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Corrections to the Transmission-Line Parameters of a Coaxial Line When the Center Conductor has Impedance

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Abstract

Transmission-line equations for a uniform coaxial structure whose center conductor has impedance are derived rigorously. These equations reduce to the usual transmission-line equations for coax when the impedance of the center conductor becomes small. In the more general case, the effect of the impedance of the center conductor is to modify the L and the C of the usual transmission-line equations.

The center conductor impedance of an equivalent uniform line is calculated approximately for the case where the center conductor is a row of conducting disks separated by homogeneous dielectric. An outline of a more exact analysis of the disk line is also given.

Contents

Figures

Tables

Acknowledgement

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I. Introduction

It is of interest, at this time, to measure the impedance per unit length of a structure that, to a first approximation, can be considered a uniform circular cylinder.

One technique for making such a measurement has been suggested by Carl Baum. He proposes to enclose a length of the test cylinder within a hollow circular cylindrical tube, and to measure the transmission-line characteristics (say $\beta(\omega)$, the propagation constant, and $Z(\omega)$, the transmission-line impedance) of the cylinder-tube structure. If we also measure the transmission-line characteristics of the structure made up of the same outer tube and a dummy test cylinder, resembling the cylinder under study but having zero impedance per unit length, and, furthermore, assume elementary transmission-line theory to be strictly applicable, we have more than enough information to determine the impedance per unit length of the test cylinder.

Alternatively, it may be possible to calculate, in some simple manner, the inductance or capacitance per unit length of the test cylinder-tube configuration. For example, if the test cylinder can really be accurately approximated by a uniform circular cylinder, the inductance or capacitance calculation is elementary. When such calculations are possible, the measurements with a dummy test cylinder would be unnecessary. There would be enough information to determine the impedance per unit length of the real cylinder without any extra measurements. It would still be necessary to assume, of course, that elementary transmission-line theory is strictly applicable.

The primary purpose of this note is to develop some way to estimate the range of validity of the above measurement technique. The main limitation on the validity would seem to be the assumption that elementary transmission-line theory is strictly applicable. Therefore, we will study the full wave solution of a certain idealized cylinder-tube configuration (a circular coaxial system) and determine under what conditions the formulas for the cylinder's impedance per unit length, in terms of easily measured quantities, go over into the simple formulas that are the result of the transmission-line assumption. Because of their immediate practical interest (and because they are simpler), we will concentrate quite a bit of attention on test cylinders whose impedance per unit length is purely reactive.

One important kind of test cylinder can be modelled as a row of closely spaced parallel disks, separated by homogeneous dielectric material. A secondary purpose of the present note is to sketch an analytical treatment of such a structure. In Section IV, we will develop an approximate uniform reactive cylinder, equivalent to the disk line, by using some heuristic arguments. A more exact approach to the analysis of such a line is outlined in Section V.

II. Transmission-Line Solution for a Uniform Line

Consider the transmission structure shown in figure 1. Elementary transmission-line considerations lead us to the equations

$$\frac{dI}{dZ} = i\omega C_o V \qquad (2.1)$$

$$\frac{dV}{dZ} = i\omega L_o I - Z_i(\omega) I \qquad (2.2)$$

where C_o and L_o are certain constants, depending on the geometry of the transmitting structure but not on ω or Z_i, and $Z_i(\omega)$ is the impedance per unit length of the inner conductor. It follows in an elementary fashion that

$$\beta^2(\omega) = \omega^2 L_o C_o + i\omega C_o Z_i(\omega) \qquad (2.3)$$

and

$$Z^2(\omega) = \frac{i\omega L_o - Z_i(\omega)}{i\omega C_o} \qquad (2.4)$$

Elementary transmission-line theory also leads us to state that

$$L_o C_o = \mu_o \varepsilon_o . \qquad (2.5)$$

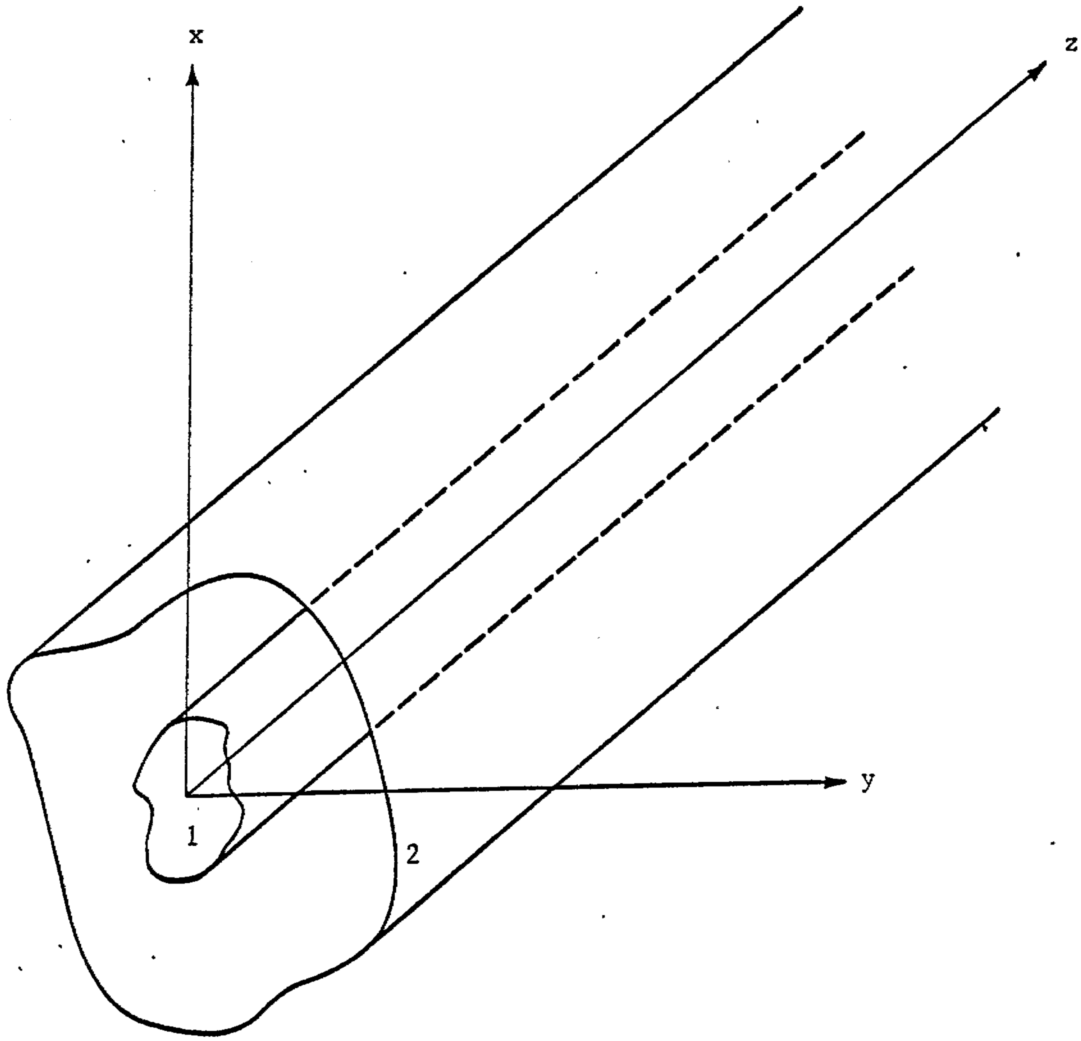
Thus, by defining $\beta_o(\omega)$ as the value of $\beta(\omega)$ when Z_i is zero, i.e.,

$$\beta_o^2(\omega) \equiv \omega^2 \mu_o \varepsilon_o , \qquad (2.6)$$

we can rewrite equation (2.3) as

$$\beta^2(\omega) = \beta_o^2(\omega) + i\omega C_o Z_i(\omega) \qquad (2.7)$$

If $Z_i(\omega)$ were zero, Z would be equal to $(L_o/C_o)^{\frac{1}{2}}$, which we will write as Z_o. With this notation, equation (2.4) becomes



Z_i = impedance per unit length of conductor 1.

C_o = capacitance per unit length between conductors 1 and 2.

L_o = inductance per unit length between conductors 1 and 2.

Figure 1. A uniform transmission structure with inner conductor impedance.

$$Z^2(\omega) = Z_o^2 + i\,\frac{Z_i(\omega)}{\omega C_o} \tag{2.8}$$

By rearranging equations (2.7) and (2.8) we can arrive at some formulas for $Z_i(\omega)$ in terms of quantities measured with Z_i present ($\beta(\omega)$ and $Z(\omega)$) and quantities determined primarily by the geometry of the line ($\beta_o(\omega)$, Z_o, C_o). Because of equations (2.5) and (2.6), just one number (together with the value of ω) is sufficient to determine all quantities of the second, geometry dependant, type. Some formulas for Z_i are

$$Z_i(\omega) = i\,\frac{\beta_o^2(\omega)-\beta^2(\omega)}{\omega C_o} \tag{2.9}$$

$$= i\omega C_o\left\{Z_o^2 - Z^2(\omega)\right\} \tag{2.10}$$

$$= i\left\{[Z_o^2 - Z^2(\omega)][\beta_o^2(\omega) - \beta^2(\omega)]\right\}^{\frac{1}{2}} \tag{2.11}$$

The technique for measuring $Z_i(\omega)$ mentioned in the introduction would involve the use of one of the above formulas. In the next section we will see how these formulas are altered, for the special case of circular coax, when a full wave treatment of the problem is considered, rather than the elementary transmission-line treatment we have considered in this section.

