation for low-frequency forms of the various moments. The Green's function (see equation 4.13) with $e^{-\gamma r}$ factored out can be expanded for $s \to 0$ as

$$\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} = \frac{e^{-\gamma r}}{4\pi r}\,\alpha^{-1/2}e^{-\gamma r[\alpha^{1/2}-1]}$$

$$= \frac{e^{-\gamma r}}{4\pi r}\left\{\alpha^{-1/2}+\frac{sr}{c}[-1+\alpha^{-1/2}]+\left(\frac{sr}{c}\right)^2\left[\frac{\alpha^{1/2}}{2}-1+\frac{\alpha^{-1/2}}{2}\right]+O_s(s^3)\right\} \tag{9.2}$$

where we use O_s and o_s to indicate order symbols for $s \to 0$ to distinguish them from O_r and o_r used for $r \to \infty$. In this section we always consider $s \to 0$ first and second consider $r \to \infty$. As defined in equation 4.14 we have

$$\alpha \equiv \left|\vec{e}_r-\frac{\vec{r}'}{r}\right|^2 = \left[\frac{|\vec{r}-\vec{r}'|}{r}\right]^2 = 1 - 2\,\frac{\vec{e}_r\cdot\vec{r}'}{r} + \left(\frac{r'}{r}\right)^2 \tag{9.3}$$

The gradient of the Green's function (see equation 4.17) can be written for $s \to 0$ as

$$\nabla\left[\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\right] = -\frac{\vec{r}-\vec{r}'}{4\pi|\vec{r}-\vec{r}'|^3}[1+\gamma|\vec{r}-\vec{r}'|]e^{-\gamma|\vec{r}-\vec{r}'|}$$

$$= -\frac{e^{-\gamma r}}{4\pi r^2}\left[\vec{e}_r-\frac{\vec{r}'}{r}\right][\alpha^{-3/2}+\gamma r\alpha^{-1}]e^{-\gamma r[\alpha^{1/2}-1]}$$

$$= -\frac{e^{-\gamma r}}{4\pi r^2}\left[\vec{e}_r-\frac{\vec{r}'}{r}\right]\left\{\alpha^{-3/2}+\frac{sr}{c}\,\alpha^{-3/2}+\left(\frac{sr}{c}\right)^2\left[-\frac{\alpha^{-1/2}}{2}+\frac{\alpha^{-3/2}}{2}\right]\right.$$

$$\left.+O_s(s^3)\right\} \tag{9.4}$$

These expansions for $s \to 0$ can be substituted into the integrals for the potentials and fields.

Consider first the scalar potential. From equation 4.1 the scalar potential for $s \to 0$ can be written as

$$\tilde{\Phi}(\vec{r}) = \frac{1}{\varepsilon_o} \int_{V'} \tilde{\rho}(\vec{r}\,') \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} \, dV'$$

$$= \frac{1}{\varepsilon_o} \frac{e^{-\gamma r}}{4\pi r} \int_{V'} \left\{ \alpha^{-1/2} + \frac{sr}{c}[-1+\alpha^{-1/2}] + O_s(s^2) \right\} \tilde{\rho}(\vec{r}\,')dV'$$

$$= \tilde{\Phi}_{s_o}(\vec{r}) + \tilde{\Phi}_{s_1}(r) + e^{-\gamma r} O_s\left(s^2 f_\rho(s)\right) \qquad (9.5)$$

where we have labelled the terms by use of subscripts s_n where n is the power of s in the expansion in the integrand so as to avoid confusing these terms with the first, second, etc. order terms in the expansion for large r. Evaluating these terms we consider the behavior of each term for large r. The s_o term is

$$\tilde{\Phi}_{s_o}(\vec{r}) = \frac{1}{\varepsilon_o} \frac{e^{-\gamma r}}{4\pi r} \int_{V'} \alpha^{-1/2} \tilde{\rho}(\vec{r}\,')dV' = \frac{e^{-\gamma r}}{\varepsilon_o} \int_{V'} \frac{\tilde{\rho}(\vec{r}\,')dV'}{4\pi|\vec{r}-\vec{r}\,'|} \qquad (9.6)$$

Thus the s_o term is the electrostatic term with a delay included. Expanding $\alpha^{-1/2}$ from equation 5.7 as $r \to \infty$ gives the result

$$\tilde{\Phi}_{s_o}(\vec{r}) = \frac{1}{\varepsilon_o} \frac{e^{-\gamma r}}{4\pi r} \int_{V'} \left\{ 1 + \frac{\vec{e}_r \cdot \vec{r}\,'}{r} + \frac{1}{r^2}\left[\frac{3}{2}(\vec{e}_r \cdot \vec{r}\,')^2 - \frac{r'^2}{2}\right] + O_r(r^{-3}) \right\} \tilde{\rho}(\vec{r}\,')dV'$$

$$= e^{-\gamma r} \left\{ \frac{1}{4\pi r} \frac{1}{\varepsilon_o} \tilde{Q} + \frac{1}{4\pi r^2} \frac{1}{\varepsilon_o} \vec{e}_r \cdot \tilde{\vec{p}} + \frac{1}{4\pi r^3} \frac{1}{\varepsilon_o} \left[\frac{3}{2} \vec{e}_r \cdot [\vec{e}_r \cdot \tilde{\vec{\vec{Q}}}] - \frac{1}{2} \tilde{q}'\right] \right.$$

$$\left. + f_\rho(s) O_r(r^{-4}) \right\}$$

$$= e^{-\gamma r} \left\{ \frac{1}{4\pi r^2} \frac{1}{\varepsilon_o} \vec{e}_r \cdot \tilde{\vec{p}} + f_\rho(s) O_r(r^{-3}) \right\} \qquad (9.7)$$

where the r^{-1} term is zero because the monopole charge $\tilde{Q}$ is zero. Note that we include the frequency dependence $f_\rho(s)$ of ρ with the O_r term to show the frequency dependence of that term as $s \to 0$ as well as its behavior for large r, i.e. as $s \to 0$ this term is $O_s(f_\rho(s))$. The s_1 term for $r \to \infty$ is just

$$\tilde{\Phi}_{s_1}(\vec{r}) = Z_o s \frac{e^{-\gamma r}}{4\pi r} \int_{V'} r[-1+\alpha^{-1/2}]\tilde{\rho}(\vec{r}')dV'$$

$$= Z_o s \frac{e^{-\gamma r}}{4\pi r} \int_{V'} \left\{ \vec{e}_r \cdot \vec{r}' + \frac{1}{r}\left[\frac{3}{2}(\vec{e}_r \cdot \vec{r}')^2 - \frac{r'^2}{2}\right] + O_r(r^{-2}) \right\} \tilde{\rho}(\vec{r}')dV'$$

$$= se^{-\gamma r}\left\{ \frac{1}{4\pi r} Z_o \vec{e}_r \cdot \tilde{\vec{p}} + \frac{1}{4\pi r^2} Z_o \left[\frac{3}{2}\, \vec{e}_r \cdot [\vec{e}_r \cdot \tilde{\vec{\vec{Q}}}] - \frac{1}{2}\, \tilde{q}'\right] + f_\rho(s) O_r(r^{-3}) \right\}$$

$$= se^{-\gamma r}\left\{ \frac{1}{4\pi r} Z_o \vec{e}_r \cdot \tilde{\vec{p}} + f_\rho(s) O_r(r^{-2}) \right\} \tag{9.8}$$

As $s \to 0$ we have

$$\tilde{\Phi}_{s_o}(\vec{r}) = O_s(f_\rho(s)) \ , \quad \tilde{\Phi}_{s_1} = O_s(sf_\rho(s)) \tag{9.9}$$

As $s \to 0$ the static scalar potential (the s_o term) is dominant for all fixed $r > r_o$; as $r \to \infty$ the electric dipole moment dominates the static scalar potential. All that is needed is for $\vec{p}_\infty \neq \vec{0}$. The leading term in the static scalar potential goes like r^{-2}. The electric dipole moment also dominates the s_1 term which goes like r^{-1} but is proportional to $s\tilde{\vec{p}}$. The results for $s \to 0$ might be summarized as

$$\tilde{\Phi}(\vec{r}) = e^{-\gamma r}\left\{ \left\{ \frac{1}{4\pi r^2} \frac{1}{\varepsilon_o} \vec{e}_r \cdot \tilde{\vec{p}} + f_\rho(s) O_r(r^{-3}) \right\} \right.$$

$$\left. + s\left\{ \frac{1}{4\pi r} Z_o \vec{e}_r \cdot \tilde{\vec{p}} + f_\rho(s) O_r(r^{-2}) \right\} + O_s\left(s^2 f_\rho(s)\right) \right\} \tag{9.10}$$

where we take $r \to \infty$ after first taking $s \to 0$. Note that we have included the frequency dependence $f_\rho(s)$ of the charge density for $s \to 0$ along with the remainder terms for large r.

Next consider the vector potential. From equations 4.4 for $s \to 0$ the vector potential can be written as

$$\tilde{\vec{A}}(\vec{r}) = \mu_o \int_{V'} \tilde{\vec{J}}(\vec{r}\,') \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} \, dV'$$

$$= \mu_o \frac{e^{-\gamma r}}{4\pi r} \int_{V'} \left\{ \alpha^{-1/2} + \frac{sr}{c}[-1+\alpha^{-1/2}] + O(s^2) \right\} \tilde{\vec{J}}(\vec{r}\,') \, dV'$$

$$= \tilde{\vec{A}}_{s_o}(\vec{r}) + \tilde{\vec{A}}_{s_1}(\vec{r}) + e^{-\gamma r} O_s\left(s^2 f_J(s)\right) \tag{9.11}$$

Again the subscript s_n is used to identify the terms in the expansion for $s \to 0$ with the powers of s as listed in the integrand; it does not necessarily indicate the power of s appearing as a coefficient with a moment because in transforming $\tilde{\vec{J}}$ into a form which uses ρ for the electric moments an additional factor of s enters, basically from the continuity equation. The s_n subscript is merely a convenient label to consider different terms in the asymptotic expansion for $s \to 0$.

The s_o term for the vector potential is

$$\tilde{\vec{A}}_{s_o}(\vec{r}) = \mu_o \frac{e^{-\gamma r}}{4\pi r} \int_{V'} \alpha^{-1/2} \tilde{\vec{J}}(\vec{r}\,') \, dV' = \mu_o e^{-\gamma r} \int_{V'} \frac{\tilde{\vec{J}}(\vec{r}\,') \, dV'}{4\pi|\vec{r}-\vec{r}\,'|} \tag{9.12}$$

This can be considered as a magnetostatic vector potential with a delay factor included, except that for magnetostatic problems one normally considers a divergenceless current density ($\tilde{\vec{J}}_h$ only) so that there is no time-changing charge density. Thus the s_o term for the vector potential is only static in its form which is similar to equation 9.6 for the scalar potential. Expanding $\alpha^{-1/2}$ in inverse powers of r we can separate the various parts of the s_o term into electric and magnetic moments as

$$\tilde{\vec{A}}_{s_o}(\vec{r}) = \mu_o \frac{e^{-\gamma r}}{4\pi r} \int_{V'} \left\{ 1 + \frac{\vec{e}_r \cdot \vec{r}\,'}{r} + O_r(r^{-2}) \right\} \tilde{\vec{J}}(\vec{r}\,') \, dV'$$

$$= e^{-\gamma r} \left\{ \frac{\mu_o}{4\pi r} s\tilde{\vec{p}} + \frac{\mu_o}{4\pi r^2} \left[\frac{s}{2} \vec{e}_r \cdot \tilde{\vec{\vec{Q}}} - \vec{e}_r \times \tilde{\vec{m}} \right] + f_J(s) O_r(r^{-3}) \right\}$$

$$= e^{-\gamma r} \left\{ \frac{\mu_o}{4\pi r} s\tilde{\vec{p}} - \frac{\mu_o}{4\pi r^2} \vec{e}_r \times \tilde{\vec{m}} + s f_\rho(s) O_r(r^{-2}) + f_J(s) O_r(r^{-3}) \right\} \tag{9.13}$$

Thus the s_o term has both electric and magnetic moments where the first electric moments have a factor of s in their coefficients. Note that the electric dipole term with its r^{-1} dependence is associated with the far vector potential while the magnetic dipole term can be considered the first term in the static vector potential.

The s_1 term for the vector potential is

$$\tilde{\vec{A}}_{s_1} = \frac{\mu_o}{c} s \frac{e^{-\gamma r}}{4\pi r} \int_{V'} r[-1+\alpha^{-1/2}]\tilde{\vec{J}}(\vec{r}\,')dV'$$

$$= \frac{\mu_o}{c} s \frac{e^{-\gamma r}}{4\pi r} \int_{V'} \left\{ \vec{e}_r\cdot\vec{r}\,'+O_r(r^{-1}) \right\} \tilde{\vec{J}}(\vec{r}\,')dV'$$

$$= se^{-\gamma r}\left\{ \frac{1}{4\pi r} \frac{\mu_o}{c}\left[\frac{s}{2}\, \vec{e}_r\cdot\tilde{\vec{\vec{Q}}}-\vec{e}_r\times\tilde{\vec{m}}\right]+f_J(s)O_r(r^{-2}) \right\}$$

$$= se^{-\gamma r}\left\{ -\frac{1}{4\pi r} \frac{\mu_o}{c}\, \vec{e}_r\times\tilde{\vec{m}}+sf_\rho(s)O_r(r^{-1})+f_J(s)O_r(r^{-2}) \right\} \tag{9.14}$$

Thus the s_1 term has both electric and magnetic moments also. Combining the results for the s_o and s_1 terms the results for $s \to 0$ can be summarized

$$\tilde{\vec{A}}(\vec{r}) = e^{-\gamma r}\Bigg\{ \left\{ -\frac{\mu_o}{4\pi r^2}\, \vec{e}_r\times\tilde{\vec{m}}+f_J(s)O_r(r^{-3}) \right\}$$

$$+s\left\{ \frac{\mu_o}{4\pi r}\,\tilde{\vec{p}}-\frac{1}{4\pi r}\frac{\mu_o}{c}\,\vec{e}_r\times\tilde{\vec{m}}+f_\rho(s)O_r(r^{-2})+f_J(s)O_r(r^{-2}) \right\}$$

$$+O_s\left(s^2 f_\rho(s)\right)+O_s\left(s^2 f_J(s)\right)\Bigg\} \tag{9.15}$$

The magnetic dipole times r^{-2} dominates the first term while both electric and magnetic dipole moments times r^{-1} dominate the second term with a coefficient s.

Now go on to the electric field. For $s \to 0$ the electric field can be written from equation 4.21 as

$$\tilde{\vec{E}}(\vec{r}) = -\frac{1}{\varepsilon_o}\int_{V'}\tilde{\rho}(\vec{r}')\nabla\left[\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\right]dV' - \mu_o s\int_{V'}\tilde{\vec{J}}(\vec{r}')\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\,dV'$$

$$= \frac{1}{\varepsilon_o}\frac{e^{-\gamma r}}{4\pi r^2}\int_{V'}\left[\vec{e}_r-\frac{\vec{r}'}{r}\right]\left\{\alpha^{-3/2}+\frac{sr}{c}\,\alpha^{-3/2}+\left(\frac{sr}{c}\right)^2\left[-\frac{\alpha^{-1/2}}{2}+\frac{\alpha^{-3/2}}{2}\right]\right.$$

$$\left.+O_s(s^3)\right\}\tilde{\rho}(\vec{r}')dV'$$

$$-\ \mu_o\frac{e^{-\gamma r}}{4\pi r}\int_{V'}\left\{s\alpha^{-1/2}+\frac{s^2r}{c}[-1+\alpha^{-1/2}]+O_s(s^2)\right\}\tilde{\vec{J}}(\vec{r}')dV'$$

$$= \tilde{\vec{E}}_{s_o}(\vec{r})+\tilde{\vec{E}}_{s_1}(\vec{r})+\tilde{\vec{E}}_{s_2}(\vec{r})+e^{-\gamma r}\left[O_s\left(s^3f_\rho(s)\right)+O_s\left(s^3f_J(s)\right)\right] \quad (9.16)$$

where again s_n refers to the power of s in the integrands above.

The s_o term for the electric field is

$$\tilde{\vec{E}}_{s_o}(\vec{r}) = \frac{1}{\varepsilon_o}\frac{e^{-\gamma r}}{4\pi r^2}\int_{V'}\left[\vec{e}_r-\frac{\vec{r}'}{r}\right]\alpha^{-3/2}\tilde{\rho}(\vec{r}')dV'$$

$$= \frac{1}{\varepsilon_o}e^{-\gamma r}\int_{V'}\frac{\vec{r}-\vec{r}'}{4\pi|\vec{r}-\vec{r}'|^3}\,\tilde{\rho}(\vec{r}')dV' \quad (9.17)$$

This is the electrostatic field with a delay factor. From equation 5.10 expand $\alpha^{-3/2}$ for $r \to \infty$ to give

$$\tilde{\vec{E}}_{s_o}(\vec{r}) = \frac{1}{\varepsilon_o}\frac{e^{-\gamma r}}{4\pi r^2}\int_{V'}\left[\vec{e}_r-\frac{\vec{r}'}{r}\right]\left\{1+\frac{3}{r}\,\vec{e}_r\cdot\vec{r}'+\frac{1}{r^2}\left[\frac{15}{2}(\vec{e}_r\cdot\vec{r}')^2-\frac{3}{2}\,r'^2\right]\right.$$

$$\left.+O_r(r^{-3})\right\}\tilde{\rho}(\vec{r}')dV'$$

$$= e^{-\gamma r}\left\{\frac{1}{4\pi r^2}\,\frac{1}{\varepsilon_o}\,\vec{e}_r\tilde{Q}+\frac{1}{4\pi r^3}\,\frac{1}{\varepsilon_o}[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{p}}]-\tilde{\vec{p}}]\right.$$

$$\left.+\frac{1}{4\pi r^4}\,\frac{1}{\varepsilon_o}\left[\frac{15}{2}\,\vec{e}_r[\vec{e}_r\cdot[\vec{e}_r\cdot\tilde{\vec{\vec{Q}}}]]-3\vec{e}_r\cdot\tilde{\vec{\vec{Q}}}-\frac{3}{2}\,\vec{e}_r\tilde{q}'\right]+f_\rho(s)O_r(r^{-5})\right\}$$

$$= e^{-\gamma r}\left\{\frac{1}{4\pi r^3}\,\frac{1}{\varepsilon_o}[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{p}}]-\tilde{\vec{p}}]+f_\rho(s)O_r(r^{-4})\right\} \qquad (9.18)$$

where the monopole charge $\tilde{Q}$ is zero. The s_1 term for $r\to\infty$ is

$$\tilde{\vec{E}}_{s_1}(\vec{r}) = Z_o s\,\frac{e^{-\gamma r}}{4\pi r}\int_{V'}\left[\vec{e}_r-\frac{\vec{r}'}{r}\right]\alpha^{-3/2}\tilde{\rho}(\vec{r}')\,dV'$$

$$-\mu_o s\,\frac{e^{-\gamma r}}{4\pi r}\int_{V'}\alpha^{-1/2}\tilde{\vec{J}}(\vec{r}')\,dV'$$

$$= Z_o s\,\frac{e^{-\gamma r}}{4\pi r}\int_{V'}\left[\vec{e}_r-\frac{\vec{r}'}{r}\right]\left\{1+\frac{3}{r}\,\vec{e}_r\cdot\vec{r}'+O_r(r^{-2})\right\}\tilde{\rho}(\vec{r}')\,dV'$$

$$-\mu_o s\,\frac{e^{-\gamma r}}{4\pi r}\int_{V'}\left\{1+\frac{\vec{e}_r\cdot\vec{r}'}{r}+O_r(r^{-2})\right\}\tilde{\vec{J}}(\vec{r}')\,dV'$$

$$= se^{-\gamma r}\left\{\frac{Z_o}{4\pi r}\,\vec{e}_r\tilde{Q}+\frac{Z_o}{4\pi r^2}[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{p}}]-\tilde{\vec{p}}]+f_\rho(s)O_r(r^{-3})\right.$$

$$\left.-\frac{\mu_o}{4\pi r}\,s\tilde{\vec{p}}-\frac{\mu_o}{4\pi r^2}\left[\frac{s}{2}\,\vec{e}_r\cdot\tilde{\vec{\vec{Q}}}-\vec{e}_r\times\tilde{\vec{m}}\right]+f_J(s)O_r(r^{-3})\right\}$$

$$= se^{-\gamma r}\left\{\frac{Z_o}{4\pi r^2}[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{p}}]-\tilde{\vec{p}}]-\frac{\mu_o}{4\pi r}\,s\tilde{\vec{p}}+\frac{\mu_o}{4\pi r^2}\,\vec{e}_r\times\tilde{\vec{m}}\right.$$

$$\left.+f_\rho(s)O_r(r^{-3})+sf_\rho(s)O_r(r^{-2})+f_J(s)O_r(r^{-3})\right\} \qquad (9.19)$$

The s_2 term for $r \to \infty$ is

$$\tilde{\vec{E}}_{s_2}(\vec{r}) = \mu_o s^2 \frac{e^{-\gamma r}}{4\pi} \int_{V'} \left[\vec{e}_r - \frac{\vec{r}'}{r}\right]\left[-\frac{\alpha^{-1/2}}{2} + \frac{\alpha^{-3/2}}{2}\right]\tilde{\rho}(\vec{r}')dV'$$

$$- \frac{\mu_o}{c} s^2 \frac{e^{-\gamma r}}{4\pi} \int_{V'} [-1+\alpha^{-1/2}]\tilde{\vec{J}}(\vec{r}')dV'$$

$$= \mu_o s^2 \frac{e^{-\gamma r}}{4\pi} \int_{V'} \left[\vec{e}_r - \frac{\vec{r}'}{r}\right]\left\{\frac{\vec{e}_r\cdot\vec{r}'}{r} + \frac{1}{r^2}\left[3(\vec{e}_r\cdot\vec{r}')^2 - \frac{r'^2}{2}\right]\right.$$

$$\left. + O_r(r^{-3})\right\}\tilde{\rho}(\vec{r}')dV'$$

$$- \frac{\mu_o}{c} s^2 \frac{e^{-\gamma r}}{4\pi} \int_{V'} \left\{\frac{\vec{e}_r\cdot\vec{r}'}{r} + O_r(r^{-2})\right\}\tilde{\vec{J}}(\vec{r}')dV'$$

$$= s^2 e^{-\gamma r}\left\{\frac{\mu_o}{4\pi r}\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{p}}] + \frac{\mu_o}{4\pi r^2}\left[3\vec{e}_r[\vec{e}_r\cdot[\vec{e}_r\cdot\tilde{\vec{\vec{Q}}}]] - \vec{e}_r\cdot\tilde{\vec{\vec{Q}}} - \frac{\tilde{q}'}{2}\right]\right.$$

$$\left. + f_\rho(s)O_r(r^{-3}) - \frac{1}{4\pi r}\frac{\mu_o}{c}\left[\frac{s}{2}\vec{e}_r\cdot\tilde{\vec{\vec{Q}}} - \vec{e}_r\times\tilde{\vec{m}}\right] + f_J(s)O_r(r^{-2})\right\}$$

$$= s^2 e^{-\gamma r}\left\{\frac{\mu_o}{4\pi r}\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{p}}] + \frac{1}{4\pi r}\frac{\mu_o}{c}\vec{e}_r\times\tilde{\vec{m}}\right.$$

$$\left. + f_\rho(s)O_r(r^{-2}) + sf_\rho(s)O_r(r^{-1}) + f_J(s)O_r(r^{-2})\right\} \qquad (9.20)$$

Combining the results for equations 9.16 through 9.20 the electric field for $s \to 0$ (followed by $r \to \infty$) can be summarized by rearranging the terms as

$$\tilde{\vec{E}}(\vec{r}) = e^{-\gamma r}\left\{\left\{\frac{1}{4\pi r^3}\,\frac{1}{\varepsilon_o}[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{p}}]-\tilde{\vec{p}}]+f_\rho(s)O_r(r^{-4})\right\}\right.$$

$$+s\left\{\frac{Z_o}{4\pi r^2}[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{p}}]-\tilde{\vec{p}}]+\frac{\mu_o}{4\pi r^2}\,\vec{e}_r\times\tilde{\vec{m}}+f_\rho(s)O_r(r^{-3})+f_J(s)O_r(r^{-3})\right\}$$

$$+s^2\left\{\frac{\mu_o}{4\pi r}\,\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{p}}]+\frac{1}{4\pi r}\,\frac{\mu_o}{c}\,\vec{e}_r\times\tilde{\vec{m}}+f_\rho(s)O_r(r^{-2})+f_J(s)O_r(r^{-2})\right\}$$

$$\left.+O_s\left(s^3f_\rho(s)\right)+O_s\left(s^3f_J(s)\right)\right\} \tag{9.21}$$

The electric dipole times r^{-3} dominates the first term, both electric and magnetic dipoles times r^{-2} dominate the second term with a coefficient s, and both electric and magnetic dipoles times r^{-1} dominate the third term with a coefficient s^2.

