Sensor and Simulation Notes

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Electrically-Small Ellipsoidal Antennas

K.S.H. Lee
The Dikewood Corporation, Westwood Research Branch
Los Angeles, California

Abstract

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Analytical and numerical results are obtained for the equivalent area, radiation resistance, polarizabilities, and field enhancement factors of an ellipsoid or a half ellipsoid symmetrically resting on an infinite ground plane. It is shown that simple relationship exists among these quantities.

Keywords: electromagnetic pulses, antennas, radiation, sensors, electric fields

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I. Introduction

There are four problem areas of practical interest that can be described geometrically as a half ellipsoid symmetrically resting on an infinite ground plane. They are: (1) blade antennas on an aircraft; (2) electric-field sensors; (3) field distortions by a ground plane of finite thickness; (4) protrusions on a large aeronautical system or within an EMP simulator. This note places particular emphasis on problem (1), whereas the other three problems will be discussed briefly and with sufficient quantitative results.

When we began our investigation into the response of aircraft deliberate antennas to EMP, it was thought useful to report the results of our investigation in two different formats. One format would be a series of technical memos which contain, for any particular antenna chosen for study, (1) elaborate drawings of the antenna depicting clearly its physical shape and the electrical connections (e.g., the driving point, the conducting paths), (2) a brief description as to how the antenna works, and (3) numerical and graphical results for the effective height (or the equivalent area) and the input impedance at the nearest accessible terminals of the antenna. Between the accessible terminals and the "terminals" usually referred to in the antenna theory there are always many different kinds of matching (compensating) networks which often require ingenuity and antenna engineer's experience to figure out what they are intended to match with.

Another format to present the results of our investigation would be to ferret out the features that are common to a class of antennas and then a detailed elaborate analysis for each feature would be carried out, including derivations of all the formulas and a parametric study of the antenna geometry. The results presented in this format would then be applicable to areas not necessarily restricted to aircraft deliberate antennas. The present note is the first of a series of notes in this format, while several technical memos have already been completed and are now being circulated.

This note deals with the electromagnetic response of a half ellipsoid resting on a infinite ground plane or, if one wishes, a whole ellipsoid in free space. For aerodynamic reasons, aircraft antennas are often of the shape of a blade. A blade is nothing more than just one of the many limiting forms of an ellipsoid, or simply an elliptic disk. An ellipsoid can be said to be the most general three-dimensional object which can degenerate into a prolate spheroid, a sphere, an oblate spheroid, an elliptic disk, and a circular disk. Apart from being flexible in shape, an ellipsoid lends itself to analytical treatment because the Laplacian operator is separable in ellipsoidal corrdinates and because elliptic integrals and elliptic functions, which always crop up in this analysis, have been well studied and even tabulated. In this note the Laplace equation is all that needs to be solved, since we will concern ourselves only with electrically small ellipsoids.

The "complementary" problem of our present problem (i.e., an infinite ground plane with an ellipsoidal boss) is that of an ellipsoidal depression in an infinite ground plane (i.e., an ellipsoidal bowl with an infinite flange). Unfortunately, it is complementary only in the sense of geometry but not in their respective electromagnetic characteristics, and so one cannot really relate the electrical properties of the two. The "complementary" antenna is often used in aircraft and warrants a detailed analysis in the future. The analysis involved for such an antenna will be more complex and much more numerical than the problem considered in this note. It belongs to the class of mixed boundary-value problems.

The boundary-value problems that we will solve below are simple enough that we can afford to do several things which are otherwise impossible in a note form. First, the analysis will be self-contained. Second, we will get as much as possible by performing different simple operations on the solution of the problem and arrive at different useful quantities, such as the equivalent area, the radiation resistance, the polarizabilities, the field enhancement factors, etc. Third, all possible parameters will be varied to cover all possible geometric shapes. Fourth, extensive tables and curves will be given. Fifth, we will point out several related problems and concepts.