III. Wave Solution for a Uniform Line

Consider the special transmission structure that is defined by the inner conductor being a circular cylinder of radius "a" and the outer conductor being a circular cylinder of radius "b". We will consider the propagation of the lowest TM mode. This mode goes over into the usual TEM mode as the impedance of the inner conductor approaches zero, and this mode will be the only propagating mode at the frequencies of interest. It is an axially symmetric mode, and, employing the usual notation for cylindrical coordinates ([1], p. 51), the only field components present will be $E_\rho(\rho,z)$, $E_z(\rho,z)$, and $H_\phi(\rho,z)$. Two relevant components of Maxwell's equations may be written

$$\frac{\partial H_\phi(\rho,z)}{\partial z} = i\omega\varepsilon_o E_\rho(\rho,z) \tag{3.1}$$

$$\frac{\partial E_\rho(\rho,z)}{\partial z} - \frac{\partial E_z(\rho,z)}{\partial \rho} = i\omega\mu_o H_\phi(\rho,z). \tag{3.2}$$

Let us integrate these equations with respect to ρ between the limits a and b. The result is

$$\frac{\partial}{\partial z}\int_a^b H_\phi(\rho,z)d\rho = i\omega\varepsilon_o \int_a^b E_\rho(\rho,z)d\rho \tag{3.3}$$

$$\frac{\partial}{\partial z}\int_a^b E_\rho(\rho,z)d\rho = i\omega\mu \int_a^b H_\phi(\rho,z)d\rho + E_z(\rho,z)\Big|_a^b \tag{3.4}$$

Let us make the identifications

$$V(z) \equiv \int_a^b E_\rho(\rho,z)d\rho \tag{3.5}$$

$$I(z) \equiv 2\pi a H_\phi(a,z) \tag{3.6}$$

$$E_z(a,z) = Z_i I(z) \tag{3.7}$$

Then, recalling that $E_z(b,z)$ is zero, equations (3.3) and (3.4) may be written as

$$\frac{\partial I(z)}{\partial z} = i\omega C_o F V(z) \tag{3.8}$$

$$\frac{\partial V(z)}{\partial z} = \frac{i\omega L_o}{F} I(z) - Z_i I(z) \tag{3.9}$$

where

$$C_o = \frac{2\pi}{\varepsilon_o \ln(b/a)} \tag{3.10}$$

$$L_o = \frac{\mu_o}{2\pi} \ln(b/a) \tag{3.11}$$

are the usual coaxial transmission-line inductance and capacitance per unit length, while

$$F = a \ln(b/a) \frac{H_\phi(a,z)}{\int_a^b H_\phi(\rho,z)d\rho} . \tag{3.12}$$

Since, for a single mode, $H_\phi(\rho,z)$ has the same z-dependance as $H_\phi(a,z)$, F is independant of z. Thus, by defining an inductance and capacitance per unit length through

$$C \equiv C_o F \tag{3.13}$$

$$L \equiv L_o/F, \tag{3.14}$$

we can write some transmission-line equations, for the V and I we have defined, in the forms

$$\frac{\partial I}{\partial z} = i\omega CV \tag{3.15}$$

$$\frac{\partial V}{\partial z} = i\omega LI - Z_i I \tag{3.16}$$

By noting that

$$\omega^2 LC = \omega^2 L_o C_o = \beta_o^2(\omega) \tag{3.17}$$

$$\frac{L}{C} = \frac{1}{F^2}\frac{L_o}{C_o} = \frac{1}{F^2} Z_o^2, \tag{3.18}$$

we can use the similarity of form between equations (2.1), (2.2) and (3.15), (3.16) to say, from equations (2.9), (2.10), and (2.11), that

$$Z_i(\omega) = i\left\{\frac{\beta_o^2(\omega)-\beta^2(\omega)}{\omega C_o F}\right\} \tag{3.19}$$

$$= i\omega C_o F\left\{\frac{Z_o^2}{F^2} - Z^2(\omega)\right\} \tag{3.20}$$

$$= i\left\{\left[\frac{Z_o^2}{F^2} - Z^2(\omega)\right]\left[\beta_o^2(\omega) - \beta^2(\omega)\right]\right\}^{\frac{1}{2}} \tag{3.21}$$

It is clear that these equations reduce to the more elementary equations of the previous section if F is equal to unity. This will be true if Z_i is very small, since in that case

$$H_\phi(\rho,z) \simeq (a/\rho)H_\phi(a,z), \tag{3.22}$$

and substituting equation (3.22) into equation (3.12) leads quickly to the result that F is unity.

We must now determine how F changes as Z_i increases, and thus find out how much we can trust equations (2.9), (2.10) and (2.11).

For the mode we are interested in, the three field components may be written in the forms

$$E_z(\rho,z) = E_o\{Y_o(\alpha r)J_o(\alpha\rho/a) - J_o(\alpha r)Y_o(\alpha\rho/a)\}e^{i\beta_o\sqrt{1-(\alpha/\beta_o a)^2}\, z} \tag{3.23}$$

$$\frac{H_\phi(\rho,z)}{E_o} = -\frac{i\omega\varepsilon_o a}{\alpha}\{Y_o(\alpha r)J_1(\alpha\rho/a) - J_o(\alpha r)Y_1(\alpha\rho/a)\}e^{i\beta_o\sqrt{1-(\alpha/\beta_o a)^2}\, z} \tag{3.24}$$

$$E_\rho(\rho,z) = -\frac{ia}{\alpha}\sqrt{\beta_o^2-\alpha^2/a^2}\, E_o\{Y_o(\alpha r)J_1(\alpha\rho/a) - J_o(\alpha r)Y_1(\alpha\rho/a)\}e^{i\beta_o\sqrt{1-(\alpha/\beta_o a)^2}\, z} \tag{3.25}$$

where the usual notation for Bessel functions ([2], Chap. 9) has been used, and we have already satisfied the condition that $E_z(b,z)$ be zero. Note that, in writing the above equations, we have used the notation

$$r \equiv b/a.$$

The α of equations (3.23), (3.24), and (3.25) must be determined from the impedance condition at the inner cylinder, i.e.,

$$\frac{1}{Z_i} = \frac{2\pi a H_\phi(a,z)}{E_z(a,z)} = -\frac{i\omega\varepsilon_o(2\pi a^2)}{\alpha}\left\{\frac{Y_o(\alpha r)J_1(\alpha)-J_o(\alpha r)Y_1(\alpha)}{Y_o(\alpha r)J_o(\alpha)-J_o(\alpha r)Y_o(\alpha)}\right\} \tag{3.26}$$

In this note we are primarily interested in reactive Z_i, and so we will write

$$Z_i = \frac{S_r}{-i\omega\varepsilon_o\pi a^2}, \tag{3.27}$$

where S_r is the elastance per unit length of the inner cylinder normalized to the free space elastance per unit length of that cylinder. Note that this elastance notation is mainly a matter of convenience; S_r could actually be a function of frequency without disturbing any of the following arguments. Now, from equations (3.26) and (3.27), we can rewrite the α equation in the form

$$\alpha^2 = 2S_r\alpha\left\{\frac{Y_o(\alpha r)J_1(\alpha)-J_o(\alpha r)Y_1(\alpha)}{Y_o(\alpha r)J_o(\alpha)-J_o(\alpha r)Y_o(\alpha)}\right\} \tag{3.28}$$

Now let us see what F is. From equations (3.12) and (3.24) it follows that

$$F = \frac{a \ln r}{(a/\alpha)} \left\{ \frac{Y_o(\alpha r)J_1(\alpha) - J_o(\alpha r)Y_1(\alpha)}{Y_o(\alpha r)J_o(\alpha) - J_o(\alpha r)Y_o(\alpha)} \right\}, \tag{3.29}$$

and thus, making use of equation (3.28),

$$F = \frac{\alpha^2 \ln r}{2S_r} \tag{3.30}$$

So we see that for each value of r and S_r we must solve equation (3.28) (possibly by an iteration based on the equation resulting from taking the square root of both sides of that equation), and then compute F from equation (3.30). A rather pleasant way to compute the Bessel function cross products in equation (3.28) is given in the appendix, where it is shown that

$$\alpha \left\{ \frac{Y_o(\alpha r)J_1(\alpha) - J_o(\alpha r)Y_1(\alpha)}{Y_o(\alpha r)J_o(\alpha) - J_o(\alpha r)Y_o(\alpha)} \right\} = \frac{2}{r^2-1} \frac{\sum_{k=1}^{\infty} T_k}{\sum_{k=1}^{\infty} (T_k/k)} \tag{3.31}$$

where

$$T_1 = 1 \tag{3.32}$$

$$T_2 = \frac{r^2-1}{r^2} \tag{3.33}$$

and

$$T_{k+1} = \frac{r^2-1}{r^2} \left\{ T_k - \frac{\alpha^2(r^2-1)}{4} \cdot \frac{T_{k-1}}{k(k-1)} \right\} \tag{3.34}$$

From the appendix it also follows that, as α approaches zero, the right hand side of equation (3.31) approaches $(1/\ln r)$. Thus, for small S_r (which corresponds to small α), equation (3.28) gives $\alpha^2 = (2\, S_r/\ln r)$, and thus equation (3.30) leads to $F = 1$, as we would hope. The values of F for several other values of S_r and r are given in table 1, along with the corresponding values of α and α^2. This data is also presented graphically in figures 2, 3, 4, and 5.

In Table 2 and figure 6 we give some data on $(\alpha b/a)$ vs. (a/b) for a few values of S_r. This information is presented as a matter of interest. If there were no center conductor, $(\alpha b/a)$ would be the first root of $J_o(x)$, i.e., 2.4048. We approach this value more rapidly, as (a/b) approaches zero, if S_r is large.

Table 1 enables us to tell if the S_r we calculate from some measurements, by the technique described in the introduction, is low enough for the technique to be accurate (i.e., low enough for F to be close to unity). Another way to look at this data is given in figure 7, where the maximum S_r that keeps F greater than .99, .95 and .9 is plotted vs. (a/b).

We note from equations (2.9) and (3.19) that what we actually compute, when we use equation (2.9), is not $Z_i(\omega)$ but $FZ_i(\omega)$. But F is itself a function of Z_i (and r). We may recover Z_i from FZ_i by using the data of table 1. The result is a corrected version of equation (2.9) which, for the case where Z_i is equal to $i(S_r/\omega\varepsilon_o\pi a^2)$, we have displayed graphically as figure 8. By using figure 8, a measurement of the propagation constant of the line is enough to give us an accurate value of S_r, without the approximation involved in assuming elementary transmission-line theory to be strictly applicable.

Before closing this section, it might be well to say again explicitly why we have gone to the trouble of tabulating F in various ways. The reason is that, for the coaxial line, 1 - F is a measure of the error incurred by using equations such as (2.9), (2.10), and (2.11) rather than the accurate relations (3.19), (3.20), and (3.21). Thus 1 - F should also be a measure of the error due to using elementary transmission-line formulas when the measurement setup is almost, but not exactly, coaxial. The fact that 1 - F is a measure of the error for the coaxial case is clear, and rigorous, when one uses equation (2.9) instead of Equation (3.19) for, denoting the approximate Z_i by $\tilde{Z}_i$, we have, from equations (2.9) and (3.19),

$$\frac{Z_i-\tilde{Z}_i}{Z_i} = 1 - F.$$

But, even if one of the other formulas for Z_i were used, 1 - F can still be used to get a first order estimate of the relative error in Z_i in the coaxial

Table 1. Correction factor vs. relative elastance for various a/b.

