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Inductance and Current Density of a Cylindrical Shell

by

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Abstract

The current distribution on a perfectly conducting, infinitely thin, cylindrical shell is calculated under the conditions that the current density has only an azimuthal component and that the total current is nonzero. The total inductance corresponding to this current distribution is also computed. The question of approximating this continuous current distribution by the discrete current distribution of N loops is discussed. Results are presented in graphical and/or tabular form.

I. Introduction

The figure of merit introduced by Baum is a useful quantity in the design of EM sensors.[1] It is defined as the ratio of the equivalent volume of a sensor to the smallest geometric volume inside which the sensor can be enclosed. The equivalent volume, V_{eq}, is a measure of the total energy that the sensor can extract with a resistive load from a pulsed incident wave. In the case of a $\dot{B}$ sensor, which we will discuss exclusively in this note, V_{eq} is directly proportional to the square of the equivalent area, A_{eq}, and inversely proportional to the inductance, L, of the sensor. Or, equivalently, V_{eq} is proportional to A_{eq} squared times the upper frequency response. For proof of these statements the reader is referred to Ref. 1. Thus, one can improve the response characteristics of a sensor by increasing V_{eq}.

In this note, we will fit N identical loops of radius a into a specified cylindrical volume so that the figure of merit, η, is maximum. Here, A_{eq} is roughly equal to $N\pi a^2$ and hence η is proportional to N^2/L. Maximizing η then is tantamount to minimizing L for a fixed N. The problem at hand can now be stated precisely as follows: Given N identical loops of radius a and a cylindrical volume of radius a and half-length h, it is required to space these loops in such a volume so that the total inductance is minimum.

The approach used here is first to calculate, in a cylindrical shell of zero thickness, the current distribution that corresponds to minimum inductance and then to approximate this continuous current distribution by a given number of loops. In a forthcoming note, we will attack the problem directly without making use of this continuous current distribution.

In section II, we formulate an integral equation for the current induced in a perfectly conducting, infinitely thin, cylindrical shell by a harmonic plane wave with the magnetic field parallel to the axis of the shell. The low-frequency limit of this integral equation is then taken, and the resulting equation is the formulation of our problem and will be derived again, in section III, from the requirement of minimum inductance. The numerical method used to solve this resulting integral equation is discussed in section IV, and the numerical results are presented in section V in both graphical and tabular form.

II. Formulation via Scattering

Consider the situation depicted in Fig. 1 where a perfectly conducting cylindrical shell of total length 2h and radius a is immersed in a plane electromagnetic wave. We wish to calculate the induced current distribution in the shell, this current having only an ϕ component. We will then let ka tend to zero and obtain the static current distribution. The current distribution deduced in this way will correspond to minimum magnetostatic energy and hence minimum inductance, as will be shown in the next section. Referring to Fig. 1 and suppressing the time dependence $e^{-i\omega t}$ throughout, we write for the incident wave

$$\underline{E}^{inc} = E_o \underline{e}_y e^{ikx} = E_o(\underline{e}_\rho \sin\phi + \underline{e}_\phi \cos\phi) e^{ik\rho\cos\phi}$$

Expressing the scattered electric field $\underline{E}^{sc}$ in terms of the vector potential $\underline{A}$ as

$$\underline{E}^{sc} = i\omega\underline{A} - \frac{1}{i\omega\mu\varepsilon}\nabla\nabla\cdot\underline{A}$$

and requiring that

$$E_\phi^{inc} + E_\phi^{sc} = 0, \text{ for } \rho = a, \quad |z| \le h$$

we have

$$i\omega A_\phi - \frac{1}{i\omega\mu\varepsilon a}\frac{\partial}{\partial\phi}\nabla\cdot\underline{A} = -E_o \cos\phi e^{ika\cos\phi}, \text{ for } \rho = a, \quad |z| \le h \tag{1}$$

Integrating (1) with respect to ϕ from 0 to 2π and dividing by 2π we get

$$\frac{1}{2\pi}\int_o^{2\pi} A_\phi d\phi = -\frac{1}{\omega}E_o J_1(ka) \tag{2}$$

Here, J_1 is the Bessel function of the first kind of order one and is defined by (Formula 9.1.21 of Ref. 2)

$$J_1(x) = \frac{i^{-1}}{2\pi} \int_0^{2\pi} \cos\phi e^{ix\cos\phi}\, d\phi$$

Now A_ϕ and the current density K_ϕ are related by[3*]

$$A_\phi(\rho,z,\phi) = \mu \int_{-h}^{h} K_\phi(z',\phi')dz' \int_0^{2\pi} \cos(\phi - \phi') \frac{e^{ikR}}{4\pi R}\, a d\phi' \qquad (3)$$

where

$$R^2 = (z - z')^2 + \rho^2 + a^2 - 2\rho a \cos(\phi - \phi')$$

Setting $\rho = a$ in (3) and then substituting the resulting expression into (2) we obtain

$$\int_{-h}^{h} \overline{K}_\phi(z')dz' \int_0^{2\pi} \cos\phi \frac{e^{ikR}}{4\pi R}\, d\phi = - \frac{1}{\omega\mu a} E_o J_1(ka), \quad |z| \le h \qquad (4)$$

where

$$R^2 = (z - z')^2 + 2a^2 - 2a^2 \cos\phi$$

$$\overline{K}_\phi(z) = \frac{1}{2\pi} \int_0^{2\pi} K_\phi(z,\phi)d\phi$$

We now go to the static limit $ka \to 0$ in (4). Noting that $J_1(x) = x/2 + O(x^3)$ $\sqrt{\varepsilon/\mu}\, E_o = H_o$ we obtain

$$\int_{-h}^{h} i_\phi(z')dz' \cdot \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos\phi d\phi}{\sqrt{(z-z')^2+2a^2-2a^2\cos\phi}} = - H_o, \quad |z| \le h \qquad (5)$$

where i_ϕ is the static limit of $\overline{K}_\phi$. This is the integral equation for $i_\phi(z)$ we set out to seek at the beginning of this section and will be obtained again in the next section from the principle of minimum magnetostatic energy.

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*The other component of the induced current K_z in the shell does not give rise to any A_ϕ.

III. Formulation via Calculus of Variations

Given a total DC current flowing in the cylindrical shell (Fig. 1) in the azimuthal direction, the current will distribute itself along the shell in such a way that the magnetic energy is a minimum. In this section we will determine the current distribution from this minimum energy requirement and show that this current distribution must satisfy equation (5), as we have claimed in section II.

The magnetic energy is proportional to the square of the total current and the proportionality is exactly one-half the inductance L, i.e.,

$$L = \frac{\text{Total mangetic energy}}{\frac{1}{2}(\text{Total current})^2} = \frac{2\pi a \int_{-h}^{h} A_\phi i_\phi dz}{\left(\int_{-h}^{h} i_\phi dz\right)^2}$$

$$= \frac{\int_{-h}^{h}\int_{-h}^{h} i_\phi(z)K(z,z')i_\phi(z')dz'dz}{\left(\int_{-h}^{h} i_\phi(z)dz\right)^2} \qquad (6)$$

where i_ϕ is the current density in amperes per unit length as defined before, and K is $\pi a^2 \mu$ times the kernel in (5).

To find the $i_\phi(z)$ that makes L the minimum for a nonzero total current we set the variation of L equal to zero, i.e., $\delta L = 0$. After some standard manipulations in the calculus of variations we obtain from (6)

$$\int_{-h}^{h}\int_{-h}^{h} \delta i_\phi(z)K(z,z')i_\phi(z')dz'dz - \left[L\int_{-h}^{h} i_\phi(z)dz\right]\int_{-h}^{h} \delta i_\phi(z)dz = 0 \qquad (7)$$

where we have used $K(z,z') = K(z',z)$ and $\int_{-h}^{h} i_\phi(z)dz \neq 0$. With λ denoting the quantity in the square bracket, equation (7) can be rewritten as

$$\int_{-h}^{h}\left\{\int_{-h}^{h} K(z,z')i_\phi(z')dz' - \lambda\right\}\delta i_\phi(z)dz = 0 \qquad (8)$$

Since equation (8) holds for arbitrary δi_ϕ, it follows that

$$\int_{-h}^{h} K(z,z')i_\phi(z')dz' = \lambda, \text{ for } |z| \le h \tag{9}$$

This equation could also have been obtained by constructing a functional for the magnetic energy and treating the constraint that $\int_{-h}^{h} i_\phi dz$ = constant by the method of Lagrange multipliers. In fact, the parameter λ in (8) and (9) is the Lagrange multiplier in this method. Equations (5) and (9) are of the same form except for a multiplicative constant which has no significance whatsoever in the current distribution.