It should be pointed out that here, we will not make any calculations on the antenna capacitance, although capacitance (sometimes, inductance) is one of the two important parameters that characterize an electrically small antenna, the other parameter being the equivalent (effective) height or the equivalent area.[1] The capacitance value is very sensitive to the size of the antenna's excitation gap and is usually very large and needs to be tuned out for the frequency range of interest. The capacitance problem can be easily formulated but will require considerable numerical work. However, once the capacitance problem has been solved all other parameters of an electrically small antenna can be obtained.[2]

This note may as well be entitled "Electrically-Small Ellipsoidal EMP Sensors."

II. An Ellipsoid in Uniform Fields

We will first give some pertinent formulas that connect the rectangular coordinates (x,y,z) and the ellipsoidal coordinates (ξ,η,ζ), and then derive the induced charge density on an ellipsoidal boss on an infinite ground plane in a uniform electric field.

For the two coordinate systems mentioned above we have (Fig. 1)[3,4,5]

$$x^2 = \frac{(\xi+a^2)(\eta+a^2)(\zeta+a^2)}{(b^2-a^2)(c^2-a^2)}$$

$$y^2 = \frac{(\xi+b^2)(\eta+b^2)(\zeta+b^2)}{(c^2-b^2)(a^2-b^2)} \tag{1}$$

$$z^2 = \frac{(\xi+c^2)(\eta+c^2)(\zeta+c^2)}{(a^2-c^2)(b^2-c^2)}$$

$$\frac{x^2}{a^2+\xi} + \frac{y^2}{b^2+\xi} + \frac{z^2}{c^2+\xi} = 1 \qquad \begin{pmatrix}\text{ellipsoids} \\ \xi \geq -c^2\end{pmatrix}$$

$$\frac{x^2}{a^2+\eta} + \frac{y^2}{b^2+\eta} + \frac{z^2}{c^2+\eta} = 1 \qquad \begin{pmatrix}\text{hyperboloids of one sheet} \\ -c^2 \geq \eta \geq -b^2\end{pmatrix} \tag{2}$$

$$\frac{x^2}{a^2+\zeta} + \frac{y^2}{b^2+\zeta} + \frac{z^2}{c^2+\zeta} = 1 \qquad \begin{pmatrix}\text{hyperboloids of two sheets} \\ -b^2 \geq \zeta \geq -a^2\end{pmatrix}$$

where, without loss of generality, we have assumed $a \geq b \geq c \geq 0$. In the following discussions it will be helpful if one can picture the geometry traced out by constant values of ξ, η and ζ in the three planes defined respectively by $x = 0$, $y = 0$, and $z = 0$.

In the $x = 0$ plane we have

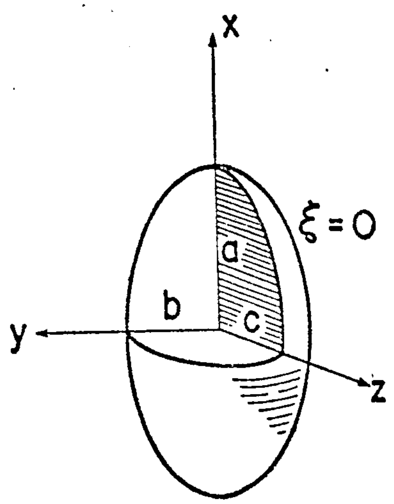

Figure 1a. Ellipsoid.

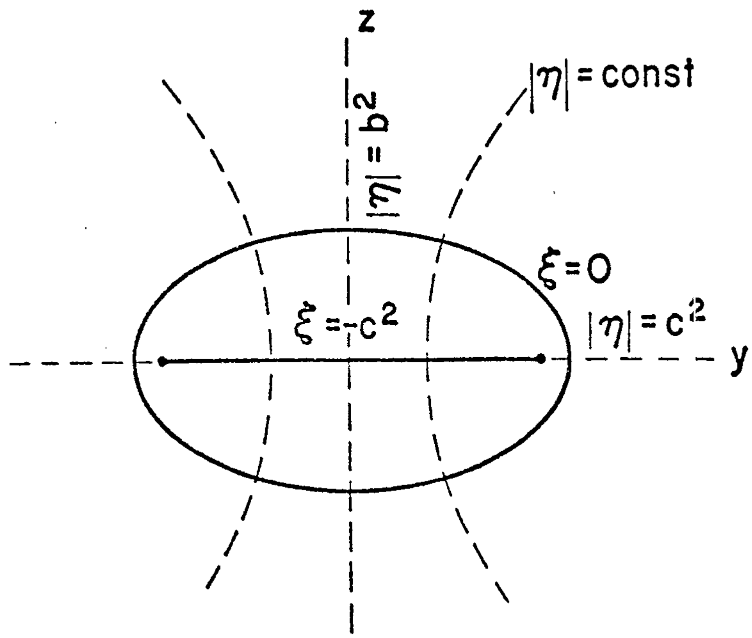

Figure 1b. The x = 0 plane.