$a/b = .1$

S_r	α	α^2	F	S_r	α	α^2	F
.01	.08954	.00802	.92297	.1	.20242	.04097	.47171
.02	.12157	.01478	.85071	.2	.22300	.04973	.28626
.03	.14294	.02043	.78405	.3	.23029	.05303	.20352
.04	.15854	.02513	.72340	.4	.23396	.05474	.15754
.05	.17042	.02904	.66878	.5	.23615	.05577	.12841
.06	.17973	.03230	.61983	.6	.23761	.05646	.10833
.07	.18718	.03504	.57624	.7	.23865	.05695	.09367
.08	.19324	.03734	.53737	.8	.23943	.05733	.08250
.09	.19823	.03930	.50270	.9	.24003	.05761	.07371
.10	.20242	.04097	.47171	1.0	.24052	.05785	.06660

$a/b = .2$

S_r	α	α^2	F	S_r	α	α^2	F
.01	.10981	.01206	.97035	.1	.30453	.09274	.74630
.02	.15298	.02340	.94165	.2	.37846	.14323	.57631
.03	.18458	.03407	.91392	.3	.41543	.17258	.46294
.04	.20999	.04410	.88715	.4	.43718	.19112	.38450
.05	.23134	.05352	.86134	.5	.45132	.20369	.32783
.06	.24974	.06237	.83648	.6	.46120	.21271	.28528
.07	.26586	.07068	.81258	.7	.46846	.21946	.25228
.08	.28017	.07850	.78959	.8	.47401	.22469	.22601
.09	.29298	.08584	.76751	.9	.47839	.22885	.20462
.10	.30453	.09274	.74630	1.0	.48193	.23226	.18689

$a/b = .3$

S_r	α	α^2	F	S_r	α	α^2	F
.01	.12782	.01634	.98359	.1	.37593	.14132	.85073
.02	.17929	.03214	.96752	.2	.49297	.24302	.73149
.03	.21779	.04743	.95179	.3	.56333	.31734	.63677
.04	.24944	.06222	.93639	.4	.61064	.37288	.56117
.05	.27663	.07652	.92131	.5	.64451	.41539	.50012
.06	.30059	.09036	.90656	.6	.66986	.44871	.45019
.07	.32209	.10374	.89213	.7	.68947	.47537	.40881
.08	.34159	.11668	.87802	.8	.70506	.49711	.37406
.09	.35945	.12921	.86422	.9	.71772	.51512	.34455
.10	.37593	.14132	.85073	1.0	.72821	.53029	.31922

Table 1 (Continued)

a/b = .4

S_r	α	α^2	F	S_r	α	α^2	F
.01	.14698	.02160	.98972	.1	.44408	.19720	.90348
.02	.20679	.04276	.97958	.2	.59830	.35797	.82000
.03	.25197	.06349	.96959	.3	.69983	.48977	.74795
.04	.28947	.08379	.95973	.4	.77374	.59867	.68570
.05	.32200	.10368	.95002	.5	.83035	.68947	.63176
.06	.35095	.12316	.94044	.6	.87516	.76590	.58483
.07	.37716	.14225	.93100	.7	.91355	.83458	.54488
.08	.40118	.16094	.92170	.8	.94152	.88646	.50766
.09	.42339	.17926	.91252	.9	.96674	.93459	.47575
.10	.44408	.19720	.90348	1.0	.98819	.97651	.44739

a/b = .5

S_r	α	α^2	F	S_r	α	α^2	F
.01	.16929	.02866	.99324	.1	.51948	.26986	.93527
.02	.23860	.05693	.98654	.2	.71124	.50586	.87659
.03	.29124	.08482	.97991	.3	.84427	.71279	.82344
.04	.33517	.11234	.97335	.4	.94594	.89480	.77529
.05	.37348	.13949	.96684	.5	1.02738	1.05551	.73162
.06	.40776	.16619	.96040	.6	1.09451	1.19795	.69197
.07	.43897	.19269	.95403	.7	1.15096	1.32471	.65588
.08	.46772	.21876	.94771	.8	1.19917	1.43801	.62297
.09	.49445	.24448	.94146	.9	1.24083	1.53966	.59289
.10	.51948	.26986	.93527	1.0	1.27719	1.63121	.56533

a/b = .6

S_r	α	α^2	F	S_r	α	α^2	F
.01	.19743	.03898	.99553	.1	.61198	.37452	.95658
.02	.27858	.07761	.99108	.2	.84690	.71724	.91596
.03	.34043	.11589	.98667	.3	1.01549	1.03123	.87796
.04	.39222	.15383	.98228	.4	1.14861	1.31932	.84243
.05	.43754	.19144	.97793	.5	1.25859	1.58404	.80917
.06	.47824	.22871	.97360	.6	1.35193	1.82772	.77804
.07	.51541	.26565	.96930	.7	1.43262	2.05239	.74887
.08	.54979	.30227	.96503	.8	1.50331	2.25993	.72152
.09	.58185	.33855	.96079	.9	1.56588	2.45198	.69585
.10	.61198	.37452	.95658	1.0	1.62174	2.63002	.67174

Table 1 (Continued)

a/b = .7

S_r	α	α^2	F	S_r	α	2	F
.01	.23646	.05591	.99714	.1	.73823	.54499	.97191
.02	.33392	.11151	.99428	.2	1.02947	1.05982	.94503
.03	.40839	.16678	.99144	.3	1.24356	1.54643	.91929
.04	.47089	.22174	.98862	.4	1.41656	2.00665	.89465
.05	.52573	.27639	.98580	.5	1.56274	2.44217	.87106
.06	.57508	.33072	.98300	.6	1.68956	2.85460	.84847
.07	.62028	.38475	.98021	.7	1.80151	3.24543	.82683
.08	.66217	.43846	.97743	.8	1.90159	3.61606	.80610
.09	.70134	.49188	.97467	.9	1.99199	3.96779	.78623
.10	.73823	.54499	.97191	1.0	2.07409	4.30183	.76718

a/b = .8

S_r	α	α^2	F	S_r	α	α^2	F
.01	.29918	.08951	.99867	.1	.93889	.88151	.98352
.02	.42268	.17866	.99667	.2	1.31690	1.73423	.96746
.03	.51725	.26754	.99501	.3	1.59977	2.55927	.95181
.04	.59677	.35613	.99335	.4	1.83240	3.35769	.93656
.05	.66665	.44442	.99170	.5	2.03237	4.13051	.92170
.06	.72967	.53242	.99006	.6	2.20878	4.87873	.90721
.07	.78748	.62013	.98842	.7	2.36713	5.60328	.89310
.08	.84116	.70755	.98678	.8	2.51099	6.30510	.87934
.09	.89144	.79467	.98513	.9	2.64292	6.98503	.86592
.10	.93889	.88151	.98352	.1	2.76477	7.64395	.85285

a/b = .9

S_r	α	α^2	F	S_r	α	α^2	F
.01	.43553	.18968	.99926	.1	1.37268	1.88426	.99263
.02	.61570	.37909	.99852	.2	1.93413	3.74087	.98535
.03	.75380	.56821	.99778	.3	2.36016	5.57034	.97816
.04	.87009	.75705	.99704	.4	2.71535	7.37312	.97104
.05	.97243	.94561	.99630	.5	3.02484	9.14964	.96401
.06	1.06485	1.13390	.99557	.6	3.30158	10.90042	.95706
.07	1.14974	1.32191	.99483	.7	3.55329	12.62589	.95019
.08	1.22867	1.50963	.99410	.8	3.78503	14.32648	.94340
.09	1.30272	1.69708	.99336	.9	4.00033	16.00263	.93669
.10	1.37268	1.88426	.99263	1.0	4.20176	17.65478	.93006

Table 1 (Continued)

a/b = 3/4

S_r	α	α^2	F	S_r	α	α^2	F
.01	.26338	.06937	.99777	.1	.82461	.67998	.97809
.02	.37205	.13842	.99556	.2	1.15348	1.33052	.95692
.03	.45517	.20718	.99335	.3	1.39754	1.95311	.93646
.04	.52500	.27562	.99114	.4	1.59661	2.54916	.91668
.05	.58631	.34376	.98895	.5	1.76636	3.12002	.89757
.06	.64156	.41161	.98676	.6	1.91494	3.66699	.87911
.07	.69220	.47915	.98458	.7	2.04726	4.19126	.86125
.08	.73918	.54639	.98241	.8	2.16657	4.69401	.84399
.09	.78315	.61333	.98025	.9	2.27515	5.17631	.82730
.10	.82461	.67998	.97809	1.0	2.37470	5.63922	.81115

a/b = 2/3

S_r	α	α^2	F	S_r	α	α^2	F
.01	.22172	.04916	.99665	.1	.69075	.47713	.96730
.02	.31303	.09799	.99333	.2	.96104	.92359	.93621
.03	.38275	.14650	.99001	.3	1.15830	1.34166	.90666
.04	.44123	.19468	.98672	.4	1.31661	1.73346	.87857
.05	.49249	.24255	.98344	.5	1.44946	2.10095	.85186
.06	.53860	.29009	.98018	.6	1.56395	2.44595	.82646
.07	.58079	.33732	.97694	.7	1.66438	2.77017	.80229
.08	.61987	.38423	.97371	.8	1.75361	3.07513	.77929
.09	.65638	.43084	.97049	.9	1.83365	3.36228	.75738
.10	.69075	.47713	.96730	1.0	1.90603	3.63294	.73652

a/b = 1/3

S_r	α	α^2	F	S_r	α	α^2	F
.01	.13398	.01795	.98609	.1	.39839	.15871	.87181
.02	.18816	.03541	.97242	.2	.52812	.27891	.76605
.03	.22886	.05238	.95901	.3	.60903	.37092	.67917
.04	.26244	.06887	.94583	.4	.66516	.44244	.60759
.05	.29140	.08492	.93291	.5	.70639	.49899	.54820
.06	.31704	.10051	.92022	.6	.73789	.54448	.49847
.07	.34012	.11568	.90777	.7	.76267	.58167	.45645
.08	.36115	.13043	.89555	.8	.78264	.61253	.42058
.09	.38048	.14477	.88357	.9	.79904	.63846	.38968
.10	.39839	.15871	.87181	1.0	.81275	.66055	.36285