IV. Numerical Method

We now go on to discuss, in sufficient detail, the numerical method that will be used to solve equation (5). Let us first substitute into (5) the following

$$
\begin{aligned}
z &= hx \\
z' &= hx' \\
\alpha &= a/h \\
i_\phi(hx) &= -\frac{\alpha H_o}{\ln 2} \frac{F(x)}{\sqrt{1-x^2}}
\end{aligned}
\qquad (10)
$$

Then, (5) becomes

$$\int_{-1}^{1} G(x,x') \frac{F(x')}{\sqrt{1-x'^2}} dx' = \frac{\pi \ln 2}{2}, \qquad |x| \le 1 \qquad (11)$$

where[3]

$$
\begin{aligned}
G(x,x') &= \frac{\alpha}{4} \int_o^{2\pi} \frac{\cos \phi d\phi}{\sqrt{(x-x')^2+4\alpha^2 \sin^2(\phi/2)}} = \frac{1}{k}\left\{(1 - \frac{1}{2} k^2)K(k) - E(k)\right\} \\
k &= \frac{2\alpha}{\sqrt{4\alpha^2+(x-x')^2}}
\end{aligned}
\qquad (12)
$$

Here, K and E are complete elliptic integrals of the first and second kind, respectively.* The reason for choosing i_ϕ to have the form (10) is that i_ϕ has the square-root singularity at both ends of the shell.

Let us now examine the behavior of G when x is very near x'. As $x \to x'$, $k' = \sqrt{1 - k^2} \to 0$ and[4]

$$K(k) \sim \ln \frac{4}{k'} + 0(k'^2 \ln k')$$

$$E(k) \sim 1 + 0(k'^2 \ln k')$$

Thus, as $x \to x'$

$$G(x,x') \sim -\frac{1}{2} \ln|x - x'| + \frac{1}{2} \ln(8\alpha e^{-2}) + 0(k'^2 \ln k') \qquad (13)$$

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*In Ref. 2, p. 590, the parameter m, instead of k, is used.

In view of (13) we rewrite (11) in the form

$$\int_{-1}^{1} \left[G(x,x') + \frac{1}{2}\ln|x - x'|\right] \frac{F(x')}{\sqrt{1-x'^2}}\, dx' + \int_{-1}^{1} \left[F(x) - F(x')\right] \frac{\ln|x-x'|}{2\sqrt{1-x'^2}}\, dx'$$

$$- \frac{1}{2} F(x) \int_{-1}^{1} \frac{\ln|x-x'|}{\sqrt{1-x'^2}}\, dx' = \frac{\pi \ln 2}{2}, \quad |x| \le 1 \qquad (14)$$

An application of Chebyshev-Gauss quadrature formula (Formula 25.4.38 of Ref. 2) to (14) gives

$$\frac{1}{2}\ln(8\alpha e^{-2}) w_i F_i + \sum_{j\neq i}^{n} \left\{G_{ij} + \frac{1}{2}\ln|x_i - x_j|\right\} w_j F_j$$

$$+ \frac{1}{2}\sum_{j\neq i}^{n} (F_i - F_j) w_j \ln|x_i - x_j| + \frac{\pi \ln 2}{2} F_i = \frac{\pi \ln 2}{2}, \quad i = 1,2,\ldots n \qquad (15)$$

where

$$F_i = F(x_i)$$

$$G_{ij} = G(x_i, x_j)$$

$$w_i = \frac{\pi}{n}$$

$$x_i = \cos\left[\frac{(2i-1)\pi}{2n}\right]$$

In arriving at (15) from (14) we have used the easily derived formula

$$\int_{-1}^{1} \frac{\ln|x-x'|}{\sqrt{1-x'^2}}\, dx' = -\pi \ln 2, \qquad \text{for } |x| \le 1$$

Rearranging (15) we get

$$\left[1 + \frac{1}{n \ln 2}\sum_{j\neq i}^{n} \ln|x_i - x_j| + \frac{1}{n \ln 2}\ln(8\alpha e^{-2})\right] F_i$$

$$+ \frac{2}{n \ln 2}\sum_{j\neq i}^{n} G_{ij} F_j = 1, \quad i = 1, 2, \ldots n \qquad (16)$$

The sum in the square bracket can be summed in the following way. Noting that the x_i's are the zeros of the Chebyshev polynomial T_n of order n and $T_n(x) = 2^{n-1} \prod_{i=1}^{n} (x - x_i)$, we have[5]

$$\sum_{j\neq i}^{n} \ln|x_i - x_j| = \ln \prod_{j\neq i}^{n} |x_i - x_j| = \ln\left\{2^{-n+1}\left|\frac{dT_n(x)}{dx}\right|_{x=x_i}\right\} = \ln\left[\frac{n2^{-n+1}}{\sqrt{1-x_i^2}}\right] \quad (17)$$

With (17) we can simplify (16) in the form

$$\frac{1}{n \ln 2} \ln\left[\frac{16\alpha n e^{-2}}{\sqrt{1-x_i^2}}\right] F_i + \frac{2}{n \ln 2} \sum_{j\neq i}^{n} G_{ij}F_j = 1, \quad i = 1,2,\ldots n \quad (18)$$

The dimension of this matrix equation can be reduced by a factor of 2 if the following symmetry conditions are used

$$x_i = \cos\left[\frac{(2i-1)\pi}{2n}\right] = - x_{n-i+1}$$

$$F_i = F(x_i) = F(-x_i) = F(x_{n-i+1}) = F_{n-i+1}$$

After some manipulations on (18) with $m = n/2$, we arrive at the final matrix equation

$$\left\{\frac{1}{2m \ln 2} \ln\left[\frac{32\alpha m e^{-2}}{\sqrt{1-x_i^2}}\right] + \frac{1}{m \ln 2} G_i\right\} F_i$$

$$+ \frac{1}{m \ln 2} \sum_{j\neq i}^{m} (G_{ij}^{+} + G_{ij}^{-})F_j = 1, \quad i = 1,2,\ldots m \quad (19)$$

where

$$x_i = \cos\left[\frac{(2i-1)\pi}{4m}\right]$$

$$G_i = \frac{1}{k_i}\left[(1 - \frac{1}{2} k_i^2)K(k_i) - E(k_i)\right]$$

$$G_{ij}^{\pm} = \frac{1}{k_{ij}^{\pm}} \left[\left\{ 1 - \frac{1}{2} k_{ij}^{(\pm)2} \right\} K(k_{ij}^{\pm}) - E(k_{ij}^{\pm}) \right]$$

$$k_i = \frac{\alpha}{\sqrt{\alpha^2 + x_i^2}}$$

$$k_{ij}^{\pm} = \frac{2\alpha}{\sqrt{4\alpha^2 + (x_i \pm x_j)^2}}$$

Equation (19) was solved by an electronic computer and the numerical results will be presented in the next section.

V. Results

The solution of equation (19) required less than 30 seconds of CDC 6600 computation time for four-place accuracy for 13 different a/h values. In this section we present the numerical results in the normalized coordinates of the cylindrical shell (Fig. 2).

Figure 3 shows the normalized current density J, defined by

$$J(x) = \frac{i_\phi(x)}{\int_0^1 i_\phi(x)dx}, \qquad (20)$$

as function of x with a/h as a parameter. These curves agree very well with those reported in Ref. 6. Figure 4 shows the total current I, defined by

$$I(x) = \int_0^x J(x')dx', \qquad (21)$$

as function of x with a/h as a parameter. These curves give some idea about the locations of the division points when one tries to approximate the current density in Fig. 3 by a given number of current loops. We will return to this point shortly and discuss the division points in great detail.

In the limiting case where $a/h \to \infty$ (i.e., $\alpha \to \infty$) one can easily show from (12) and (11) that F(x) is a constant for $|x| \leq 1$. Hence, as $\alpha \to \infty$

$$J(x) \to \frac{(1-x^2)^{-\frac{1}{2}}}{\int_0^1 (1-x^2)^{-\frac{1}{2}}dx} = \frac{2}{\pi}\frac{1}{\sqrt{1-x^2}}$$

$$I(x) \to \frac{2}{\pi}\int_0^x (1 - x'^2)^{-\frac{1}{2}}\, dx' = \frac{2}{\pi}\sin^{-1} x$$

These asymptotic forms are shown as dashed curves in Figs. 3 and 4. In the other limiting case where $a/h \to 0$ (i.e., $\alpha \to 0$) one has, as expected from the curves in Figs. 3 and 4,

$$J(x) \to 1 \begin{cases} \text{almost everywhere except at } x = 1 \text{ where } J(x) \\ \text{has a square-root singularity.} \end{cases}$$

$$I(x) \to x$$

These asymptotic forms are shown as dashed curves in Figs. 3 and 4.