Finally consider the magnetic field which for $s \to 0$ can be written from equation 4.15 as

$$\tilde{\vec{H}}(\vec{r}) = \int_{V'}\nabla\left[\frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}\right]\times\tilde{\vec{J}}(\vec{r}\,')\,dV'$$

$$= -\frac{e^{-\gamma r}}{4\pi r^2}\int_{V'}\left\{\alpha^{-3/2}+\frac{sr}{c}\,\alpha^{-3/2}+\left(\frac{sr}{c}\right)^2\left[-\frac{\alpha^{-1/2}}{2}+\frac{\alpha^{-3/2}}{2}\right]\right.$$

$$\left.+O_s(s^3)\right\}\left[\vec{e}_r-\frac{\vec{r}\,'}{r}\right]\times\tilde{\vec{J}}(\vec{r}\,')\,dV'$$

$$= \tilde{\vec{H}}_{s_o}(\vec{r})+\tilde{\vec{H}}_{s_1}(\vec{r})+\tilde{\vec{H}}_{s_2}(\vec{r})+e^{-\gamma r}O_s\left(s^3f_J(s)\right) \tag{9.22}$$

The s_o term for the magnetic field is

$$\tilde{\vec{H}}_{s_o}(\vec{r}) = -\frac{e^{-\gamma r}}{4\pi r^2}\int_{V'} \alpha^{-3/2}\left[\vec{e}_r - \frac{\vec{r}\,'}{r}\right]\times\tilde{\vec{J}}(\vec{r}\,')\,dV'$$

$$= -e^{-\gamma r}\int_{V'} \frac{\vec{r}-\vec{r}\,'}{4\pi|\vec{r}-\vec{r}\,'|^3}\times\tilde{\vec{J}}(\vec{r}\,')\,dV' \tag{9.23}$$

which can be considered the magnetostatic field with a delay factor provided $\vec{J}$ is a divergenceless current density ($\vec{J}_h$ only) so that there is no time-changing charge density. Expanding the integrand for $r \rightarrow \infty$ we have

$$\tilde{\vec{H}}_{s_o}(\vec{r}) = -\frac{e^{-\gamma r}}{4\pi r^2}\int_{V'}\left\{\vec{e}_r - \frac{\vec{r}\,'}{r} + \frac{3}{r}(\vec{e}_r\cdot\vec{r}\,')\vec{e}_r + O_r(r^{-2})\right\}\times\tilde{\vec{J}}(\vec{r}\,')\,dV'$$

$$= e^{-\gamma r}\left\{-\frac{1}{4\pi r^2}\,s\vec{e}_r\times\tilde{\vec{p}} + \frac{1}{4\pi r^3}\left[2\tilde{\vec{m}} - \frac{3}{2}\,s\vec{e}_r\times[\vec{e}_r\cdot\tilde{\vec{\vec{Q}}}] + 3\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{m}}]\right]\right.$$

$$\left. + f_J(s)O_r(r^{-4})\right\}$$

$$= e^{-\gamma r}\left\{-\frac{1}{4\pi r^2}\,s\vec{e}_r\times\tilde{\vec{p}} + \frac{1}{4\pi r^3}[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{m}}] - \tilde{\vec{m}}] + sf_\rho(s)O_r(r^{-3})\right.$$

$$\left. + f_J(s)O_r(r^{-4})\right\} \tag{9.24}$$

The s_1 term for $r \rightarrow \infty$ is

$$\tilde{\vec{H}}_{s_1}(\vec{r}) = -\frac{s}{c}\,\frac{e^{-\gamma r}}{4\pi r}\int_{V'}\alpha^{-3/2}\left[\vec{e}_r - \frac{\vec{r}\,'}{r}\right]\times\tilde{\vec{J}}(\vec{r}\,')\,dV'$$

$$= -\frac{s}{c}\,\frac{e^{-\gamma r}}{4\pi r}\int_{V'}\left\{\vec{e}_r - \frac{\vec{r}\,'}{r} + \frac{3}{r}(\vec{e}_r\cdot\vec{r}\,')\vec{e}_r + O_r(r^{-2})\right\}\times\tilde{\vec{J}}(\vec{r}\,')\,dV'$$

$$= se^{-\gamma r}\left\{-\frac{1}{4\pi r}\frac{s}{c}\,\vec{e}_r\times\tilde{\vec{p}}+\frac{1}{4\pi r^2}\frac{1}{c}\left[2\tilde{\vec{m}}-\frac{3}{2}\,s\vec{e}_r\times[\vec{e}_r\cdot\tilde{\vec{\vec{Q}}}]+3\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{m}}]\right]\right.$$

$$\left.+f_J(s)O_r(r^{-3})\right\}$$

$$= se^{-\gamma r}\left\{-\frac{1}{4\pi r}\frac{s}{c}\,\vec{e}_r\times\tilde{\vec{p}}+\frac{1}{4\pi r^2}\frac{1}{c}[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{m}}]-\tilde{\vec{m}}]+sf_\rho(s)O_r(r^{-2})\right.$$

$$\left.+f_J(s)O_r(r^{-3})\right\} \tag{9.25}$$

The s_2 term for $r \to \infty$ is

$$\tilde{\vec{H}}_{s_2}(\vec{r}) = -\left(\frac{s}{c}\right)^2\frac{e^{-\gamma r}}{4\pi}\int_{V'}\left[-\frac{\alpha^{-1/2}}{2}+\frac{\alpha^{-3/2}}{2}\right]\left[\vec{e}_r-\frac{\vec{r}'}{r}\right]\times\tilde{\vec{J}}(\vec{r}')\,dV'$$

$$= -\left(\frac{s}{c}\right)^2\frac{e^{-\gamma r}}{4\pi}\int_{V'}\left\{\frac{\vec{e}_r\cdot\vec{r}'}{r}+O_r(r^{-2})\right\}\left[\vec{e}_r-\frac{\vec{r}'}{r}\right]\times\tilde{\vec{J}}(\vec{r}')\,dV'$$

$$= s^2e^{-\gamma r}\left\{-\frac{1}{4\pi r}\frac{1}{c^2}\left[\frac{s}{2}\,\vec{e}_r\times[\vec{e}_r\cdot\tilde{\vec{\vec{Q}}}]-\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{m}}]\right]+f_J(s)O_r(r^{-2})\right\}$$

$$= s^2e^{-\gamma r}\left\{\frac{1}{4\pi r}\frac{1}{c^2}\,\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{m}}]+sf_\rho(s)O_r(r^{-1})+f_J(s)O_r(r^{-2})\right\} \tag{9.26}$$

Combining the results for equations 9.22 through 9.26 and re-arranging terms the magnetic field can be summarized for $s \to 0$ (followed by $r \to \infty$) as

$$\tilde{\vec{H}}(\vec{r}) = e^{-\gamma r}\left\{\left\{\frac{1}{4\pi r^3}[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{m}}]-\tilde{\vec{m}}]+f_J(s)O_r(r^{-4})\right\}\right.$$

$$+s\left\{-\frac{1}{4\pi r^2}\,\vec{e}_r\times\tilde{\vec{p}}+\frac{1}{4\pi r^2}\frac{1}{c}[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{m}}]-\tilde{\vec{m}}]+f_\rho(s)O_r(r^{-3})+f_J(s)O_r(r^{-3})\right\}$$

$$+s^2\left\{-\frac{1}{4\pi r}\frac{1}{c}\vec{e}_r\times\tilde{\vec{p}}+\frac{1}{4\pi r}\frac{1}{c^2}\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{m}}]+f_\rho(s)O_r(r^{-2})+f_J(s)O_r(r^{-2})\right\}$$

$$+O_s\left(s^3f_\rho(s)\right)+O_s\left(s^3f_J(s)\right)\Bigg\} \tag{9.27}$$

The magnetic dipole times r^{-3} dominates the first term, both electric and magnetic dipoles times r^{-2} dominate the second term with a coefficient s, and both electric and magnetic dipoles times r^{-1} dominate the third term with a coefficient s^2. Note the similarity of this result to that for the electric field in equation 9.21. The roles of electric and magnetic dipoles are switched with some sign changes; f_ρ and f_J are also switched.

Equations 9.10, 9.15, 9.21, and 9.27 summarize the results for the potentials and fields for low frequencies, the potentials being written out through terms proportional to s and the fields written out through terms proportional to s^2. Each of these terms is dominated by the electric and/or magnetic dipole moments provided that the electric dipole moment has the same low-frequency behavior as the charge density (i.e. provided $\vec{p}_\infty \neq 0$ for $f_\rho(s) \neq 0$) and that the magnetic dipole moment has the same low-frequency behavior as the current density (i.e. provided $\vec{m}_\infty \neq 0$ for $f_J(s) \neq 0$). The low-frequency behavior of the current and charge densities is as assumed in equations 6.53 but is not completely independent as seen from equation 6.59. There is, of course, the possibility of certain discrete directions ($\vec{e}_r$) for which electric and magnetic dipole terms do not dominate some of the low-frequency field terms at large r because of the manner in which $\vec{e}_r$ combines with $\vec{p}$ and $\vec{m}$.

In section V we found that at large r for all s with $|s|$ less than some positive constant the potentials and fields had order $O_r(r^{-1})$ after $e^{-\gamma r}$ had been factored out. In this section we have first expanded for $s \to 0$. Notice, however, that each of the terms for $s \to 0$ that we have considered (two for the potentials and three for the fields) has been $O_r(r^{-1})$ as $r \to \infty$ (for $|s|$ with an upper bound) or even smaller (i.e. $O_r(r^{-2}$ and $O_r(r^{-3}))$. If we subtract these first two or three terms from the complete integral expressions for the potentials and fields the remainders will also be $O_r(r^{-1})$ as $r \to \infty$. Therefore the remainders after two terms in equations 9.10 and 9.15 which are $O_s(s^2f_\rho(s))$ as $s \to 0$ for the scalar potential and $O_s(s^2f_\rho(s)) + O_s(s^2f_J(s))$ as $s \to 0$ for the vector potential are also $O_r(r^{-1})$ as $r \to \infty$ in both cases; this might be indicated for the scalar potential by using $s^2f_\rho(s)O_r(r^{-1})$ for the remainder and for the vector potential by using $s^2f_\rho(s)O_r(r^{-1}) + s^2f_J(s)O_r(r^{-1})$. Where two order symbols are used the above considerations can be applied to each remainder term separately. Note that this requires both all $|s| < s_o$ (a positive constant) and $r \to \infty$; the constant for bounding the magnitude of the remainder is implied in O_r for all $r > r_{min}$ (a positive constant $> r_o$). Likewise the remainders after three terms in equations 9.21 and 9.27 which are $O_s(s^3 f_\rho(s)) + O_s(s^3 f_J(s))$ as $s \to 0$ for both fields are also $O_r(r^{-1})$ as $r \to \infty$ and could be indicated $s^3 f_\rho(s) O_r(r^{-1}) + s^3 f_J(s) O_r(r^{-1})$ thereby indicating the magnitude of the remainder for both $s \to 0$ and $r \to \infty$.

In summary the potentials and fields at large r and low frequencies are dominated by the electric and magnetic dipole moments provided they have the same low-frequency behavior as the charge and current densities respectively, at least for the dominant low-frequency terms. Considering both large r and low frequency together one needs only the first two terms in the expansion for $s \to 0$ or the expansion for $r \to \infty$ to represent the potentials, and one needs only the first three terms in the expansion for $s \to 0$ or the expansion for $r \to \infty$ to represent the fields. There is a transition frequency $\omega_1 = c/r$ for any large r which governs which of these first two or three terms are dominant depending on whether ω is less than or greater than ω_1; this can also be looked at in terms of a transition radius $r_1 = c/\omega$ for any small ω.

Up to this point we have considered the electric and magnetic dipole terms together in the expansions for $s \to 0$ and $r \to \infty$. One might consider whether or not one of the two types was dominant over the other. However these two types of moments are somewhat independent since an electric dipole needs a charge density ρ (basically a $\vec{J}_e$ type term) while a magnetic dipole does not need a charge density and can be produced with a purely solenoidal current density. With this in mind we have included both f_ρ and f_J with the low-frequency expansions. In subsequent sections we consider some of the characteristics of electric and magnetic dipoles and some features of radiating antennas which are dominantly electric or magnetic dipoles at low frequencies. We consider electric and magnetic dipoles first singly and then in combination.

X. Electric Dipole Parameters

In this section we consider some of the characteristics of radiating antennas which are basically electric dipoles, especially at low frequencies and for large distances to an observer. To begin this discussion we consider what can be defined as the electric dipole potentials and fields based on an infinitesimal or point electric dipole. To do this let there be one charge q at $\vec{r}' = \vec{e}_p\, d/2$ and a second charge $-q$ at $\vec{r}' = -\vec{e}_p\, d/2$ where $\vec{e}_p$ is an arbitrary unit vector (which for the moment is made independent of time); this gives an electric dipole moment as

$$\vec{p}(t) = \int_{V'} \vec{r}'\rho(\vec{r}',t)dV' = q(t)d\vec{e}_p \qquad (10.1)$$

where we also define

$$\vec{p}(t) \equiv p(t)\vec{e}_p \; , \quad p(t) = q(t)d \qquad (10.2)$$

and consider the limit $d \to 0$ with $\vec{p}$ fixed, so that q goes as p/d. Associated with these two charges we establish a uniform current filament I on a straight line between the two charges parallel to $\vec{e}_p$ with positive convention in the direction of $\vec{e}_p$ and related to q as

$$I(t) = \frac{dq(t)}{dt} \qquad (10.3)$$

which satisfies the requirement of the equation of continuity. The time derivative of the electric dipole moment can be written in terms of this current as

$$\frac{\partial}{\partial t}\vec{p}(t) = I(t)d\vec{e}_p \qquad (10.4)$$

Consider first the scalar potential in the Laplace domain for finite $|s|$ and let $d \to 0$. For $r' \to 0$ (maximum r' of interest is $d/2$) with fixed γ and r the Green's function from equations 5.2 through 5.8 is

$$\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} = \frac{e^{-\gamma r}}{4\pi r}\left[1+\frac{\vec{e}_r\cdot\vec{r}'}{r}+O(r'^2)\right]\exp\left(\gamma\vec{e}_r\cdot\vec{r}'+O(r'^2)\right)$$

$$= e^{-\gamma r}\left\{\frac{1}{4\pi r^2}\ \vec{e}_r\cdot\vec{r}'+\frac{1}{4\pi r}[1+\gamma\vec{e}_r\cdot\vec{r}']+O(r'^2)\right\} \qquad (10.5)$$

From equation 4.1 with $d \to 0$ the scalar potential then is

$$\tilde{\Phi}_p(r) = \lim_{d\to 0}\frac{1}{\varepsilon_o}\int_{V'}\tilde{\rho}(\vec{r}')\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\,dV'$$

$$= e^{-\gamma r}\left\{\frac{1}{4\pi r^2}\ \frac{1}{\varepsilon_o}\ \vec{e}_r\cdot\vec{\tilde{p}}+\frac{Z_o}{4\pi r}\ s\vec{e}_r\cdot\vec{\tilde{p}}\right\} \qquad (10.6)$$

$$\Phi_p(\vec{r},t) = \frac{1}{4\pi r^2}\ \vec{e}_r\cdot\vec{p}\left(t-\frac{r}{c}\right)+\frac{1}{4\pi r}\ \frac{1}{c}\ \vec{e}_r\cdot\left[\frac{\partial}{\partial t}\ \vec{p}\left(t-\frac{r}{c}\right)\right]$$

where we have used the Laplace domain solution to give the time domain solution and where we have used a subscript p to designate the electric-dipole scalar potential (and similarly for the vector potential and fields). Likewise for the vector potential from equations 4.4 as $d \to 0$ we have

$$\vec{\tilde{A}}_p(\vec{r}) = \lim_{d\to 0}\mu_o\int_{V'}\vec{\tilde{J}}(\vec{r}')\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\,dV' = \frac{e^{-\gamma r}}{4\pi r}\ \mu_o s\vec{\tilde{p}}$$

$$(10.7)$$

$$\vec{A}_p(\vec{r},t) = \frac{\mu_o}{4\pi r}\ \frac{\partial}{\partial t}\ \vec{p}\left(t-\frac{r}{c}\right)$$

Consider now the fields associated with a point electric dipole. First expand the gradient of the Green's function for $r' \to 0$ from equations 5.3 through 5.12 as

$$\nabla\left[\frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\right] = -\frac{e^{-\gamma r}}{4\pi r^2}\left[\vec{e}_r-\frac{\vec{r}'}{r}\right]\left\{1+\frac{3}{r}\ \vec{e}_r\cdot\vec{r}'+\gamma r+2\gamma\vec{e}_r\cdot\vec{r}'+O(r'^2)\right\}$$

$$\exp\left(\gamma\vec{e}_r\cdot\vec{r}'+O(r'^2)\right)$$

$$= \frac{e^{-\gamma r}}{4\pi r^2}\left[-\vec{e}_r+\frac{\vec{r}\,'}{r}\right]\left\{\frac{3}{r}\,\vec{e}_r\cdot\vec{r}\,'+1+3\gamma\vec{e}_r\cdot\vec{r}\,'+\gamma r[1+\gamma\vec{e}_r\cdot\vec{r}\,']+O(r'^2)\right\}$$

$$= e^{-\gamma r}\left\{\frac{1}{4\pi r^3}[-3\vec{e}_r[\vec{e}_r\cdot\vec{r}\,']+r']+\frac{1}{4\pi r^2}\left[-\vec{e}_r-\gamma[3\vec{e}_r[\vec{e}_r\cdot\vec{r}\,']-\vec{r}\,']\right]\right.$$

$$\left.+\frac{1}{4\pi r}\left[-\gamma\vec{e}_r-\gamma^2\vec{e}_r[\vec{e}_r\cdot\vec{r}\,']\right]+O(r'^2)\right\} \tag{10.8}$$

From equation 4.21 with $d \to 0$ the electric field is then

$$\tilde{\vec{E}}_p(\vec{r}) = \lim_{d\to 0}\left\{-\frac{1}{\varepsilon_o}\int_{V'}\tilde{\rho}(\vec{r}\,')\nabla\left[\frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}\right]dV'\right.$$

$$\left.-\mu_o s\int_{V'}\tilde{\vec{J}}(\vec{r}\,')\frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}\,dV'\right\}$$

$$= e^{-\gamma r}\left\{\frac{1}{4\pi r^3}\,\frac{1}{\varepsilon_o}[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{p}}]-\tilde{\vec{p}}]+\frac{Z_o}{4\pi r^2}\,s\,[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{p}}]-\tilde{\vec{p}}]\right.$$

$$\left.+\frac{\mu_o}{4\pi r}\,s^2\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{p}}]\right\} \tag{10.9}$$

$$\vec{E}_p(\vec{r},t) = \frac{1}{4\pi r^3}\,\frac{1}{\varepsilon_o}\left[3\vec{e}_r\left[\vec{e}_r\cdot\vec{p}\left(t-\frac{r}{c}\right)\right]-\vec{p}\left(t-\frac{r}{c}\right)\right]$$

$$+\frac{Z_o}{4\pi r^2}\left[3\vec{e}_r\left[\vec{e}_r\cdot\left[\frac{\partial}{\partial t}\,\vec{p}\left(t-\frac{r}{c}\right)\right]\right]-\frac{\partial}{\partial t}\,\vec{p}\left(t-\frac{r}{c}\right)\right]$$

$$+\frac{\mu_o}{4\pi r}\,\vec{e}_r\times\left[\vec{e}_r\times\left[\frac{\partial^2}{\partial t^2}\,\vec{p}\left(t-\frac{r}{c}\right)\right]\right]$$

Likewise for the magnetic field from equation 4.15 as $d \to 0$ we have

$$\vec{H}_p(\vec{r}) = \lim_{d\to 0} \int_{V'} \nabla\left[\frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}\right]\times\vec{\tilde{J}}(\vec{r}\,')\,dV'$$

$$= e^{-\gamma r}\left\{-\frac{1}{4\pi r^2}\, s\vec{e}_r\times\tilde{\vec{p}} - \frac{1}{4\pi r}\,\frac{s^2}{c}\,\vec{e}_r\times\tilde{\vec{p}}\right\} \qquad (10.10)$$

$$\vec{H}_p(\vec{r},t) = -\frac{1}{4\pi r^2}\,\vec{e}_r\times\left[\frac{\partial}{\partial t}\,\vec{p}\left(t-\frac{r}{c}\right)\right] - \frac{1}{4\pi r}\,\frac{1}{c}\,\vec{e}_r\times\left[\frac{\partial}{\partial t}\,\vec{p}\left(t-\frac{r}{c}\right)\right]$$

Equations 10.6, 10.7, 10.9, and 10.10 give the results for the potentials and fields from an electric dipole and as such we define these terms as the electric-dipole potentials and fields and can pick out these terms as such from a more general charge and current density distribution in V'. Equations 9.10, 9.15, 9.21, and 9.27 give the low-frequency (followed by large r) expansions of the potentials and fields from such a charge and current density distribution. Note that all the electric-dipole terms in these expansions correspond one for one with the electric-dipole potentials and fields defined in this section, and thus could all be replaced by these terms with subscript p. Also note that with $e^{-\gamma r}$ factored out the electric-dipole terms have only a few different factors of s^n: the scalar potential has s^o and s^1, the vector potential has s^1, the electric field has s^o, s^1, and s^2, and the magnetic field has s^1 and s^2. The low-frequency expansions for the general charge and current density distributions have all these powers of s with electric dipole moments, but if we were to take more terms in the low frequency expansion and expand those terms for large r we would get no more electric-dipole contributions for the simple reason that they do not appear in the electric-dipole potentials and fields which basically only depend on the volume integral of $\vec{r}\,'\rho$. The particular form of charge and current we used here is only a convenience. The important thing is that the higher order terms in r' (as $r' \to 0$) give higher order moments. We could have had magnetic-dipole terms as well had we not positioned $\pm q$ and I symmetrically about $\vec{r}\,' = \vec{0}$ such that in the limit as we brought the charges together they went to $\vec{r}\,' = \vec{0}$, and also had both $q = O(r'^{-1})$ and $I = O(r'^{-1})$ for small r'. Note also that although we have derived the results for a point dipole with a fixed direction $\vec{e}_p$ the direction of $\vec{p}(t)$ can be arbitrary, even changing with time or frequency, simply by superposition of three electric dipoles with orthogonal directions.

Now that we have the potentials and fields from a point electric dipole placed at $\vec{r}\,' = \vec{0}$, we can also find the potentials and fields from a point electric dipole at some other fixed position, say $\vec{r} = \vec{r}_p$. Simply replace $\vec{r}$ by $\vec{r} - \vec{r}_p$ (or r by $|\vec{r} - \vec{r}_p|$ and $\vec{e}_r$ by $(\vec{r} - \vec{r}_p)/|\vec{r} - \vec{r}_p|$) in equations 10.6, 10.7, 10.9, and 10.10. If this is expanded for large r so that $r^{-\gamma r}$ and inverse powers of r appear for $r \to \infty$ then various other moments (an infinite sequence of them) will appear. For example the associated magnetic dipole moment for a point electric dipole $\vec{p}(t)$ at $\vec{r} = \vec{r}_c$ is

$$\vec{m}_p(t) = \frac{1}{2}\int_{V'} \vec{r}'\times\vec{J}(\vec{r}')\,dV' = \frac{1}{2}\,\vec{r}_p\times\int_{V'}\vec{J}(\vec{r}')\,dV' = \frac{1}{2}\,\vec{r}_p\times\left[\frac{\partial}{\partial t}\,\vec{p}(t)\right] \tag{10.11}$$

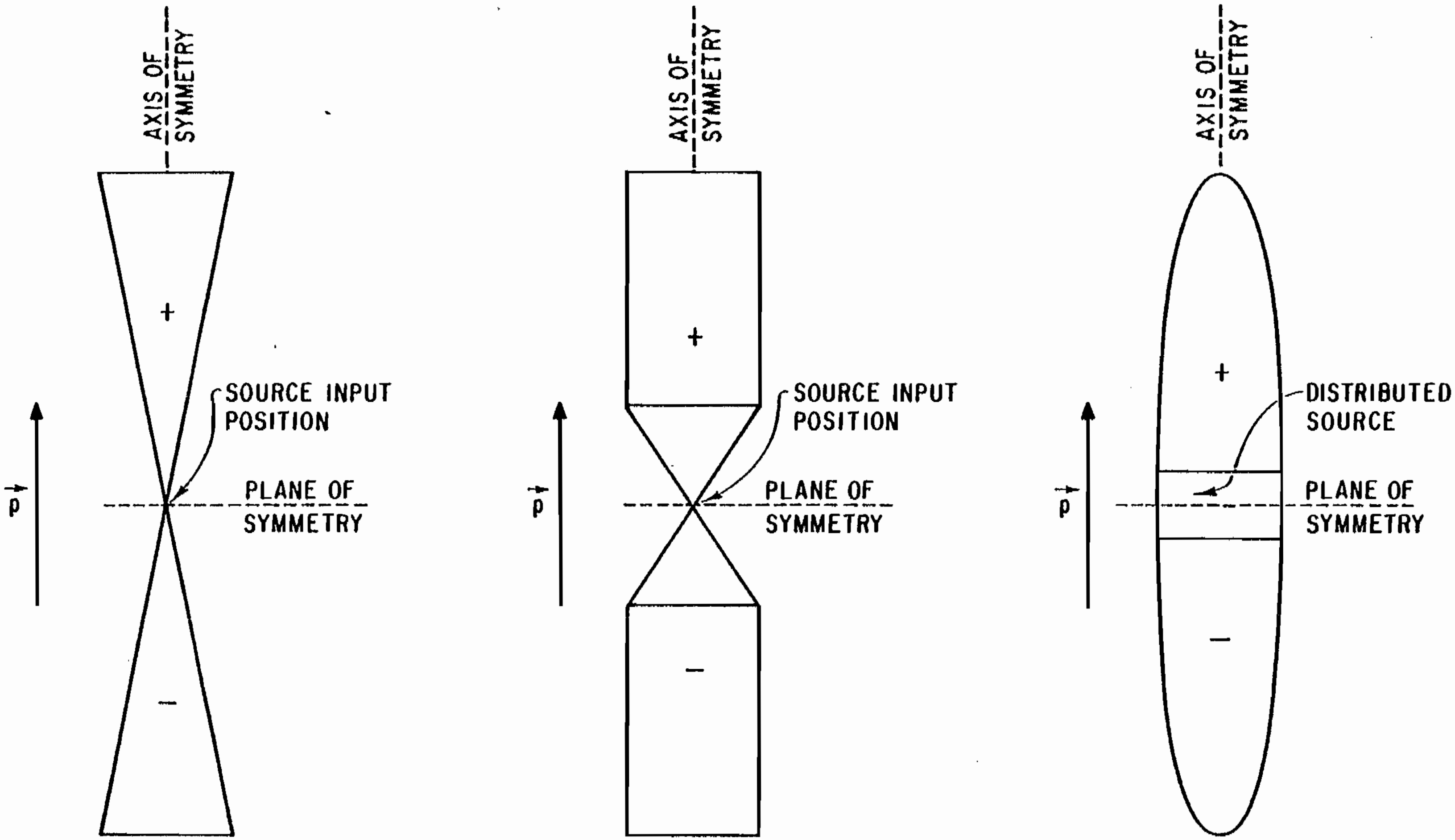
In the low-frequency limit for large r the electric dipole terms would still dominate. Note that we are still discussing an electric dipole with a fixed position in space. If one wishes to have a point electric dipole move on some path through space the results would be more complex.

So far we have considered what are the potentials and fields associated with a point or ideal electric dipole with three components as arbitrary functions of frequency or time, thereby giving the potentials and fields in equations 10.6, 10.7, 10.9, and 10.10 which apply for arbitrary $\vec{p}(t)$ or $\tilde{\vec{p}}$. Now real antennas can be designed which attempt to maximize the electric dipole moment by separating large charges at large distances apart. However in doing this one does not in general have a point or ideal dipole, simply because r' of interest includes values that are greater than zero. Thus a real antenna may still be categorized as an electric dipole because its dominant potentials and fields at large r and low frequencies are the electric-dipole potentials and fields. However other moments can be significant at high frequencies and/or small r unless the charge and current density distributions are constrained so as to give only an electric dipole moment.

Figure 2 shows a few typical shapes of real electric dipoles. Real electric dipoles are designed such that charge is transferred through one or more generators so as to produce a charge separation between or among two or more sections of antenna. The antenna may be composed of conductors, resistors, dielectrics, etc. to most appropriately control the charge separation. The generators need not even be distinct from the antenna, allowing for more distributed antenna and generator combinations. For the electric dipole moment to dominate at low frequencies and large r we require that as $s \to 0$

$$f_J(s) = o(f_\rho(s)) \tag{10.12}$$

where $f_J(s)$ and $f_\rho(s)$ are the frequency dependences of the current density and charge density as $s \to 0$ as defined in equations



A. BICONE

B. CYLINDER WITH BICONICAL FEED FROM SOURCE

C. PROLATE SPHEROID WITH DISTRIBUTED SOURCE

Figure 2. SOME CONFIGURATIONS OF ELECTRIC DIPOLE ANTENNAS WITH AXIAL AND LENGTHWISE SYMMETRY

6.53. In particular this makes the electric dipole dominate the magnetic dipole for $s \rightarrow 0$ provided $\vec{p}_\infty \neq \vec{0}$. If one just separates charge then at low frequencies there is a scalar potential developed; this represents a capacitance, and so an electric dipole is basically a capacitor at low frequency provided other electrical elements are not connected to the antenna elements so as to appear in parallel with the generator. One definitely avoids a conducting loop across the generator of a type which gives a magnetic dipole moment, violating the requirement of equation 10.12; such types of antennas are considered in the next section. For the typical case of a capacitive electric dipole the currents go like s times the charge at low frequencies which easily meets the restriction of equation 10.12.

Returning to figure 2 note a few things of interest about the antenna geometries shown. While it is not necessary for an electric dipole antenna to have a symmetry plane between two halves with opposite charges, it is often convenient to do so. This simplifies the analysis somewhat. For cases that there is no important mechanical reason for placing the generator(s) asymmetrically along the length of the antenna, then for typical real generators one can normally maximize the low-frequency dipole moment with a symmetric generator distribution. Normally one considers a single generator (or groups of generators) located near the center of the antenna. Such is the case for the examples listed in figure 2. With such a plane of symmetry for the generator(s) and antenna there is another advantage gained. This symmetry plane can be replaced by a perfectly conducting plane and the antenna structure on one side of the conducting plane can be driven in conjunction with the ground plane by any generators in the antenna structure completely on the side of the symmetry plane of interest; if there is a generator located on the symmetry plane (centered on it) then that generator is replaced by one of half the source voltage (open circuit) with half the source impedance (or with the same short-circuit current and half the source impedance). The fields from such a dipole with ground plane (or its response as a sensor or receiving antenna if appropriate) can be calculated by considering the equivalent dipole with full lengthwise symmetry. The potentials and fields calculated apply in only one half space but the results of the previous sections for large r, low frequency, etc. all apply. At low frequencies and large r the electric dipole moment still dominates the potentials and fields but the dipole moment one would use would be the one for the equivalent antenna with full lengthwise symmetry about the perfectly conducting plane. If one were to calculate the dipole moment as the volume integral of $\vec{r}'\rho$ one would only get one half the dipole moment of the equivalent full lengthwise symmetric antenna. Note also that there is charge on the ground plane extending to arbitrarily large r' to terminate the fields at the ground plane; the charge on the ground plane is just the negative of that on the antenna. The electric dipole moment must also be perpendicular to the symmetry plane, or perfectly conducting plane as the case may be. Note that a real ground plane will not be perfect but have some loss, altering the ideal results depending on the quality of the ground plane.