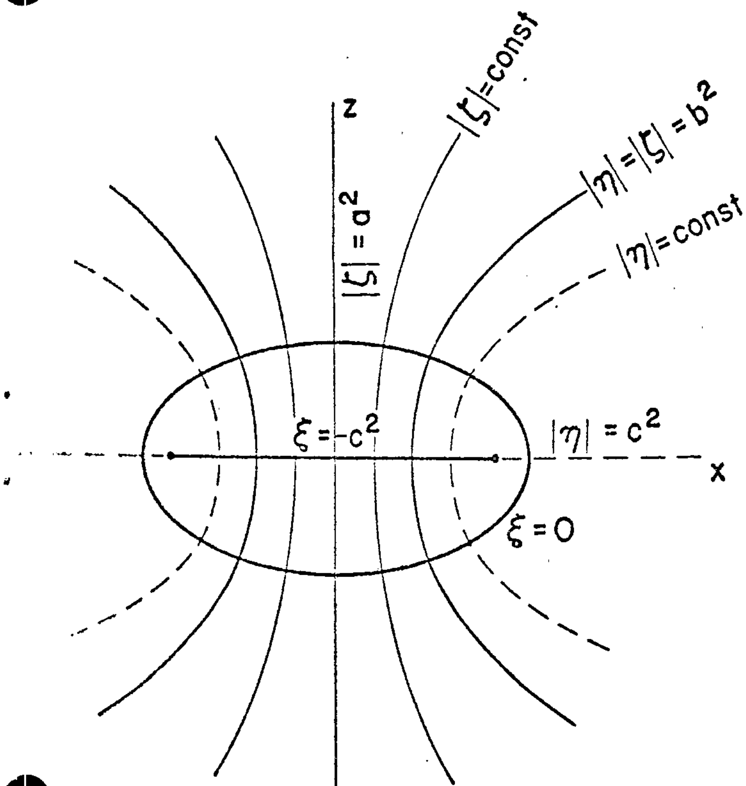

Figure 1c. The y = 0 plane.

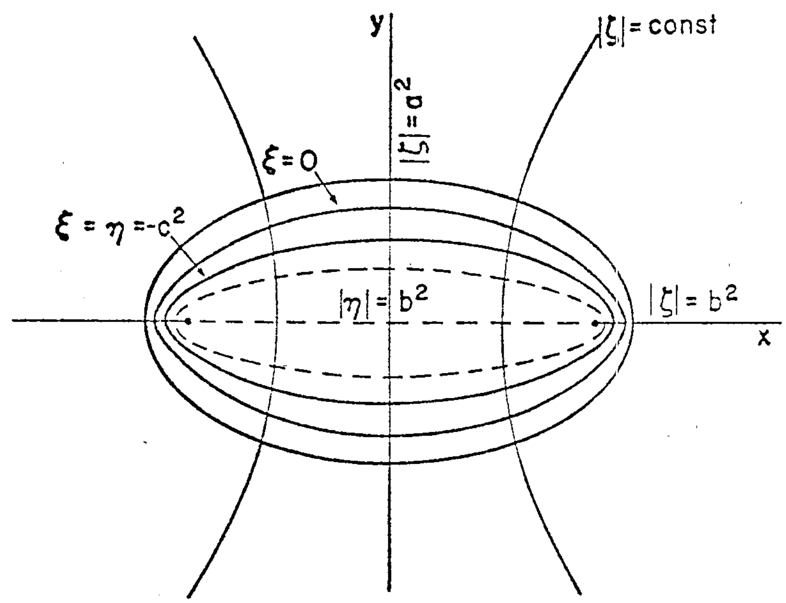

Figure 1d. The z = 0 plane.