The relative (or normalized) inductance L_r is shown in Fig. 5 as function of h/a and also tabulated in Table I. L_r is defined by the right-hand side of equation (6) divided by $\mu\pi a^2/(2h)$; that is,

$$L_r = \frac{L}{\mu\pi a^2/(2h)} = \frac{2h}{\mu\pi a^2} \cdot \frac{2\pi a \int_{-h}^{h} A_\phi i_\phi dz}{\left(\int_{-h}^{h} i_\phi dz\right)^2} = \frac{(h/a)\ln 2}{\int_o^1 F(x)(1-x^2)^{-\frac{1}{2}} dx} \tag{22}$$

where equations (3), (5) and (10) have been used.

We now return to the question of approximating the continuous current density distribution given in Fig. 3 by a discrete current distribution of N current loops, each loop having the same total current. To do this we divide the shell into N intervals (see Fig. 2) so that the total current within each interval is the same. More precisely, the division points, x_i, are determined from the equation

$$\int_o^{x_i} J(x)dx = \frac{2i}{N}, \qquad i = 1,2,\ldots. \frac{N}{2} \quad \text{for N even}$$

$$= \frac{2i-1}{N}, \qquad i = 1,2,\ldots. \frac{N+1}{2} \quad \text{for N odd} \tag{23}$$

Due to the symmetry of the problem (Fig. 2) we have

$$x_i = x_{-i}$$

$$x_{N/2} = 1 \qquad \text{(N even)}$$

$$x_{N+1/2} = 1 \qquad \text{(N odd)}$$

$$x_o = 0$$

Tables II through V give the division points, x_i, for even N with x_o and $x_{N/2}$ omitted. For instance, when N = 4 one loop should be placed between $x_o(=0)$ and x_1, one between x_1 and $x_2(=1)$, and of course the other two between x_o and $-x_1$, $-x_1$ and $-x_2$. In Tables VI through IX the division points, x_i, are given for odd N. In this case one loop is at $x_o(=0)$, but there is no loop between x_o and x_1.

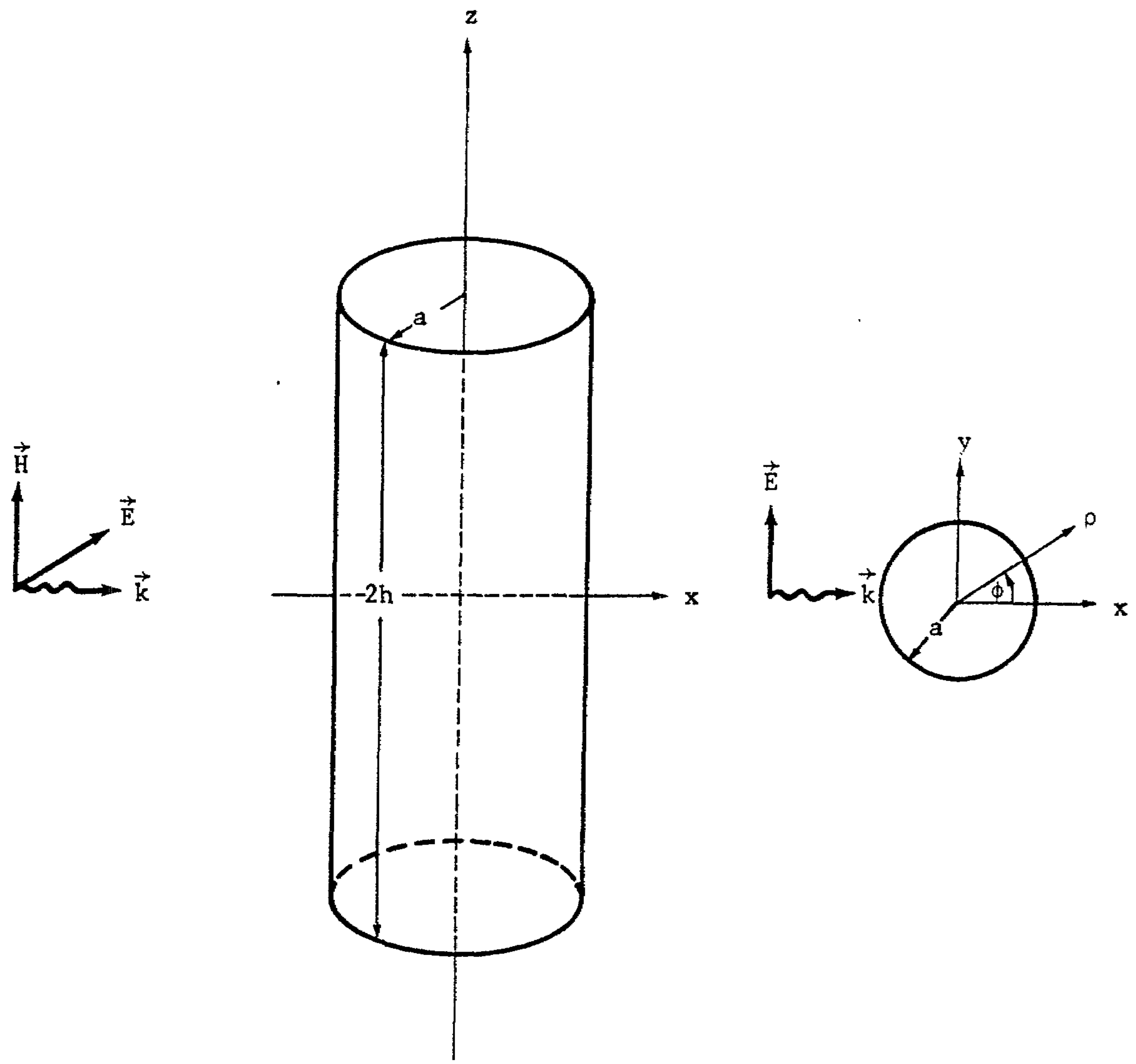

Figure 1. A cylindrical shell in a plane wave.

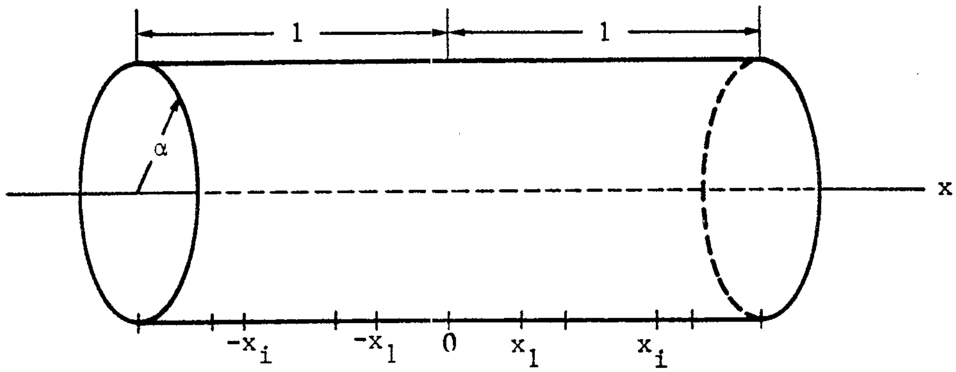

Figure 2. Normalized dimensions and division points for the cylindrical shell.