Besides lengthwise symmetry about a symmetry plane the examples of electric dipole antennas in figure 2 are chosen to have axial symmetry about a symmetry axis (also called rotational symmetry) where the symmetry axis is perpendicular to the symmetry plane. If one wishes to make the radiation from the electric dipole antenna more directional so as to radiate more power in one preferred direction along the symmetry plane, then one would not have this axial symmetry. However, one should note that at low frequencies the electric dipole moment completely dominates and has no such directionality. Such directionality can occur at higher frequencies with higher order moments involved. Here we are interested in large electric dipole moments and axial symmetry is no limitation in this respect so we choose it for some of its convenient properties.

With axial and lengthwise symmetry the calculation of the first moments for an electric dipole antenna simplifies somewhat. Consider the general electric dipole antenna with cartesian (x', y', z'), cylindrical (Ψ', ϕ', z'), and spherical (r', θ', ϕ') coordinates as illustrated in figure 3. For this type of antenna we restrict our attention to one whose materials have permittivity ε, permeability μ, and conductivity σ all arrayed in space as even functions of z' and independent of ϕ'; we also assume these parameters are scalars although they could be matrices if their components had appropriate symmetries as expressed in the cylindrical (Ψ', ϕ', z') coordinate system. With appropriate source symmetry we have the following symmetry with respect to z':

electromagnetic quantity	symmetry with respect to z'
$J_{z'}$, $A_{z'}$, $E_{z'}$, $H_{\phi'}$	even
$J_{\Psi'}$, ρ, $A_{\Psi'}$, Φ, $E_{\Psi'}$	odd

where expressed in the cylindrical system all these quantities are independent of ϕ'; the remaining quantities $J_{\phi'}$, $A_{\phi'}$, $E_{\phi'}$, $H_{\Psi'}$, $H_{z'}$ are all zero. This, of course, requires that the sources have the same symmetry for this to hold, as for example a source electric field would need to have its z' component even in z', Ψ' component odd in Ψ', and ϕ' component zero. While this type of symmetry is unnecessarily restrictive for the general case it still applies to many practical electric dipole designs. Referring back to figure 2, all these examples (and more) including cones, cylinders, prolate spheroids, etc. with discrete or distributed sources can be made with this axial and lengthwise symmetry, at least to a good approximation.

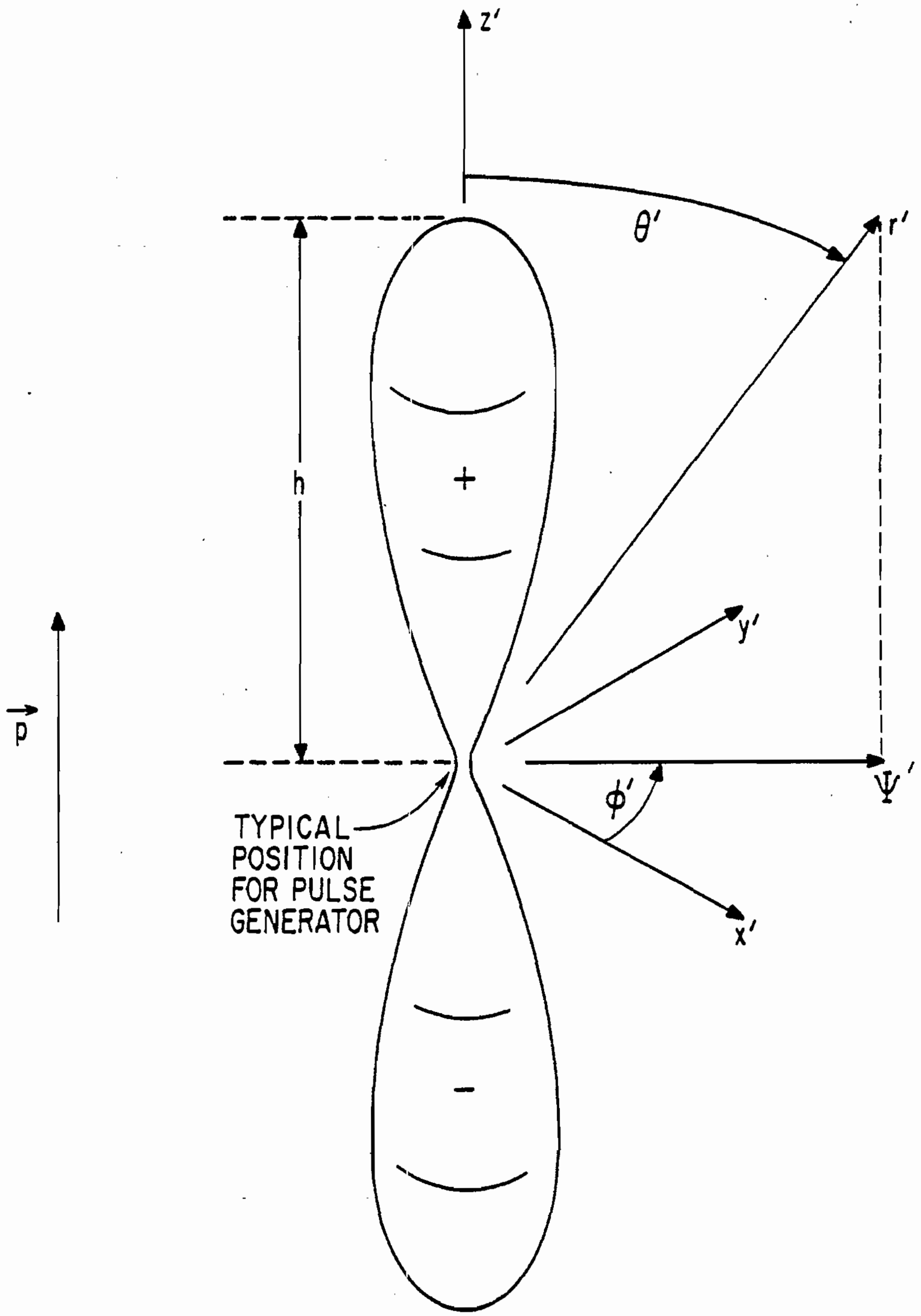


Figure 3. AXIALLY AND LENGTHWISE SYMMETRIC ELECTRIC DIPOLE ANTENNA WITH COORDINATES

Figure 3 shows a general case of an electric dipole antenna with this symmetry.

With the form of special axial and lengthwise symmetry for an electric dipole antenna as constrained above the calculation of the first few moments simplifies considerably. Consider the charge and current density at $\vec{r}'$ and at $-\vec{r}'$. Going from $\vec{r}'$ to $-\vec{r}'$ keeps Ψ' the same but reverses the direction of $\vec{e}_{\psi'}$ and reverses the sign of z' with $\vec{e}_{z'}$ staying the same. Using the ϕ' independence of the cylindrical components the current and charge densities then have the symmetries with respect to $\vec{r}'$ as

$$\vec{J}(-\vec{r}',t) = \vec{J}(\vec{r}',t) \quad , \quad \rho(-\vec{r}',t) = -\rho(\vec{r}',t) \tag{10.13}$$

Divide the volume of integration V' into V_+ for $z' > 0$ and V_- for $z' < 0$. As shown in figure 3 the maximum z' for the electric dipole antenna is h and the minimum z' is $-h$. Note that shifting from $\vec{r}'$ in V_+ to $-\vec{r}'$ maps each point in V_+ to V_-, and vice versa. An integral over V_- can be shifted to one over V_+ by replacing $\vec{r}'$ by $-\vec{r}'$ and dV' by $-dV'$ (since $dx'dy'dz'$ goes to $-dx'dy'dz'$), and by multiplying the integral by -1 since the limits on x', y', and z' all must be reversed; this can be summarized as

$$\int_{V_-} f(\vec{r}')\,dV' = \int_{V_+} f(-\vec{r}')\,dV' \tag{10.14}$$

where f might be a scalar, vector, dyadic, etc.

Applying this result and the symmetries in the current densities and charge densities gives the electric dipole moment as

$$\begin{aligned} \vec{p}(t) &= \int_{V'} \vec{r}'\rho(\vec{r}',t)\,dV' = \int_{V_+} \vec{r}'\rho(\vec{r}',t)\,dV' + \int_{V_-} \vec{r}'\rho(\vec{r}',t)\,dV' \\ &= \int_{V_+} \vec{r}'\rho(\vec{r}',t)\,dV' + \int_{V_+} [-\vec{r}']\rho(-\vec{r}',t)\,dV' \\ &= 2\int_{V_+} \vec{r}'\rho(\vec{r}',t)\,dV' \end{aligned} \tag{10.15}$$

while the electric quadrupole moment becomes

$$\overset{\leftrightarrow}{Q}(t) = \int_{V'} \vec{r}'\vec{r}'\rho(\vec{r}',t)dV' = \int_{V_+} \vec{r}'\vec{r}'\rho(\vec{r}',t)dV' + \int_{V_-} \vec{r}'\vec{r}'\rho(\vec{r}',t)dV'$$

$$= \int_{V_+} \vec{r}'\vec{r}'\rho(\vec{r}',t)dV' + \int_{V_+} [-\vec{r}'][-\vec{r}']\rho(-\vec{r}',t)dV'$$

$$= \overset{\leftrightarrow}{0} \tag{10.16}$$

and another scalar moment of a form similar to the electric quadrupole moment becomes

$$q'(t) = \int_{V'} r'^2\rho(\vec{r}')dV' = \int_{V_+} \vec{r}'\cdot\vec{r}'\rho(\vec{r}')dV' + \int_{V_-} \vec{r}'\cdot\vec{r}'\rho(\vec{r}')dV'$$

$$= \int_{V_+} \vec{r}'\cdot\vec{r}'\rho(\vec{r}')dV' + \int_{V_+} [-\vec{r}']\cdot[-\vec{r}']\rho(-\vec{r}')dV'$$

$$= 0 \tag{10.17}$$

and the magnetic dipole moment becomes

$$\vec{m}(t) = \frac{1}{2}\int_{V'} \vec{r}'\times\vec{J}(\vec{r},t)dV' = \frac{1}{2}\int_{V_+} \vec{r}'\times\vec{J}(\vec{r}',t)dV' + \frac{1}{2}\int_{V_-} \vec{r}'\times\vec{J}(\vec{r}',t)dV'$$

$$= \frac{1}{2}\int_{V_+} \vec{r}'\times\vec{J}(\vec{r},t)dV' + \frac{1}{2}\int_{V_+} [-\vec{r}']\times\vec{J}(-\vec{r}',t)dV'$$

$$= \vec{0} \tag{10.18}$$

Thus for the electric dipole antenna with our chosen form of axial and lengthwise symmetry the first few moments other than the electric dipole moment are zero. This makes the asymptotic expansions of the potentials and fields in sections 6 through 9 for large r and low frequency somewhat simpler and more accurate in the sense that several other terms beyond those involving the electric dipole moment are zero. In other words the first correction terms beyond those involving the electric dipole moment are zero so that the corrections become of higher order in s and/or inverse powers of r. The symmetry of ρ and $\vec{J}$

with respect to $\vec{r}\,'$ in equations 10.13 has made the moments involving integrals over ρ with second powers of $\vec{r}\,'$ and over $\vec{J}$ with first powers of $\vec{r}\,'$ zero. Higher order moments, such as those involving integrals over ρ with third powers of $\vec{r}\,'$ and over $\vec{J}$ with second powers of $\vec{r}\,'$ are not necessarily zero with this symmetry if another sign change is introduced in going from V_- to V_+. Note that the three moments above that are zero require that the coordinate origin be taken at the geometric center of the antenna, and not just that the antenna have axial and lengthwise symmetry unassociated with a coordinate system.

The electric dipole moment in equation 10.15 can be further simplified by utilizing the fact that ρ is independent of ϕ'. This makes $\vec{p}$ have only a z component which can be written as a two dimensional integral in the form

$$\vec{p}(t) = \int_{V'} \vec{r}\,'\rho(\vec{r}\,',t)dV' = p(t)\vec{e}_z \tag{10.19}$$

where

$$\begin{aligned} p(t) &= \int_{V'} z'\rho(\vec{r}\,',t)dV' = 2\pi\int_{-h}^{h} z'\int_{0}^{\Psi_0'(z')} \Psi'\rho(\Psi',z',t)d\Psi'dz' \\ &= 4\pi\int_{0}^{h} z'\int_{0}^{\Psi_0'(z')} \Psi'\rho(\Psi',z',t)d\Psi'dz' \end{aligned} \tag{10.20}$$

where $\Psi_0'(z')$ is simply a value of Ψ' which keeps all the charge density at each z' within a circle of radius Ψ_0'. In many cases of interest the charge density is a surface charge density as on the surface of a conductor so that equation 10.20 often reduces to a one-dimensional integral over z' or some arc length parameter.

The potentials and fields at large r and low frequencies from an electric dipole antenna driven by some pulse generator(s) are dominated by the electric dipole moment. The electric dipole moment at low frequencies can be related to a few properties of the antenna plus the characteristics of the generator(s) at low frequencies. Consider the simple case that the antenna is driven by a single pulse generator as discussed in a previous note.[8] The antenna is then characterized at low frequencies by two important parameters: the capacitance C_a and the mean charge separation distance $\vec{h}_a$. These are related to how much charge is separated on the antenna and how far apart it is separated; together these give the electric dipole moment at low frequencies. By low frequencies here we mean that corresponding wavelengths are large compared to antenna dimensions (such as h) and that the charge distribution has reached its low-frequency asymptotic form including the effects of loading elements (such as resistors with an RC time for charge flow through them) which could slow down or otherwise alter the distribution of charge around the antenna. The parameters we are now considering depend on a static charge distribution with negligible current; these are the asymptotic low-frequency parameters of an electric dipole antenna.

Consider the mean charge separation distance $\vec{h}_a$ which contributes to the dipole moment through the definition (applying in the static limit)

$$\vec{h}_a \equiv \frac{1}{\tilde{Q}_a} \tilde{\vec{p}} = \frac{1}{\tilde{Q}_a} \tilde{p}\vec{e}_z \tag{10.21}$$

where Q_a is the charge transferred through the single generator from the lower to upper portions of the antenna (with respect to the $\vec{e}_z$ direction). For our case of lengthwise symmetry and a single generator the charge is just

$$\tilde{Q}_a \equiv \int_{V_+} \tilde{\rho}(\vec{r}\,')dV' \tag{10.22}$$

Taking the low-frequency asymptotic form of the charge density from equations 6.53 as

$$\tilde{\rho}(\vec{r}\,') = f_\rho(s)\rho_\infty(\vec{r}\,') + o(f_\rho(s)) \tag{10.23}$$

this gives a low-frequency electric dipole moment of the form

$$\tilde{\vec{p}} = f_\rho(s)\vec{p}_\infty + o(f_\rho(s))$$

$$\vec{p}_\infty = \int_{V'} r'\rho_\infty(\vec{r}\,')dV' = p_\infty\vec{e}_z \tag{10.24}$$

$$p_\infty = \int_{V'} z'\rho_\infty(\vec{r}\,')dV'$$

We then also have for $s \to 0$

$$\tilde{Q}_a = f_\rho(s) Q_\infty + o(f_\rho(s))$$

$$Q_\infty = \int_{V_+} \rho_\infty(\vec{r}\,') dV' \tag{10.25}$$

The mean charge separation distance is then

$$\vec{h}_a = h_a \vec{e}_z$$

$$h_a = \frac{1}{Q_\infty} p_\infty \tag{10.26}$$

As discussed in reference 8 we have $\vec{h}_a = \vec{h}_{eq}$ so that the equivalent height of the electric dipole antenna as an electric field sensor is the same as the mean charge separation distance which applies to the antenna as a radiator.

Suppose that our axially and lengthwise symmetric electric dipole antenna has $\Psi_o' \ll h$ so that it is long and thin. If we approximate the antenna as a uniform transmission line, which would apply most readily to the biconical antenna in figure 2, then we can roughly say

$$h_a \simeq h \tag{10.27}$$

as is discussed in a previous note.[9] This is only a rough approximation, applying to thin electric dipole antennas. However, it does point out the simple and important design consideration that for a big low-frequency electric dipole moment make the antenna long.

Now consider the antenna capacitance C_a. By definition this is the capacitance between the two parts of the antenna separated by the pulse generator near the center of the lengthwise symmetric antenna; this capacitance is taken in the low-frequency limit and is thus an electrostatic parameter. The capacitance is simply the charge divided by the voltage difference between the two antenna portions (separated by the assumed single generator). Thus consider the static situation with charge density ρ, charge Q_a, scalar potential Φ and potential difference V between the two antenna portions. For a pulsed electric dipole antenna this situation applies to both the case that the antenna is slowly charged prior to its fast-transient discharge by a switch between the antenna sections, and the case that a generator transiently transfers charge between the two antenna sections and leaves them charged at late times when the charge density distribution has reached equilibrium.

For this static situation we can summarize some of the results from a previous note.[4] The electrostatic energy stored by the antenna is just

$$U_e = \frac{1}{2} C_a V^2 = \frac{1}{2} \int_{V_\infty} \vec{D}(\vec{r}\,') \cdot \vec{E}(\vec{r}\,') dV'$$

$$= \frac{1}{2} \int_{V'} \Phi(\vec{r}\,') \rho(\vec{r}\,') dV' = \frac{1}{2\varepsilon_o} \int_{V'} \int_{V'} \frac{\rho(\vec{r}\,')\rho(\vec{r}\,'')}{4\pi|\vec{r}\,'-\vec{r}\,''|} dV'' dV' \qquad (10.28)$$

Using

$$V = \frac{Q_a}{C_a} = \frac{1}{C_a} \int_{V_+} \rho(\vec{r}\,') dV' \qquad (10.29)$$

the capacitance can be written in terms of the charge density distribution as

$$C_a = \varepsilon_o \left\{ \int_{V_+} \rho(\vec{r}\,') dV' \right\}^2 \left\{ \int_{V'} \int_{V'} \frac{\rho(\vec{r}\,')\rho(\vec{r}\,'')}{4\pi|\vec{r}\,'-\vec{r}\,''|} dV'' dV' \right\}^{-1} \qquad (10.30)$$

Note that $\vec{r}\,''$ and dV'' are just used instead of $\vec{r}\,'$ and dV' for a second set of integration variables for the double integrals over V'.

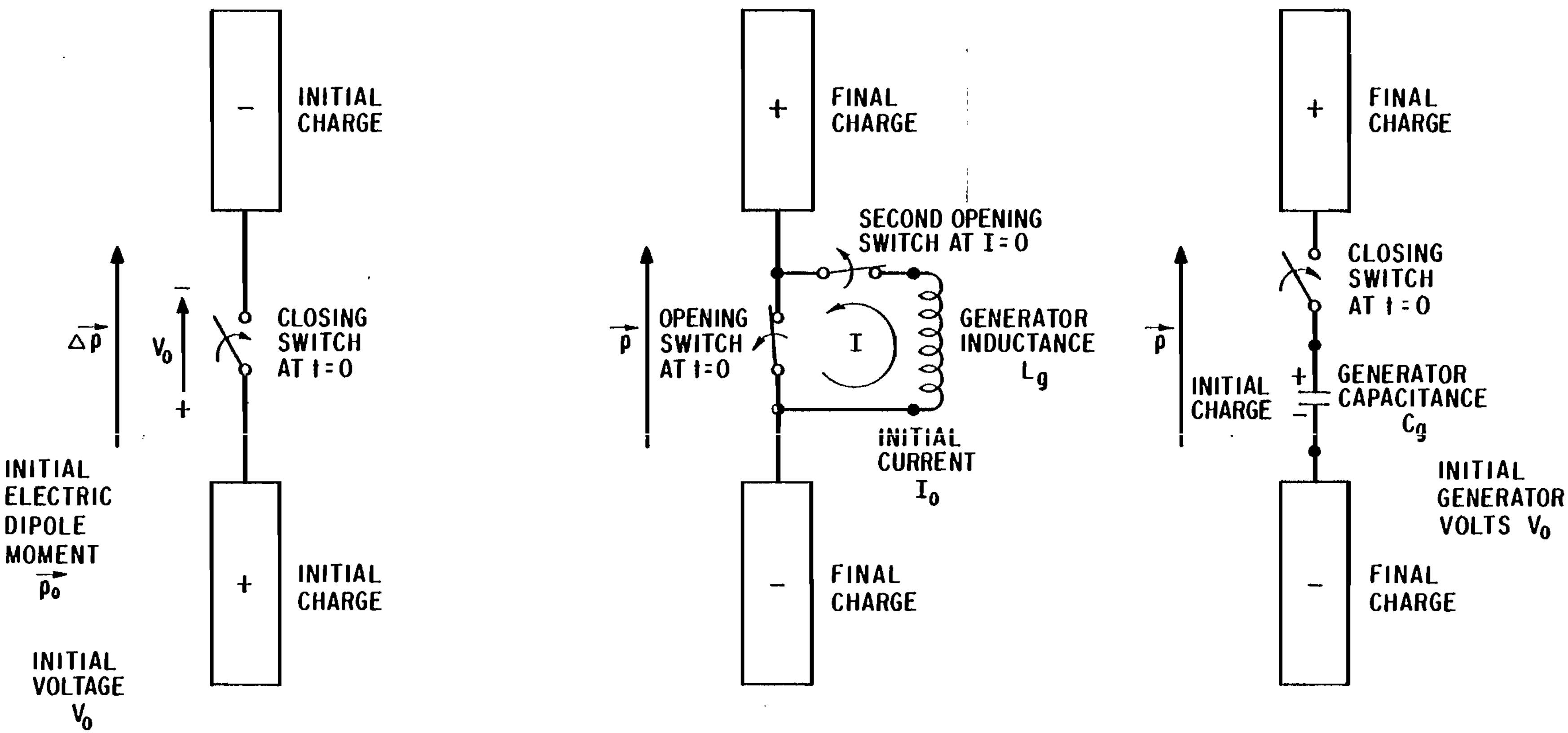
The calculation of C_a depends, of course, on the specific antenna geometry under consideration. In order to have a rough feel for the capacitance of such antennas consider the case of a long, thin electric dipole antenna driven at its center. If such an antenna has a radius $\Psi_o' \simeq a$ the simple transmission line model based on a biconical approximation gives[9]

$$C_a \simeq \pi\varepsilon_o h \left[\ln\left(\frac{2h}{a}\right) \right]^{-1} \qquad (10.31)$$

With $h >> a$ then C_a is roughly proportional to h. For a given low-frequency voltage on the antenna the charge is proportional to C_a and thus to h. Of course C_a can load the generator and thereby influence the voltage on the antenna. In general, however, increasing C_a increases the magnitude of the charge and thus the low-frequency electric dipole moment. Note that increasing h increases both h_a and C_a thereby increasing the low-frequency electric dipole moment through both of these factors.

In simulating the nuclear EMP one often desires to have the low-frequency content of the pulse waveform (in the Fourier transform sense) not roll off but hold up to quite low frequencies. For electric dipole antennas which are intended to radiate a pulse to large r (large compared to h) this poses a rather difficult problem. At low frequencies the electric-dipole fields (equations 10.9 and 10.10) are dominant. As pointed out in previous notes[5,8] the far fields (1/r terms) have fundamental low frequency limitations. Even a step-function voltage applied to the antenna gives far fields proportional to s at low frequencies. Increasing the low-frequency content of the far fields to a constant value would require an antenna voltage proportional to s^{-2} at low frequencies which in the time domain would be basically a ramp function for long times of interest. This would greatly increase the energy required to charge the antenna as in equation 10.28 since the magnitude of the antenna voltage would need to increase (comparatively slowly) much beyond the value associated with the fast transient part of the waveform, or equivalently decrease from such a large value if the pulse is achieved by antenna discharge. Considering the complete electric dipole fields which include r^{-2} and r^{-3} terms one has more low frequencies at large r tha- just the far-field terms. However, the contribution of these additional terms relative to the far fields decreases with increasing r. Note that both electric and magnetic fields have r^{-2} terms proportional to s^o for an electric dipole moment proportional to 1/s (step voltage on the antenna at low frequencies); the r^{-3} term gives a term proportional to 1/s for the electric field. Thus there are some low frequency contributions to the fields but they can still be much smaller than one would like, particularly if $r >> h$. For r not too large compared to h (say on the same order as h) these low frequency contributions can be quite significant.

There are various types of pulse generation techniques that one might consider using with electric dipole antennas where one was trying to maximize the low-frequency fields at large r. Figure 4 shows a few choices schematically. One possibility is to slowly charge the antenna to give an initial dipole moment $\vec{p}_o$ as shown in figure 4A. At $t = 0$ a switch is closed allowing the charge on the antenna to flow through the switch and produce a fast transient waveform. If the charging time were much larger than times of interest then one might just consider the transient change of the electric dipole moment as



A. DISCHARGE OF SLOWLY CHARGED ANTENNA

B. INDUCTIVE GENERATOR PLUS A SECOND OPENING SWITCH

C. CAPACITIVE GENERATOR

Figure 4. SOME SCHEMATIC PULSE-GENERATOR CONFIGURATIONS WITH ELECTRIC DIPOLE ANTENNAS

$$\Delta\vec{p}(t) \equiv \vec{p}(t) - \vec{p}_o \tag{10.32}$$

and consider the waveform associated with $\Delta\vec{p}$. If before $t = 0$ the antenna is charged to a voltage V_o the electrostatic energy stored is

$$U_o = \frac{1}{2} C_a V_o^2 \tag{10.33}$$

and as $t \to \infty$ the change in the electric dipole moment is

$$\Delta\vec{p}(\infty) = V_o C_a h_a \vec{e}_z \tag{10.34}$$

This method of pulse generation has one advantage in that none of the charge initially stored on the antenna is left as stored charge; in the case of a capacitive generator switched onto the antenna some fraction of the initial stored charge remains in the capacitive generator and does not contribute to the dipole moment. A significant question concerning this technique is the matter of prepulse. While the antenna can be slowly charged and thus have only very low frequencies in the fields produced during the charging of the antenna, nevertheless at these low frequencies the prepulse does significantly contribute to the fields at large r through the r^{-3} and r^{-2} terms (which include $s^0\tilde{\vec{p}}$ and $s^1\tilde{\vec{p}}$ respectively in the terms). How significant the prepulse is depends on the size of r. For r sufficiently large then the far fields (r^{-1} terms including $s^2\tilde{\vec{p}}$) are dominant and the low frequencies are much reduced. However for r of the order of h (or even less) the prepulse fields can be very large at low frequencies, even larger in comparison to the high frequencies of interest than desired; the near electric field in particular in the prepulse can be large indeed at low frequencies.

Figure 4B shows the case of an electric dipole antenna driven by an inductive generator with a second opening switch. An inductive generator would typically store magnetic energy $L_g I_o^2/2$ where L_g is the generator inductance and I_o the peak current just before switching. The first switch is opened at $t = 0$ and the current diverted to flow into the antenna thereby producing the electric dipole moment. For maximum low-frequency output one would like a maximum dipole moment and thus maximum charge displacement at late times. To achieve this one can put a second opening switch which opens when the current through the generator reaches zero and thereby prevents charge flowing through the inductor which at low frequencies is a low impedance. The actual charge transferred depends on the details of the antenna design and not just on C_a (a low-frequency parameter) due to the transient requirement of the second switch opening time. A significant disadvantage of this type of generator scheme is that an opening switch is much slower than a closing switch in the present state of the art in pulse power technology; switching speed is needed for good high frequency performance. Furthermore two such switches are used so as to maximize the low-frequency performance. While this generator scheme is not attractive for a pulsed electric dipole antenna, it is interesting as the dual of a scheme involving a capacitive generator with a magnetic dipole antenna which is considered in section XI.