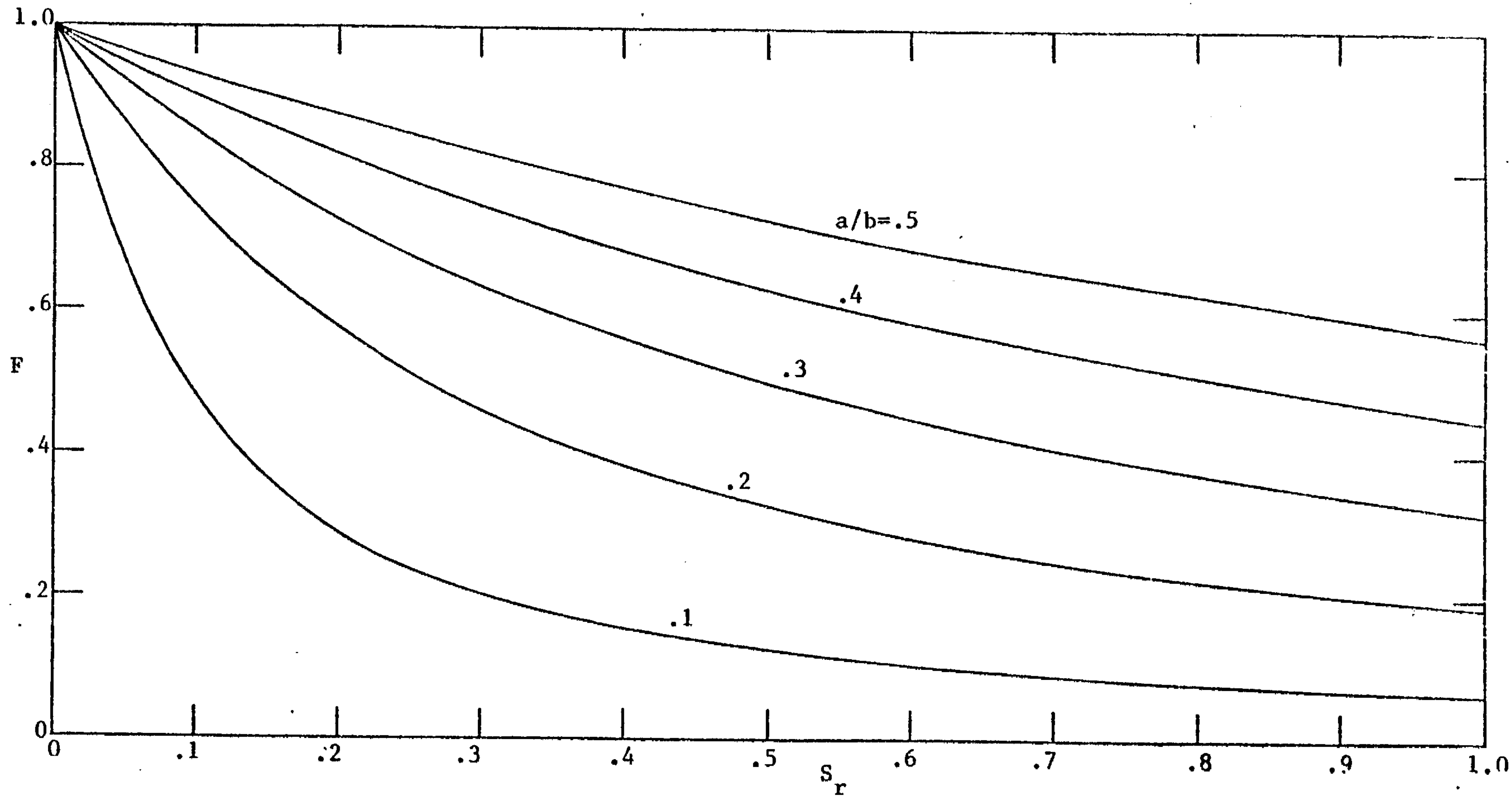


Figure 2. Correction factor vs. relative elastance for thin inner conductors.

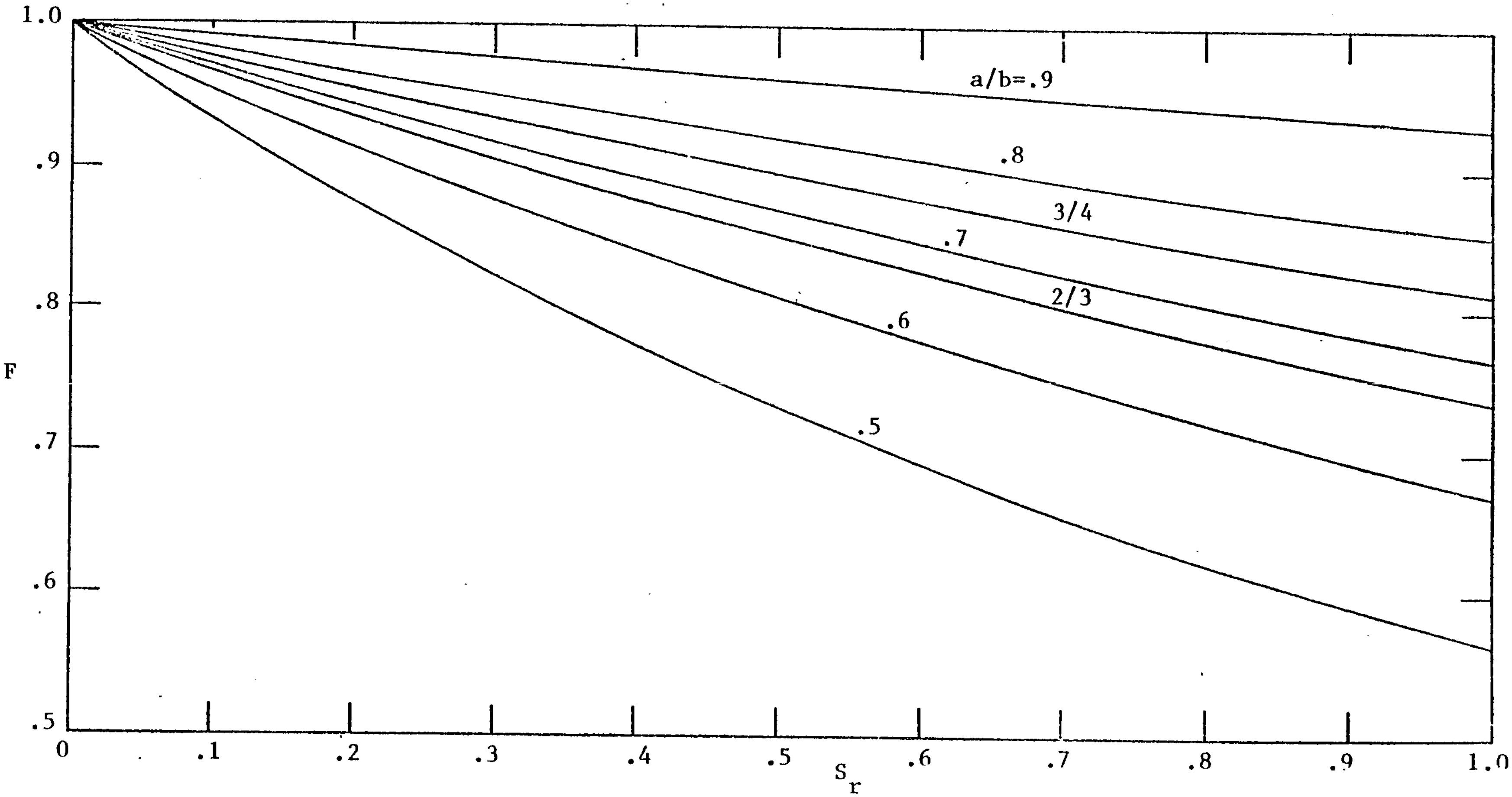


Figure 3. Correction factor vs. relative elastance for thick inner conductors.

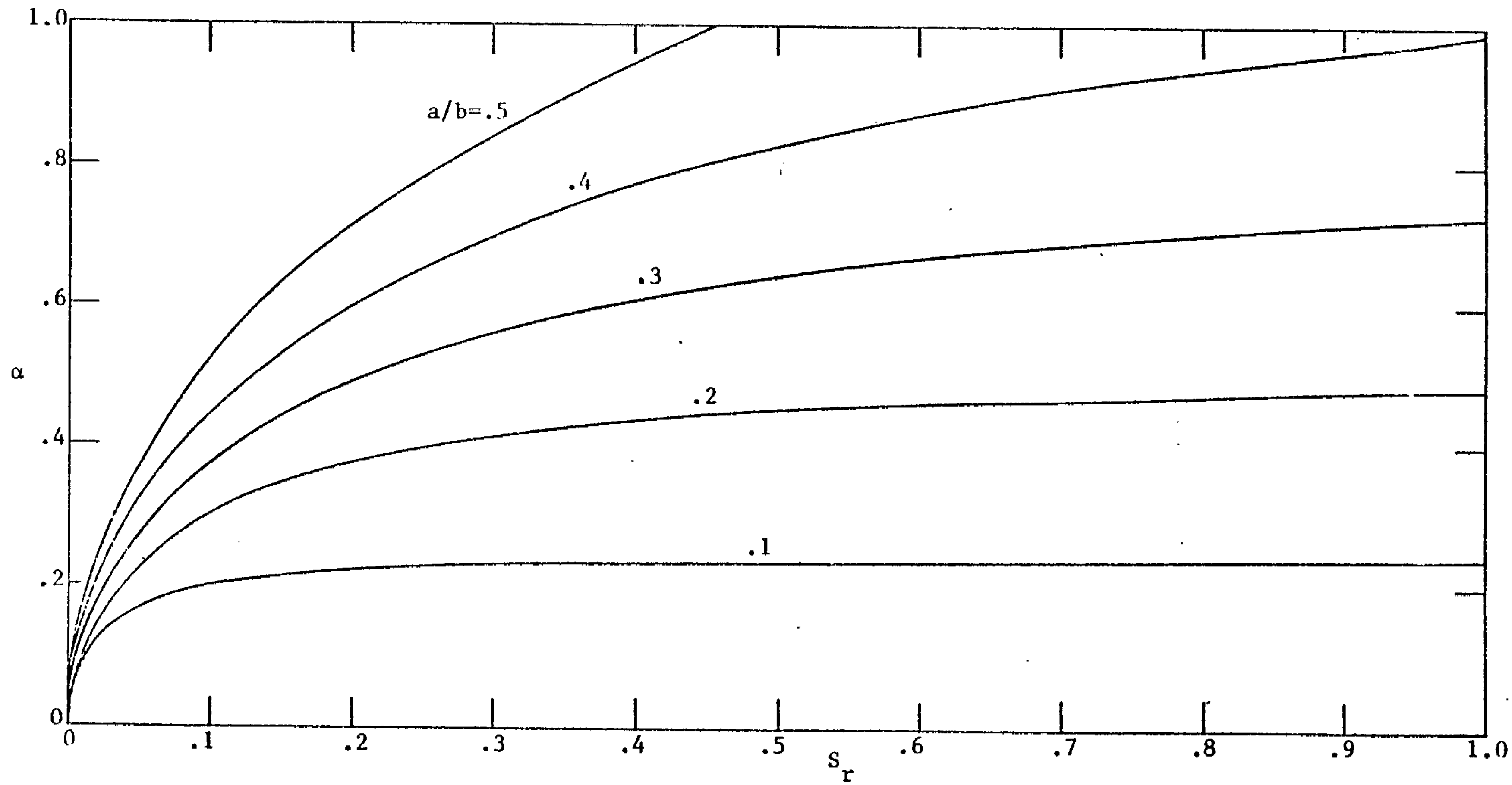


Figure 4. α vs. relative elastance for thin inner conductors.