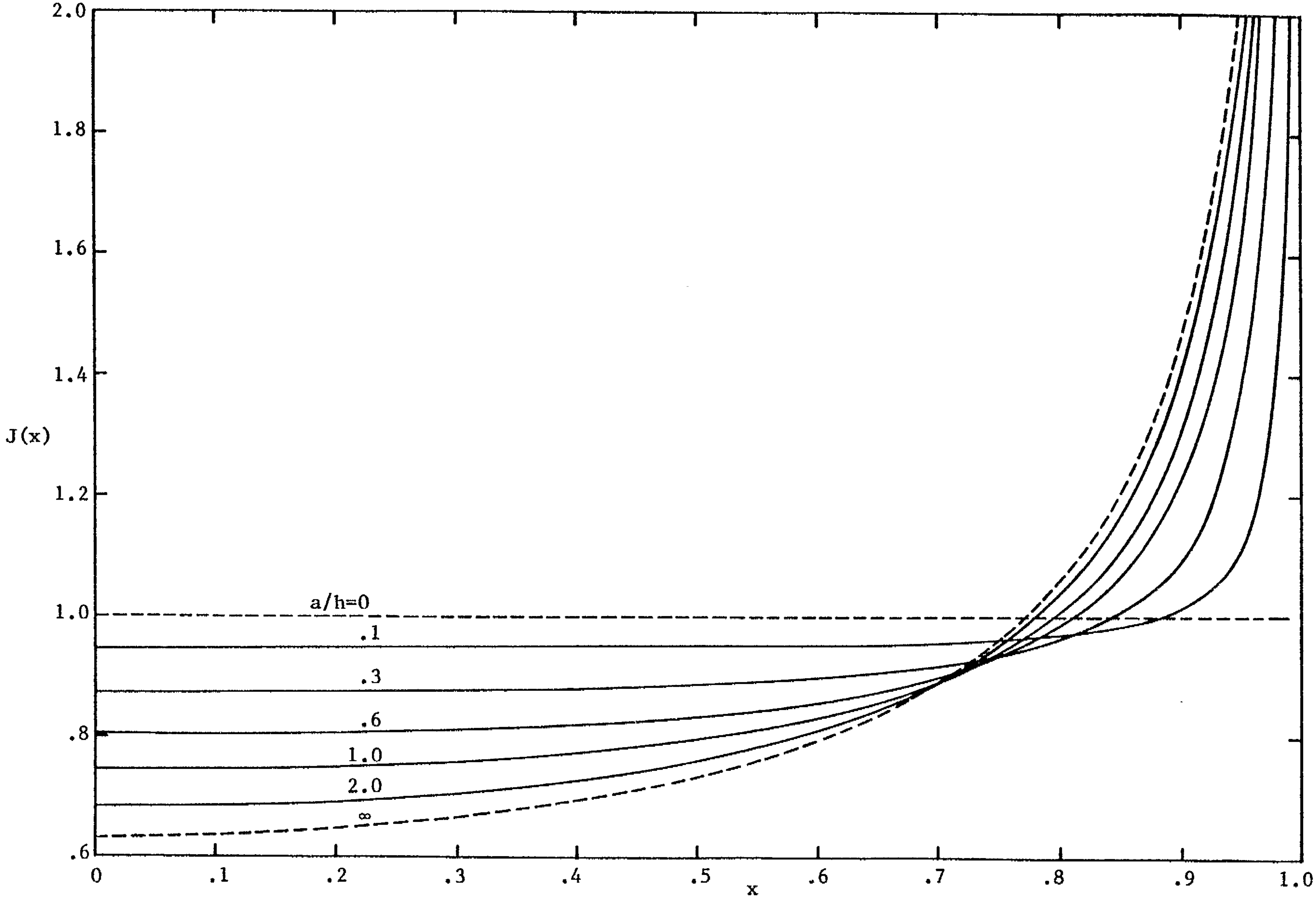

Figure 3. Current density as function of x.

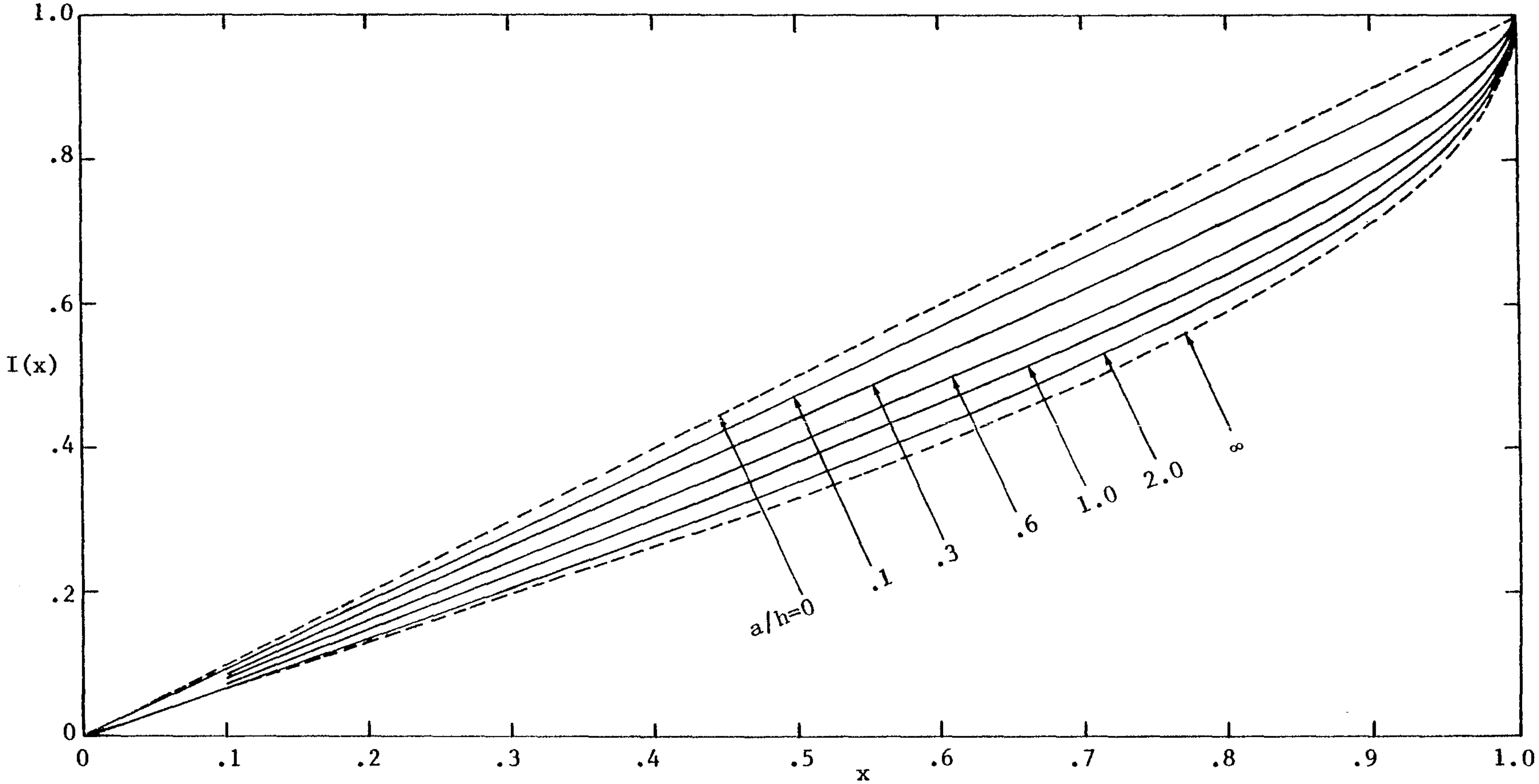

Figure 4. Integral of current density as function of x.

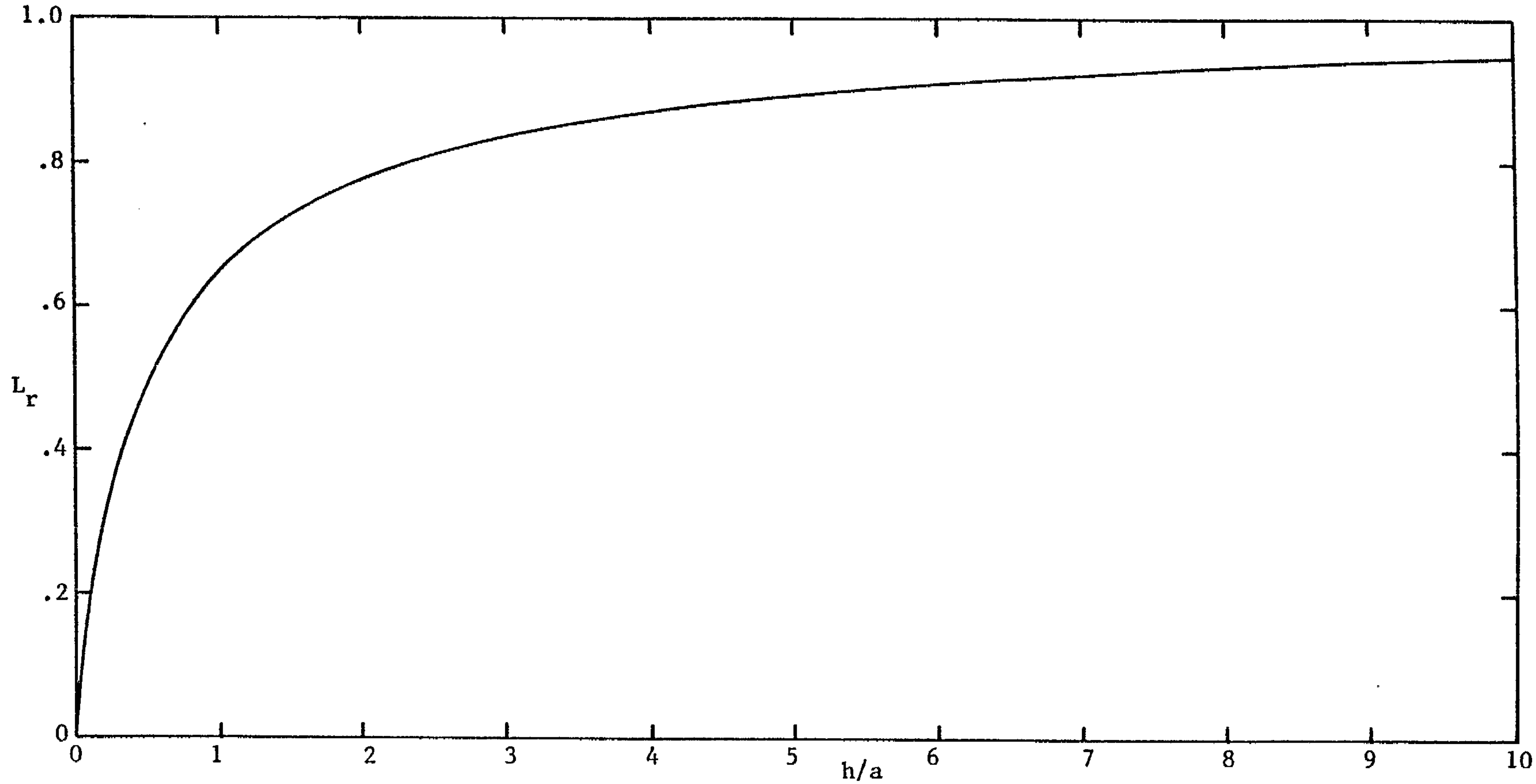

Figure 5. Normalized inductance as function of h/a.