Figure 4C shows the case of an electric dipole antenna driven by a capacitive generator. The generator with capacitance C_g and initial voltage V_o has a stored energy $C_g V_o^2/2$. At $t = 0$ a switch is closed to connect the generator voltage onto the antenna. Charge flows into the antenna giving voltage, charge, and electric dipole moment in the late-time limit as[8]

$$V_a(\infty) = V_o \frac{C_g}{C_a+C_g}$$

$$Q_a(\infty) = V_a(\infty) C_g = V_o \frac{C_a C_g}{C_a+C_g} \tag{10.35}$$

$$\vec{p}(\infty) = Q_a(\infty)\vec{h}_a = V_o \frac{C_a C_g}{C_a+C_g} h_a \vec{e}_z$$

These late-time parameters apply for late times of interest provided the antenna is allowed to discharge (say through a resistive path around the generator capacitors) at times much larger than late times (and corresponding low frequencies) of interest. As discussed before the late-time dipole moment is important for maximizing the low-frequency performance of the antenna. Note that the capacitive generator does this rather simply, requiring a closing switch which can be made to give a fast rise time. Note also there is a capacitive divider effect so that not all the generator charge goes out onto the antenna. This implies a tradeoff between generator capacitance and voltage and antenna size (affecting both C_a and h_a) for large simulators of this type trying to maximize $\vec{p}(\infty)$. For this type of capacitive generator there is no prepulse before $t = 0$ which gives this generator some advantage over that in figure 4A. However, some practical generators of this type do give some prepulse before the main switch closes because of other coupling (e.g. capacitive) to the antenna, but this is generally small.

Here we have briefly considered a few types of pulse generators that might be used with an electric dipole antenna to produce pulsed fields at large r. Of these capacitive type generators tend to combine with the capacitive antenna more simply than inductive generators in cases that low-frequency performance is an important design consideration. There are, of course, various other types of pulse generators one might consider, such as those involving high explosives and/or more complex electrical circuits than those shown in figure 4. Similar considerations can be applied to these with regard to the low-frequency content of the resulting waveforms.

With pulsed electric dipoles designed to maximize the low-frequency content of the fields the late-time electric dipole moment $\vec{p}(\infty)$ is quite important. For simplicity, consider the case where $\vec{p}(t) = \vec{0}$ for $t < 0$ and let the late-time electric dipole moment be non zero so that it is a step function at low frequencies. Thus we take the case where the charge density as $s \to 0$ is given by

$$\tilde{\rho}(\vec{r}\,') = f_\rho(s)\rho_\infty(\vec{r}\,') + o(f_\rho(s))$$

$$f_\rho(s) = \frac{1}{s} \tag{10.36}$$

so that the electric dipole moment as $s \to 0$ is

$$\tilde{\vec{p}} = \frac{1}{s}\vec{p}_\infty + o\left(\frac{1}{s}\right) = \frac{1}{s}\vec{p}(\infty) + o\left(\frac{1}{s}\right) \tag{10.37}$$

This case is consistent with the results for the capacitive generator above (equations 10.35) which has finite energy. From equations 10.6 and 10.7 the electric-dipole potentials for $s \to 0$ are then

$$\tilde{\Phi}_p(\vec{r}) = e^{-\gamma r}\left\{\frac{1}{4\pi r^2}\frac{1}{\varepsilon_0}\left[\frac{1}{s}\vec{e}_r\cdot\vec{p}(\infty)+o\left(\frac{1}{s}\right)\right]\right.$$

$$\left.+\frac{Z_0}{4\pi r}[\vec{e}_r\cdot\vec{p}(\infty)+o(1)]\right\} \tag{10.38}$$

$$\tilde{\vec{A}}_p(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r}\mu_0[\vec{p}(\infty)+o(1)]$$

and the associated fields from equations 10.9 and 10.10 are

$$\tilde{\vec{E}}_p(\vec{r}) = e^{-\gamma r}\Bigg\{\frac{1}{4\pi r^3}\,\frac{1}{\varepsilon_o}\left[\frac{1}{s}\left[3\vec{e}_r[\vec{e}_r\cdot\vec{p}(\infty)]-\vec{p}(\infty)\right]+o\left(\frac{1}{s}\right)\right]$$

$$+\frac{Z_o}{4\pi r^2}\left[3\vec{e}_r[\vec{e}_r\cdot\vec{p}(\infty)]-\vec{p}(\infty)+o(1)\right]$$

$$+\frac{\mu_o}{4\pi r}\left[s\vec{e}_r\times[\vec{e}_r\times\vec{p}(\infty)]+o(s)\right]\Bigg\} \tag{10.39}$$

$$\tilde{\vec{H}}_p(\vec{r}) = e^{-\gamma r}\Bigg\{-\frac{1}{4\pi r^2}[\vec{e}_r\times\vec{p}(\infty)+o(1)]$$

$$-\frac{1}{4\pi r}\,\frac{1}{c}\,[s\vec{e}_r\times\vec{p}(\infty)+o(s)]\Bigg\}$$

This generalizes the results of a previous note on pulsed electric dipole antennas.[5] Driven by a step-function excitation (for low frequencies) both electric and magnetic fields have r^{-1} terms proportional to s as $s \to 0$. This term is the far-field term; the associated far-field waveform (or first order waveform) must have at least one zero crossing for any not-identically-zero field component. Considering the second order terms (proportional to r^{-2}) both electric and magnetic fields have such terms and they are proportional to s^o. The associated second order waveforms need not have any zero crossing, but must return to zero value in the late time limit. Considering the third order terms (proportional to r^{-3}) only the electric field has one (not counting higher order moments) and it is proportional to s^{-1}. The associated third order waveforms need not go to zero in the late-time limit, but can have step function characteristics at late time; some third order electric field component must be non zero at late time. For sufficiently large r the time-domain waveforms are dominated by the far fields, at least at those times corresponding to high frequencies. However if one goes far out in time there is a non zero electric field associated with the third order term which dominates the electric field at that time. The second order terms make the complete time integrals of the electric and magnetic fields non zero as well. However if one is at large enough r these contributions of the second and third order terms to the time-domain waveforms can be quite small in comparison to the far fields, say in their peak values.

8A

XI. Magnetic Dipole Parameters

Now we consider some characteristics of radiating antennas which are basically megnatic dipoles, especially at low frequencies and large r. In many respects the magnetic dipole antenna can be considered the dual of the electric dipole antenna considered in the previous section. Together the electric and magnetic dipoles give the leading terms to the potentials and fields for large r and low frequencies as discussed in sections VI through IX.

Consider first what can be defined as the magnetic dipole potentials and fields based on an infinitesimal or point magnetic dipole. For this it is convenient to take a circular current path of radius d centered on the origin with a uniform current I(t) flowing on this current path in a direction $\vec{e}_I$ (a unit vector). This circular current path lies in a plane which passes through the origin ($\vec{r}\,' = \vec{0}$) and is perpendicular to the unit vector $\vec{e}_m$ where

$$\vec{e}_I = \vec{e}_m \times \vec{e}_r\,' = \vec{e}_m \times \left(\frac{\vec{r}\,'}{d}\right) \tag{11.1}$$

where $\vec{e}_r\,'$ is the unit vector in the $\vec{r}\,'$ direction when $\vec{r}\,'$ is on the current path. The directions of $\vec{e}_r\,'$ and $\vec{e}_I$ change, of course, as one moves around the current path. The area of the circular current path is just

$$\vec{A}_d = \pi d^2 \vec{e}_m \tag{11.2}$$

where we have assigned a direction to this area based on the normal vector to the plane of the area. The magnetic dipole moment of this uniform circular current filament is

$$\vec{m}(t) = \frac{1}{2}\int_{V'} \vec{r}\,' \times \vec{J}(\vec{r}\,',t)\,dV' = I(t)\pi d^2 \vec{e}_m = I(t)\vec{A}_d \tag{11.3}$$

where we can also define

$$\vec{m}(t) \equiv m(t)\vec{e}_m \;, \quad m(t) = I(t)\pi d^2 = I(t)|\vec{A}_d| \tag{11.4}$$

We will consider the limit as $d \rightarrow 0$ with $\vec{m}$ fixed so that I goes as $m/(\pi d^2)$ and use this to define the magnetic-dipole potentials and fields. Note that since we have chosen a uniform closed current path we have a solenoidal current distribution (a $\vec{J}_h$ type of distribution). Thus the time rate of change of the charge density must be zero. Then for convenience we choose the charge density to be identically zero; this automatically makes any moments of the charge density (electric dipole, electric quadrupole, etc.) zero. While we choose $\vec{m}$ as above having a fixed direction the results for the magnetic-dipole potentials and fields apply to arbitrary time or frequency dependence of the direction of $\vec{m}$ (i.e. $\vec{e}_m$) simply by superposition of three magnetic dipoles with orthogonal directions.

The magnetic-dipole scalar potential from equation 4.1 is an integral over the charge density which is zero so that we have

$$\tilde{\Phi}_m(\vec{r}) = \lim_{d\to 0} \frac{1}{\varepsilon_o} \int_{V'} \tilde{\rho}(\vec{r}\,') \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} \, dV' = 0$$

$$\Phi_m(\vec{r},t) = 0 \tag{11.5}$$

We use a subscript m to designate the magnetic-dipole scalar potential (and similarly for the vector potential and fields). The vector potential from equations 4.4 as $d \to 0$ is

$$\tilde{\vec{A}}_m(\vec{r}) = \lim_{d\to 0} \mu_o \int_{V'} \tilde{\vec{J}}(\vec{r}\,') \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} \, dV'$$

$$= e^{-\gamma r}\left\{-\frac{\mu_o}{4\pi r^2} \vec{e}_r\times\tilde{\vec{m}} - \frac{1}{4\pi r}\frac{\mu_o}{c} s\vec{e}_r\times\tilde{\vec{m}}\right\} \tag{11.6}$$

$$\vec{A}_m(\vec{r},t) = -\frac{\mu_o}{4\pi r^2} \vec{e}_r\times\vec{m}\left(t-\frac{r}{c}\right) - \frac{1}{4\pi r}\frac{\mu_o}{c} \vec{e}_r\times\left[\frac{\partial}{\partial t} \vec{m}\left(t-\frac{r}{c}\right)\right]$$

where we have used the expansion of the Green's function for $r' \to 0$ as in equation 10.5. As with the case of the electric dipole we use the Laplace domain results to obtain the time domain results.

From equation 4.21 with $d \to 0$ the magnetic-dipole electric field is

$$\tilde{\vec{E}}_m(\vec{r}) = \lim_{d\to 0}\left\{-\frac{1}{\varepsilon_o}\int_{V'}\tilde{\rho}(\vec{r}\,')\nabla\frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}\,dV'\right.$$

$$\left. - \mu_o s\int_{V'}\tilde{\vec{J}}(\vec{r}\,')\frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}\,dV'\right\}$$

$$= e^{-\gamma r}\left\{\frac{\mu_o}{4\pi r^2}\, s\vec{e}_r\times\tilde{\vec{m}}+\frac{1}{4\pi r}\,\frac{\mu_o}{c}\, s^2\vec{e}_r\times\tilde{\vec{m}}\right\} \tag{11.7}$$

$$\vec{E}_m(\vec{r},t) = \frac{\mu_o}{4\pi r^2}\,\vec{e}_r\times\left[\frac{\partial}{\partial t}\,\vec{m}\left(t-\frac{r}{c}\right)\right]+\frac{1}{4\pi r}\,\frac{\mu_o}{c}\,\vec{e}_r\times\left[\frac{\partial^2}{\partial t^2}\,\vec{m}\left(t-\frac{r}{c}\right)\right]$$

The magnetic-dipole magnetic field comes from equation 4.15 with $d \to 0$ as

$$\tilde{\vec{H}}_m(\vec{r}) = \lim_{d\to 0}\int_{V'}\nabla\left[\frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}\right]\times\tilde{\vec{J}}(\vec{r}\,')\,dV'$$

$$= e^{-\gamma r}\left\{\frac{1}{4\pi r^3}[3\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{m}}]+2\tilde{\vec{m}}]+\frac{1}{4\pi r^2}\,\frac{s}{c}[3\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{m}}]+2\tilde{\vec{m}}]\right.$$

$$\left.+\frac{1}{4\pi r}\,\frac{s^2}{c^2}\,\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{m}}]\right\}$$

$$= e^{-\gamma r}\left\{\frac{1}{4\pi r^3}[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{m}}]-\tilde{\vec{m}}]+\frac{1}{4\pi r^2}\,\frac{s}{c}[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{m}}]-\tilde{\vec{m}}]\right.$$

$$\left.+\frac{1}{4\pi r}\,\frac{s^2}{c^2}\,\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{m}}]\right\} \tag{11.8}$$

$$\vec{H}_m(\vec{r},t) = \frac{1}{4\pi r^3}\left[3\vec{e}_r\left[\vec{e}_r\cdot\vec{m}\left(t-\frac{r}{c}\right)\right]-\vec{m}\left(t-\frac{r}{c}\right)\right]$$

$$+\frac{1}{4\pi r^2}\frac{1}{c}\left[3\vec{e}_r\left[\vec{e}_r\cdot\left[\frac{\partial}{\partial t}\vec{m}\left(t-\frac{r}{c}\right)\right]\right]-\frac{\partial}{\partial t}\vec{m}\left(t-\frac{r}{c}\right)\right]$$

$$+\frac{1}{4\pi r}\frac{1}{c^2}\vec{e}_r\times\left[\vec{e}_r\times\left[\frac{\partial^2}{\partial t^2}\vec{m}\left(t-\frac{r}{c}\right)\right]\right]$$

where we have used the expansion of the gradient of the Green's function for $r' \to 0$ as in equation 10.8.

Refer to equations 9.10, 9.15, 9.21, and 9.27 for the low frequency (followed by large r) expansions for the potentials and fields from a general current density and charge density distribution. The magnetic dipole terms in these expansions correspond one for one with the magnetic-dipole potentials and fields defined in this section. Between the electric-dipole potentials and fields of the previous section and the magnetic-dipole potentials and fields of this section we have all the dominant terms in the low-frequency large r expansions. Note that all the magnetic-dipole terms in the potential and field expansions are included there simply because they contain all the terms in the magnetic-dipole potentials and fields of this section. The magnetic-dipole terms depend on integrals over terms that are of the form $\vec{r}'\vec{J}$. The higher order terms in the expansion all involve integrals with higher powers of $\vec{r}'$ in the integrand. The form of the closed uniform current path used in this section is only a convenience. The important thing is that as $r' \to 0$ for the path we have I proportional to $1/r'^2$. The hither order moments then go to zero and the magnetic dipole moment is just I times the area of the current path taken in a vector sense which is what is calculated when one evaluates the integral over V' of $\vec{r}' \times \vec{J}/2$.

One can observe a certain duality between the electric-dipole fields in equations 10.9 and 10.10 and the magnetic-dipole fields considered in this section. If one makes the substitutions

electric-dipole quantities		magnetic-dipole quantities
$\vec{p}$	$\leftrightarrow$	$\frac{\vec{m}}{c}$
$\vec{E}_p$	$\leftrightarrow$	$Z_o\vec{H}_m$
$\vec{H}_p$	$\leftrightarrow$	$-\frac{1}{Z_o}\vec{E}_m$

one can convert the electric-dipole fields to magnetic-dipole fields and vice versa. This is a rather interesting duality property of electric and magnetic dipoles which can lead to some interesting and useful results which are considered in the next section of this note.

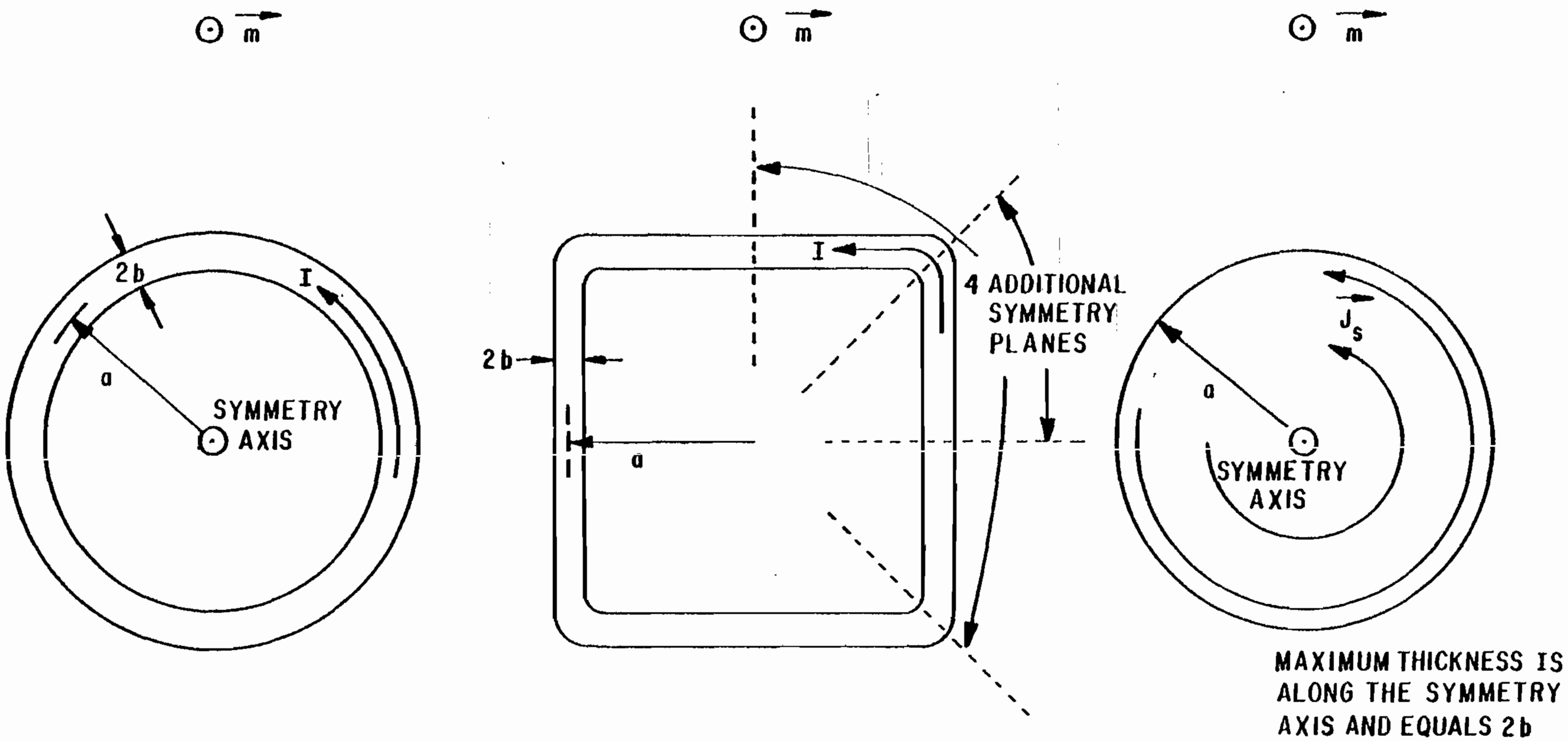
Now that we have the potentials and fields associated with a point magnetic dipole positioned at $\vec{r}' = \vec{0}$ these results can be applied to a point magnetic dipole at any other $\vec{r}'$, say $\vec{r}' = \vec{r}_m$. Simply replace $\vec{r}$ by $\vec{r} - \vec{r}_m$, r by $|\vec{r} - \vec{r}_m|$, and $\vec{e}_r$ by $(\vec{r} - \vec{r}_m)/|\vec{r} - \vec{r}_m|$. Of course, if one expands these results for large r such that $e^{-\gamma r}$ times inverse powers of r appear in the expansions then other than magnetic-dipole terms will appear. However no moments written as integrals over the charge density such as electric dipole, electric quadrupole, etc. will appear in such expansions because the point magnetic dipole has been defined with the charge density equal to zero. In the low-frequency limit for large r the magnetic dipole term is still dominant.

Thus far we have considered the potentials and fields for a point magnetic dipole as given in equations 11.5 through 11.8 and applying for arbitrary $\vec{m}(t)$ or $\tilde{\vec{m}}$. Now consider some characteristics of practical magnetic dipoles. To maximize the magnetic dipole moment real magnetic dipoles can be designed to make large currents flow around large areas. In so doing the dimensions become large and other moments (both electric and magnetic) can become significant, especially at high frequencies, depending on the actual geometry of the magnetic dipole with its electrical generators. Figure 5 shows a few possible shapes for magnetic dipoles, but without including the generators. Note that one can have various symmetry axes and symmetry planes in the magnetic-dipole geometry. However the generator positions must also be included before one can consider the possible symmetries of the charge and current density distributions. Figure 5 shows a few different approaches to the magnetic dipole geometry. One might have a thin loop such as the circular or square loops shown in figures 5A and 5B, or one might have a more distributed current density as for example on the surface of an oblate spheroid (or circular disk in the limiting case) as in figure 5C.

For a real magnetic dipole we require that as $s \to 0$

$$f_\rho(s) = o(f_J(s)) \qquad (11.9)$$

where $f_\rho(s)$ and $f_J(s)$ are the frequency dependences as $s \to 0$ of the charge and current densities as defined in equations 6.53. This restriction assures that as $s \to 0$ the magnetic dipole dominates the electric dipole provided the current density distribution is such as to make $\vec{m}_\infty \neq \vec{0}$. This restriction is the dual



A. CIRCULAR LOOP

B. SQUARE LOOP

C. OBLATE SPHEROID OR DISK

FIGURE 5. SOME GEOMETRIC SHAPES OF MAGNETIC DIPOLE ANTENNAS WITH A SYMMETRY PLANE PARALLEL TO THE PAGE

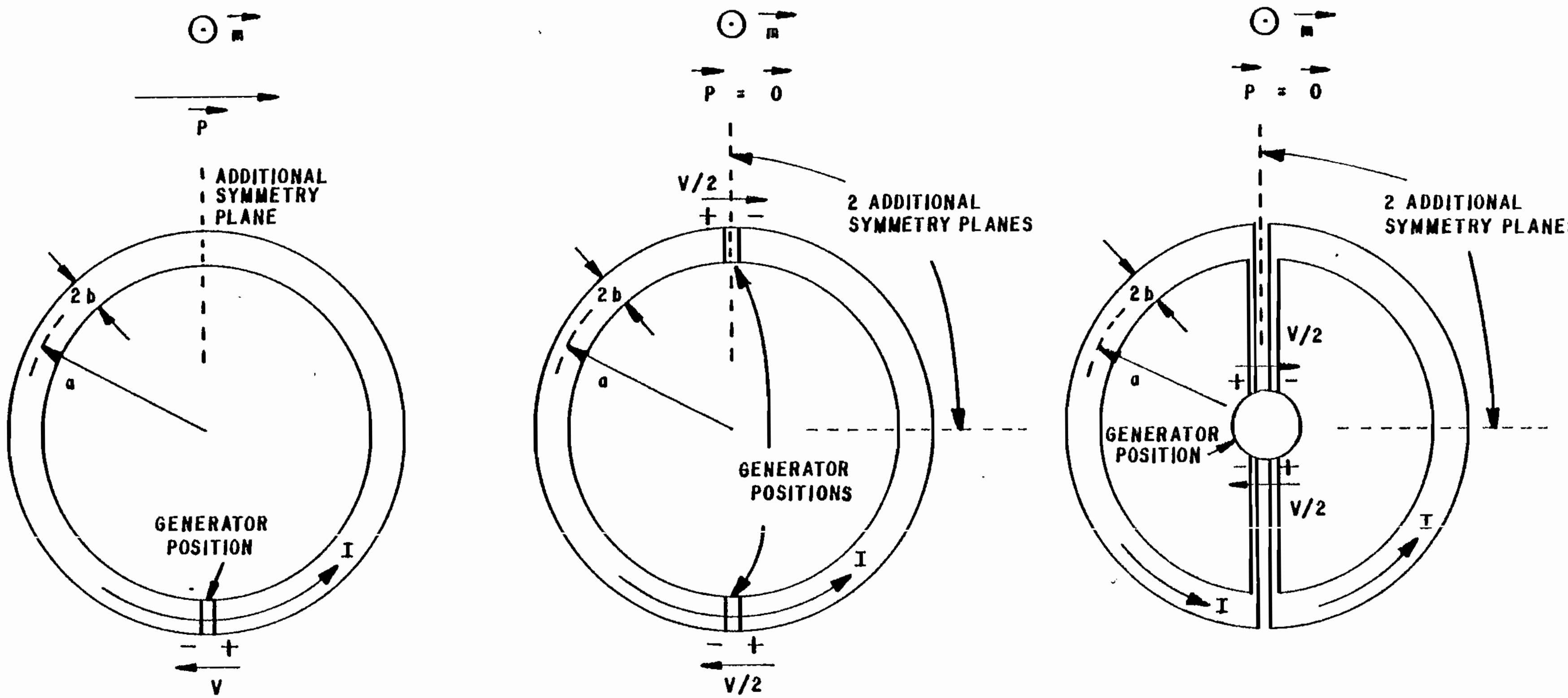
of the one we assumed for the electric dipole in equation 10.12. At low frequencies a magnetic dipole has a current generating a local vector potential with negligible charge and scalar potential because the current is made to flow around a highly conducting (low impedance) path. Thus a practical magnetic dipole is basically an inductor for its low-frequency impedance. Compare this to the case of an electric dipole which is a capacitor at low frequencies, another duality. Note that if the current path were basically a frequency-independent resistor at low frequencies then the voltage and current would have the same frequency dependence and thus so would the charge on the resistor surfaces; this would violate the restriction of equation 11.9 and depending on the antenna geometry could give a significant electric dipole moment. However for a perfectly conducting current path the resulting inductance would make the voltage (and thus the charge on the conductor surfaces) be proportional to s times the current, thereby satisfying the restriction of equation 11.9.

The magnetic dipole geometries in figure 5 have various symmetries. They each have a symmetry plane which is parallel to the desired current flow directions for the desired magnetic dipole moment; this symmetry plane is then perpendicular to the desired magnetic dipole moment. In figure 5 this symmetry plane is parallel to the page and splits the magnetic dipole geometries into two halves which are reflections of one another about this plane. This particular symmetry plane would also apply to the current and charge density distributions resulting when appropriate generators are added. For the loops in figures 5A and 5B typical generators would simply be inserted at particular positions along the loop path so as to drive current around the loop. It is a fairly simple matter to make the generator put out a current density which is symmetric with respect to this plane (at least to a good approximation). More distributed geometries such as an oblate spheroid as in figure 5C can also be driven so that the current and charge densities are symmetric with respect to this plane by appropriate symmetry in the distribution of the generators and any connections to the conductors forming such a distributed magnetic dipole geometry. If one wishes he could think of such a distributed geometry as the combination of many conducting loops (made circular in this example) located and driven symmetrically with respect to the chosen symmetry plane.

There are other symmetries in the magnetic dipole geometries shown in figure 5. The circular loop in figure 5A has a summetry axis in addition to its symmetry plane, giving it a rather high order of symmetry. The square loop in figure 5B has somewhat less symmetry in that it does not have such a symmetry axis associated with rotational (or axial) symmetry; instead it has 4 additional symmetry planes. This could be extended to loops shaped as regular polygons of any number of sides and the number of additional symmetry planes equals the number of sides. The more distributed geometry in figure 5C has a symmetry axis but there are various other parameters such as the thickness 2b of the oblate spheroid and the variation of the surface current density $\vec{J}_s$ with distance from the symmetry axis which can still be chosen in some optimal manner.

The question of symmetry in a magnetic-dipole antenna is important if one wishes to minimize the effect of other moments on the fields produced by the antenna. However the distribution of the sources in the antenna geometry is also important in determining the symmetry of the current and charge density distributions and thus the various moments. To illustrate the influence of source configurations on the moments other than the desired magnetic dipole moment consider a few different configurations of sources at discrete positions with respect to a circular loop antenna as shown in figure 6. Figure 6A shows the case of a single generator position. For this case we have one additional symmetry plane which divides the loop, passing through the generator position; this is a symmetry plane for the charge and current densities in that the various scalars and vector components are either even or odd on making a reflection to a symmetric position on the opposite side of the plane. However with a single generator position we still get an electric dipole moment on the antenna. This is easy to see by referring to figure 6A. Except in the low frequency limit where the charge goes to zero (in comparison to the current as in equation 11.9) and the current is uniform around the loop, there is a charge separation across the additional symmetry plane associated with the single generator transiently moving charge across this plane. This gives an electric dipole moment perpendicular to the additional symmetry plane as indicated on the figure. Since the loop is an inductor at low frequencies then this electric dipole moment is proportional in magnitude to s times the magnetic dipole moment as $s \to 0$. Thus the magnetic dipole moment still dominates at low frequencies. While one may not like to have an electric dipole moment present for certain applications, for other applications the presence of the lower order electric dipole moment may be less important than advantages which may be gained through the use of a single generator and single generator connection position on the loop.