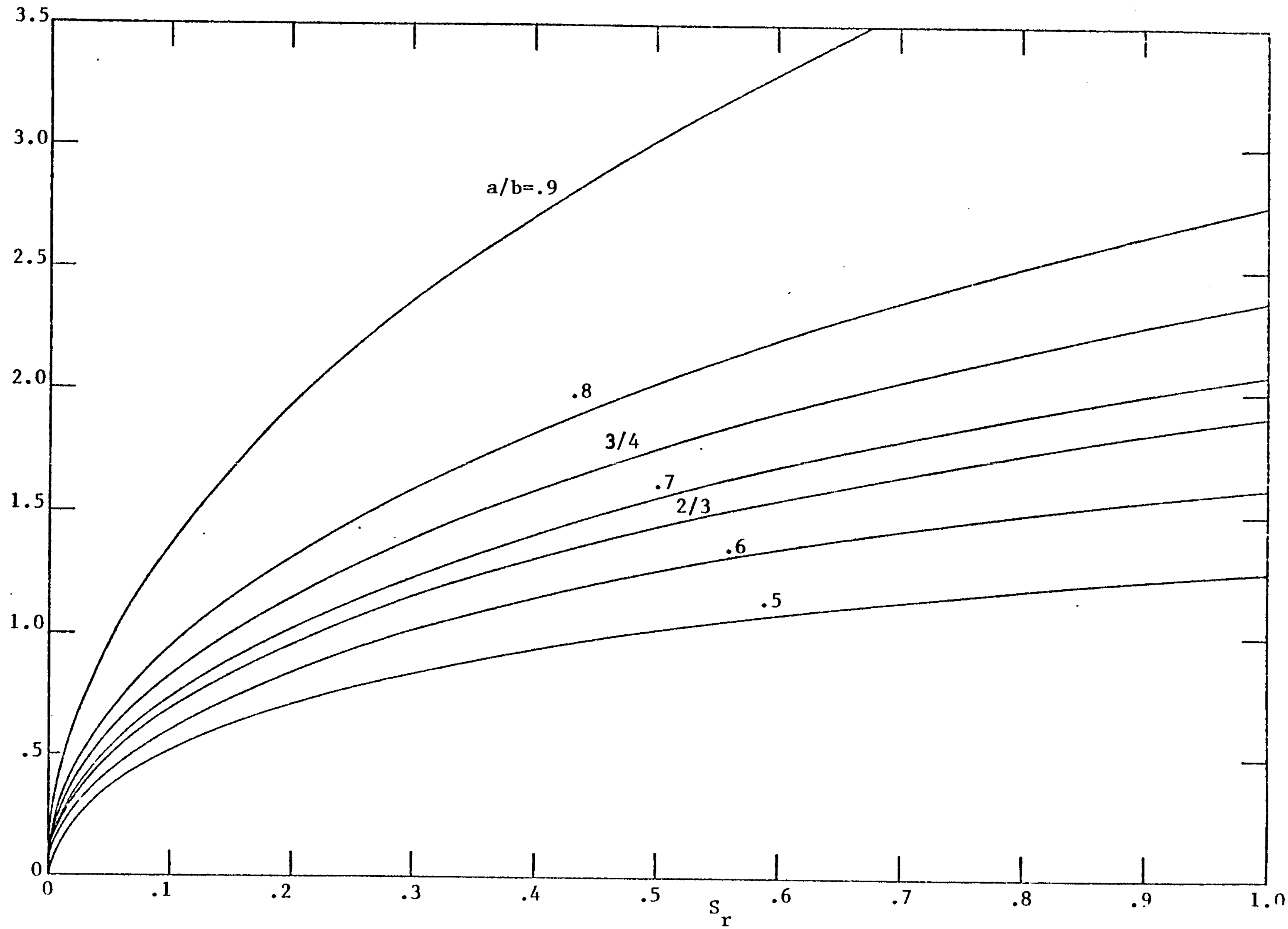


Figure 5. α vs. relative elastance for thick inner conductors.

Table 2. $\alpha b/a$ vs. a/b for a few values of relative elastance.

$S_r = .01$

a/b	αb/a
0	2.4048
.1	.8954
.2	.5491
.3	.4261
.4	.3675
.5	.3386
.6	.3291
.7	.3378
.8	.3740
.9	.4839

$S_r = .1$

a/b	αb/a
0	2.4048
.1	2.0242
.2	1.5227
.3	1.2531
.4	1.1102
.5	1.0390
.6	1.0200
.7	1.0546
.8	1.1736
.9	1.5252

$S_r = .02$

a/b	αb/a
0	2.4048
.1	1.2157
.2	.7649
.3	.5976
.4	.5170
.5	.4772
.6	.4643
.7	.4770
.8	.5284
.9	.6841

$S_r = .2$

a/b	αb/a
0	2.4048
.1	2.2300
.2	1.8923
.3	1.6432
.4	1.4958
.5	1.4225
.6	1.4115
.7	1.4707
.8	1.6461
.9	2.1490

$S_r = .05$

a/b	αb/a
0	2.4048
.1	1.7042
.2	1.1567
.3	.9221
.4	.8050
.5	.7470
.6	.7292
.7	.7510
.8	.8333
.9	1.0805

$S_r = 1.0$

a/b	αb/a
0	2.4048
.1	2.4052
.2	2.4097
.3	2.4274
.4	2.4705
.5	2.5544
.6	2.7029
.7	2.9630
.8	3.4560
.9	4.6686

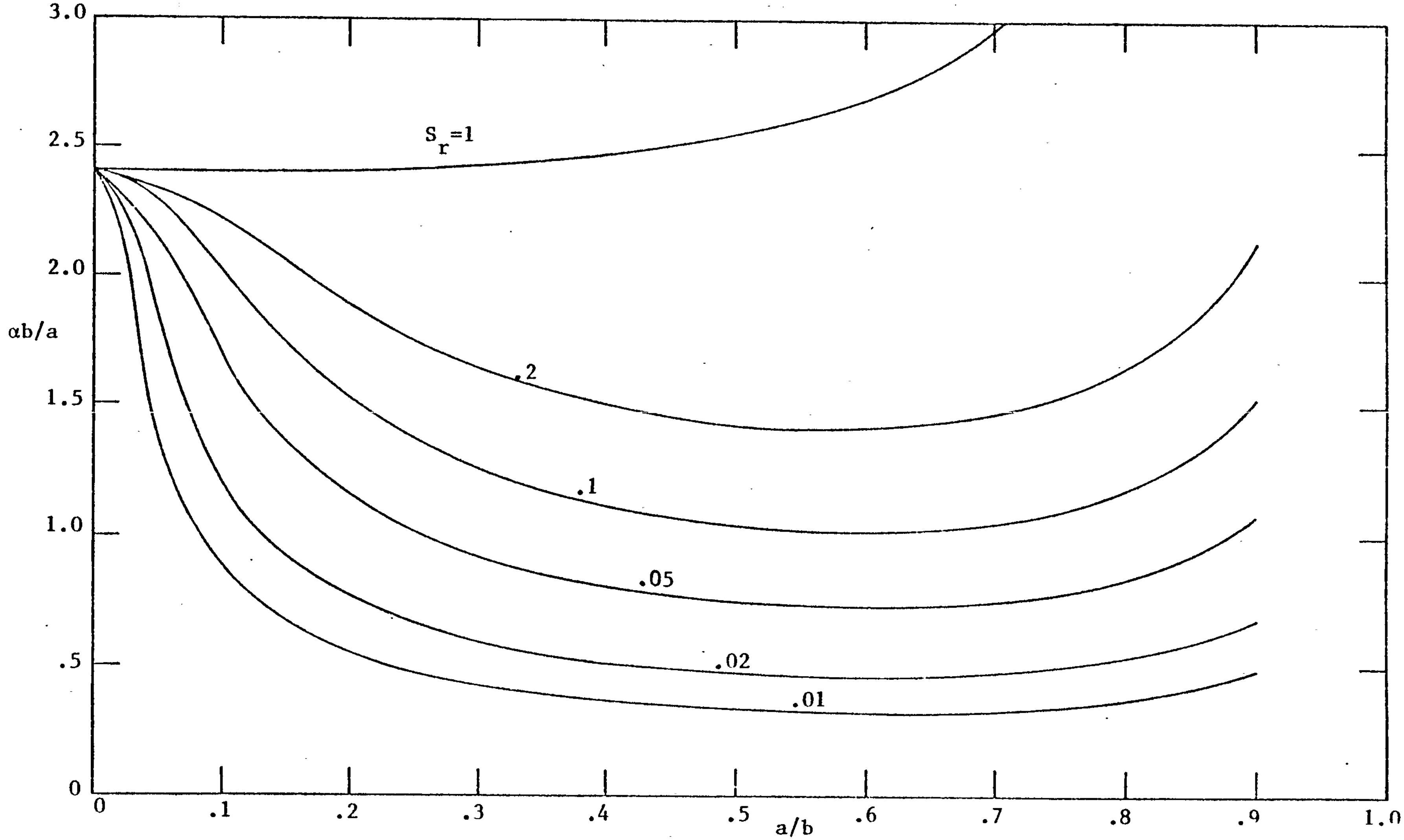


Figure 6. (αb/a) vs. a/b for a few values of relative elastance.

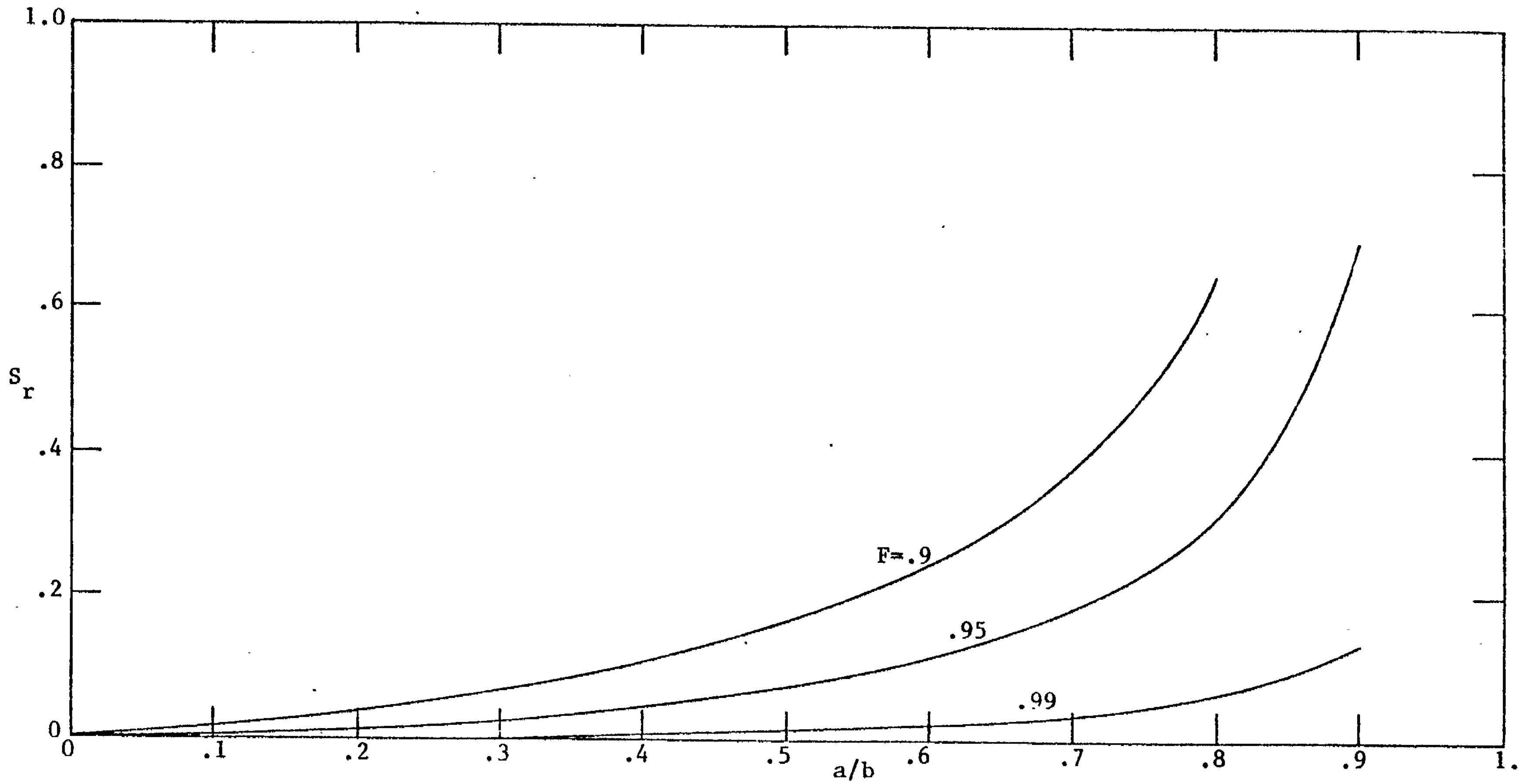


Figure 7. S_r vs. a/b to give a given F.