Table I. Normalized Inductance

a/h	L_r	$L_r^{(o)}$	Δ
.1	.9433	.9588	1.6%
.2	.8943	.9200	2.8%
.3	.8514	.8839	3.7%
.4	.8133	.8498	4.3%
.5	.7792	.8181	4.8%
.6	.7484	.7884	5.1%
.7	.7205	.7609	5.3%
.8	.6949	.7351	5.5%
.9	.6713	.7109	5.6%
1.0	.6496	.6884	5.6%
2.0	.4972	.5255	5.4%
5.0	.3062	.3198	4.3%
10.0	.1962	.2034	3.5%

In Table I, $L_r^{(o)}$ is computed by assuming a uniform current distribution in the shell, and Δ is defined by

$$\Delta = \frac{L_r^{(o)} - L_r}{L_r^{(o)}} \times 100\%$$

A more extensive table for $L_r^{(o)}$ can be found in Ref. 7.

Table II. Division Points x_i (no loop at the center)

N	i \ a/h	.1	.2	.3	.4	.5	.6	.7	.8	.9	1	2	5	10
4	1	.527	.550	.569	.585	.599	.612	.622	.631	.639	.646	.680	.700	.705
6	1	.352	.367	.381	.392	.403	.413	.421	.428	.435	.441	.473	.493	.498
	2	.702	.729	.752	.770	.784	.795	.805	.812	.819	.824	.849	.862	.865
8	1	.264	.276	.286	.295	.303	.310	.317	.323	.328	.333	.360	.377	.381
	2	.527	.550	.569	.585	.599	.612	.622	.631	.639	.646	.680	.700	.705
	3	.788	.817	.838	.854	.866	.875	.882	.888	.893	.896	.913	.921	.923
10	1	.211	.220	.229	.236	.243	.249	.254	.259	.263	.267	.289	.304	.307
	2	.422	.440	.456	.470	.482	.493	.503	.511	.519	.525	.560	.581	.585
	3	.632	.658	.679	.697	.712	.725	.735	.744	.751	.757	.787	.804	.807
	4	.840	.868	.887	.900	.910	.917	.922	.926	.929	.932	.944	.949	.951
20	1	.106	.110	.114	.118	.121	.125	.127	.130	.132	.134	.146	.154	.156
	2	.211	.220	.229	.236	.243	.249	.254	.259	.263	.267	.289	.304	.307
	3	.317	.331	.343	.353	.363	.372	.380	.387	.393	.398	.428	.447	.452
	4	.422	.440	.456	.470	.482	.493	.503	.511	.519	.525	.560	.581	.585
	5	.527	.550	.569	.585	.599	.612	.622	.631	.639	.646	.680	.700	.705
	6	.632	.658	.679	.697	.712	.725	.735	.744	.751	.757	.787	.804	.807
	7	.737	.765	.787	.804	.818	.829	.837	.844	.850	.855	.876	.887	.890
	8	.840	.868	.887	.900	.910	.917	.922	.926	.929	.932	.944	.949	.951
	9	.939	.957	.967	.972	.975	.978	.979	.981	.982	.982	.986	.987	.988

Table III. Division Points x_i (no loop at the center)

N	i \ a/h	.1	.2	.3	.4	.5	.6	.7	.8	.9	1	2	5	10
	1	.053	.055	.057	.059	.061	.062	.064	.065	.066	.067	.073	.077	.078
	2	.106	.110	.114	.118	.121	.125	.127	.130	.132	.134	.146	.154	.156
	3	.158	.165	.172	.177	.182	.187	.191	.195	.198	.201	.218	.229	.232
	4	.211	.220	.229	.236	.243	.249	.254	.259	.263	.267	.289	.304	.307
	5	.264	.276	.286	.295	.303	.310	.317	.323	.328	.333	.360	.377	.381
	6	.317	.331	.343	.353	.363	.372	.380	.387	.393	.398	.428	.447	.452
	7	.369	.385	.399	.412	.423	.433	.442	.449	.456	.462	.495	.515	.520
	8	.422	.440	.456	.470	.482	.493	.503	.511	.519	.525	.560	.581	.585
	9	.475	.495	.512	.528	.541	.553	.563	.572	.580	.586	.622	.642	.647
40	10	.527	.550	.569	.585	.599	.612	.622	.631	.639	.646	.680	.700	.705
	11	.580	.604	.624	.642	.656	.669	.680	.689	.696	.703	.736	.754	.758
	12	.632	.658	.679	.697	.712	.725	.735	.744	.751	.757	.787	.804	.807
	13	.684	.712	.734	.752	.766	.778	.788	.796	.802	.808	.834	.848	.851
	14	.737	.765	.787	.804	.818	.829	.837	.844	.850	.855	.876	.887	.890
	15	.788	.817	.838	.854	.866	.875	.882	.888	.893	.896	.913	.921	.923
	16	.840	.868	.887	.900	.910	.917	.922	.926	.929	.932	.944	.949	.951
	17	.890	.915	.931	.940	.947	.951	.955	.957	.959	.961	.968	.971	.972
	18	.939	.957	.967	.972	.975	.978	.979	.981	.982	.982	.986	.987	.988
	19	.981	.988	.991	.993	.994	.994	.995	.995	.995	.996	.996	.997	.997
	1	.035	.037	.038	.039	.041	.042	.042	.043	.044	.045	.049	.051	.052
	2	.070	.074	.076	.079	.081	.083	.085	.087	.088	.089	.097	.103	.104
	3	.106	.110	.114	.118	.121	.125	.127	.130	.132	.134	.146	.154	.156
	4	.141	.147	.153	.157	.162	.166	.170	.173	.176	.179	.194	.204	.207
	5	.176	.184	.191	.197	.202	.207	.212	.216	.220	.223	.242	.254	.257
	6	.211	.220	.229	.236	.243	.249	.254	.259	.263	.267	.289	.304	.307
	7	.246	.257	.267	.275	.283	.290	.296	.302	.307	.311	.336	.353	.356
	8	.281	.294	.305	.314	.323	.331	.338	.344	.350	.355	.383	.400	.405
	9	.317	.331	.343	.353	.363	.372	.380	.387	.393	.398	.428	.447	.452
	10	.352	.367	.381	.392	.403	.413	.421	.428	.435	.441	.473	.493	.498
	11	.387	.404	.418	.431	.443	.453	.462	.470	.477	.483	.517	.537	.542
	12	.422	.440	.456	.470	.482	.493	.503	.511	.519	.525	.560	.581	.585
	13	.457	.477	.494	.509	.522	.533	.543	.552	.559	.566	.601	.622	.627
	14	.492	.513	.531	.547	.561	.573	.583	.592	.600	.606	.642	.662	.667
60	15	.527	.550	.569	.585	.599	.612	.622	.631	.639	.646	.680	.700	.705
	16	.562	.586	.606	.623	.638	.650	.661	.670	.678	.684	.718	.737	.741
	17	.597	.622	.643	.660	.675	.688	.698	.707	.715	.721	.754	.771	.775
	18	.632	.658	.679	.697	.712	.725	.735	.744	.751	.757	.787	.804	.807
	19	.667	.694	.716	.734	.749	.761	.771	.779	.786	.792	.819	.834	.837
	20	.702	.729	.752	.770	.784	.795	.805	.812	.819	.824	.849	.862	.865
	21	.737	.765	.787	.804	.818	.829	.837	.844	.850	.855	.876	.887	.890
	22	.771	.800	.821	.838	.851	.860	.868	.874	.879	.883	.901	.911	.913
	23	.806	.834	.855	.870	.881	.890	.896	.901	.905	.909	.924	.931	.933
	24	.840	.868	.887	.900	.910	.917	.922	.926	.929	.932	.944	.949	.951
	25	.874	.900	.917	.928	.935	.941	.945	.948	.950	.952	.961	.965	.966
	26	.907	.930	.944	.952	.957	.961	.964	.966	.968	.969	.975	.977	.978
	27	.939	.957	.967	.972	.975	.978	.979	.981	.982	.982	.986	.987	.988
	28	.968	.980	.985	.987	.989	.990	.991	.991	.992	.992	.994	.994	.994
	29	.991	.995	.996	.997	.997	.997	.998	.998	.998	.998	.998	.999	.999