If one wishes to eliminate the electric dipole moment from the loop then one could use two identical generators at opposite positions on the loop as shown in figure 6B. Alternatively one might symmetrically feed a single generator to opposite positions on the loop as shown in figure 6C. For convenience we have chosen the generator(s) so as to put a voltage V/2 onto the loop or connections to the loop to be consistent with the case in figure 6A with a total voltage V for the full loop. The cases with two symmetric generator inputs to the loop each have two additional symmetry planes as shown in figures 6B and 6C. With this feature the electric dipole moment is zero. To see this first observe that the current density and charge density have symmetry with respect to $\vec{r}\,'$ as



A. SINGLE PULSE GENERATOR

B. TWO IDENTICAL PULSE GENERATORS AT SYMMETRICAL POSITIONS

C. SINGLE PULSE GENERATOR FEEDING TWO SYMMETRICAL POSITIONS

FIGURE 6. SOME CONFIGURATIONS OF MAGNETIC-DIPOLE CIRCULAR LOOP ANTENNAS INCLUDING SOURCES WITH A SYMMETRY PLANE PARALLEL TO THE PAGE

$$\vec{J}(-\vec{r}',t) = -\vec{J}(r',t) \ , \quad \rho(-\vec{r}',t) = \rho(\vec{r}',t) \tag{11.10}$$

where $\vec{r}' = 0$ is taken in the geometric center of the circular loop on the symmetry plane which is perpendicular to $\vec{m}$ as shown. Note that to change $-\vec{r}'$ to $\vec{r}'$ on the loop in figures 6B and 6C simply rotate the loop by π radians around $\vec{r}' = 0$ keeping the loop parallel to the symmetry plane perpendicular to $\vec{m}$; then invert positions between opposite sides of this symmetry plane. The same loop, generators, etc. is obtained again. Note that the symmetries in equations 11.10 for the magnetic dipole antenna with special symmetries are precisely opposite those for the axially and lengthwise symmetric electric dipole antenna given in equation 10.13. Using the same techniques as in the previous section (equation 10.14) the loop can be divided into two volumes of integration V_+ and V_-, the division occuring at one of the symmetry planes of the loop (out of the 3 symmetry planes identified thus far). The electric dipole moment is then just

$$\begin{aligned}
\vec{p}(t) &= \int_{V'} \vec{r}'\rho(\vec{r}',t)dV' = \int_{V_+} \vec{r}'\rho(\vec{r}',t)dV' + \int_{V_-} \vec{r}'\rho(\vec{r}',t)dV' \\
&= \int_{V_+} \vec{r}'\rho(\vec{r}',t)dV' + \int_{V_-} [-\vec{r}']\rho(-\vec{r}',t)dV' \\
&= \vec{0}
\end{aligned} \tag{11.11}$$

Similarly the magnetic dipole moment can be written as

$$\begin{aligned}
\vec{m}(t) &= \frac{1}{2}\int_{V'} \vec{r}'\times\vec{J}(\vec{r}',t)dV' = \frac{1}{2}\int_{V_+} \vec{r}'\times\vec{J}(\vec{r}',t)dV' + \frac{1}{2}\int_{V_-} \vec{r}'\times\vec{J}(\vec{r}')dV' \\
&= \frac{1}{2}\int_{V_+} \vec{r}'\times\vec{J}(\vec{r}',t)dV' + \frac{1}{2}\int_{V_+} [-\vec{r}']\times\vec{J}(-\vec{r}',t)dV' \\
&= \int_{V_+} \vec{r}'\times\vec{J}(\vec{r},t)dV'
\end{aligned} \tag{11.12}$$

Thus the splitting of the integral over V' on a symmetry plane can reduce the volume of integration to calculate the magnetic dipole moment. By splitting the volume V_+ along the remaining symmetry planes the volume of integration for $\vec{m}$ can be even further reduced. While the electric dipole moment is zero this is not in general the case for the electric quadrupole moment as

$$\overset{\leftrightarrow}{Q}(t) = \int_{V'} \vec{r}\,'\vec{r}\,'\rho(\vec{r}\,',t)\,dV' = \int_{V_+} \vec{r}\,'\vec{r}\,'\rho(\vec{r}\,',t)\,dV' + \int_{V_-} \vec{r}\,'\vec{r}\,'\rho(\vec{r}\,',t)\,dV'$$

$$= \int_{V_+} \vec{r}\,'\vec{r}\,'\rho(\vec{r}\,',t)\,dV' + \int_{V_+} [-\vec{r}\,'][-\vec{r}\,']\rho(-\vec{r}\,',t)\,dV'$$

$$= 2\int_{V_+} \vec{r}\,'\vec{r}\,'\rho(\vec{r}\,',t)\,dV' \qquad (11.13)$$

Just looking at figures 6B and 6C with the generator feed polarities one can see a characteristic electric quadrupole charge pattern and thus would not necessarily expect the electric quadrupole moment to be zero. Similarly the scalar moment similar to the electric quadrupole moment is

$$q'(t) = \int_{V'} r'^2\rho(\vec{r}\,',t)\,dV' = \int_{V_+} \vec{r}\,'\cdot\vec{r}\,'\rho(\vec{r}\,',t)\,dV' + \int_{V_-} \vec{r}\,'\cdot\vec{r}\,'\rho(\vec{r}\,',t)\,dV'$$

$$= \int_{V_+} \vec{r}\,'\cdot\vec{r}\,'\rho(\vec{r}\,',t)\,dV' + \int_{V_+} [-\vec{r}\,']\cdot[-\vec{r}\,']\rho(-\vec{r}\,',t)\,dV'$$

$$= 2\int_{V_+} r'^2\rho(\vec{r}\,',t)\,dV' \qquad (11.14)$$

By subsequent division of q' along the additional symmetry planes it can be shown to be zero for our cases under consideration, but $\overset{\leftrightarrow}{Q}$ is not in general zero for these cases. Thus the introduction of the two additional symmetry planes by the use of two symmetrically placed identical generators or generator connections the electric dipole moment is made zero, but not the electric quadrupole moment. Such a loop then has the advantage of giving a closer approximation to the magnetic-dipole fields at low frequencies and large r than a loop driven at one position on its circumference. Of course one could use three or more equally spaced generator connection positions on a circular conducting loop to obtain even more symmetry for the current density and charge density distributions.

Another feature of the symmetry planes of magnetic dipole antennas is that certain of these planes (those perpendicular to the electric field) can be replaced by perfectly conducting planes without affecting the fields, currents, etc. By this procedure the analysis for certain magnetic dipole antennas in free space can be applied to magnetic dipoles in immediate proximity to large conducting ground planes. In the three examples of loops in figure 6 perfectly conducting planes can be placed along what are labelled as additional symmetry planes. Along such planes there is no tangential electric field and these planes are parallel to the magnetic dipole moment. The loops can be split (including splitting the generators as appropriate) along these symmetry planes and one half placed against a conducting ground plane in place of the symmetry plane. The resulting magnetic dipole moment is one half that of the full loop and is parallel to the conducting plane. Compare this to the case of an electric dipole with a conducting plane where the electric dipole moment is perpendicular to the conducting plane. Even though the magnetic dipole moment is cut in half we are only considering fields in a half space (one side of the conducting plane) so that we can consider the "half" loop driving the half space or the full loop (by including the image) driving the full space. Note that we cannot replace the symmetry plane (plane of the loops and parallel to the page in figure 6) by a conducting plane because on this plane the electric field is parallel to the plane by symmetry.

Note some of the possibilities for half circular loops on conducting planes that can be constructed by placing conducting planes along what are called additional symmetry planes in figure 6. From figure 6A we have one possibility of a half circular loop driven at one of its tie points to the conducting plane. This case has a non zero electric dipole moment perpendicular to the conducting plane. On the other hand it is rather simple, having only one generator connected at one position and should then be comparatively easy to construct if one is trying to build a very large pulsed magnetic dipole antenna on a perfectly conducting ground plane (or even a finitely conducting ground plane). The loop in figure 6B could be split along two different symmetry planes. In one case a single generator is placed equidistant from both tie points of the loop to the ground plane; if the ground plane is earth and the semicircular loop is large then the generator is placed high up in the air. In the second case the loop is driven by two identical generators in a push-pull fashion at the tie points of the loop to the ground plane. The loop in figure 6C could also be split along two different symmetry planes. In one case there is a single generator located on or near the ground plane midway between the tie points of the loop to the ground plane and connected to the loop by two conducting leads running perpendicular to the conducting plane. In the second case there is a single generator located on or near the conducting plane (with or without a balanced electrical contact to the ground plane);

the half circular loop has its ends near the ground plane (without contact to it) on opposite sides and equidistant from the generator and connected to it by conducting leads parallel to the ground plane. While these cases allow a single generator near the ground plane to drive the half circular loop in a manner which avoids an electric dipole moment there are disadvantages in the requirement of the conducting leads to connect the generator to the loop and the extra inductance added by these leads. While most of our discussion has been centered on full and half circular loops the same symmetry considerations also apply to other shapes. For example instead of half circular loops adjacent to a conducting plane one might use a rectangular or isosceles triangular shape which could be connected to generators in a manner similar to the half circular loop and give zero electric dipole moment if desired. For very large loop structures this could be a significant consideration because some shapes will in general be easier to construct than others.

The fields (and vector potential) at large r and low frequencies from a magnetic dipole antenna driven by one or more pulse generators are dominated by the magnetic dipole moment. If the antenna is driven by a single pulse generator (or by multiple generators with appropriate symmetry) then the antenna is characterized at low frequencies by two important parameters: an inductance L_a and an area $\vec{A}_a$ which is taken as a vector. The inductance is related to how much current is flowing in the antenna at low frequencies; the area is an area around which the current flows such that the current times the vector area is the magnetic dipole moment. By low frequencies we require that wavelengths be large compared to the antenna dimensions (such as a) and that the current distribution has reached its asymptotic low-frequency form including any effects of loading elements (such as resistors and inductors with associated L/R times) which might be used to modify the current distribution at higher frequencies. For simple loop geometries (such as in figure 6) the area $\vec{A}_a$ is easy to visualize. For more distributed current distribution geometries (such as in figure 5C) the definition of the area (as well as the inductance) depends on how the generators are connected to the antenna at various positions on the somewhat distributed antenna. For simplicity we consider the case of a single generator driving a loop antenna, at least for the examples. The general results apply to more complicated cases as well.

Consider the antenna area $\vec{A}_a$ which we might call the current circulation area of the antenna. This is defined in the static limit by

$$\vec{A}_a \equiv \frac{1}{\tilde{I}} \tilde{\vec{m}} = \frac{1}{\tilde{I}} \tilde{m}\vec{e}_m \qquad (11.15)$$

where I is the current flowing around the loop as driven by the generator(s), say as in figure 6A. The current I is of course the integral of $\vec{J}$ over an area which breaks the loop at some position along its closed path. In many cases it is more appropriate to consider a surface current density $\vec{J}_s$ on the surface of the conducting loop so that I is an integral of $\vec{J}_s$ crossing a line boundary around the loop conductor. Taking the low-frequency asymptotic form of the current density from equations 6.53 as

$$\tilde{\vec{J}}(\vec{r}\,') = f_J(s)\vec{J}_\infty(\vec{r}\,') + o(f_J(s)) \tag{11.16}$$

this gives a low-frequency magnetic dipole moment of the form

$$\begin{aligned}\tilde{\vec{m}} &= f_J(s)\vec{m}_\infty + o(f_J(s)) \\ \vec{m}_\infty &= \frac{1}{2}\int_{V'} \vec{r}\,'\times\vec{J}_\infty(\vec{r}\,')\,dV' = m_\infty\vec{e}_m\end{aligned} \tag{11.17}$$

Note that $\vec{e}_m$ is a unit vector in the direction of the magnetic dipole moment which for a simple loop as in figure 6A with a symmetry plane parallel to the loop has $\vec{e}_m$ perpendicular to this symmetry plane (as shown in the figure). Now $\vec{e}_m$ may be a coordinate axis. Just for the present discussion let $\vec{e}_m$ be parallel to the z' axis and consider a cylindrical (Ψ', ϕ', z') coordinate system based on the center of the loop; this coordinate system is illustrated for the case of an axially and lengthwise symmetric electric dipole in figure 3. Then $\vec{e}_m = \vec{e}_{z'}$ so that the z' components of $\vec{r}\,'$ and $\vec{J}_\infty$ do not contribute to the calculation of $\vec{m}_\infty$. Using $\vec{r}\,' = \Psi'\vec{e}_{\Psi'} + z'\vec{e}_{z'}$ we then have

$$m_\infty = \frac{1}{2}\int_{V'} \Psi' J_{\infty_{\phi'}}(\vec{r}\,')\,dV' \tag{11.18}$$

In later sections when we combine electric and magnetic dipoles $\vec{e}_m$ will not necessarily be in the z' direction.

For $s \to 0$ the current can also be written as

$$\tilde{I} = f_J(s)I_\infty + o(f_J(s)) \tag{11.19}$$

where I_∞ is an integral of $\vec{J}_\infty$ over an area which breaks the loop at some position along its closed path. The current circulation area is then

$$\vec{A}_a = A_a \vec{e}_m$$

$$A_a = \frac{m_\infty}{I} \tag{11.20}$$

As discussed in reference 4 we have $\vec{A}_a = \vec{A}_{eq}$ so that the current circulation area (or area giving the magnetic dipole moment) is the same as the equivalent area of the magnetic dipole antenna as a magnetic field sensor.

Considering the case of a circular loop magnetic dipole antenna as in figure 6A let the major radius be much larger than the minor radius (i.e. a >> b). Since the current is then confined to a thin conductor path (compared to a) we can then approximately calculate the area as

$$A_a \simeq \pi a^2 \tag{11.21}$$

As this area is an important factor in the low-frequency magnetic dipole moment then clearly for big low-frequency magnetic dipole moments make the antenna big.

Now consider the antenna inductance L_a which is taken in the low-frequency limit as a magnetostatic parameter. The inductance is simply the magnetic flux (linking the current) divided by the current. Thus we consider a static situation with current density $\vec{J}$, current I, vector potential $\vec{A}$, and magnetic flux ϕ_a (surface integral of $\vec{B}$ or $\mu_0\vec{H}$) linking the current I. For a transient pulse this static situation applies to two cases. In the first case the current is slowly built up to some peak value prior to its transient interruption by an opening switch in place of a pulse generator which stores energy. In the second case a generator transiently switches current into the antenna and keeps this current flowing in the antenna at late times (by minimizing losses in the antenna and generator in the low-frequency limit and/or continuing to supply energy, perhaps at a reduced rate) until the current distribution has reached equilibrium.

For this magnetostatic situation we can summarize the results of a previous note.[4] The magnetostatic energy stored by the antenna is

$$U_m = \frac{1}{2} L_a I^2 = \frac{1}{2} \int_{V_\infty} \vec{B}(\vec{r}\,') \cdot \vec{H}(\vec{r}\,') dV'$$

$$= \frac{1}{2} \int_{V'} \vec{A}(\vec{r}\,') \cdot \vec{J}(\vec{r}\,') dV' = \frac{\mu_o}{2} \int_{V'} \int_{V'} \frac{\vec{J}(\vec{r}\,') \cdot \vec{J}(\vec{r}\,'')}{4\pi |\vec{r}\,' - \vec{r}\,''|} dV'' dV' \quad (11.22)$$

Using the definition of the inductance as

$$L_a \equiv \frac{\phi_a}{I} \quad (11.23)$$

the flux linking the antenna can be written as

$$\phi_a = \frac{\mu_o}{I} \int_{V'} \int_{V'} \frac{\vec{J}(\vec{r}\,') \cdot \vec{J}(\vec{r}\,'')}{4\pi |\vec{r}\,' - \vec{r}\,''|} dV'' dV' \quad (11.24)$$

while the inductance can be written as

$$L_a = \frac{\mu_o}{I^2} \int_{V'} \int_{V'} \frac{\vec{J}(\vec{r}\,') \cdot \vec{J}(\vec{r}\,'')}{4\pi |\vec{r}\,' - \vec{r}\,''|} dV'' dV' \quad (11.25)$$

where again $\vec{r}\,''$ and dV'' have been used as a second set of integration variables for the double integrals over V'.

The calculation of the inductance L_a depends of course on the antenna geometry being considered. To get an approximate expression for this inductance consider the circular loop antenna as in figure 6A. For $b << a$ the inductance is approximately[10]

$$L_a \simeq \mu_o a \left[\ln\left(\frac{8a}{b}\right) - 2 \right] \quad (11.26)$$

where we have assumed a highly conducting loop with the current concentrated in the form of a surface current density on the surface of the loop conductor. If the current is distributed throughout the volume of the loop conductor then there is a little additional inductance associated with the magnetic field inside the loop conductor. With $a >> b$ then the inductance is roughly proportional to a. For a given low-frequency voltage from the generator driving the antenna the current is proportional to $1/L_a$ and thus to $1/a$. Of course the size of L_a influences the generator voltage at low frequencies. In general, however, decreasing the inductance increases the magnitude of the current which is a factor in the magnetic dipole moment. On the other hand one should not reduce a to increase the magnetic dipole moment because in addition to the current there is the area which is proportional to a^2. Increasing a, while it may decrease the current still tends to increase the low-frequency magnetic dipole moment.

Suppose now that one wishes to maximize the low-frequency content of the pulsed fields from a magnetic dipole antenna. At low frequencies and large r the magnetic-dipole fields are dominant, so the question is one of maximizing the magnetic dipole moment. Note that this problem is the dual of the electric-dipole-antenna problem where the problem is to maximize the low-frequency electric dipole moment. The same considerations apply to the magnetic dipole antenna simply by interchanging dual quantities (fields and dipole moments) as outlined earlier in this section. For the magnetic dipole antenna one would then try to maximize the low-frequency or late-time current (instead of charge as for the electric dipole antenna). Consistent with a finite energy per pulse from the generator we see from equation 11.22 that one can have a current (and thus a magnetic dipole moment) that behaves like a step function for late times or low frequencies (for which it is proportional to $1/s$). Similarly if the current were slowly built up in the antenna a finite energy implies a finite magnitude for I so that in stepping I the transient current still has step-function characteristics. Looking at the magnetic-dipole fields in equations 11.7 and 11.8 let $\tilde{m}$ have its best frequency dependence of $1/s$ at low frequencies for finite energy in a pulse. The far fields are proportional to s implying far-field components with waveforms with at least one zero crossing and a complete time integral equal to zero. The second order fields (r^{-2} terms) are proportional to s^o for step excitation and thus the second order waveforms go to zero at late times and their components can have waveforms without zero crossings. For third order fields (r^{-3} terms) we only have a magnetic field and it is proportional to $1/s$ for step excitation; this is basically the magnetostatic field. If one wishes to have a non zero complete time integral for a waveform then besides the far fields the second order fields must be considered as well; for the magnetic field the third order term further increases the low-frequency content. However for sufficiently large r the second and third order fields can be quite small compared to the far fields for low frequencies of interest. On the other hand for r not too large compared to a the low-frequency contribution can be quite significant.

There are various ways one might try to make a pulsed magnetic dipole antenna give an optimum low-frequency content to the fields so that for low frequencies of interest the magnetic dipole moment has basically a step function behavior (proportional to 1/s). Figure 7 shows a few generator configurations which give this kind of low-frequency performance. These examples are chosen to be duals of the generator configurations used with electric dipole antennas as shown in figure 4. One possibility is to slowly build up the current in the antenna, storing energy U_m in the magnetic field near the antenna, and then at some time, say t = 0, open the switch to stop the current flowing in the antenna as shown in figure 7A. This produces transient fields associated with the change in current and associated magnetic dipole moment. If one has a current I_o and magnetic dipole moment $\vec{m}_o$ just before t = 0 then one can consider

$$\Delta\vec{m}(t) \equiv \vec{m}(t) - \vec{m}_o \tag{11.27}$$

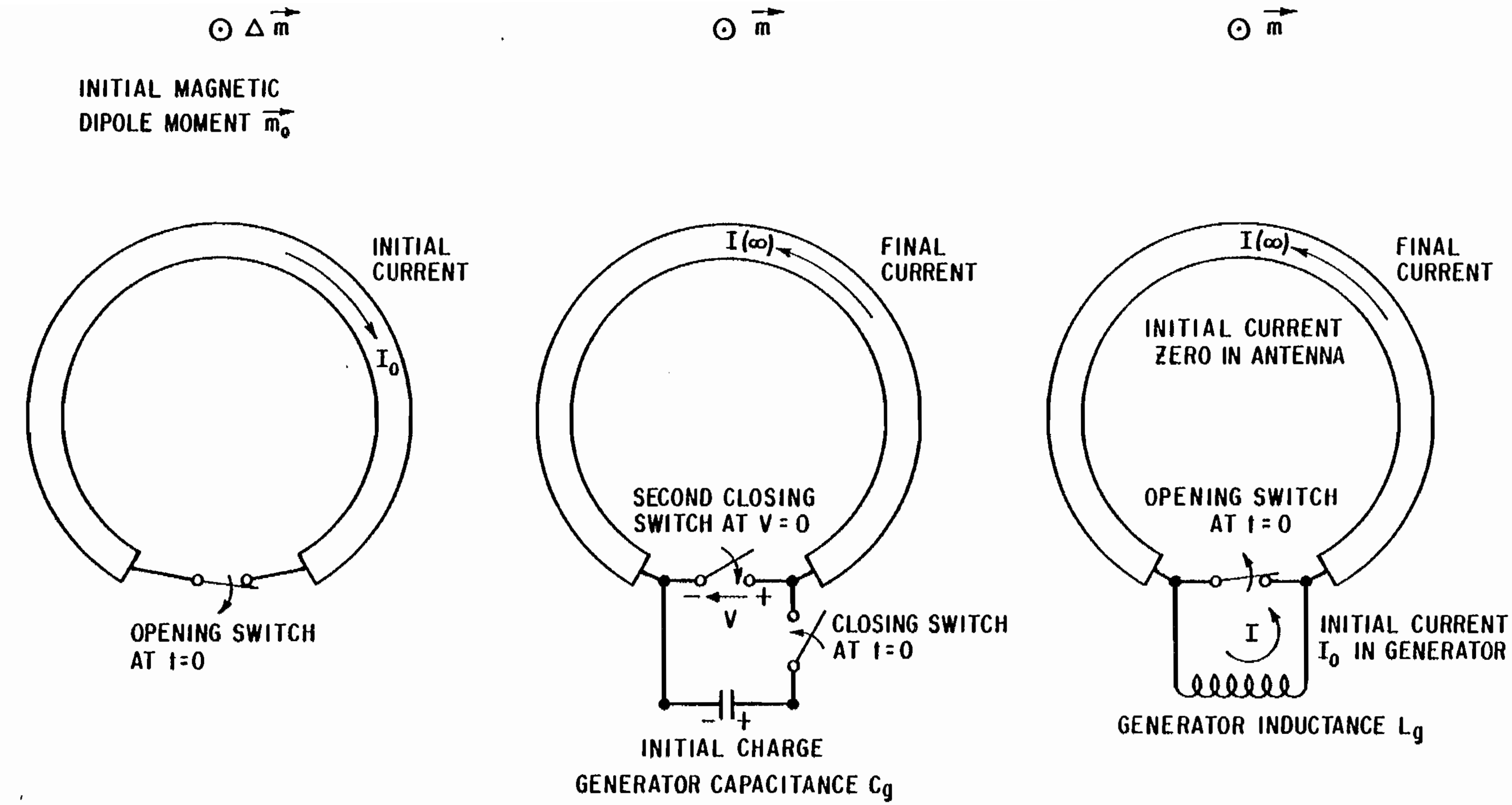
as the change of the magnetic dipole moment and consider the fields, potentials, etc. associated with this change. The magnetostatic energy stored just before t = 0 is

$$U_o = \frac{1}{2} L_a I_o^2 \tag{11.28}$$

and as $t \to \infty$ the change in the magnetic dipole moment is

$$\Delta\vec{m}(\infty) = I_o \vec{A}_a \tag{11.29}$$

For this scheme of pulse generation note that all of the current built up in the antenna contributes to $\Delta\vec{m}(\infty)$. One disadvantage of this scheme is the use of an opening switch which, in the present state of the art, is much slower than a closing switch as is typically used with a capacitive generator. On the other hand in the present state of the art much higher inductive energies (compared to capacitive energies) can be easily built up and stored at least briefly. As with the dual of this system, the discharge of an initially charged electric dipole as in figure 4A, this kind of system where the energy is initially stored by the antenna can have a significant prepulse. If the current on the antenna is built up very slowly then for sufficiently large r the prepulse can be negligible at low frequencies of interest; for r on the same order as the antenna dimensions the low-frequency content of the prepulse can be large indeed and is associated with the r^{-2} and r^{-3} terms in the fields. Considering just the far fields at low frequencies of interest but large enough r that the 1/r terms are dominant, note that the current build up time on the antenna must be long



A. STOPPING OF CURRENT WHICH HAD BEEN SLOWLY BUILT UP

B. CAPACITIVE GENERATOR PLUS A SECOND CLOSING SWITCH (CROWBAR)

C. INDUCTIVE GENERATOR

Figure 7. SOME SCHEMATIC PULSE-GENERATOR CONFIGURATIONS WITH MAGNETIC DIPOLE ANTENNAS

enough that these low frequencies of interest are not affected and the $\Delta\vec{m}$ formulation adequately describe the fields at these low frequencies.

If one would like to drive a magnetic dipole antenna with a capacitive generator, say because of faster switching for the high-frequency performance, then one could use the generator configuration shown in figure 7B. Such a capacitive generator would store an electrostatic energy $C_g V_0^2/2$ where C_g is the generator capacitance and V_o is the voltage on the generator just before switching it onto the antenna at t = 0. Since a capacitor represents an open circuit at low frequencies and we want current flowing in the antenna at late times for a large $\vec{m}(\infty)$ then some additional feature is needed. One solution as shown in figure 7B is to provide a second closing switch (called a crowbar switch) around the generator which is closed when the voltage across the generator has gone to zero so that all the stored energy has been delivered to the antenna. The current then keeps on flowing because of the antenna inductance. Just how much late-time current is flowing in the antenna with this type of generator depends on antenna parameters other than L_a which in turn depends on the particular antenna design and so is not considered here. One disadvantage of this type of generator is, of course, the requirement of the second switch which complicates the generator somewhat. On the other hand this type of generator does use much presently existing capacitive generator technology. Note that this generator scheme is just the dual of that in figure 4B where an inductive generator with two opening switches drives an electric dipole antenna.

Finally in figure 6C we have the example of an inductive generator driving a magnetic dipole antenna. The generator inductance L_g and the initial current I_o in the generator just before t = 0 combine to give a stored energy $L_g I_0^2/2$. By various means, such as by disconnecting the antenna from the generator during current buildup in the generator, the initial current (prepulse) in the antenna can be made negligible. At t = 0 the switch carrying the generator current is opened thereby transiently feeding current into the antenna because of the generator inductance. In the late-time limit the current and magnetic dipole moment are

$$I(\infty) = I_o \frac{L_g}{L_a+L_g}$$

$$\vec{m}(\infty) = I(\infty)\vec{A}_a = I_o \frac{L_g}{L_a+L_g} A_a \vec{e}_m \qquad (11.30)$$

which can be found by considering a step current source of magnitude I_o replacing the switch and driving L_a in parallel with L_g at low frequencies or late times. Actually the current will decay at late times due to resistance in the generator and/or antenna at low frequencies. This decay time needs to be made much longer than times of interest for these late-time results in equations 11.30 to apply. For large magnetic dipole antennas for radiating pulses with large low-frequency content the presence of a late-time magnetic dipole moment is of dominant importance. This type of generator scheme gives such a late-time magnetic dipole moment rather simply, using basically an inductor and a switch. However all the energy in the generator is not converted so as to maximize $\vec{m}(\infty)$ because of an inductive divider effect. In designing a large magnetic dipole antenna with such a pulse generator there is a tradeoff between antenna size (affecting both A_a and L_a) and generator size (as in I_o and L_g) in trying to maximize $\vec{m}(\infty)$ for a large simulator. Note that since this generator also uses an opening switch to transfer current there is a disadvantage in the slower switching speed (compared to a closing switch) in the present state of the art. This generator-antenna combination is the dual of that shown in figure 4C where a capacitive generator is switched with a closing switch onto an electric dipole antenna.