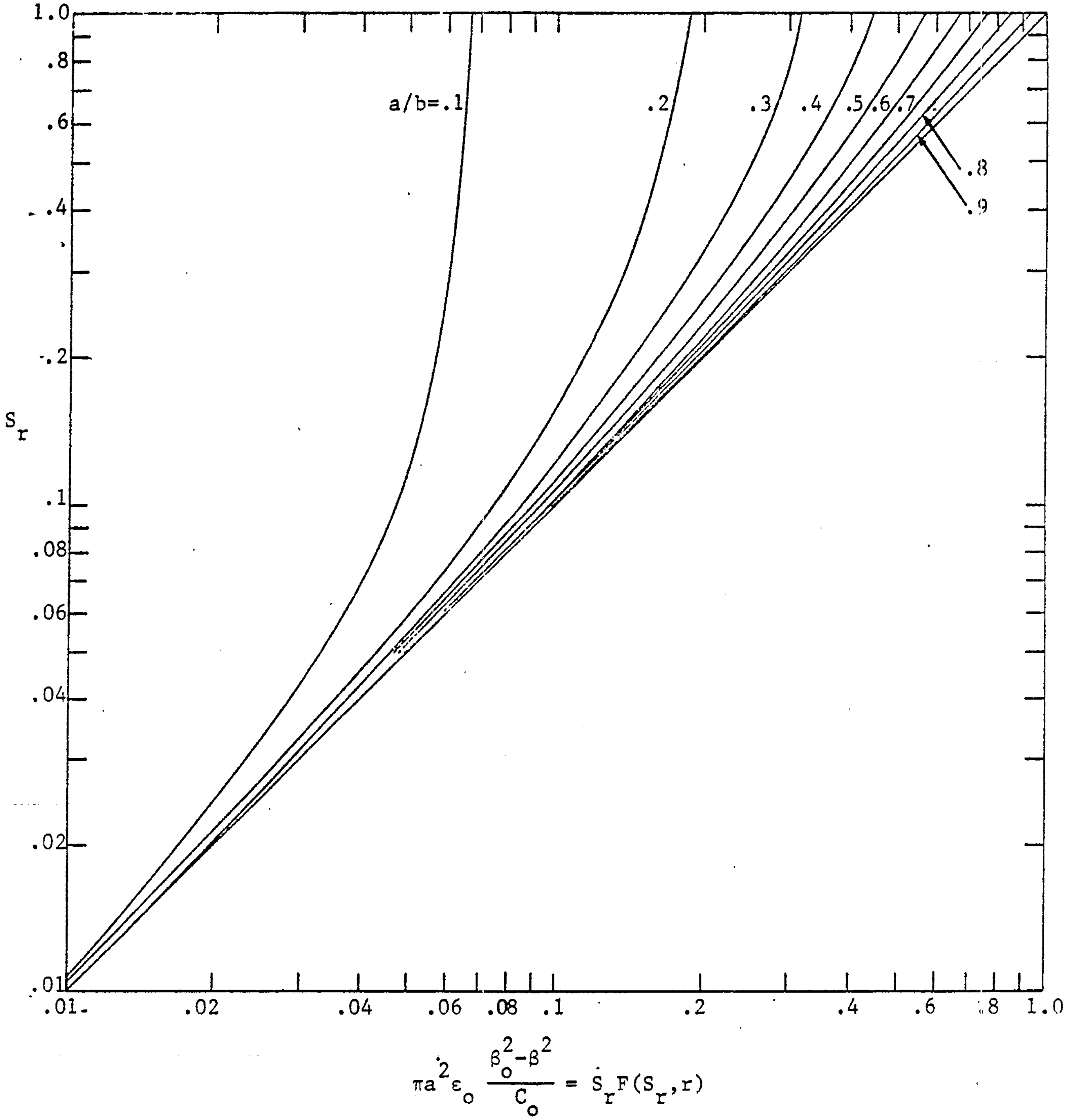


Figure 8. A corrected version of equation (2.9).

case. For example, if equation (2.11) were used instead of equation (3.21), we have

$$\frac{Z_i-\widetilde{Z}_i}{Z_i} = 1 - F\left\{\frac{Z_o^2-Z^2}{Z_o^2-F^2Z^2}\right\}^{\frac{1}{2}}$$

or, if $1 - F \equiv \delta$,

$$\frac{Z_i-\widetilde{Z}_i}{Z_i} = 1 - F\left\{\frac{Z_o^2-Z^2}{Z_o^2-Z^2+2\delta Z^2-\delta^2 Z^2}\right\}^{\frac{1}{2}}$$

$$= \delta \cdot \frac{1}{1-(Z/Z_o)^2} + O(\delta^2)$$

All relevant quantities in the final expression are easily obtainable from measurements or the tables, and so we can quickly determine if equation (2.11) is adequate in any given case.

IV. Uniform Line Equivalent of a Disk Line

Consider the transmitting structure shown in figure 9. The center "conductor" of this structure is made up of a row of uniformly spaced metallic disks, separated by a homogeneous dielectric. This structure is an idealized model of a structure that is of practical importance. Let us see what can be done toward an analytical treatment of this structure. In this section, we will briefly derive an approximate Z_i for the row of disks, based on some rather handwaving arguments, and present some numerical data on this approximate Z_i. In the next section, we will outline a more exact approach.

If the disks of figure 9 are close together, compared to their radii, then between each pair of disks the dominant electric field will be along the z-axis, and will vary very little with z. Thus, approximate equations for the fields between the disks are

$$E_z(\rho,z) \simeq E_o J_o(\eta\beta_o\rho) \tag{4.1}$$

$$\frac{H_\phi(\rho,z)}{E_o} \simeq \frac{-i}{Z_1} J_1(\eta\beta_o\rho) \tag{4.2}$$

where

$$Z_1 \equiv \frac{1}{\eta}\sqrt{\frac{\mu_o}{\varepsilon_o}} \tag{4.3}$$

$$\eta^2 \equiv \varepsilon/\varepsilon_o \tag{4.4}$$

$$\beta_o^2 \equiv \omega^2\mu_o\varepsilon_o \tag{4.5}$$

If equations (4.1) and (4.2) were good approximations all the way out to $\rho = a$ (and these approximations improve as Δ/a decreases) then, from the definition of Z_i, we could say immediately that

$$Z_i = \frac{J_o(\eta\beta_o a)(\delta/\Delta)}{2\pi a(-i/Z_1)J_1(\eta\beta_o a)} \tag{4.6}$$

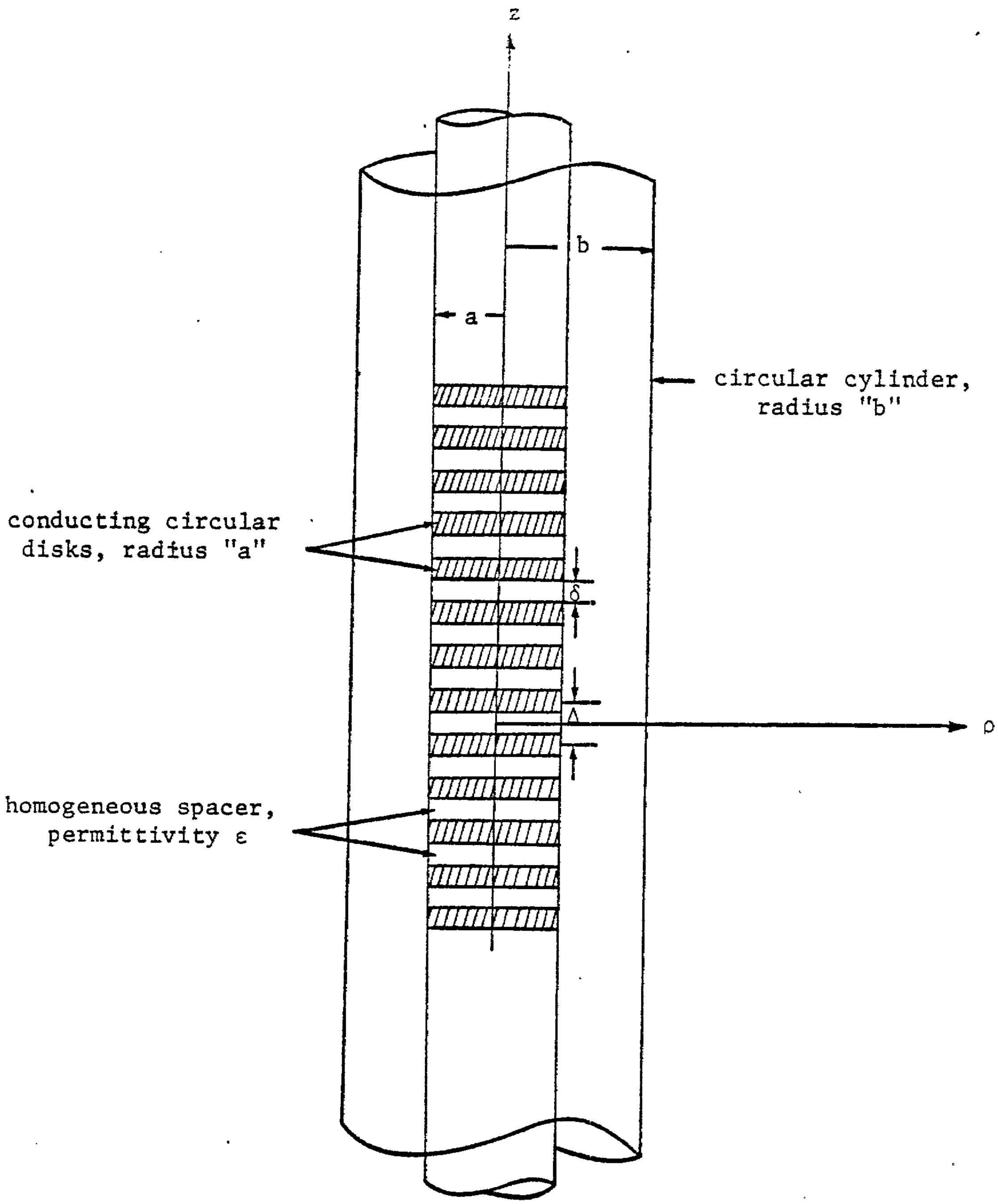


Figure 9. A disk line.

where the E_z field at $\rho = a$, which is nonzero only between the disks, has been averaged over a full Δ period of the structure.