Table IV. Division Points x_i (no loop at the center)

N	i \ a/h	.1	.2	.3	.4	.5	.6	.7	.8	.9	1	2	5	10
	1	.026	.028	.029	.030	.030	.031	.032	.032	.033	.034	.037	.039	.039
	2	.053	.055	.057	.059	.061	.062	.064	.065	.066	.067	.073	.077	.078
	3	.079	.083	.086	.089	.091	.093	.096	.097	.099	.101	.109	.115	.117
	4	.106	.110	.114	.118	.121	.125	.127	.130	.132	.134	.146	.154	.156
	5	.132	.138	.143	.148	.152	.156	.159	.162	.165	.168	.182	.192	.194
	6	.158	.165	.172	.177	.182	.187	.191	.195	.198	.201	.218	.229	.232
	7	.185	.193	.200	.207	.212	.218	.223	.227	.231	.234	.254	.267	.270
	8	.211	.220	.229	.236	.243	.249	.254	.259	.263	.267	.289	.304	.307
	9	.237	.248	.257	.265	.273	.280	.286	.291	.296	.300	.325	.340	.344
	10	.264	.276	.286	.295	.303	.310	.317	.323	.328	.333	.360	.377	.381
	11	.290	.303	.314	.324	.333	.341	.348	.355	.361	.366	.394	.412	.417
	12	.317	.331	.343	.353	.363	.372	.380	.387	.393	.398	.428	.447	.452
	13	.343	.358	.371	.383	.393	.402	.411	.418	.425	.430	.462	.482	.486
	14	.369	.385	.399	.412	.423	.433	.442	.449	.456	.462	.495	.515	.520
	15	.396	.413	.428	.441	.453	.463	.472	.480	.488	.494	.528	.548	.553
	16	.422	.440	.456	.470	.482	.493	.503	.511	.519	.525	.560	.581	.585
	17	.448	.468	.484	.499	.512	.523	.533	.542	.549	.556	.591	.612	.617
	18	.475	.495	.512	.528	.541	.553	.563	.572	.580	.586	.622	.642	.647
	19	.501	.522	.541	.556	.570	.582	.593	.602	.610	.616	.651	.672	.677
80	20	.527	.550	.569	.585	.599	.612	.622	.631	.639	.646	.680	.700	.705
	21	.553	.577	.596	.613	.628	.641	.651	.660	.668	.675	.709	.728	.732
	22	.580	.604	.624	.642	.656	.669	.680	.689	.696	.703	.736	.754	.758
	23	.606	.631	.652	.670	.685	.697	.708	.717	.724	.730	.762	.780	.783
	24	.632	.658	.679	.697	.712	.725	.735	.744	.751	.757	.787	.804	.807
	25	.658	.685	.707	.725	.740	.752	.762	.770	.777	.783	.811	.827	.830
	26	.684	.712	.734	.752	.766	.778	.788	.796	.802	.808	.834	.848	.851
	27	.711	.738	.761	.778	.793	.804	.813	.821	.827	.832	.856	.868	.871
	28	.737	.765	.787	.804	.818	.829	.837	.844	.850	.855	.876	.887	.890
	29	.763	.791	.813	.830	.843	.853	.860	.867	.872	.876	.895	.905	.907
	30	.788	.817	.838	.854	.866	.875	.882	.888	.893	.896	.913	.921	.923
	31	.814	.842	.863	.878	.889	.897	.903	.908	.912	.915	.929	.936	.938
	32	.840	.868	.887	.900	.910	.917	.922	.926	.929	.932	.944	.949	.951
	33	.865	.892	.909	.921	.929	.935	.939	.943	.945	.947	.957	.961	.962
	34	.890	.915	.931	.940	.947	.951	.955	.957	.959	.961	.968	.971	.972
	35	.915	.937	.950	.957	.962	.966	.968	.970	.972	.973	.978	.980	.981
	36	.939	.957	.967	.972	.975	.978	.979	.981	.982	.982	.986	.987	.988
	37	.961	.975	.981	.984	.986	.987	.988	.989	.990	.990	.992	.993	.993
	38	.981	.988	.991	.993	.994	.994	.995	.995	.995	.996	.996	.997	.997
	39	.995	.997	.998	.998	.998	.999	.999	.999	.999	.999	.999	.999	.999

Table V. Division Points x_i (no loop at the center)

N	i \ a/h	.1	.2	.3	.4	.5	.6	.7	.8	.9	1	2	5	10
	1	.021	.022	.023	.024	.024	.025	.025	.026	.026	.027	.029	.031	.031
	2	.042	.044	.046	.047	.049	.050	.051	.052	.053	.054	.058	.062	.062
	3	.063	.066	.069	.071	.073	.075	.076	.078	.079	.081	.088	.092	.094
	4	.084	.088	.092	.095	.097	.100	.102	.104	.106	.107	.117	.123	.125
	5	.106	.110	.114	.118	.121	.125	.127	.130	.132	.134	.146	.154	.156
	6	.127	.132	.137	.142	.146	.149	.153	.156	.158	.161	.175	.184	.186
	7	.148	.154	.160	.165	.170	.174	.178	.182	.185	.188	.204	.214	.217
	8	.169	.176	.183	.189	.194	.199	.204	.207	.211	.214	.232	.244	.247
	9	.190	.198	.206	.212	.218	.224	.229	.233	.237	.241	.261	.274	.277
	10	.211	.220	.229	.236	.243	.249	.254	.259	.263	.267	.289	.304	.307
	11	.232	.243	.252	.260	.267	.273	.279	.285	.289	.294	.318	.333	.337
	12	.253	.265	.274	.283	.291	.298	.305	.310	.315	.320	.346	.362	.366
	13	.274	.287	.297	.307	.315	.323	.330	.336	.341	.346	.373	.391	.395
	14	.296	.309	.320	.330	.339	.347	.355	.361	.367	.372	.401	.419	.424
	15	.317	.331	.343	.353	.363	.372	.380	.387	.393	.398	.428	.447	.452
	16	.338	.352	.365	.377	.387	.396	.404	.412	.418	.424	.455	.475	.480
	17	.359	.374	.388	.400	.411	.421	.429	.437	.444	.449	.482	.502	.507
	18	.380	.396	.411	.424	.435	.445	.454	.462	.469	.475	.508	.529	.534
	19	.401	.418	.433	.447	.459	.469	.478	.487	.494	.500	.534	.555	.560
	20	.422	.440	.456	.470	.482	.493	.503	.511	.519	.525	.560	.581	.585
	21	.443	.462	.479	.493	.506	.517	.527	.536	.543	.550	.585	.606	.611
	22	.464	.484	.501	.516	.529	.541	.551	.560	.568	.574	.609	.630	.635
	23	.485	.506	.524	.539	.553	.565	.575	.584	.592	.598	.634	.654	.659
100	24	.506	.528	.546	.562	.576	.588	.599	.608	.616	.622	.657	.678	.682
	25	.527	.550	.569	.585	.599	.612	.622	.631	.639	.646	.680	.700	.705
	26	.548	.571	.591	.608	.622	.635	.645	.654	.662	.669	.703	.722	.727
	27	.569	.593	.613	.630	.645	.658	.668	.677	.685	.692	.725	.744	.748
	28	.590	.615	.635	.653	.668	.680	.691	.700	.708	.714	.747	.765	.769
	29	.611	.636	.657	.675	.690	.703	.713	.722	.730	.736	.767	.784	.788
	30	.632	.658	.679	.697	.712	.725	.735	.744	.751	.757	.787	.804	.807
	31	.653	.679	.701	.719	.734	.746	.757	.765	.772	.778	.807	.822	.825
	32	.674	.701	.723	.741	.756	.768	.778	.786	.792	.798	.825	.840	.843
	33	.695	.722	.744	.762	.777	.789	.798	.806	.812	.818	.843	.856	.859
	34	.716	.744	.766	.784	.798	.809	.818	.825	.831	.837	.860	.872	.875
	35	.737	.765	.787	.804	.818	.829	.837	.844	.850	.855	.876	.887	.890
	36	.757	.786	.808	.825	.838	.848	.856	.862	.868	.872	.892	.902	.904
	37	.778	.807	.828	.845	.857	.866	.874	.880	.884	.888	.906	.915	.917
	38	.799	.827	.848	.864	.875	.884	.891	.896	.900	.904	.920	.927	.929
	39	.819	.848	.868	.882	.893	.901	.907	.912	.915	.918	.932	.939	.940
	40	.840	.868	.887	.900	.910	.917	.922	.926	.929	.932	.944	.949	.951
	41	.860	.887	.905	.917	.925	.931	.936	.939	.942	.944	.954	.959	.960
	42	.880	.906	.922	.933	.940	.945	.949	.952	.954	.956	.964	.967	.968
	43	.900	.924	.939	.947	.953	.957	.960	.963	.964	.966	.972	.975	.976
	44	.920	.942	.953	.960	.965	.968	.971	.972	.974	.975	.979	.982	.982
	45	.939	.957	.967	.972	.975	.978	.979	.981	.982	.982	.986	.987	.988
	46	.957	.972	.978	.982	.984	.986	.987	.988	.988	.989	.991	.992	.992
	47	.973	.983	.987	.990	.991	.992	.992	.993	.993	.994	.995	.995	.996
	48	.987	.992	.994	.995	.996	.996	.997	.997	.997	.997	.998	.998	.998
	49	.997	.998	.999	.999	.999	.999	.999	.999	.999	.999	.999	.999	.999