Here we have considered a few types of generator schemes to drive simple magnetic dipole antennas to produce pulsed fields with maximum low-frequency content at large r. For magnetic dipole antennas inductive energy sources combine more simply with the antennas to give the desired low-frequency performance. However the associated opening switches have speed limitations in the present state of the art. Thus if high frequencies are also important one might consider capacitive generators with crowbars as an alternative. Inductive storage has another advantage in the ease with which very large energies can be stored. There are other types of generators one might consider for driving magnetic dipole antennas. For example one might use a high explosive magnetic compression generator which can deliver much energy and is a low-inductance short after the energy has been delivered to the antenna. One might combine large pulse transformers with inductive type generators in an attempt to better impedance match the generator and antenna so as to maximize the energy delivered to the antenna. In order to speed up the output of the inductive generators one might use peaking capacitors with a closing switch placed between the opening switch and the antenna. Note for all these schemes that $\vec{m}(\infty)$ is maintained by the antenna inductance. The antenna resistance in the low frequency limit should be kept sufficiently small that the inductive-resistive decay time is much larger than late times of interest.

With pulsed magnetic dipoles designed to maximize the low-frequency content of the fields the late-time magnetic dipole moment $\vec{m}(\infty)$ is quite important. For simplicity let $\vec{m}(t) = \vec{0}$

for $t < 0$ and let $\vec{m}(\infty) \neq \vec{0}$ so that $\vec{m}$ is a step function for its low frequency content. Consistent with this we take the case where the current density is proportional to 1/s at low frequencies. Thus for $s \to 0$ the current density is

$$\tilde{\vec{J}}(\vec{r}\,') = f_J(s)\vec{J}_\infty(\vec{r}\,') + o(f_J(s))$$

$$f_J(s) = \frac{1}{s} \tag{11.31}$$

so that the magnetic dipole moment for $s \to 0$ is

$$\tilde{\vec{m}} = \frac{1}{s}\,\vec{m}_\infty + o\left(\frac{1}{s}\right) = \frac{1}{s}\,\vec{m}(\infty) + o\left(\frac{1}{s}\right) \tag{11.32}$$

From equations 11.5 and 11.6 the magnetic-dipole potentials for $s \to 0$ are then

$$\tilde{\Phi}_m(\vec{r}) = 0$$

$$\tilde{\vec{A}}_m(\vec{r}) = e^{-\gamma r}\left\{-\frac{\mu_o}{4\pi r^2}\left[\frac{1}{s}\,\vec{e}_r\times\vec{m}(\infty)+o\left(\frac{1}{s}\right)\right]\right.$$

$$\left. -\frac{1}{4\pi r}\,\frac{\mu_o}{c}[\vec{e}_r\times\vec{m}(\infty)+o(1)]\right\} \tag{11.33}$$

and the associated fields from equations 11.7 and 11.8 are

$$\tilde{\vec{E}}_m(\vec{r}) = e^{-\gamma r}\left\{\frac{\mu_o}{4\pi r^2}[\vec{e}_r\times\vec{m}(\infty)+o(1)]\right.$$

$$\left. +\frac{1}{4\pi r}\,\frac{\mu_o}{c}[s\vec{e}_r\times\vec{m}(\infty)+o(s)]\right\}$$

$$\tilde{\vec{H}}_m(\vec{r}) = e^{-\gamma r}\left\{\frac{1}{4\pi r^3}\left[\frac{1}{s}[3\vec{e}_r[\vec{e}_r\cdot\vec{m}(\infty)]-\vec{m}(\infty)]+o\left(\frac{1}{s}\right)\right]\right. \tag{11.34}$$

$$+\frac{1}{4\pi r^2}\,\frac{1}{c}\left[3\vec{e}_r[\vec{e}_r\cdot\vec{m}(\infty)]-\vec{m}(\infty)+o(1)\right]$$

$$\left.+\frac{1}{4\pi r}\,\frac{1}{c^2}\left[s\vec{e}_r\times[\vec{e}_r\times\vec{m}(\infty)]+o(s)\right]\right\}$$

One can compare these low-frequency fields for step-function $\vec{m}$ with those for a step-function electric dipole in equations 10.39 and note the duality. With the fields interchanged the low frequency content is the same as the electric dipole case and the same applies to the implications regarding the time-domain waveforms that can be inferred from the low-frequency results. The far fields (r^{-1} terms) are proportional to s as $s \to 0$ and the waveforms of the far field components have a complete time integral equal to zero and have at least one zero crossing. The second order fields (r^{-2} terms) are proportional to s^0 as $s \to 0$ and the time-domain waveforms of the second order field components go to zero in the late-time limit and need not have a zero crossing. Only the magnetic field has a third order term (r^{-3} term) and it is proportional to s^{-1} as $s \to 0$; the associated time domain waveform need not go to zero in the late time limit and at least for some third order magnetic field component it does not go to zero in the late-time limit. Again for sufficiently large r the time-domain waveforms are dominated by the far fields, at least at those times corresponding to high frequencies. In the late-time limit there is a non zero magnetic field even at large r associated with the third order magnetic field. Also the second order fields make the complete time integrals of the field waveforms non zero even at large r. If, however, low frequencies of interest are restricted to some minimum ω then for large enough r the contribution of the second and third order terms is rather small compared to the far fields; similar considerations apply if late times of interest are limited to some maximum time so that at sufficiently large r the time-domain waveforms are dominated by the far fields.

XII. Combined Electric and Magnetic Dipole Field Distributions

Now that we have considered some of the characteristics of electric and magnetic dipole antennas, let us consider some of the characteristics of the combined potentials and fields of electric and magnetic dipole antennas. To do this consider the potentials and fields produced by ideal or point electric and magnetic dipoles in combination. Thus we define dipole potentials and dipole fields by adding together the results for point dipoles which we called electric dipole potentials and fields and magnetic dipole potentials and fields in the two previous sections. Using a subscript d for this combination we have

$$\begin{aligned}
\Phi_d(\vec{r},t) &= \Phi_p(\vec{r},t) + \Phi_m(\vec{r},t) = \Phi_p(\vec{r},t) \\
\vec{A}_d(\vec{r},t) &= \vec{A}_p(\vec{r},t) + \vec{A}_m(\vec{r},t) \\
\vec{E}_d(\vec{r},t) &= \vec{E}_p(\vec{r},t) + \vec{E}_m(\vec{r},t) \\
\vec{H}_d(\vec{r},t) &= \vec{H}_p(\vec{r},t) + \vec{H}_m(\vec{r},t)
\end{aligned} \tag{12.1}$$

Note that only Φ_p contributes to Φ_d while the remaining three quantities have both electric and magnetic dipole contributions. The ideal electric and magnetic dipoles are both located at $\vec{r}' = \vec{0}$ in our definition of these quantities; this restriction is loosened further on in this section for some additional considerations.

Besides the dipole moments and their unit vectors

$$\vec{p}(t) = p(t)\vec{e}_p \ , \quad \vec{m}(t) = m(t)\vec{e}_m \tag{12.2}$$

where $\vec{e}_p$ and $\vec{e}_m$ may possibly be functions of time, and the unit vector $\vec{e}_r$ pointing from the origin to any position $\vec{r} = r\vec{e}_r$ of interest, let us define a fixed unit vector $\vec{e}_o$ which specifies some particular direction of interest from the dipoles at $\vec{r}' = \vec{0}$. We are going to consider the fields near this direction, i.e. for $\vec{e}_r \simeq \vec{e}_o$ and for convenience $\vec{e}_o$ might be taken as a cartesian unit vector, say $\vec{e}_x$, for problems of interest. Note that $\vec{e}_p$ and $\vec{e}_m$ as used in equations 12.2 are the unit vectors for the time-domain dipole moments and could be changing direction with time, in which case they would not apply directly for the Laplace transformed dipole moments but would have to be redefined in terms of the Laplace transform of the dipole moments. For many problems of interest these unit vectors do have time independent directions and thus also apply directly to the Laplace domain. One case of interest involves dipoles on a ground plane which we idealize as perfectly conducting and use as a symmetry plane or image plane as discussed in the two previous sections; defining the dipole moments as including the images then $\vec{e}_p$ must be perpendicular to the plane and $\vec{e}_m$ parallel to the plane so that $\vec{e}_p$ is perpendicular to $\vec{e}_m$ making $\vec{e}_p \cdot \vec{e}_m = 0$. In this case $\vec{e}_p$ can only have a fixed direction (except for sign reversal which can better be included in the sign of p(t)). As discussed in the following section this case of electric and magnetic dipoles with a ground plane has interesting possibilities.

The potentials and fields from ideal electric and magnetic dipoles placed at $\vec{r}\,' = \vec{0}$ are summarized from the previous section as

$$\tilde{\Phi}_p(\vec{r}) = e^{-\gamma r}\left\{\frac{1}{4\pi r^2}\,\frac{1}{\varepsilon_o}\,\vec{e}_r\cdot\tilde{\vec{p}}+\frac{Z_o}{4\pi r}\,s\vec{e}_r\cdot\tilde{\vec{p}}\right\}$$

$$\tilde{\Phi}_m(\vec{r}) = 0$$

$$\tilde{\vec{A}}_p(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r}\,\mu_o s\tilde{\vec{p}}$$

$$\tilde{\vec{A}}_m(\vec{r}) = e^{-\gamma r}\left\{-\frac{\mu_o}{4\pi r^2}\,\vec{e}_r\times\tilde{\vec{m}}-\frac{1}{4\pi r}\,\frac{\mu_o}{c}\,s\vec{e}_r\times\tilde{\vec{m}}\right\}$$

$$\tilde{\vec{E}}_p(\vec{r}) = e^{-\gamma r}\left\{\frac{1}{4\pi r^3}\,\frac{1}{\varepsilon_o}[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{p}}]-\tilde{\vec{p}}]+\frac{Z_o}{4\pi r^2}\,s[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{p}}]-\tilde{\vec{p}}] +\frac{\mu_o}{4\pi r}\,s^2\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{p}}]\right\}$$

(12.3)

$$\tilde{\vec{E}}_m(\vec{r}) = e^{-\gamma r}\left\{\frac{\mu_o}{4\pi r^2}\,s\vec{e}_r\times\tilde{\vec{m}}+\frac{1}{4\pi r}\,\frac{\mu_o}{c}\,s^2\vec{e}_r\times\tilde{\vec{m}}\right\}$$

$$\tilde{\vec{H}}_p(\vec{r}) = e^{-\gamma r}\left\{-\frac{1}{4\pi r^2}\, s\vec{e}_r\times\tilde{\vec{p}} - \frac{1}{4\pi r}\,\frac{s^2}{c}\,\vec{e}_r\times\tilde{\vec{p}}\right\}$$

$$\tilde{\vec{H}}_m(\vec{r}) = e^{-\gamma r}\left\{\frac{1}{4\pi r^3}\left[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{m}}]-\tilde{\vec{m}}\right]+\frac{1}{4\pi r^2}\,\frac{s}{c}\left[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{m}}]-\tilde{\vec{m}}\right]\right.$$

$$\left. +\frac{1}{4\pi r}\,\frac{s^2}{c^2}\,\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{m}}]\right\}$$

For convenience we have listed these potentials and fields in terms of their Laplace transforms; the time domain forms can be immediately obtained from these.

With a subscript n for the terms proportional to $e^{-\gamma r}r^{-n}$ as used before let us consider the dipole potentials and fields. We have only a finite number of such terms which we can separate out as

$$\tilde{\Phi}_d(\vec{r}) = \Phi_{d_1}(\vec{r}) + \Phi_{d_2}(\vec{r})$$

$$\tilde{\vec{A}}_d(\vec{r}) = \tilde{\vec{A}}_{d_1}(\vec{r}) + \tilde{\vec{A}}_{d_2}(\vec{r})$$

$$\tilde{\vec{E}}_d(\vec{r}) = \tilde{\vec{E}}_{d_1}(\vec{r}) + \tilde{\vec{E}}_{d_2}(\vec{r}) + \tilde{\vec{E}}_{d_3}(\vec{r}) \tag{12.4}$$

$$\tilde{\vec{H}}_d(\vec{r}) = \tilde{\vec{H}}_{d_1}(\vec{r}) = \tilde{\vec{H}}_{d_2}(\vec{r}) + \tilde{\vec{H}}_{d_3}(\vec{r})$$

Consider first the far fields for which we have

$$\tilde{\vec{E}}_{d_1}(\vec{r}) = \tilde{\vec{E}}_{p_1}(\vec{r}) + \tilde{\vec{E}}_{m_1}(\vec{r})$$

$$= \frac{e^{-\gamma r}}{4\pi r}\, s^2\left\{\mu_o\vec{e}_r\times[\vec{e}_r\times\tilde{\vec{p}}]+\frac{\mu_o}{c}\,\vec{e}_r\times\tilde{\vec{m}}\right\}$$

$$= -Z_o \vec{e}_r \times \tilde{\vec{H}}_{d_1}(\vec{r}) \tag{12.5}$$

$$\tilde{\vec{H}}_{d_1}(\vec{r}) = \tilde{\vec{H}}_{p_1}(\vec{r}) + \tilde{\vec{H}}_{m_1}(\vec{r})$$

$$= \frac{e^{-\gamma r}}{4\pi r} s^2 \left\{ -\frac{1}{c} \vec{e}_r \times \tilde{\vec{p}} + \frac{1}{c^2} \vec{e}_r \times [\vec{e}_r \times \tilde{\vec{m}}] \right\}$$

$$= \frac{1}{Z_o} \vec{e}_r \times \tilde{\vec{E}}_{d_1}(\vec{r})$$

At high frequencies (or for fast times) the far fields dominate the second and third order fields. Suppose that for some particular observer located at $\vec{r} = r\vec{e}_o$ (i.e. with $\vec{e}_r = \vec{e}_o$) one wishes to maximize the far-field magnitudes for fixed magnitudes of $\tilde{\vec{p}}$ and $\tilde{\vec{m}}$. This is a question of optimum orientation of $\tilde{\vec{p}}$ and $\tilde{\vec{m}}$ for a fixed direction to the observer, $\vec{e}_o$. Consider the electric-dipole contribution. For the far magnetic field then maximize $|\vec{e}_o \times \tilde{\vec{p}}|$ which for fixed $|\tilde{\vec{p}}|$ implies $\vec{e}_o \cdot \tilde{\vec{p}} = 0$; for the far electric field maximize $|\vec{e}_o \times [\vec{e}_o \times \tilde{\vec{p}}]|$ which leads to the same result. Considering the magnetic-dipole contribution one would maximize $|\vec{e}_o \times \tilde{\vec{m}}|$ and $|\vec{e}_o \times [\vec{e}_o \times \tilde{\vec{m}}]|$ for fixed $|\tilde{\vec{m}}|$ which implies $\vec{e}_o \cdot \tilde{\vec{m}} = 0$ in both cases. Thus we have both electric and magnetic dipole moments perpendicular to the direction to the fixed observer.

In order to maximize the far-field magnitudes one can also look at the relative orientation of the electric and magnetic dipole moments. Consider this question in the time domain where s^2 is replaced by the second partial derivative with respect to time. The dipole moments are real valued vectors and at any time these second time derivatives of $\vec{p}$ and $\vec{m}$ have directions perpendicular to $\vec{e}_o$ from our previous considerations. In order to maximize say the far electric field in the direction $\vec{e}_o$ then for fixed magnitudes of $\ddot{\vec{p}}$ and $\ddot{\vec{m}}$ (using dots above for partial derivatives with respect to time) one would like $\vec{e}_o \times [\vec{e}_o \times \ddot{\vec{p}}]$ and $\vec{e}_o \times \ddot{\vec{m}}$ to have the same direction; this requires that $\ddot{\vec{p}}$ and $\ddot{\vec{m}}$ are perpendicular, i.e. $\ddot{\vec{p}} \cdot \ddot{\vec{m}} = 0$ and further that they are oriented such that $\vec{e}_o \times \ddot{\vec{p}}$ is in the same direction as $\ddot{\vec{m}}$. Considering the far magnetic field one would like $-\vec{e}_o \times \ddot{\vec{p}}$ and $\vec{e}_o \times [\vec{e}_o \times \ddot{\vec{m}}]$ to have the same direction; this leads to the same result as implied by the far electric field. Besides the far fields one might also try to maximize their first and second time integrals because of the relation of these quantities to the low-frequency content of the far fields. If the dipole moments and their time derivatives are zero before t = 0 this would imply $\dot{\vec{p}} \cdot \dot{\vec{m}} = 0$ and $\vec{p} \cdot \vec{m} = 0$ with the orientation of $\vec{e}_o \times \dot{\vec{p}}$ parallel to $\dot{\vec{m}}$ and $\vec{e}_o \times \vec{p}$ parallel to $\vec{m}$, including the same sign (i.e. $(\vec{e}_o \times \vec{p}) \cdot \vec{m} > 0$ etc.). Thus we choose $\vec{e}_o \times \vec{e}_p = \vec{e}_m$ where $p(t)$ and $m(t)$ have the same sign (both plus or both minus at the same t) except for the problem that $p(t)$ and $m(t)$ (or their derivatives, etc.) may have different waveforms so that at certain times they may not have the same sign. Thus for a special case choose $p(t)$ and $m(t)$ to have the same waveform, i.e. $m(t) = \text{constant} \times p(t)$ where the constant is positive (with units).

For our unit vectors $\vec{e}_o$, $\vec{e}_p$, and $\vec{e}_m$ we then have the cyclic relations

$$\vec{e}_o \times \vec{e}_p = \vec{e}_m \ , \quad \vec{e}_p \times \vec{e}_m = \vec{e}_o \ , \quad \vec{e}_m \times \vec{e}_o = \vec{e}_p \qquad (12.6)$$

While $\vec{e}_o$ is a fixed unit vector, $\vec{e}_p$ and $\vec{e}_m$ may be functions of time, rotating around $\vec{e}_o$ while perpendicular to $\vec{e}_o$ and to each other. To summarize where we stand so far for fixed magnitudes of $\vec{p}$, $\vec{m}$, their successive integrals, and a preferred observer direction the dipole far-field magnitudes are maximized by having $\vec{p}$, $\vec{m}$, and $\vec{e}_o$ oriented in the sense of a right handed coordinate system. While $\vec{e}_p$ and $\vec{e}_m$ may be functions of time, an interesting case to consider is with them time independent which corresponds to numerous cases of typical real electric and magnetic dipoles; note that $p(t)$ and $m(t)$ as scalars could be made to change sign together. With these unit vectors all time independent then they could be chosen for convenience in terms of the cartesian unit vectors. For example, if one chose $\vec{e}_p = \vec{e}_z$ as was done in section X, then one might choose $\vec{e}_o = \vec{e}_x$ as the direction of propagation to the preferred observer, and then one would need $\vec{e}_m = -\vec{e}_y$ to fulfill the requirements of equations 12.6. Furthermore with $\vec{e}_p$ and $\vec{e}_m$ fixed one has a stationary symmetry plane perpendicular to $\vec{e}_p$ and parallel to $\vec{e}_o$ and $\vec{e}_m$. The equation of this plane is $\vec{e}_p \cdot \vec{r} = 0$; for our example chosen above with $\vec{e}_p = \vec{e}_z$ etc. this plane is the x y plane. Thus the case of $\vec{e}_p$ and $\vec{e}_m$ being time independent can be used for electric and magnetic dipoles mounted on a conducting plane where $\vec{p}$ is perpendicular to the plane and m parallel to it as required.

Carrying on the development further look at the far fields in the direction opposite to the preferred observer for which $\vec{e}_r = -\vec{e}_o$. From equations 12.5 the far electric field here in the time domain is

$$\vec{E}_{d_1}(-r\vec{e}_o,t) = \frac{1}{4\pi r}\frac{\partial^2}{\partial t^2}\left\{\mu_o\vec{e}_o\times\left[\vec{e}_o\times\vec{p}\left(t-\frac{r}{c}\right)\right]-\frac{\mu_o}{c}\vec{e}_o\times\vec{m}\left(t-\frac{r}{c}\right)\right\}$$

$$= \frac{1}{4\pi r}\frac{\partial^2}{\partial t^2}\left\{\mu_o\vec{e}_p\left[-p\left(t-\frac{r}{c}\right)+\frac{1}{c}\,m\left(t-\frac{r}{c}\right)\right]\right\} \tag{12.7}$$

Thus with our constraints on the relation of $\vec{e}_o$, $\vec{e}_p$, and $\vec{e}_m$ the far electric field in the direction $\vec{e}_r = -\vec{e}_o$, as well as its time integrals and derivatives, can be made zero by requiring

$$p(t) = \frac{1}{c}\,m(t) \tag{12.8}$$

In vector form we can write this special constraint as

$$\vec{m}(t) = c\vec{e}_o\times\vec{p}(t)\ ,\qquad \vec{p}(t) = -\frac{1}{c}\,\vec{e}_o\times\vec{m}(t)$$

$$\vec{e}_o\cdot\vec{p}(t) = 0\ ,\qquad \vec{e}_o\cdot\vec{m}(t) = 0 \tag{12.9}$$

With this constraint we have a directional radiator which has maximum magnitude in the direction $\vec{e}_r = \vec{e}_o$ and zero in the direction $\vec{e}_r = -\vec{e}_o$. Note that the result of equation 12.8 is consistent with the previously mentioned special case where $p(t)$ is equal to some positive constant times $m(t)$ so that their contributions to the far fields add in the direction of the preferred observer. By making this positive constant equal to $1/c$ we also maximize the directional radiation characteristics of the combined electric and magnetic dipoles. Note that this directionality applies at all frequencies for the far field.

For special study then let us constrain the electric and magnetic dipole moments together with the special fixed direction $\vec{e}_o$ as summarized in equations 12.9. Denote this special case by using a subscript c in place of d for the potentials and fields. From equations 12.5 the far fields for this special case are

$$\tilde{\vec{E}}_{c_1}(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r} \mu_o s^2\{-\tilde{\vec{p}}[1+\vec{e}_r\cdot\vec{e}_o]+[\vec{e}_r+\vec{e}_o][\vec{e}_r\cdot\tilde{\vec{p}}]\}$$

$$= \frac{e^{-\gamma r}}{4\pi r} \frac{\mu_o}{c} s^2\{[\vec{e}_r\times\tilde{\vec{m}}][1+\vec{e}_r\cdot\vec{e}_o]-[\vec{e}_r\times\vec{e}_o][\vec{e}_r\cdot\tilde{\vec{m}}]\}$$

$$\tilde{\vec{H}}_{c_1}(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r} \frac{s^2}{c^2}\{-\tilde{\vec{m}}[1+\vec{e}_r\cdot\vec{e}_o]+[\vec{e}_r+\vec{e}_o][\vec{e}_r\cdot\tilde{\vec{m}}]\} \qquad (12.10)$$

$$= \frac{e^{-\gamma r}}{4\pi r} \frac{s^2}{c}\{-[\vec{e}_r\times\tilde{\vec{p}}][1+\vec{e}_r\cdot\vec{e}_o]+[\vec{e}_r\times\vec{e}_o][\vec{e}_r\cdot\tilde{\vec{p}}]\}$$

These results give the far fields in terms of $\vec{p}$ and $\vec{m}$ separately by substituting from equations 12.9 into equations 12.5 and expanding the vector combinations using the relations in reference 3. These are only a few of the forms; others, for example, can easily be generated by substituting from equations 12.9 to replace $\vec{p}$ by $\vec{m}$ and conversely in equations 12.10. In the preferred observer direction where $\vec{e}_r = \vec{e}_o$ the far fields have the simple forms

$$\tilde{\vec{E}}_{c_1}(r\vec{e}_o) = \frac{e^{-\gamma r}}{4\pi r} \mu_o s^2\{-2\tilde{\vec{p}}\}$$

$$\tilde{\vec{H}}_{c_1}(r\vec{e}_o) = \frac{e^{-\gamma r}}{4\pi r} \frac{s^2}{c^2}\{-2\tilde{\vec{m}}\} \qquad (12.11)$$

For our special case specified by equations 12.9 the electric and magnetic dipoles contribute equally to the far fields in the preferred observer direction. Comparing equations 12.11 to 12.10 one can readily see how the far fields vary near $\vec{e}_r = \vec{e}_o$ and that for small $\vec{e}_r \times \vec{e}_o$ and small changes in r (compared to r) the far fields are quite uniform when the delay $e^{-\gamma r}$ is taken into account. Also note in equations 12.10 that expressing the electric field in terms of $Z_o\tilde{\vec{p}}$ and the magnetic field in terms of $\tilde{\vec{m}}/c$ the two forms are the same.

The second order fields for our special case are

$$\tilde{\vec{E}}_{c_2}(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^2}\, s\left\{Z_o[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{p}}]-\tilde{\vec{p}}]+\mu_o\vec{e}_r\times\tilde{\vec{m}}\right\}$$

$$= \frac{e^{-\gamma r}}{4\pi r^2}\, Z_o s\{-\tilde{\vec{p}}[1+\vec{e}_r\cdot\vec{e}_o]+[3\vec{e}_r+\vec{e}_o][\vec{e}_r\cdot\tilde{\vec{p}}]\}$$

(12.12)

$$\tilde{\vec{H}}_{c_2}(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^2}\, s\left\{-\vec{e}_r\times\tilde{\vec{p}}+\frac{1}{c}[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{m}}]-\tilde{\vec{m}}]\right\}$$

$$= \frac{e^{-\gamma r}}{4\pi r^2}\, \frac{s}{c}\{-\tilde{\vec{m}}[1+\vec{e}_r\cdot\vec{e}_o]+[3\vec{e}_r+\vec{e}_o][\vec{e}_r\cdot\tilde{\vec{m}}]\}$$

As with the far fields the second order fields are symmetric in that the electric field in terms of $Z_o\tilde{\vec{p}}$ has exactly the same form as the magnetic field in terms of $\tilde{\vec{m}}/c$. Unlike the far fields, the second order fields have r components so they are not as simply related to each other as the far fields as indicated in equations 12.5. However the transverse parts (perpendicular to $\vec{e}_r$) are perpendicular to each other and are related in the same way as the far fields in equations 12.5. In the direction to the preferred observer we have the simple forms

$$\tilde{\vec{E}}_{c_2}(r\vec{e}_o) = \frac{e^{-\gamma r}}{4\pi r^2}\, Z_o s\{-2\tilde{\vec{p}}\}$$

(12.13)

$$\tilde{\vec{H}}_{c_2}(r\vec{e}_o) = \frac{e^{-\gamma r}}{4\pi r^2}\, \frac{s}{c}\{-2\tilde{\vec{m}}\}$$

Note the similarity of this result to that for the far fields in equations 12.11. Also note that there are no r components in the preferred observer direction and the second order fields are TEM there with

$$\vec{e}_o\times\tilde{\vec{E}}_{c_2}(r\vec{e}_o) = Z_o\tilde{\vec{H}}_{c_2}(r\vec{e}_o)\ ,\quad \vec{e}_o\times\tilde{\vec{H}}_{c_2}(r\vec{e}_o) = -\frac{1}{Z_o}\,\tilde{\vec{E}}_{c_2}(r\vec{e}_o) \quad (12.14)$$

In the direction opposite to the preferred observer the second order fields are zero, just like the far fields.

The third order fields for our special case are

$$\tilde{\vec{E}}_{c_3}(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^3}\frac{1}{\epsilon_o}\{3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{p}}]-\tilde{\vec{p}}\}$$

$$\tilde{\vec{H}}_{c_3}(\vec{r}) = \frac{e^{-\gamma r}}{4\pi r^3}\{3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{m}}]-\tilde{\vec{m}}\} \tag{12.15}$$

These third order fields are also symmetric in that $Z_o\tilde{\vec{p}}$ is replaced by $\tilde{\vec{m}}/c$ in going from electric to magnetic fields. The third order fields have r components and their transverse parts are not in general perpendicular to each other. In the direction to the preferred observer we have

$$\tilde{\vec{E}}_{c_3}(r\vec{e}_o) = \frac{e^{-\gamma r}}{4\pi r^3}\frac{1}{\epsilon_o}\{-\tilde{\vec{p}}\}$$

$$\tilde{\vec{H}}_{c_3}(r\vec{e}_o) = \frac{e^{-\gamma r}}{4\pi r^3}\{-\tilde{\vec{m}}\} \tag{12.16}$$

so that in the preferred observer direction the third order fields also have the TEM relation

$$\vec{e}_o\times\tilde{\vec{E}}_{c_3}(r\vec{e}_o) = Z_o\tilde{\vec{H}}_{c_3}(r\vec{e}_o)\ ,\quad \vec{e}_o\times\tilde{\vec{H}}_{c_3}(r\vec{e}_o) = -\frac{1}{Z_o}\tilde{\vec{E}}_{c_3}(r\vec{e}_o) \tag{12.17}$$

In the direction opposite the preferred observer the third order fields are

$$\tilde{\vec{E}}_{c_3}(-r\vec{e}_o) = \frac{e^{-\gamma r}}{4\pi r^3}\frac{1}{\epsilon_o}\{-\tilde{\vec{p}}\}$$

$$\tilde{\vec{H}}_{c_3}(-r\vec{e}_o) = \frac{e^{-\gamma r}}{4\pi r^3}\{-\tilde{\vec{m}}\} \tag{12.18}$$

which is the same result as in equations 12.16 for the preferred observer direction.