In order to make use of the tables of the previous section, it would be more useful to write an equivalent S_r from the above Z_i. It is not hard to see that this equivalent S_r is just

$$S_r = Z_i(-i\omega\varepsilon_o\pi a^2) = \omega\varepsilon_o a Z_1 \frac{J_o(\eta\beta_o a)}{2J_1(\eta\beta_o a)} (\delta/\Delta) \tag{4.7}$$

$$= \frac{(\delta/\Delta)}{\eta^2} g(\eta\beta_o a) \tag{4.8}$$

where

$$g(x) = \frac{xJ_o(x)}{2J_1(x)} \tag{4.9}$$

We have tabulated g(x) in table 3 and drawn it in figure 10. From this data, and the tables of the previous section, one can determine the approximate behavior of a disk line.

Table 3. Auxiliary function for computing the effective S_r of a disk line.

x	g(x)	x	g(x)	x	g(x)
.1	.99875	3.4	-3.45543	6.7	-10.01623
.2	.99499	3.5	-4.84230	6.8	-15.28040
.3	.98871	3.6	-7.38680	6.9	-29.46685
.4	.97987	3.7	-13.71950	7.0	-224.28387
.5	.96842	3.8	-59.65653	7.1	42.20694
.6	.95431	3.9	28.76085	7.2	19.55293
.7	.93746	4.0	12.02695	7.3	12.74061
.8	.91778	4.1	7.71519	7.4	9.40301
.9	.89515	4.2	5.70348	7.5	7.38477
1.0	.86944	4.3	4.51535	7.6	6.00505
1.1	.84050	4.4	3.71329	7.7	4.98063
1.2	.80812	4.5	3.12136	7.8	4.17215
1.3	.77210	4.6	2.65488	7.9	3.50275
1.4	.73217	4.7	2.26790	8.0	2.92625
1.5	.68802	4.8	1.93307	8.1	2.41286
1.6	.63929	4.9	1.63288	8.2	1.94218
1.7	.58551	5.0	1.35538	8.3	1.49931
1.8	.52619	5.1	1.09183	8.4	1.07265
1.9	.46068	5.2	.83547	8.5	.62560
2.0	.38821	5.3	.58064	8.6	.23053
2.1	.30783	5.4	.32219	8.7	-.20197
2.2	.21826	5.5	.05512	8.8	-.65372
2.3	.11831	5.6	-.22588	8.9	-1.13472
2.4	.00578	5.7	-.52684	9.0	-1.65708
2.5	-.12167	5.8	-.85503	9.1	-2.23631
2.6	-.26729	5.9	-1.21974	9.2	-2.89335
2.7	-.43547	6.0	-1.63340	9.3	-3.65791
2.8	-.63228	6.1	-2.11338	9.4	-4.57423
2.9	-.86635	6.2	-2.68515	9.5	-5.71214
3.0	-1.15047	6.3	-3.38804	9.6	-7.18939
3.1	-1.50438	6.4	-4.28652	9.7	-9.22252
3.2	-1.96026	6.5	-5.49469	9.8	-12.25929
3.3	-2.57446	6.6	-7.23590	9.9	-17.40078

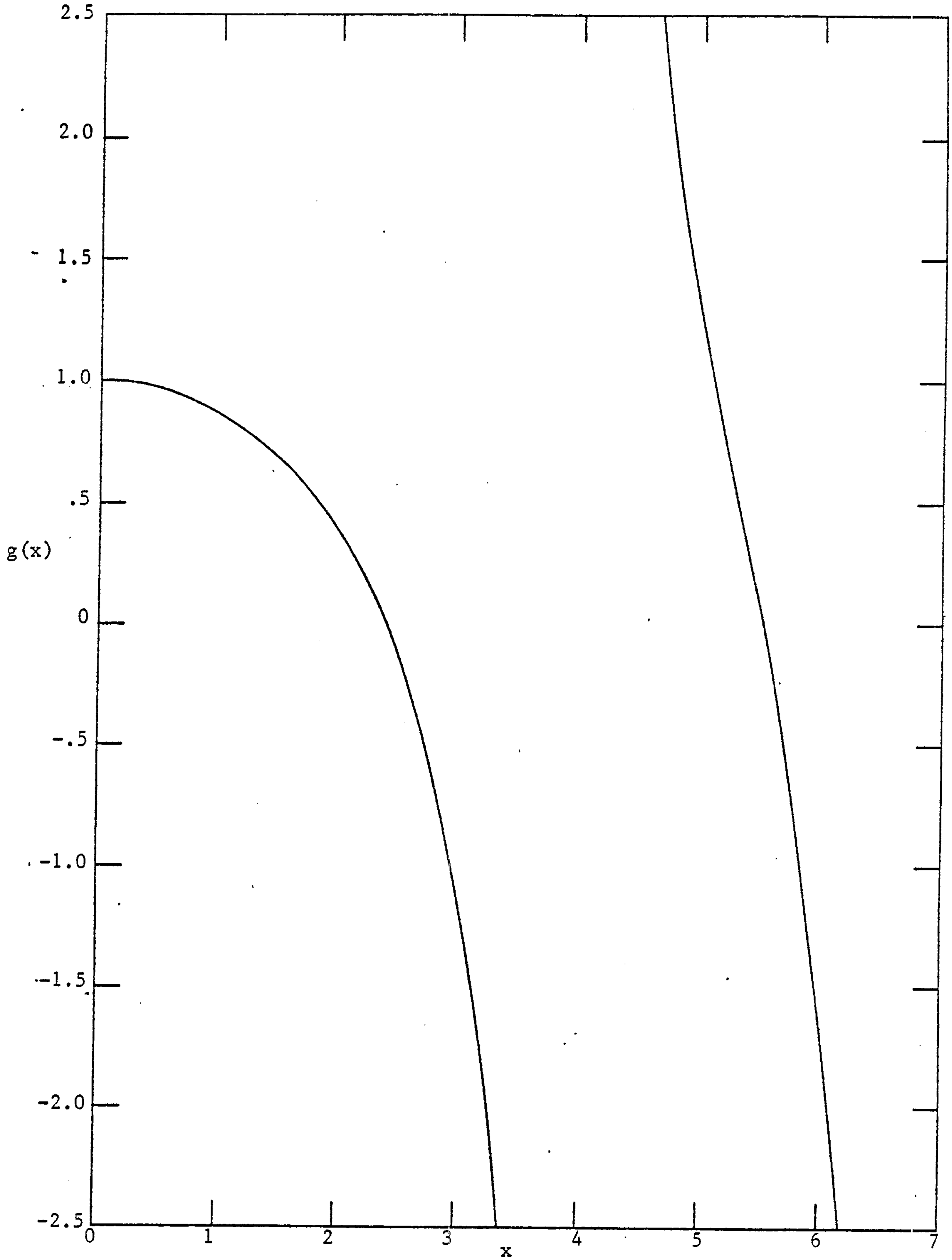


Figure 10. Auxiliary function for computing effective S_r of a disk line.

V. An Exact Approach for a Disk Line

Since the line whose idealized model is given as figure 9 is of more than passing practical interest, it may be helpful to have available a more exact analysis of such a structure than can be obtained by assuming an "equivalent" uniform line, as we did in the previous section.

A structure quite similar to that of figure 9, the iris-loaded circular waveguide, has received quite a bit of attention in the literature because of its use in linear accelerators. All we need to do here is to make some slight changes in the available analyses of iris-loaded structures. The result will be an infinite determinantal equation (to be truncated and evaluated numerically) for the determination of the propagation constants of the various modes that can exist in the disk line.

We will follow closely the analysis of the iris-loaded line by Walkinshaw [3] (a previous, better known, paper by Walkinshaw gave an approximate analysis of the iris-loaded guide; reference [3] gives a shorter, more accurate analysis). For the sake of simplicity, we will treat only the case of infinitesimally thin disks ($\delta = \Delta$).

In the outer ($a < \rho < b$) region of the disk line, we will write the fields in the form

$$E_z^{II}(\rho,z) = \sum_{m=-\infty}^{\infty} e_m \frac{F_o(\alpha_m;r,\pi/a)}{F_o(\alpha_m;r,1)} e^{i(\beta+\frac{2\pi m}{\Delta})z} \tag{5.1}$$

$$H_\phi^{II}(\rho,z) = -i\omega\varepsilon_o a \sum_{m=-\infty}^{\infty} \frac{e_m}{\alpha_m} \frac{F_1(\alpha_m;r,\rho/a)}{F_o(\alpha_m;r,1)} e^{i(\beta+\frac{2\pi m}{\Delta})z} \tag{5.2}$$

where

$$r \equiv b/a \tag{5.3}$$

$$\left(\frac{\alpha_m}{a}\right)^2 = \beta_o^2 - (\beta + \frac{2\pi m}{\Delta})^2 \tag{5.4}$$

and

$$F_o(\alpha_m;r,\rho/a) \equiv Y_o(\alpha_m r)J_o(\alpha_m\rho/a) - J_o(\alpha_m r)Y_o(\alpha_m\rho/a) \tag{5.5}$$

$$F_1(\alpha_m;r,\rho/a) \equiv Y_o(\alpha_m r)J_1(\alpha_m\rho/a) - J_o(\alpha_m r)Y_1(\alpha_m\rho/a) \tag{5.6}$$

Equations (5.1) and (5.2) are the forms established by Floquet's theorem and the boundary condition on E_z at the surface $\rho = b$.