Table VI. Division Points x_i (one loop at the center)

N	i \ a/h	.1	.2	.3	.4	.5	.6	.7	.8	.9	1	2	5	10
5	1	.211	.220	.229	.236	.243	.249	.254	.259	.263	.267	.289	.304	.307
	2	.632	.658	.679	.697	.712	.725	.735	.744	.751	.757	.787	.804	.807
7	1	.151	.158	.163	.169	.173	.178	.182	.185	.189	.191	.208	.219	.221
	2	.452	.472	.488	.503	.516	.527	.537	.546	.554	.560	.595	.616	.621
	3	.751	.780	.802	.819	.832	.842	.851	.857	.863	.867	.887	.898	.900
9	1	.117	.123	.127	.131	.135	.138	.141	.144	.147	.149	.162	.171	.173
	2	.352	.367	.381	.392	.403	.413	.421	.428	.435	.441	.473	.493	.498
	3	.586	.610	.630	.648	.663	.675	.686	.695	.703	.709	.742	.760	.764
	4	.817	.845	.866	.880	.891	.899	.905	.910	.914	.917	.931	.938	.939
19	1	.056	.058	.060	.062	.064	.066	.067	.068	.070	.071	.077	.081	.082
	2	.167	.174	.181	.186	.192	.197	.201	.205	.208	.211	.229	.241	.244
	3	.278	.290	.301	.310	.319	.327	.334	.340	.345	.350	.378	.395	.400
	4	.389	.406	.420	.433	.445	.455	.646	.472	.479	.485	.519	.540	.545
	5	.500	.521	.539	.555	.569	.581	.591	.600	.608	.615	.650	.670	.675
	6	.610	.635	.656	.674	.689	.702	.712	.721	.728	.735	.766	.783	.787
	7	.720	.748	.770	.788	.802	.813	.822	.829	.835	.840	.864	.876	.878
	8	.829	.857	.877	.891	.901	.908	.914	.919	.922	.925	.938	.944	.945
	9	.934	.953	.963	.969	.973	.975	.977	.979	.980	.981	.984	.986	.986
39	1	.027	.028	.029	.030	.031	.032	.033	.033	.034	.034	.037	.040	.040
	2	.081	.085	.088	.091	.093	.096	.098	.100	.102	.103	.112	.118	.120
	3	.135	.141	.147	.151	.156	.160	.163	.166	.169	.172	.187	.196	.199
	4	.189	.198	.205	.212	.218	.223	.228	.232	.237	.240	.260	.273	.277
	5	.244	.254	.264	.272	.280	.287	.293	.298	.303	.308	.333	.349	.353
	6	.298	.311	.322	.332	.342	.350	.357	.364	.370	.375	.404	.422	.427
	7	.352	.367	.381	.392	.403	.413	.421	.428	.435	.441	.473	.493	.498
	8	.406	.423	.439	.452	.464	.475	.484	.492	.499	.506	.540	.561	.566
	9	.460	.480	.497	.512	.525	.536	.546	.555	.563	.569	.604	.625	.630
	10	.514	.536	.554	.570	.584	.597	.607	.616	.624	.631	.666	.686	.691
	11	.568	.591	.611	.629	.643	.656	.667	.676	.683	.690	.723	.742	.747
	12	.621	.647	.668	.686	.701	.714	.724	.733	.740	.746	.777	.794	.798
	13	.675	.702	.724	.742	.757	.769	.779	.787	.794	.799	.826	.841	.844
	14	.729	.757	.779	.796	.810	.821	.830	.837	.843	.848	.870	.882	.884
	15	.782	.810	.832	.848	.860	.870	.877	.883	.887	.891	.909	.917	.919
	16	.835	.862	.882	.896	.905	.913	.918	.922	.926	.929	.941	.947	.948
	17	.886	.912	.927	.937	.944	.949	.952	.955	.957	.959	.966	.970	.971
	18	.936	.955	.965	.971	.974	.977	.978	.980	.981	.981	.985	.987	.987
	19	.980	.988	.991	.992	.993	.994	.994	.995	.995	.995	.996	.997	.997

Table VII. Division Points x_i (one loop at the center)

N	i \ a/h	.1	.2	.3	.4	.5	.6	.7	.8	.9	1	2	5	10
	1	.018	.019	.019	.020	.021	.021	.022	.022	.022	.023	.025	.026	.026
	2	.054	.056	.058	.060	.062	.063	.065	.066	.067	.068	.074	.078	.079
	3	.089	.093	.097	.100	.103	.106	.108	.110	.112	.114	.124	.130	.132
	4	.125	.131	.136	.140	.144	.148	.151	.154	.157	.159	.173	.182	.184
	5	.161	.168	.174	.180	.185	.190	.194	.198	.201	.204	.222	.233	.236
	6	.197	.206	.213	.220	.226	.232	.237	.242	.246	.249	.270	.284	.287
	7	.233	.243	.252	.260	.267	.274	.280	.285	.290	.294	.318	.334	.337
	8	.268	.280	.291	.300	.308	.316	.322	.328	.334	.339	.365	.383	.387
	9	.304	.317	.329	.340	.349	.357	.365	.372	.377	.383	.412	.431	.435
	10	.340	.355	.368	.379	.390	.399	.407	.414	.421	.426	.458	.478	.482
	11	.376	.392	.406	.419	.430	.440	.449	.457	.464	.470	.503	.523	.528
	12	.411	.429	.445	.458	.470	.481	.490	.499	.506	.512	.547	.568	.572
	13	.447	.466	.483	.497	.510	.522	.532	.540	.548	.554	.589	.610	.615
	14	.483	.503	.521	.537	.550	.562	.572	.581	.589	.596	.631	.651	.656
59	15	.518	.540	.559	.575	.590	.602	.612	.621	.629	.636	.671	.691	.695
	16	.554	.577	.597	.614	.629	.641	.652	.661	.669	.675	.709	.728	.733
	17	.590	.614	.635	.652	.667	.680	.690	.699	.707	.713	.746	.764	.768
	18	.625	.651	.672	.690	.705	.717	.728	.736	.744	.750	.781	.797	.801
	19	.661	.687	.709	.727	.742	.754	.764	.772	.779	.785	.813	.828	.832
	20	.696	.723	.746	.764	.778	.790	.799	.807	.813	.819	.844	.857	.860
	21	.731	.759	.782	.799	.813	.824	.832	.840	.845	.850	.872	.884	.886
	22	.767	.795	.817	.833	.846	.856	.864	.870	.875	.879	.898	.908	.910
	23	.802	.830	.851	.866	.878	.886	.893	.898	.902	.906	.921	.929	.931
	24	.836	.864	.884	.897	.907	.914	.919	.924	.927	.930	.942	.948	.949
	25	.871	.897	.914	.925	.933	.939	.943	.946	.949	.951	.959	.963	.964
	26	.905	.928	.942	.950	.956	.960	.963	.965	.967	.968	.974	.977	.977
	27	.937	.956	.966	.971	.975	.977	.979	.980	.981	.982	.985	.987	.987
	28	.967	.979	.984	.987	.988	.990	.990	.991	.991	.992	.993	.994	.994
	29	.990	.994	.996	.997	.997	.997	.998	.998	.998	.998	.998	.999	.999

Table VIII. Division Points x_i (one loop at the center)