Combining these results the fields for the special case of electric and magnetic dipole moments are

$$\tilde{\vec{E}}_c(\vec{r}) = e^{-\gamma r}\left\{\frac{1}{4\pi r^3}\frac{1}{\varepsilon_o}[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{p}}]-\tilde{\vec{p}}] + \frac{Z_o}{4\pi r^2}s[-\tilde{\vec{p}}[1+\vec{e}_r\cdot\vec{e}_o]+[3\vec{e}_r+\vec{e}_o][\vec{e}_r\cdot\tilde{\vec{p}}]] + \frac{\mu_o}{4\pi r}s^2[-\tilde{\vec{p}}[1+\vec{e}_r\cdot\vec{e}_o]+[\vec{e}_r+\vec{e}_o][\vec{e}_r\cdot\tilde{\vec{p}}]]\right\}$$

$$\tilde{\vec{H}}_c(\vec{r}) = e^{-\gamma r}\left\{\frac{1}{4\pi r^3}[3\vec{e}_r[\vec{e}_r\cdot\tilde{\vec{m}}]-\tilde{\vec{m}}] + \frac{1}{4\pi r^2}\frac{s}{c}[-\vec{m}[1+\vec{e}_r\cdot\vec{e}_o]+[3\vec{e}_r+\vec{e}_o][\vec{e}_r\cdot\tilde{\vec{m}}]] + \frac{1}{4\pi r}\frac{s^2}{c^2}[-\tilde{\vec{m}}[1+\vec{e}_r\cdot\vec{e}_o]+[\vec{e}_r+\vec{e}_o][\vec{e}_r\cdot\tilde{\vec{m}}]]\right\} \tag{12.19}$$

For the complete fields for our special case again note the symmetry in that $Z_o\tilde{\vec{p}}$ can be replaced by $\tilde{\vec{m}}/c$ (or $\tilde{\vec{p}}$ by $\varepsilon_o\tilde{\vec{m}}$) to convert the electric field to the magnetic field. In the preferred observer direction ($\vec{e}_r = \vec{e}_o$) the fields are

$$\tilde{\vec{E}}_c(r\vec{e}_o) = e^{-\gamma r}\left\{\frac{1}{4\pi r^3}\frac{1}{\varepsilon_o}[-\tilde{\vec{p}}]+\frac{Z_o}{4\pi r^2}s[-2\tilde{\vec{p}}]+\frac{\mu_o}{4\pi r}s^2[-2\tilde{\vec{p}}]\right\}$$

$$\tilde{\vec{H}}_c(r\vec{e}_o) = e^{-\gamma r}\left\{\frac{1}{4\pi r^3}[-\tilde{\vec{m}}]+\frac{1}{4\pi r^2}\frac{s}{c}[-2\tilde{\vec{m}}]+\frac{1}{4\pi r}\frac{s^2}{c^2}[-2\tilde{\vec{m}}]\right\} \tag{12.20}$$

In this direction the fields are TEM with the relations

$$\vec{e}_o \times \tilde{\vec{E}}_c(r\vec{e}_o) = Z_o \tilde{\vec{H}}_c(r\vec{e}_o) \ , \quad \vec{e}_o \times \tilde{\vec{H}}_c(r\vec{e}_o) = -\frac{1}{Z_o} \tilde{\vec{E}}_c(r\vec{e}_o) \tag{12.21}$$

Note that this holds for all frequencies or times and for all $r > 0$. Thus along $\vec{r} = r\vec{e}_o$ we have TEM fields at all frequencies including not just the far fields but the higher order terms as well. This special case then may be used to produce a TEM field distribution over a volume of space centered on a particular direction ($\vec{e}_r = \vec{e}_o$) from a "radiating" antenna at all frequencies; it just requires approximating the ideal crossed dipoles as in equations 12.9. For certain applications this could be used for producing high quality fields. In the direction opposite to the preferred observer we have

$$\tilde{\vec{E}}_c(-r\vec{e}_o) = \frac{e^{-\gamma r}}{4\pi r^3} \frac{1}{\varepsilon_o}[-\tilde{\vec{p}}]$$

$$\tilde{\vec{H}}_c(-r\vec{e}_o) = \frac{e^{-\gamma r}}{4\pi r^3}[-\tilde{\vec{m}}] \tag{12.22}$$

Only the third order fields are produced in what might be called the back direction. This special case of crossed dipoles is then a directional radiator, radiating toward $\vec{e}_r = \vec{e}_o$ while somewhat minimizing the back radiation. In terms of crossed electric and magnetic dipole sensors with appropriately matched sensitivities then reciprocity would indicate that one could use the pair as a sensor sensitive to the direction of propagation with respect to $\vec{e}_o$.

For completeness consider the potentials for our special case for which we have

$$\tilde{\Phi}_c(\vec{r}) = e^{-\gamma r}\left\{\frac{1}{4\pi r^2} \frac{1}{\varepsilon_o} \vec{e}_r \cdot \tilde{\vec{p}} + \frac{Z_o}{4\pi r} s\vec{e}_r \cdot \tilde{\vec{p}}\right\}$$

$$\tilde{\vec{A}}_c(\vec{r}) = e^{-\gamma r}\left\{-\frac{\mu_o}{4\pi r^2} \vec{e}_r \times \tilde{\vec{m}} - \frac{1}{4\pi r} \frac{\mu_o}{c} s[\vec{e}_o + \vec{e}_r] \times \tilde{\vec{m}}\right\} \tag{12.23}$$

where many other forms can of course be written. In the direction of the preferred observer we have

$$\tilde{\Phi}_c(r\vec{e}_o) = 0$$

$$\tilde{\vec{A}}_c(r\vec{e}_o) = e^{-\gamma r}\left\{\frac{Z_o}{4\pi r^2}\tilde{\vec{p}}+\frac{\mu_o}{4\pi r}s[2\tilde{\vec{p}}]\right\} \tag{12.24}$$

and in the opposite direction we have

$$\tilde{\Phi}_c(-r\vec{e}_o) = 0$$

$$\tilde{\vec{A}}_c(-r\vec{e}_o) = e^{-\gamma r}\left\{\frac{Z_o}{4\pi r^2}[-\tilde{\vec{p}}]\right\} \tag{12.25}$$

Note then some directionality of the vector potential in that there is a far vector potential at $\vec{r} = r\vec{e}_o$ but no far vector potential at $\vec{r} = -r\vec{e}_o$ (where $r > 0$).

As in previous sections consider the response at low frequencies for electric and magnetic dipole moments which are zero for $t < 0$ and go to constant values in the late time limit so that we have step-function-like excitation at low frequencies. Thus for $s \to 0$ we take (from equations 10.37 and 11.32)

$$\tilde{\vec{p}} = \frac{1}{s}\,p_\infty + O\left(\frac{1}{s}\right) = \frac{1}{s}\,\vec{p}(\infty) + O\left(\frac{1}{s}\right) = \frac{1}{s}\,p(\infty)\vec{e}_p + O\left(\frac{1}{s}\right)$$

$$\tilde{\vec{m}} = \frac{1}{s}\,\vec{m}_\infty + O\left(\frac{1}{s}\right) = \frac{1}{s}\,\vec{m}(\infty) + O\left(\frac{1}{s}\right) = \frac{1}{s}\,m(\infty)\vec{e}_m + O\left(\frac{1}{s}\right) \tag{12.26}$$

with

$$\vec{m}(\infty) = c\vec{e}_o \times \vec{p}(\infty)\ , \quad \vec{p}(\infty) = -\frac{1}{c}\,\vec{e}_o \times \vec{m}(\infty)$$

$$m(\infty) = cp(\infty) \tag{12.27}$$

The fields for our special case of crossed electric and magnetic dipoles are for $s \to 0$

$$\tilde{\vec{E}}_c(\vec{r}) = e^{-\gamma r}\left\{\frac{1}{4\pi r^3}\,\frac{1}{\varepsilon_o}\left[\frac{1}{s}[3\vec{e}_r[\vec{e}_r\cdot\vec{p}(\infty)]-\vec{p}(\infty)]+o\left(\frac{1}{s}\right)\right]\right.$$

$$+\frac{Z_o}{4\pi r^2}[-\vec{p}(\infty)[1+\vec{e}_r\cdot\vec{e}_o]+[3\vec{e}_r+\vec{e}_o][\vec{e}_r\cdot\vec{p}(\infty)]+o(1)]$$

$$\left.+\frac{\mu_o}{4\pi r^2}\left[s[-\vec{p}(\infty)[1+\vec{e}_r\cdot\vec{e}_o]+[\vec{e}_r+\vec{e}_o][\vec{e}_r\cdot\vec{p}(\infty)]]+o(s)\right]\right\}$$

(12.28)

$$\tilde{\vec{H}}_c(\vec{r}) = e^{-\gamma r}\left\{\frac{1}{4\pi r^3}\left[\frac{1}{s}[3\vec{e}_r[\vec{e}_r\cdot\vec{m}(\infty)]-\vec{m}(\infty)]+o\left(\frac{1}{s}\right)\right]\right.$$

$$+\frac{1}{4\pi r^2}\,\frac{1}{c}[-\vec{m}(\infty)[1+\vec{e}_r\cdot\vec{e}_o]+[3\vec{e}_r+\vec{e}_o][\vec{e}_r\cdot\vec{m}(\infty)]+o(1)]$$

$$\left.+\frac{1}{4\pi r}\,\frac{1}{c^2}\left[s[-\vec{m}(\infty)[1+\vec{e}_r\cdot\vec{e}_o]+[\vec{e}_r+\vec{e}_o][\vec{e}_r\cdot\vec{m}(\infty)]]+o(s)\right]\right\}$$

and the potentials for $s \to 0$ are

$$\tilde{\Phi}_c(\vec{r}) = e^{-\gamma r}\left\{\frac{1}{4\pi r^2}\,\frac{1}{\varepsilon_o}\left[\frac{1}{s}\,\vec{e}_r\cdot\vec{p}(\infty)+o\left(\frac{1}{s}\right)\right]\right.$$

$$\left.+\frac{Z_o}{4\pi r}[\vec{e}_r\cdot\vec{p}(\infty)+o(1)]\right\}$$

(12.29)

$$\tilde{\vec{A}}_c(\vec{r}) = e^{-\gamma r}\left\{-\frac{\mu_o}{4\pi r^2}\left[\frac{1}{s}\,\vec{e}_r\times\vec{m}(\infty)+o\left(\frac{1}{s}\right)\right]\right.$$

$$\left.-\frac{1}{4\pi r}\,\frac{\mu_o}{c}\,[\vec{e}_o+\vec{e}_r]\times\vec{m}(\infty)+o(1)\right\}$$

In the direction of the preferred observer the fields and potentials for $s \to 0$ are

$$\tilde{\vec{E}}_c(r\vec{e}_o) = e^{-\gamma r}\left\{\frac{1}{4\pi r^3}\,\frac{1}{\varepsilon_o}\left[-\frac{1}{s}\,\vec{p}(\infty)+o\left(\frac{1}{s}\right)\right]+\frac{Z_o}{4\pi r^2}[-2\vec{p}(\infty)+o(1)]\right.$$

$$\left.+\frac{\mu_o}{4\pi r}\left[-s[2\vec{p}(\infty)]+o(s)\right]\right\}$$

$$\tilde{\vec{H}}_c(r\vec{e}_o) = e^{-\gamma r}\left\{\frac{1}{4\pi r^3}\left[-\frac{1}{s}\,\vec{m}(\infty)+o\left(\frac{1}{s}\right)\right]+\frac{1}{4\pi r^2}\,\frac{1}{c}[-2\vec{m}(\infty)+o(1)]\right. \tag{12.30}$$

$$\left.+\frac{1}{4\pi r}\,\frac{1}{c^2}\left[-s[2\vec{m}(\infty)]+o(s)\right]\right\}$$

$$\tilde{\Phi}_c(r\vec{e}_o) = 0$$

$$\tilde{\vec{A}}_c(r\vec{e}_o) = e^{-\gamma r}\left\{\frac{Z_o}{4\pi r^2}\left[\frac{1}{s}\,\vec{p}(\infty)+o\left(\frac{1}{s}\right)\right]+\frac{\mu_o}{4\pi r}[2\vec{p}(\infty)+o(1)]\right\}$$

As discussed in the previous two sections real electric and magnetic dipoles can maximize their low-frequency outputs for finite energy supplied in a single pulse by making the electric and magnetic dipole moments respectively behave as step functions for their late time behavior. Using the results of this section for the special case of crossed electric and magnetic dipoles we see that in the vicinity of the preferred observer direction ($\vec{e}_r \simeq \vec{e}_o$) the field distribution is both uniform and approximately TEM for all frequencies and r for which the electric and magnetic dipole moments are dominant. One could then use this technique for producing an electromagnetic pulse at some distance from the source with maximum low frequency content in both electric and magnetic fields, and high field purity (at low frequency) in the sense of uniform and TEM with electric and magnetic fields related by Z_o centered on a direction $\vec{e}_r = \vec{e}_o$. The results for this special case are schematically shown in figure 8 with dipole moments and fields indicated; the field polarities are based on the far fields and show the relative polarities of the different fields.

Crossed ideal electric and magnetic dipoles can then be specified so as to give fields with some rather ideal characteristics near $\vec{e}_r = \vec{e}_o$. Of course real electric and magnetic dipole antennas do not have precisely the electric and magnetic dipole fields respectively as summarized in equations 12.3, if only because other moments are also present on real electric

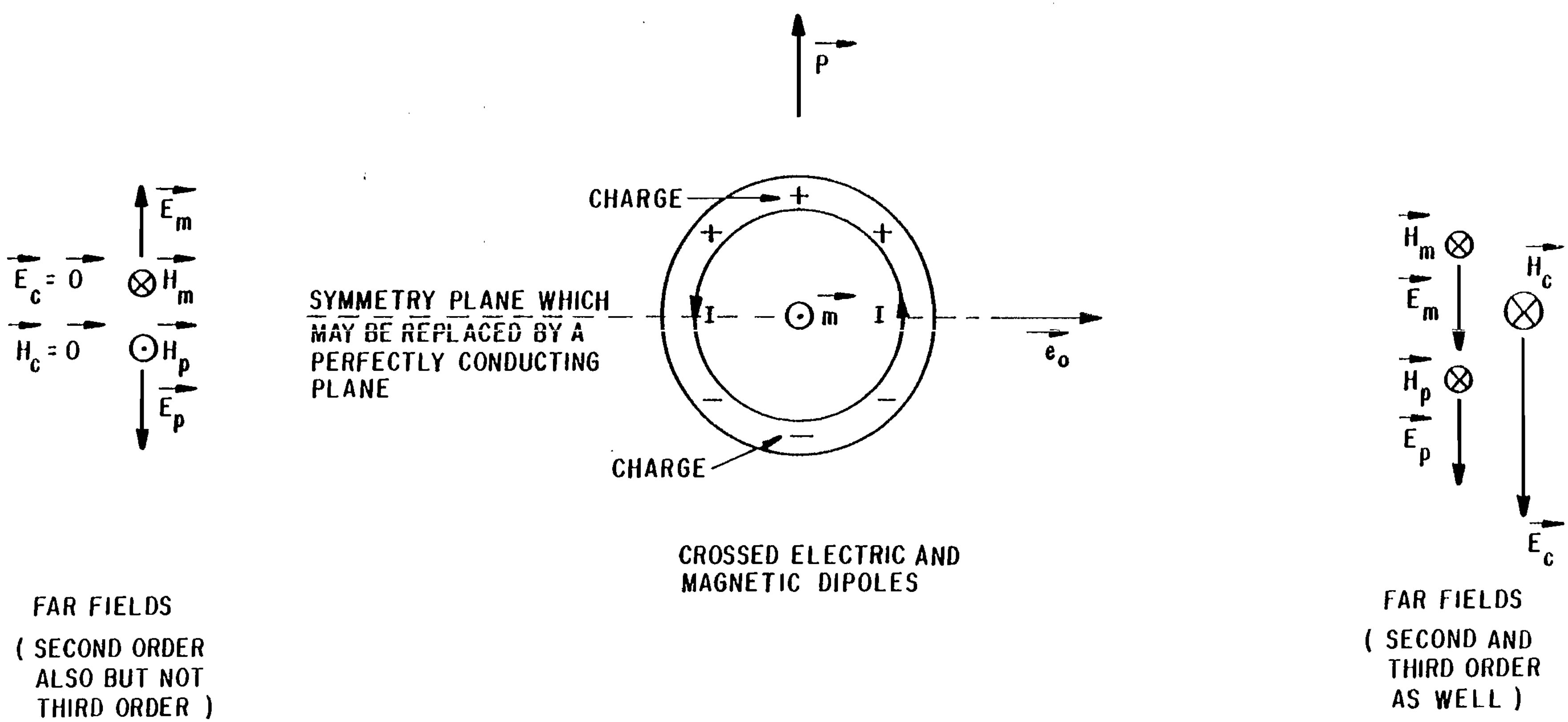


FIGURE 8. SPECIAL CASE OF CROSSED ELECTRIC AND MAGNETIC DIPOLES IN FREE SPACE FOR FIELD ADDITION IN PREFERRED DIRECTION

and magnetic dipole antennas. In addition there is the problem of making the electric and magnetic dipole moments precisely related by c as in equations 12.9 at all times or frequencies of interest. Furthermore for the ideal results for crossed dipoles the electric and magnetic dipoles have been point dipoles both situated at $\vec{r}' = \vec{0}$. In building a real set of crossed electric and magnetic dipoles one might try to build them both as one structure centered on $\vec{r}' = \vec{0}$ with various symmetries included in the design so as to make some of the higher order moments zero. Alternatively one might build the two dipoles as separate structures which are separated not too far apart but controlled together (for waveform, timing, etc.). With choice of $\vec{p}$ and $\vec{m}$ one can still try to optimize the field distribution in the vicinity of some direction $\vec{e}_r = \vec{e}_o$.

Suppose now that $\vec{p}(t)$ is located at $\vec{r}' = \vec{r}_p$ and $\vec{m}(t)$ is located at $\vec{r}' = \vec{r}_m$. In terms of a coordinate system centered on $\vec{r}' = \vec{0}$ associated with both $\vec{p}$ and $\vec{m}$ there are in general an infinite number of multipole moments, and associated with these multipole moments are various terms in an asymptotic expansion of the potentials and fields for $r \to \infty$, with an infinite number of terms of the form $e^{-\gamma r}r^{-n}$ combined with the multipole moments. Thus we could take the infinitesimal dipoles defined as limits in sections X and XI and displace their centers to $\vec{r}_p$ and $\vec{r}_m$ and calculate all the moments as required for these charge and current distributions. Then one could calculate the various r^{-n} terms as done in sections VI through VIII or expand for low frequencies as done in section IX. However in making these expansions for displaced point dipoles some simplicity is lost because with them located at $\vec{r}' = \vec{0}$ the complete fields are simply expressed using only $\vec{p}$ and $\vec{m}$ together with r^{-1} through r^{-3} terms only; the expansion is finite both in moments and in terms of the form $e^{-\gamma r}r^{-n}$.

For our present purposes let us keep the simpler form with only a few terms for the fields and potentials by taking the electric dipole potentials and fields as in equations 12.3 and everywhere replace $\vec{r}$ by $\vec{r} - \vec{r}_p$, r by $|\vec{r} - \vec{r}_p|$, and $\vec{e}_r$ by $(\vec{r} - \vec{r}_p)/|\vec{r} - \vec{r}_p|$; similarly for the magnetic dipole potentials and fields in equations 12.3 everywhere replace $\vec{r}$ by $\vec{r} - \vec{r}_m$, r by $|\vec{r} - \vec{r}_m|$, and $\vec{e}_r$ by $(\vec{r} - \vec{r}_m)/|\vec{r} - \vec{r}_m|$. Then consider what might be optimum choices for $\vec{r}_m$ and $\vec{r}_p$ so as to give what might be considered an optimum field distribution in some region of space centered on a direction $\vec{e}_r = \vec{e}_o$ where r is large compared to r_p and r_m so that we wish to best approximate in some sense the results of the special case of crossed electric and magnetic dipoles already considered in this section. For convenience we denote this case by use of a prime with the electromagnetic quantity so that we can directly carry over all the previous quantities. Where we have used subscripts 1, 2, 3 to denote inverse powers of r they now apply to inverse powers of $|\vec{r} - \vec{r}_p|$ and $|\vec{r} - \vec{r}_m|$ and sums of such inverse powers.

Consider first the change of the far fields to these new coordinate centers. For the far electric field we have

$$\tilde{\vec{E}}'_{d_1}(\vec{r}) = \tilde{\vec{E}}'_{p_1}(\vec{r}) + \tilde{\vec{E}}'_{m_1}(\vec{r})$$

$$\tilde{\vec{E}}'_{p_1}(\vec{r}) = \frac{e^{-\gamma|\vec{r}-\vec{r}_p|}}{4\pi|\vec{r}-\vec{r}_p|}\,\mu_o s^2\left(\frac{\vec{r}-\vec{r}_p}{|\vec{r}-\vec{r}_p|}\right) \times \left[\left(\frac{\vec{r}-\vec{r}_p}{|\vec{r}-\vec{r}_p|}\right) \times \tilde{\vec{p}}\right] \qquad (12.31)$$

$$\tilde{\vec{E}}'_{m_1}(\vec{r}) = \frac{e^{-\gamma|\vec{r}-\vec{r}_m|}}{4\pi|\vec{r}-\vec{r}_m|}\,\frac{\mu_o}{c}\, s^2\left(\frac{\vec{r}-\vec{r}_m}{|\vec{r}-\vec{r}_m|}\right) \times \tilde{\vec{m}}$$

As $r \to \infty$ we can neglect $\vec{r}_p$ and $\vec{r}_m$ compared with $\vec{r}$ in all except the exponential delay terms. Then as $r \to \infty$ we can write the first terms as

$$\tilde{\vec{E}}'_{p_1}(\vec{r}) = \frac{e^{-\gamma r+\gamma\vec{e}_r\cdot\vec{r}_p}}{4\pi r}\,\mu_o s^2\vec{e}_r \times [\vec{e}_r\times\tilde{\vec{p}}] + O(r^{-2})$$

$$(12.32)$$

$$\tilde{\vec{E}}'_{m_1}(\vec{r}) = \frac{e^{-\gamma r+\gamma\vec{e}_r\cdot\vec{r}_m}}{4\pi r}\,\frac{\mu_o}{c}\, s^2\vec{e}_r \times \tilde{\vec{m}} + O(r^{-2})$$

where these asymptotic expansions follow directly from those in section V involving $|\vec{r} - \vec{r}'|$ as $r \to \infty$. For $r \to \infty$ then our displaced dipoles give a far electric field just like that in equations 12.5 except for the additional exponential time-shifting (advance or delay) factors. Combining these additional scalar exponential factors with the dipole moments they have exactly the same form. Thus we can follow the same line of reasoning in going from equations 12.5 to 12.6 involved in maximizing the fields in the direction $\vec{e}_r = \vec{e}_o$, giving as a direct carryover of equations 12.6

$$\vec{e}_o \times \vec{e}_p = \vec{e}_m\,, \quad \vec{e}_p \times \vec{e}_m = \vec{e}_o\,, \quad \vec{e}_m \times \vec{e}_o = \vec{e}_p \qquad (12.33)$$

Going on to make the far electric field zero in the direction $\vec{e}_r = -\vec{e}_o$ implies from equations 12.7 and 12.8

$$p\left(t-\frac{r}{c}-\frac{\vec{e}_o\cdot\vec{r}_p}{c}\right) = \frac{1}{c}\, m\left(t-\frac{r}{c}-\frac{\vec{e}_o\cdot\vec{r}_m}{c}\right) \tag{12.34}$$

whereas for them to add in the forward direction $\vec{e}_r = \vec{e}_o$ we need

$$p\left(t-\frac{r}{c}+\frac{\vec{e}_o\cdot\vec{r}_p}{c}\right) = \frac{1}{c}\, m\left(t-\frac{r}{c}+\frac{\vec{e}_o\cdot\vec{r}_m}{c}\right) \tag{12.35}$$

which for early times (or high frequencies) is essential if one wishes to simulate a single fast-rising pulse in the direction of the preferred observer. Equations 12.34 and 12.35 imply

$$\vec{e}_o \cdot \vec{r}_p = \vec{e}_o \cdot \vec{r}_m \tag{12.36}$$

Of course one might use only the restriction of equation 12.35 but one can just as easily obtain that extra symmetry in the fields by making the back far fields zero. The results of equations 12.9 now follow directly

$$\begin{gathered} \vec{m}(t) = c\vec{e}_o \times \vec{p}(t) \; , \quad \vec{p}(t) = -\frac{1}{c}\, \vec{e}_o \times \vec{m}(t) \\ \vec{e}_o \cdot \vec{p}(t) = 0 \; , \quad \vec{e}_o \cdot \vec{m}(t) = 0 \end{gathered} \tag{12.37}$$

Without lack of generality we can extend the results of equation 12.36 by requiring

$$\vec{e}_o \cdot \vec{r}_p = \vec{e}_o \cdot \vec{r}_m = 0 \tag{12.38}$$

which corresponds to merely choosing the origin of coordinates $\vec{r}\,' = \vec{0}$ by a simple shift parallel to $\vec{e}_o$. With this choice of coordinate origin then along $\vec{r} = r\vec{e}_o$ we have as $r \to \infty$

$$\begin{gathered} |\vec{r}-\vec{r}_p| = |r\vec{e}_o-\vec{r}_p| = \left[r^2+|r_p|^2\right]^{1/2} = r+\frac{1}{2r}|\vec{r}_p|^2+O(r^{-3}) \\ |\vec{r}-\vec{r}_m| = |r\vec{e}_o-\vec{r}_m| = \left[r^2+|r_m|^2\right]^{1/2} = r+\frac{1}{2r}|\vec{r}_m|^2+O(r^{-3}) \end{gathered} \tag{12.39}$$

which are both $r + O(r^{-1})$ so that making the dot products zero has made these two terms which appear throughout the field and potential expressions equal to r to order r^{-1} instead of just to a constant term for the preferred observer direction.

The results of equation 12.35 were based on making the signals from the two dipoles arrive at the preferred observer at large r, leading to the results of equations 12.36 and 12.38. If one wishes, the arrival time can be made the same for all r on $\vec{r} = r\vec{e}_o$ by requiring

$$|r\vec{e}_o - \vec{r}_p| = |r\vec{e}_o - \vec{r}_m| \qquad (12.40)$$

which with equation 12.38 implies

$$|\vec{r}_p| = |\vec{r}_m| \qquad (12.41)$$

so that the coordinate center is equidistant from the two dipoles. With the restriction given by equation 12.40 the field and potential expressions simplify somewhat in that at the preferred observer these quantities which appear throughout the expressions are the same number for both electric and magnetic dipoles.

Further refinements in choosing $\vec{r}_p$ and $\vec{r}_m$ are possible which can be based on such things as the TEM quality of the fields on $\vec{r} = r\vec{e}_o$. Of course no matter what one does the signal arrival times from the two separate dipoles cannot be the same to all positions in space as long as the dipoles occupy different positions ($\vec{r}_m \neq \vec{r}_p$). Thus the high frequency performance cannot be optimized over all space but can be optimized along various paths in space. As one moves off these paths the fields do not add optimally at high frequencies. At large r with some fixed high frequency of interest one can depart from an optimum direction some angle before a certain amount of high frequency degradation sets in. So at sufficiently large r one can have a fairly wide volume of interest where the fields are uniform to some desired degree, even at high frequencies (with a finite maximum frequency of interest) where the problems are most difficult. As long as $r >> |\vec{r}_p|$ and $r >> |\vec{r}_m|$ then at low frequencies the results for crossed dipoles located at $\vec{r}\,' = \vec{0}$ still apply.