In the inner ($\rho < a$) region, we will choose a form for the fields that is finite on the z-axis and has zero radial field on the disks. Such a form, in the basic region ($-\Delta/2 < z < \Delta/2$) (note that in the region $((N - \frac{1}{2})\Delta < z < (N + \frac{1}{2})\Delta)$ we must multiply each B_s and D_s coefficient by $e^{iN\beta\Delta}$) is:

$$E_z^{(I)}(\rho,z) = \sum_{s=0}^{\infty} B_s \frac{J_o(\gamma_s\rho/a)}{J_o(\gamma_s)} \cos(\frac{2\pi sz}{\Delta}) + \sum_{s=1}^{\infty} D_s \frac{J_o(\gamma'_s\rho/a)}{J_o(\gamma'_s)} \sin(\frac{2s-1}{\Delta}\pi z) \tag{5.7}$$

$$\frac{H_\phi^{(I)}(\rho,z)}{-i\omega\varepsilon} = \sum_{s=0}^{\infty} \frac{B_s}{\gamma_s} \frac{J_1(\gamma_s\rho/a)}{J_o(\gamma_s)} \cos(\frac{2\pi sz}{\Delta}) + \sum_{s=1}^{\infty} \frac{D_s}{\gamma'_s} \frac{J_1(\gamma'_s\rho/a)}{J_o(\gamma'_s)} \sin(\frac{2s-1}{\Delta}\pi z) \tag{5.8}$$

where

$$\left(\frac{\gamma_s}{a}\right)^2 = \beta_o^2 - (\frac{2\pi s}{\Delta})^2 \tag{5.9}$$

$$\left(\frac{\gamma'_s}{a}\right)^2 = \beta_o^2 - (\frac{2s-1}{\Delta}\pi)^2 \tag{5.10}$$

By matching the two representations for E_z on the $\rho = a$ surface, we obtain the relations

$$B_o\Delta = \sum_{m=-\infty}^{\infty} e_m C_{om} \tag{5.11}$$

$$\frac{1}{2} B_s\Delta = \sum_{m=-\infty}^{\infty} e_m C_{sm} \tag{5.12}$$

$$\frac{1}{2} D_s \Delta = \sum_{m=-\infty}^{\infty} e_m S_{sm} \tag{5.13}$$

where

$$C_{sm} = \int_{-\Delta/2}^{\Delta/2} \cos(\frac{2\pi s z}{\Delta}) e^{i(\beta + \frac{2\pi m}{\Delta}) z} dz$$

$$= 2(-)^{m+s} \sin(\frac{\beta\Delta}{2}) \frac{\beta + (2\pi m/\Delta)}{[\beta + (2\pi m/\Delta)]^2 - (2\pi s/\Delta)^2} \tag{5.14}$$

and

$$S_{sm} = \int_{-\Delta/2}^{\Delta/2} \sin(\frac{2s-1}{\Delta} \pi z) e^{i(\beta + \frac{2\pi m}{\Delta}) z} dz$$

$$= 2i(-)^{m+s} \cos(\frac{\beta\Delta}{2}) \frac{\beta + (2\pi m/\Delta)}{[\beta + (2\pi m/\Delta)]^2 - [\pi(2s-1/\Delta)]^2} \tag{5.15}$$

Now we must also match the two representations for H_ϕ on the $\rho = a$ surface. These matching equations take the form

$$\frac{B_o \Delta}{\gamma_o} \eta^2 \frac{J_1(\gamma_o)}{J_o(\gamma_o)} = \sum_{m=-\infty}^{\infty} \frac{e_m}{\alpha_m} \frac{F_1(\alpha_m; r, 1)}{F_o(\alpha_m; r, 1)} C_{om} \tag{5.16}$$

$$\frac{B_s \Delta}{2\gamma_s} \eta^2 \frac{J_1(\gamma_s)}{J_o(\gamma_s)} = \sum_{m=-\infty}^{\infty} \frac{e_m}{\alpha_m} \frac{F_1(\alpha_m; r, 1)}{F_o(\alpha_m; r, 1)} C_{sm} \tag{5.17}$$

$$\frac{D_s \Delta}{2\gamma'_s} \eta^2 \frac{J_1(\gamma'_s)}{J_o(\gamma'_s)} = \sum_{m=-\infty}^{\infty} \frac{e_m}{\alpha_m} \frac{F_1(\alpha_m; r, 1)}{F_o(\alpha_m; r, 1)} S_{sm} \tag{5.18}$$

or, making use of equations (5.11), (5.12), and (5.13),

$$\sum_{m=-\infty}^{\infty} e_m C_{sm} \left\{ \frac{\eta^2}{\gamma_s} \frac{J_1(\gamma_s)}{J_o(\gamma_s)} - \frac{1}{\alpha_m} \frac{F_1(\alpha_m; r, 1)}{F_o(\alpha_m; r, 1)} \right\} = 0 \qquad s = 0, 1, \cdots \infty \tag{5.19}$$

$$\sum_{m=-\infty}^{\infty} e_m S_{sm} \left\{ \frac{\eta^2}{\gamma_s'} \frac{J_1(\gamma_s')}{J_o(\gamma_s')} - \frac{1}{\alpha_m} \frac{F_1(\alpha_m;r,1)}{F_o(\alpha_m;r,1)} \right\} = 0 \qquad s = 1,2,\cdots\infty \tag{5.20}$$

We may give these equations a little more presentable appearance by multiplying the first set by

$$\frac{(-)^s}{2\Delta \sin(\beta\Delta/2)}$$

the second set by

$$\frac{(-)^s}{2i\Delta \cos(\beta\Delta/2)}$$

and defining new variables e_m' by $e_m' \equiv (-)^m e_m$. The truncated version of equations equivalent to (5.19) and (5.20) may then be written in the form

$$\sum_{m=-N}^{N} M_{sm} e_m' = 0 \qquad s = 0,1,\cdots 2N \tag{5.21}$$

where

$$M_{sm} \doteq \frac{\Delta\beta+2\pi m}{(\Delta\beta+2\pi m)^2-(\pi s)^2} \left\{ \frac{\eta^2}{\Gamma_2} \frac{J_1(\Gamma_s)}{J_o(\Gamma_s)} - \frac{1}{\alpha_m} \frac{F_1(\alpha_m;r,1)}{F_o(\alpha_m;r,1)} \right\} \tag{5.22}$$

and

$$\left(\frac{\Gamma_s}{a}\right)^2 \equiv \beta_o^2 - \left(\frac{\pi s}{\Delta}\right)^2 \tag{5.23}$$

The propagation constant, β, is determined by setting the determinant of the set (5.21) equal to zero (there are several roots of course; we are interested primarily in the smallest one). The accuracy increases as we increase N. If N is zero we return to the approximation of the previous section (for $\delta = \Delta$). Walkinshaw effectively says that, in the case of the iris-loaded guide, we get about one part in a thousand error in β, if Δ/a is not too large, when N is one. There should be no great difficulty in obtaining very accurate values of β in the present case, should it become desirable to do so.

The present analysis can be readily extended to the case where the disks have finite thickness. It just gets about twice as messy.

VI. Conclusion

The main result of this note is that, with the tables and curves of Section III, we have available a way to tell whether the impedance per unit length, measured and calculated by the simple techniques of Sections I and II, is really the impedance per unit length of the structure under test.

The tables and curves of Section III could be extended, in some future note, to include complex values of Z_i, rather than the pure imaginary values of Z_i that we have examined here. The same development, including equation (3.28) and equations (3.31) to (3.34), would hold in the more general case (S_r and α would be complex). It is believed that, for the test structures that are presently of interest (which have a dominantly imaginary Z_i) the present data should be sufficient.

A secondary result of the note is the beginning of an accurate analysis of a disk line. Should it become necessary, it would be straightforward to carry out numerical work, based on the present analytical work.

Appendix

An Algorithm for Two Bessel Function Crossproducts

In this appendix we will develop expressions that are well suited to numerical evaluation for the two Bessel function crossproducts used in the note

$$B_o(\alpha,r) = (\pi/2)[Y_o(\alpha r)J_o(\alpha) - J_o(\alpha r)Y_o(\alpha)] = (\pi/2)F_o(\alpha;r,1) \tag{A.1}$$

$$B_1(\alpha,r) = (\alpha\pi/2)[Y_o(\alpha r)J_1(\alpha) - J_o(\alpha r)Y_1(\alpha)] = (\alpha\pi/2)F_1(\alpha;r,1) \tag{A.2}$$

From the multiplication theorem for Bessel functions ([2], p. 363), setting $(\alpha r) = x$, we get

$$B_o(\alpha,r) = \frac{\pi}{2}\sum_{k=0}^{\infty} \frac{(1-1/r^2)^k(x/2)^k}{k!}[Y_o(x)J_k(x) - J_o(x)Y_k(x)] \tag{A.3}$$

The first term in this series disappears. The crossproducts within the series satisfy the difference equation for Bessel functions; i.e., if

$$c_k \equiv (\pi/2)[Y_o(x)J_k(x) - J_o(x)Y_k(x)] \tag{A.4}$$

we have

$$c_{k+1} = \frac{2k}{x}c_k - c_{k-1} \tag{A.5}$$

with

$$c_1 = (1/x) = (1/\alpha r) \tag{A.6}$$

$$c_2 = (2/x^2) = (2/\alpha^2 r^2) \tag{A.7}$$

and we can write

$$B_o(\alpha,r) = \sum_{k=1}^{\infty} \left(\frac{r-1/r}{2}\right)^k \frac{\alpha^k}{k!} c_k. \tag{A.8}$$

In a similar manner it follows that

$$B_1(\alpha,r) = \frac{\alpha\pi}{2r} \sum_{k=0}^{\infty} \frac{(1-1/r^2)^k (x/2)^k}{k!} \left[Y_o(x)J_{k+1}(x) - J_o(x)Y_{k+1}(x)\right] \tag{A.9}$$

$$= \frac{\alpha}{r} \sum_{k=1}^{\infty} \left(\frac{r-1/r}{2}\right)^{k-1} \frac{\alpha^{k-1}}{(k-1)!} c_k \tag{A.10}$$

Now let us define some new coefficients through

$$T_k = \alpha r \left(\frac{r-1/r}{2}\right)^{k-1} \frac{\alpha^{k-1}}{(k-1)!} c_k \tag{A.11}$$

thus

$$B_1(\alpha,r) = \frac{1}{r^2} \sum_{k=1}^{\infty} T_k \tag{A.12}$$

and

$$B_o(\alpha,r) = \frac{r^2-1}{2r^2} \sum_{k=1}^{\infty} \frac{T_k}{k} \tag{A.13}$$

while, from equation (A.11) and the difference equation for c_k, it follows that

$$T_{k+1} = \frac{r^2-1}{r^2} \left\{ T_k - \frac{\alpha^2(r^2-1)}{4} \cdot \frac{T_{k-1}}{k(k-1)} \right\} \tag{A.14}$$

with

$$T_1 = 1 \tag{A.15}$$

$$T_2 = \frac{r^2-1}{r^2} \tag{A.16}$$

Expressions (A.12) and (A.13) are easy to evaluate, with even a desktop computer. The T_k's are generated from equation (A.14), using equations (A.15) and (A.16) to get started. Equations (A.12) to (A.16) make up the algorithm we have been aiming at.

Note that, if $\alpha = 0$, $T_k = (1 - 1/r^2)^{k-1}$, and so

$$B_1(0,r) = \frac{1}{r^2} \sum_{k=1}^{\infty} \left(\frac{r^2-1}{r^2}\right)^{k-1} = 1$$

$$B_o(0,r) = \frac{r^2-1}{2r^2} \sum_{k=1}^{\infty} \left(\frac{r^2-1}{r^2}\right)^{k-1} \cdot \frac{1}{k} = \ln r$$

References

[1] Julius Adams Stratton, Electromagnetic Theory, McGraw-Hill, New York, 1941.

[2] Milton Abromowitz and Irene A. Stegun, editors, Handbook of Mathematical Functions, National Bureau of Standards, AMS-55, 1964.

[3] W. Walkinshaw, "Notes on 'Wave Guides for Slow Waves'," Journal of Applied Physics, vol. 20, p. 634 (1949).