N	i \ a/h	.1	.2	.3	.4	.5	.6	.7	.8	.9	1	2	5	10
	1	.013	.014	.014	.015	.015	.016	.016	.016	.017	.017	.019	.020	.020
	2	.040	.042	.043	.045	.046	.047	.048	.049	.050	.051	.055	.059	.059
	3	.067	.070	.072	.075	.077	.079	.081	.082	.084	.085	.092	.097	.099
	4	.094	.098	.101	.105	.108	.110	.113	.115	.117	.119	.129	.136	.138
	5	.120	.126	.130	.135	.138	.142	.145	.148	.150	.153	.166	.175	.177
	6	.147	.154	.159	.164	.169	.173	.177	.181	.184	.187	.203	.213	.216
	7	.174	.181	.188	.194	.200	.205	.209	.213	.217	.220	.239	.251	.254
	8	.200	.209	.217	.224	.230	.236	.241	.246	.250	.254	.275	.289	.292
	9	.227	.237	.246	.254	.261	.267	.273	.278	.283	.287	.311	.326	.330
	10	.254	.265	.275	.284	.292	.299	.305	.311	.316	.321	.346	.363	.367
	11	.281	.293	.304	.313	.322	.330	.337	.343	.349	.354	.381	.399	.403
	12	.307	.321	.333	.343	.353	.361	.369	.375	.381	.387	.416	.435	.439
	13	.334	.349	.361	.373	.383	.392	.400	.407	.414	.419	.450	.470	.475
	14	.361	.376	.390	.402	.413	.423	.431	.439	.446	.452	.484	.504	.509
	15	.387	.404	.419	.432	.443	.454	.463	.471	.478	.484	.517	.538	.543
	16	.414	.432	.447	.461	.473	.484	.494	.502	.509	.516	.550	.571	.576
	17	.441	.460	.476	.491	.503	.514	.524	.533	.540	.547	.582	.603	.608
	18	.467	.487	.505	.520	.533	.545	.555	.564	.571	.578	.613	.634	.639
	19	.494	.515	.533	.549	.563	.575	.585	.594	.602	.608	.644	.664	.669
79	20	.521	.543	.561	.578	.592	.604	.615	.624	.632	.638	.673	.693	.698
	21	.547	.570	.590	.607	.621	.634	.644	.653	.661	.668	.702	.721	.726
	22	.574	.598	.618	.635	.650	.663	.673	.682	.690	.697	.730	.748	.753
	23	.600	.625	.646	.664	.679	.691	.702	.711	.718	.725	.757	.774	.778
	24	.627	.652	.674	.692	.707	.719	.730	.738	.746	.752	.782	.799	.803
	25	.653	.680	.702	.720	.734	.747	.757	.765	.772	.778	.807	.822	.826
	26	.680	.707	.729	.747	.762	.774	.783	.791	.798	.804	.830	.844	.848
	27	.706	.734	.756	.774	.788	.800	.809	.817	.823	.828	.853	.865	.868
	28	.733	.761	.783	.800	.814	.825	.834	.841	.846	.851	.873	.885	.887
	29	.759	.787	.809	.826	.839	.849	.857	.864	.869	.873	.893	.903	.905
	30	.785	.814	.835	.851	.863	.872	.880	.885	.890	.894	.911	.919	.921
	31	.811	.840	.860	.875	.886	.894	.901	.906	.910	.913	.927	.934	.936
	32	.837	.865	.884	.898	.908	.915	.920	.924	.928	.930	.942	.948	.949
	33	.863	.890	.907	.919	.927	.933	.938	.941	.944	.946	.956	.960	.961
	34	.888	.914	.929	.939	.945	.950	.954	.956	.958	.960	.967	.971	.971
	35	.913	.936	.949	.956	.961	.965	.967	.969	.971	.972	.977	.980	.980
	36	.938	.956	.966	.971	.975	.977	.979	.980	.981	.982	.985	.987	.987
	37	.960	.974	.980	.984	.986	.987	.988	.989	.989	.990	.992	.993	.993
	38	.980	.988	.991	.993	.994	.994	.995	.995	.995	.995	.996	.997	.997
	39	.995	.997	.998	.998	.998	.999	.999	.999	.999	.999	.999	.999	.999

Table IX. Division Points x_i (one loop at the center)

N	i \ a/h	.1	.2	.3	.4	.5	.6	.7	.8	.9	1	2	5	10
	1	.011	.011	.012	.012	.012	.013	.013	.013	.013	.014	.015	.016	.016
	2	.032	.033	.035	.036	.037	.038	.039	.039	.040	.041	.044	.047	.047
	3	.053	.056	.058	.060	.061	.063	.064	.066	.067	.068	.074	.078	.079
	4	.075	.078	.081	.084	.086	.088	.090	.092	.093	.095	.103	.109	.110
	5	.096	.100	.104	.107	.110	.113	.116	.118	.120	.122	.133	.140	.141
	6	.117	.123	.127	.131	.135	.138	.141	.144	.147	.149	.162	.171	.173
	7	.138	.145	.150	.155	.159	.164	.167	.170	.173	.176	.191	.201	.204
	8	.160	.167	.173	.179	.184	.189	.193	.197	.200	.203	.220	.232	.234
	9	.181	.189	.196	.203	.208	.214	.218	.223	.226	.230	.249	.262	.265
	10	.203	.212	.219	.227	.233	.239	.244	.249	.253	.257	.278	.292	.295
	11	.224	.234	.243	.250	.257	.264	.269	.275	.279	.283	.306	.322	.325
	12	.245	.256	.266	.274	.282	.289	.295	.300	.305	.310	.335	.351	.355
	13	.267	.278	.289	.298	.306	.314	.320	.326	.332	.336	.363	.380	.384
	14	.288	.301	.312	.321	.330	.338	.346	.352	.358	.363	.391	.409	.413
	15	.309	.323	.335	.345	.355	.363	.371	.378	.384	.389	.419	.437	.442
	16	.330	.345	.358	.369	.379	.388	.396	.403	.409	.415	.446	.465	.470
	17	.352	.367	.381	.392	.403	.413	.421	.428	.435	.441	.473	.493	.498
	18	.373	.389	.403	.416	.427	.437	.446	.454	.461	.467	.500	.520	.525
	19	.394	.411	.426	.439	.451	.462	.471	.479	.486	.492	.526	.547	.552
	20	.416	.434	.449	.463	.475	.486	.495	.504	.511	.517	.552	.573	.578
	21	.437	.456	.472	.486	.499	.510	.520	.529	.536	.543	.577	.598	.603
	22	.458	.478	.495	.510	.523	.534	.544	.553	.561	.567	.602	.623	.628
	23	.479	.500	.518	.533	.547	.558	.569	.577	.585	.592	.627	.648	.653
99	24	.501	.522	.540	.556	.570	.582	.593	.602	.609	.616	.651	.672	.676
	25	.522	.544	.563	.579	.593	.606	.616	.625	.633	.640	.675	.695	.699
	26	.543	.566	.585	.602	.617	.629	.640	.649	.657	.663	.698	.717	.722
	27	.564	.588	.608	.625	.640	.652	.663	.672	.680	.686	.720	.739	.743
	28	.586	.610	.630	.648	.663	.675	.686	.695	.703	.709	.742	.760	.764
	29	.607	.632	.653	.670	.685	.698	.709	.717	.725	.731	.763	.780	.784
	30	.628	.654	.675	.693	.708	.720	.731	.739	.747	.753	.783	.800	.804
	31	.649	.675	.697	.715	.730	.742	.753	.761	.768	.774	.803	.819	.822
	32	.670	.697	.719	.737	.752	.764	.774	.782	.789	.795	.822	.837	.840
	33	.691	.719	.741	.759	.773	.785	.795	.802	.809	.814	.840	.854	.857
	34	.712	.740	.762	.780	.794	.806	.815	.822	.828	.834	.858	.870	.873
	35	.733	.761	.784	.801	.815	.826	.834	.841	.847	.852	.874	.885	.888
	36	.754	.783	.805	.822	.835	.845	.853	.860	.865	.870	.890	.900	.902
	37	.775	.804	.826	.842	.854	.864	.871	.877	.882	.886	.904	.913	.915
	38	.796	.825	.846	.861	.873	.882	.889	.894	.898	.902	.918	.926	.928
	39	.817	.845	.866	.880	.891	.899	.905	.910	.914	.917	.931	.938	.939
	40	.838	.866	.885	.898	.908	.915	.920	.925	.928	.931	.943	.948	.950
	41	.858	.885	.903	.915	.924	.930	.935	.938	.941	.943	.953	.958	.959
	42	.879	.905	.921	.932	.939	.944	.948	.951	.953	.955	.963	.967	.968
	43	.899	.923	.937	.946	.952	.956	.960	.962	.964	.965	.972	.975	.975
	44	.919	.941	.953	.960	.964	.968	.970	.972	.973	.974	.979	.981	.982
	45	.938	.957	.966	.971	.975	.977	.979	.980	.981	.982	.985	. 87	.987
	46	.956	.971	.978	.981	.984	.985	.986	.987	.988	.988	.991	.992	.992
	47	.973	.983	.987	.989	.991	.992	.992	.993	.993	.993	.995	.995	.995
	48	.987	.992	.994	.995	.996	.996	.997	.997	.997	.997	.998	.998	.998
	49	.996	.998	.999	.999	.999	.999	.999	.999	.999	.999	.999	.999	.999

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