For the case where we are using a conducting ground plane with crossed electric and magnetic dipoles our choice of $\vec{r}_p$ and $\vec{r}_m$ is somewhat restricted. In this case for the dipole field results to apply the antennas and their images must be considered together and their effective centers are then on the ground plane. Thus $\vec{r}_p$ and $\vec{r}_m$ are on the ground plane, on which we also take $\vec{r}' = \vec{0}$. Now $\vec{e}_p$ is perpendicular to the ground plane (of necessity) and likewise $\vec{e}_m$ is parallel to the ground plane. Thus $\vec{r}_p$ and $\vec{r}_m$ are parallel to the ground plane and we have

$$\vec{e}_p \cdot \vec{r}_p = \vec{e}_p \cdot \vec{r}_m = 0 \tag{12.42}$$

If one maintains the restriction of equation 12.38 then for $\vec{r}_p$ and $\vec{r}_m$ to be both perpendicular to $\vec{e}_o$ and lie in the ground plane there are only two directions to choose from. If we make their magnitudes equal as in equation 12.41 then we must have $\vec{r}_m = \pm\vec{r}_p$. If they were equal they might as well both be $\vec{0}$. Thus the case of interest here has

$$\vec{r}_m = -\vec{r}_p \tag{12.43}$$

XIII. Combined Electric and Magnetic Dipoles for Complementary High-Frequency and Low-Frequency Characteristics

As we have seen one can produce uniform TEM fields in a preferred direction from crossed electric and magnetic dipoles related by $p = m/c$. However, one might also combine electric and magnetic dipoles with a view to using the best features of both in the design of large EMP simulators to radiate pulses to large distances. For large EMP simulators one might make large crossed electric and magnetic dipoles which would be positioned at some distance above the ground. Such a simulator could be rather difficult to work with in that both a large electric dipole (a long thin cylindrical antenna plus generator) and a large magnetic dipole (a large loop with a comparatively thin conductor plus generator) would have to be maneuvered together in space as well as fired together. For such an application one might try to make the electric and magnetic dipole antennas and their generators into a single unit so as to simplify the handling and control problems.

In combining real electric and magnetic dipole antennas and their associated generators the mutual interaction of the antenna and generators for the two different dipole moments should be considered. In designing a large pulse-radiating simulator of this type where a significant low-frequency content is desired one would like to have late-time dipole moments (both electric and magnetic) if the low-frequency TEM characteristics are desired, or at least one of the two dipole moments to just have something approaching the maximum low frequency content. As has been discussed in sections X and XI one can have step-function late time electric or magnetic dipoles within the constraint of finite energy supplied by the generator for each pulse. For the electric dipole this requires a capacitive energy for charge separated with a potential difference but no current. For the magnetic dipole this requires an inductive energy for the current flowing around an area but a zero volts. In combining electric and magnetic dipoles we have late time energy stored in both electric and magnetic fields, but separately since these are the only two terms required to describe the energy stored in the fields; alternatively this energy or part of it might be stored in the fields before the main pulse. However one still has the problem of both current and volts present in the static situation without requiring a continuous power flow into or out of a generator (unless power can be continuously transferred between generators). One might try to avoid this problem by having one of the dipole moments built up slowly before $t = 0$ and then stopped so as to give a transient change, while on the other hand the second dipole moment would be transiently turned on by an appropriate pulse generator. This puts the stored static electric and magnetic energies in different time regimes. However, one can make antennas which support both electric and magnetic dipole moments with no static power flow from the generators. One simply builds appropriate symmetry into the antenna structure. Another alternative is to have separate antennas with their own generators for the two dipole moments; here one is concerned with their mutual interaction but for static situations we can still have generators supplying voltage but no current or current but no voltage as appropriate.

Consider a few examples of antennas with late-time electric and magnetic dipole moments combined as illustrated in figure 9. Start with the simple case of two generator positions on a loop as shown in figure 9A. The two generators separate charge at low frequencies between two halves of the loop and thereby have a voltage across them. Furthermore the generators have current passing through them as it flows around the loop to make the low-frequency magnetic dipole moment. With this distribution of charge and current one of the generators is supplying power (VI) and the second one is absorbing an equal amount of power (assuming no losses in the loop structure in the static or low-frequency limit). If the power absorbed by the second generator could be transferred back to the first (with no losses) then the static current and charge distribution could be maintained at late times without continually feeding energy from the generators. For a simple second generator one might have a resistor to give the desired ratio of V/I. However this does require a continual power input from the first generator at late times to maintain the static dipole moments and so is not an efficient approach for making very large crossed electric and magnetic dipole moments for long times.

Now modify the antenna design by introducing two more generator positions giving four equally spaced generator positions around the loop as shown in figure 9B. The two new generators need not have any potential across them, the charge separation coming from the first two generators. These two new generators might then be inductive generators (like in figure 7C for example) which can have a late-time current flowing in a zero resistance loop. On the other hand the first two generators still have one supplying power (VI) and the other absorbing an equal power at late times, so the introduction of the two new late-time current generators has not alleviated the late-time power problem. However note the two possible current shorting paths indicated in figure 9B. If conductors were placed along this path then the first two generators could be simply providing a voltage with no current passing through them at late time; the first two generators could now be simple capacitive generators (like in figure 4C for example). The addition of the two current shorting paths allows the current supplied by the two new generators to each flow around half the full loop without passing through the first two generators at late times. This separates the late time current and late time voltage into two separate pairs of generators so that no generator needs to supply power at late times assuming the conductors have negligible resistance to the current at low frequencies.

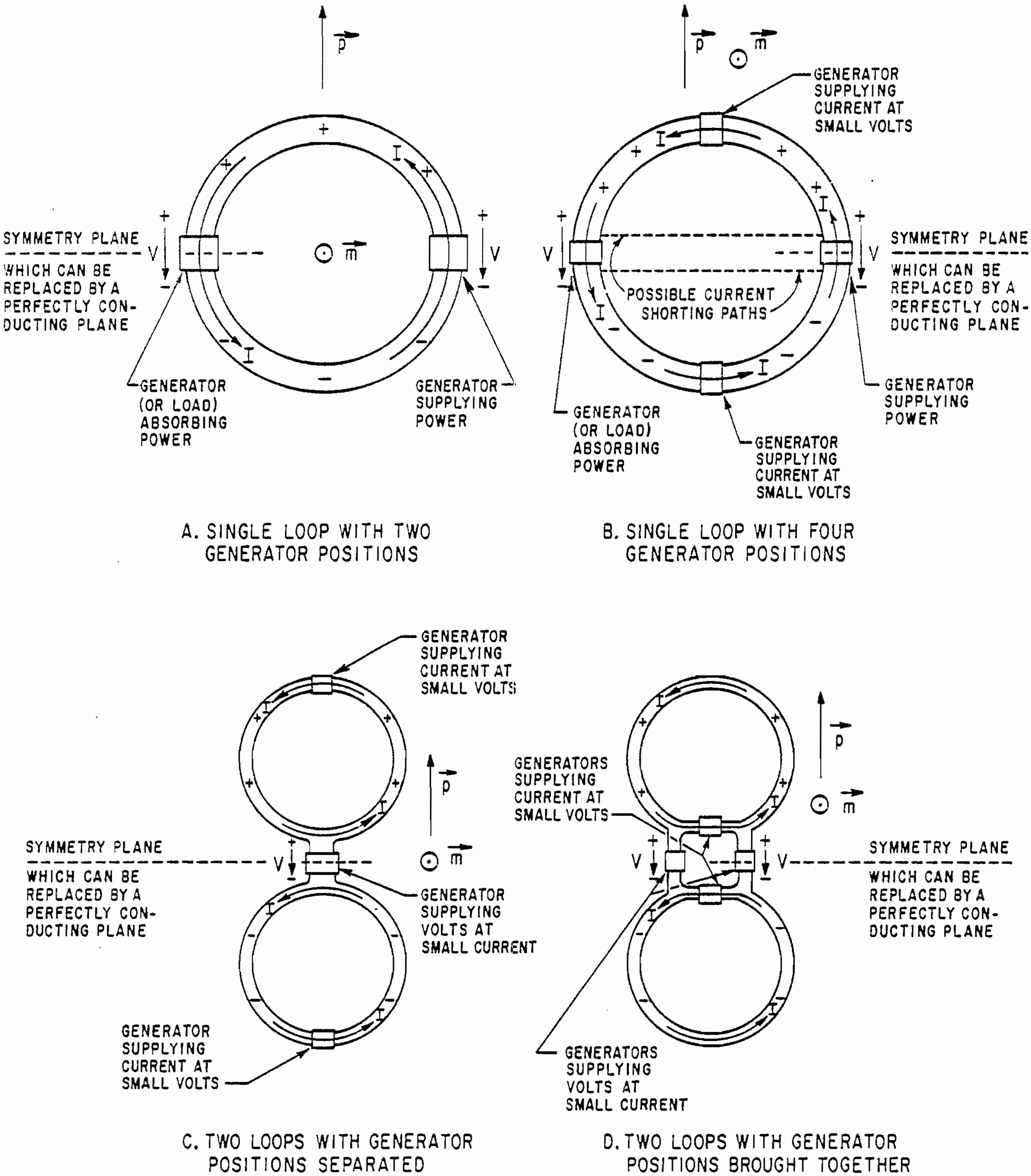


Figure 9. SOME CONFIGURATIONS OF CROSSED ELECTRIC AND MAGNETIC DIPOLES WITH GENERATORS IN ONE ANTENNA STRUCTURE

There are various modifications of this technique to reduce the number of generators and/or change their relative location. For example suppose the geometry of figure 9B is distorted by shortening the two possible current shorting paths until the two generators providing the late time voltage come together and are replaced by a single generator. This gives a three generator configuration as in figure 9C. This antenna has two loops each with a generator supplying the late-time current for the magnetic dipole moment; the generator for the late-time voltage transfers charge between the two loops for the electric dipole moment. Note that while the number of generators is reduced to three they are still separated to different positions on the antenna structure.

In order to have all the generators near one another one could make a different modification of figure 9B. Move the two new generators supplying the late-time current at zero volts from their positions on the loop as indicated in figure 9B to the two midpoints of the two possible current shorting paths shown there. With the current shorting paths included and the current generators directing current in opposite directions in these two paths, then the first two generators can supply late-time voltage at zero current. Now shrink the length of the current shorting paths to bring the two voltage generators in toward the current generators thereby giving the configuration shown in figure 9D. All four generators are now in immediate proximity and might be contained in a single generator package with four electrical connections to contact the two loops. At late times both loops have a current to give the magnetic dipole moment and the voltage between the loops displaces charge to give the electric dipole moment. Note that if one wishes one of the late-time voltage generators could be removed from this configuration and one would still have a late-time electric dipole moment. However the two late-time voltage generators do give more symmetry to the structure and could be useful for some of the high-frequency characteristics.

Note that all the examples in figure 9 have been chosen such that they have a symmetry plane along which the tangential electric field is zero so that it may be replaced by a perfectly conducting plane if desired. The electric dipole moment is perpendicular to this symmetry plane and the magnetic dipole moment is parallel to this symmetry plane. Thus all these examples can be used for crossed electric and magnetic dipoles on a ground plane. In so doing only half of the antenna structure is needed, the number of generators required is reduced in some cases, and the voltages of some generators are halved for the same effective dipole moments (including the image).

One could also combine late-time electric and magnetic dipole moments in one antenna without having the symmetry plane with zero tangential electric field (as in figure 9). By so doing one might try to decrease the number of generators. For example one might take the design in figure 9C with one late-time voltage generator and two late-time current generators and remove one of the latter. Then only one of the two loops would be contributing to driving the late-time magnetic dipole moment except for mutual inductance between the two loops. The loop without a current generator might be replaced by another shape such as a rod which would still function as far as contributing to the late-time electric dipole moment. Another way that one might try to reduce the number of late-time current generators and still maintain symmetry is by the use of transformers whereby one late-time current generator could drive two loops (inductors at low frequencies) with the transformers at the positions of one or more current generators as in figure 9D. Note that such transformers would need to withstand the late time voltage V (or some fraction of it for multiple transformers) which is the potential between the loops in figure 9D.

When discussing the pulse generators one might use with electric dipole antennas for large late-time electric dipole moments (section X) several examples were considered as shown in figure 4. A particularly appropriate generator for use with an electric dipole antenna is a simple capacitive generator as shown in figure 4C. Various generators of this general type have been made for use with electric dipole antennas or other kinds of EMP simulators. For even large voltage outputs the risetime of the associated closing output switch can be as fast as several nanoseconds at the present state of the art. An electric dipole antenna driven by a capacitive generator is then an appropriate high-frequency pulse radiator. An electric dipole antenna with a capacitive generator also can be used to give a late-time electric dipole moment. An inductive generator can for a given size store much more energy than a capacitive generator at the present state of the art but has a comparatively slow risetime of the required opening switch; there is also needed a second opening switch to give a late-time electric dipole moment. One can also charge up the electric dipole antenna and discharge it through a relatively fast closing switch. For this latter technique there is the possibility of a significant prepulse depending on the distance to the observer.

When considering the pulse generators for use with magnetic dipole antennas for large late-time magnetic dipole moments (section XI) several examples were considered as shown in figure 7. One could use a capacitive generator as in figure 7B. The first closing switch could be designed to give a fast rising radiated pulse. However to maintain a late-time magnetic dipole moment requires a second closing switch to remove the capacitor from the loop. One might prefer to use the capacitive generator with an electric dipole antenna for simplicity and still get the low frequencies from the electric dipole moment. Alternatively one could use an inductive generator as in figure 7C. Inductive generators store a lot of energy compared to capacitive generators for a given size in the present state of the art but give slower risetimes. However the much larger energy can give a much larger late-time magnetic dipole moment and the inductive generator matches well to an inductive loop at low frequencies. Instead of a normal inductive generator one might use an explosively driven magnetic-field-compression generator which also can deliver large energies but also at a comparatively slow rise time for the radiated pulse. This type of generator is also inductive at low frequencies so that the loop has a late-time magnetic dipole moment. For these inductive types of generators one might also use special pulse transformers to try to best match the generators to the antennas, but at a possible further slowing of the rise time. One can also slowly build up the current in a loop and transiently stop it by opening a switch in the loop. This last technique, however, can have a significant prepulse depending on the distance to the observer.

From the point of view of the antenna both electric and magnetic dipoles are comparable as far as radiating a fast rising pulse and supporting a large late time dipole moment. Either type of dipole moment contributes in the same manner to the low frequency far fields; it merely depends on which is larger, p or m/c. Likewise both give second order fields with the same type of frequency dependence; only in the third order fields is there a difference, one giving a static electric field and the other a static magnetic field. However, from the point of view of the pulse generators there are significant differences at the current state of the art. Capacitive generators have fast rise times and inductive (including explosive) generators have large energies. So one might ask: "Why not combine the best of both?". Use an inductive type generator for its large energy and connect it to a loop designed to be a conductor at low frequencies so as to give a large late-time magnetic dipole moment and thus large low-frequency content to the radiated fields. Simultaneously use a capacitive type generator for its fast rise time giving a large high-frequency content to the radiated fields. One might typically connect this to an electric dipole antenna to get some low frequency performance from a late time electric dipole moment but the magnetic dipole would be the principal low frequency radiator. Thus one might give up some of the low-frequency benefits of a large electric dipole moment to gain something else, for example a more directional radiation of the high-frequency energy. Some of these considerations may change with time as pulse generator technology progresses and the relative advantages of different pulse generators possibly change.

The various geometries of electric and magnetic dipoles combined in one antenna structure as in figure 9 (and many others) could be used to give the high frequencies from the electric dipole and low frequencies from the magnetic dipole as discussed above. There are still many design considerations concerning the intermediate frequency regions which we do not discuss here. Alternatively the electric and magnetic dipoles might be physically separated but controlled together. If they were suspended far above the earth's surface this could be rather difficult, particularly if they were not mechanically connected together. However on the surface of the earth using a ground plane and/or the earth's surface as an approximation of a symmetry plane (or image plane), handling and controlling large separated electric and magnetic pulsed dipole antennas (including relative firing times) should be comparatively simple.

In a previous note[11] we have considered some of the characteristics of a large electric dipole antenna (a circular cone in the case) mounted on a large ground plane and driven by a capacitive type of generator near the base of the antenna. Perhaps such an antenna could be improved by added height such as by wires running up to a balloon so as to increase the late-time electric dipole moment. Or perhaps the antenna might be slanted toward a preferred observer direction to maximize the high frequencies there.

Consider the example of crossed electric and magnetic dipoles on the earth's surface and separated from each other as illustrated in figure 10. Utilizing possibly buried conduits the relative generator firing times can be easily controlled. Note the presence of a conducting ground plane which can help to shield the buried control equipment and monitor lines. The conducting ground plane gives better high-frequency propagation characteristics near the electric dipole antenna and forms a highly conducting path at the base of the magnetic dipole antenna to complete the current loop. The generators are located where the antennas meet the ground plane, either above or below it as desired. Since the magnetic dipole antenna meets the ground plane at two widely separated positions one might use two inductive type generators operating in push-pull fashion. The antenna structures might be supported by balloons to attain large heights or could be supported by dielectric towers except that parts of towers which were also used as parts of the antennas could be metal; perhaps combinations of balloons and towers could be used. Parts of the antennas might be many wires, perhaps structured into a cage form. Provision should also be made for the insertion of impedance elements into the antennas to damp the pulse waveforms and make the intermediate part of the frequency spectrum vary smoothly with frequency. For the electric dipole this might simply be resistive loading. For the magnetic dipole they might be parallel combinations of inductors and resistors or some other combinations which is basically an inductance at low frequencies so that the late-time current in the magnetic dipole antenna does not dissipate significant power in the impedance elements. Note that the electric dipole antenna induces charge on the magnetic dipole antenna thereby reducing the net electric dipole moment. How

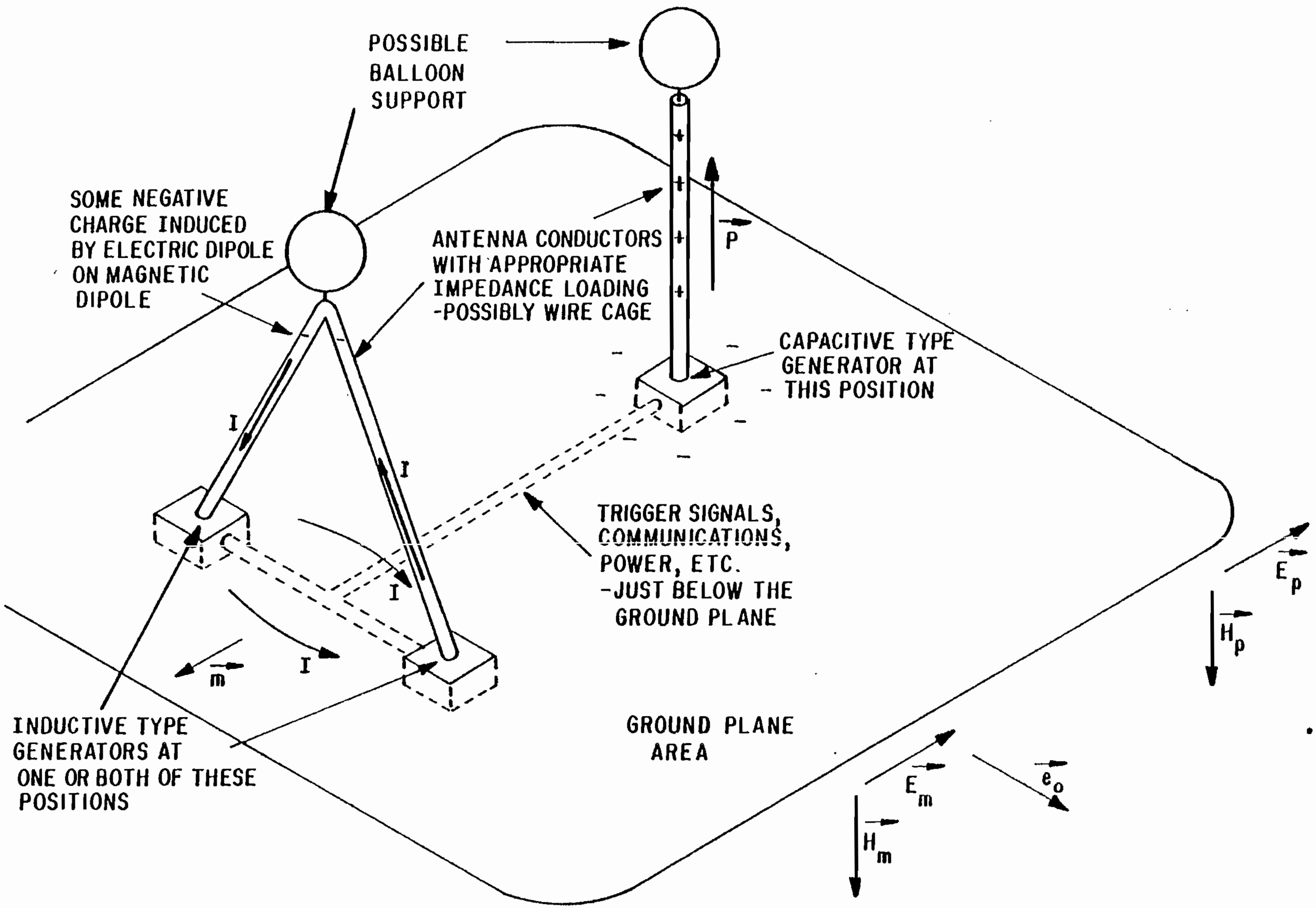


FIGURE 10. LARGE SEPARATED CROSSED ELECTRIC AND MAGNETIC DIPOLE ANTENNAS ON THE GROUND SURFACE

much of a reduction depends on the distance between the two antennas. At low-frequencies the static electric fields from the electric dipole fall off as r^{-3} and thus the reduction of the electric dipole moment can be made small in a fractional sense at low frequencies by sufficient spacing between the antennas. At higher frequencies where the fields fall off like r^{-1} the scattering can be more significant but this scattering can be reduced by the addition of lossy elements in the antennas which damp resonances etc. on the antenna structure at these higher frequencies. As indicated in figure 10 the electric and magnetic dipole moments would be chosen to make the fields add in the preferred observer direction $\vec{e}_o$. Of course if p and m do not have the same time histories then the fields may not add at all times. Besides the design of the individual dipole antennas attention should be paid to the relative generator characteristics and relative position in space of the two antennas so that in the direction $\vec{e}_o$ the resulting waveform and frequency spectrum does not have undesirable features. The example in figure 10 is only that. Various other antenna shapes could be used depending both on their desirable electromagnetic features and the mechanical ease of construction and maintenance.

XIV. Summary

Using the Helmholz theorem the current density in a volume of space with finite dimensions can be split into irrotational and solenoidal terms, the former being associated with the charge density. In expanding the fields and potentials associated with this current density the dominant terms at low frequency and large distance from the source are the electric and magnetic dipole terms. The electric dipole moment is associated with the charge density and thus with the irrotational current density. The magnetic dipole moment is basically associated with the solenoidal current density in that the charge density can be zero and still have a magnetic dipole moment. This separation is important for the late-time dipole moments in that a real electric dipole antenna can have a late-time electric dipole moment supported by a charge separation and associated voltage but zero current and thus no power; a real magnetic dipole antenna can have a late-time magnetic dipole moment supported by current flowing around an area but with no voltage drop and thus no power.

In expanding the fields at large r and low frequencies there are three dominant terms to consider, each dominated by the two dipole moments; these are the far fields (r^{-1} terms), second order fields (r^{-2}) and third order fields (r^{-3}) which all have different frequency coefficients (s^2, s, and s^o in that order) times various components of the dipole moments. Depending on how large r is then which terms dominate depend on the radian frequency ω. For $r > c/\omega$ the far fields dominate while for $r < c/\omega$ the second and third order fields dominate.

The electric and magnetic dipole fields also apply to the case of an infinitely large perfectly conducting ground plane. Real electric and magnetic dipoles can be built on ground planes and the electric dipole moment is perpendicular to the ground plane while the magnetic dipole moment is parallel to it.

Electric and magnetic dipoles can be combined to try to improve the fields produced in some way. With ideal dipoles related by $p = m/c$ and kept mutually perpendicular and perpendicular to some fixed preferred observer direction then the fields in this direction from the source have some interesting and useful properties. Besides being fairly uniform as long as the direction from the source is approximately the preferred observer direction, the fields in this region are also TEM at all frequencies on a term by term basis (r^{-1}, r^{-2}, and r^{-3}), the E/H ratio is the free space impedance, and $\vec{E} \times \vec{H}$ is in the direction of the preferred observer. The two dipole fields add in the preferred observer direction and the r^{-1} and r^{-2} terms cancel in the reverse direction leaving only the static r^{-3} term there.

For best low-frequency performance real electric and magnetic dipoles can be pulse excited by what is a step function charge or current as far as low frequencies or late times are concerned where the late time power input can be zero in both cases. Various types of pulse generators can be used with various switching arrangements to give a late-time electric or magnetic dipole moment as appropriate, some with more efficiency or simplicity than others. Since the low-frequency content of pulses radiated from real dipole antennas is significantly limited it is important to have as much of a late-time dipole moment as possible to maximize the low-frequency content of the radiated pulse.

Real electric and magnetic dipole antennas can be combined in various ways. In so doing it is possible to have both electric and magnetic dipole moments at late times without continuing to supply power at late times. Some generators can be configured to pass current into an inductive loop at low frequencies and sustain negligible volts at late time; others can be configured to separate charge and sustain voltage while passing negligible current at late times. These combined electric and magnetic dipoles can be used in free space or in combination with a ground plane. Alternatively the electric and magnetic dipole antennas (with their pulsers) can be physically separate but controlled together to try to minimize the problems of one interacting with the other.

For purposes of large EMP simulators one might not constrain $p = m/c$ for crossed electric and magnetic dipoles, but try to take better advantage of various antenna and generator designs to use each for that part of the frequency content of a radiated pulse for which it works best. For ease of operating and controlling large crossed electric and magnetic dipoles, separated large electric and magnetic dipole antennas on the earth's surface and driven by capacitive and inductive generators respectively look rather attractive.

Let us now give a name to the technique of combining these large electric and magnetic dipoles. Let us call it DILEMMA, a rather appropriate name. This name can also be considered an acronym based on <u>DI</u>pole <u>L</u>arge <u>E</u>lectric and <u>M</u>agnetic <u>M</u>ixed <u>A</u>ntennas.

"I know what you're thinking about," said Tweedledum; "but it isn't so, nohow."

"Contrariwise," continued Tweedledee, "if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic."

(Lewis Carroll, Through the Looking Glass)

References

1. Capt Carl E. Baum, Sensor and Simulation Note 100, Some Characteristics of Planar Distributed Sources for Radiating Transient Pulses, March 1970.

2. Morse and Feshbach, Methods of Theoretical Physics, McGraw Hill, 1953, pp. 52-54.

3. J. Van Bladel, Electromagnetic Fields, McGraw Hill, 1964, appendix 1, pp. 487-498.

4. Capt Carl E. Baum, Sensor and Simulation Note 74, Parameters for Electrically-Small Loops and Dipoles Expressed in Terms of Current and Charge Distributions, January 1969.

5. Capt Carl E. Baum, Sensor and Simulation Note 65, Some Limiting Low-Frequency Characteristics of a Pulse-Radiating Antenna, October 1968.

6. C. H. Papas, Theory of Electromagnetic Wave Propagation, McGraw Hill, 1965, chapter 4.

7. J. D. Jackson, Classical Electrodynamics, Wiley, 1962, chapter 9.

8. Capt Carl E. Baum, Sensor and Simulation Note 69, Design of a Pulse-Radiating Dipole Antenna as Related to High-Frequency and Low-Frequency Limits, January 1969.

9. Capt Carl E. Baum, Sensor and Simulation Note 81, Resistively Loaded Radiating Dipole Based on a Transmission-Line Model for the Antenna, April 1969.

10. W. R. Smythe, Static and Dynamic Electricity, 3rd ed., McGraw Hill, 1968, section 8.10.

11. Capt Carl E. Baum, Sensor and Simulation Note 36, A Circular Conical Antenna Simulator, March 1967.