Figure 1. Ellipsoidal and rectangular coordinates.

$$\frac{y^2}{b^2+\xi} + \frac{z^2}{c^2+\xi} = 1 \qquad \text{(ellipses: } \xi > -c^2\text{)}$$

$$\frac{y^2}{b^2-|\eta|} - \frac{z^2}{|\eta|-c^2} = 1 \qquad \text{(hyperbolas: } b^2 > |\eta| > c^2\text{)}$$

$$\frac{y^2}{b^2-|\zeta|} + \frac{z^2}{c^2-|\zeta|} = 1 \qquad \text{(no traces: } a^2 > |\zeta| > b^2\text{)}$$

The geometry is shown in Fig. 1b. In the y = 0 plane we have

$$\frac{x^2}{a^2+\xi} + \frac{z^2}{c^2+\xi} = 1 \qquad \text{(ellipses: } \xi > -c^2\text{)}$$

$$\frac{x^2}{a^2-|\eta|} - \frac{z^2}{|\eta|-c^2} = 1 \qquad \text{(hyperbolas: } b^2 > |\eta| > c^2\text{)}$$

$$\frac{x^2}{a^2-|\zeta|} - \frac{z^2}{|\zeta|-c^2} = 1 \qquad \text{(hyperbolas: } a^2 > |\zeta| > b^2\text{)}$$

The geometry is shown in Fig. 1c. In the z = 0 plane we have

$$\frac{x^2}{a^2+\xi} + \frac{y^2}{b^2+\xi} = 1 \qquad \text{(ellipses: } \xi > -c^2\text{)}$$

$$\frac{x^2}{a^2-|\eta|} + \frac{y^2}{b^2-|\eta|} = 1 \qquad \text{(ellipses: } b^2 > |\eta| > c^2\text{)}$$

$$\frac{x^2}{a^2-|\zeta|} - \frac{y^2}{|\zeta|-b^2} = 1 \qquad \text{(hyperbolas: } a^2 > |\zeta| > b^2\text{)}$$

The geometry is shown in Fig. 1d.

To complete the geometry we include the line element $d\ell$ and the metric coefficients h_1, h_2 and h_3:

$$(d\ell)^2 = (dx)^2 + (dy)^2 + (dz)^2$$

$$= h_1^2(d\xi)^2 + h_2^2(d\eta)^2 + h_3^2(d\zeta)^2$$

$$h_1^2 = \frac{(\xi-\eta)(\xi-\zeta)}{4R_\xi^2}$$

$$h_2^2 = \frac{(\eta-\zeta)(\eta-\xi)}{4R_\eta^2} \tag{3}$$

$$h_3^2 = \frac{(\zeta-\xi)(\zeta-\eta)}{4R_\zeta^2}$$

$$R_s^2 = (s + a^2)(s + b^2)(s + c^2), \qquad (s = \xi,\eta,\zeta)$$

We now have all the necessary formulas and geometry we need for the following calculations and discussions.

As evident from (2), the surface $\xi = 0$ corresponds to the surface defined by $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, which is the surface of our ellipsoidal antenna in Fig. 1a.

Let us now consider the ellipsoid in Fig. 1a immersed in a uniform electric field $\underline{E}$ parallel to the x-axis or, equivalently, a half ellipsoidal boss resting on an infinite ground plane (Fig. 2). The incident potential ϕ_o is

$$\phi_o = -Ex = -E\sqrt{\frac{(\xi+a^2)(\eta+a^2)(\zeta+a^2)}{(b^2-a^2)(c^2-a^2)}} \tag{4}$$

where (1) has been used. Outside the ellipsoid the total potential ϕ is given by $\phi = \phi_o + \phi_s$, ϕ_s being the scattered potential. Noticing that ϕ_o and ϕ_s are solutions of the Laplace equation and ϕ_o is explicitly given by (4) one can then easily find ϕ_s by the method of variation of parameters. In that method one writes $\phi_s = \phi_o F(\xi)$, where the unknown function F depends only on ξ,

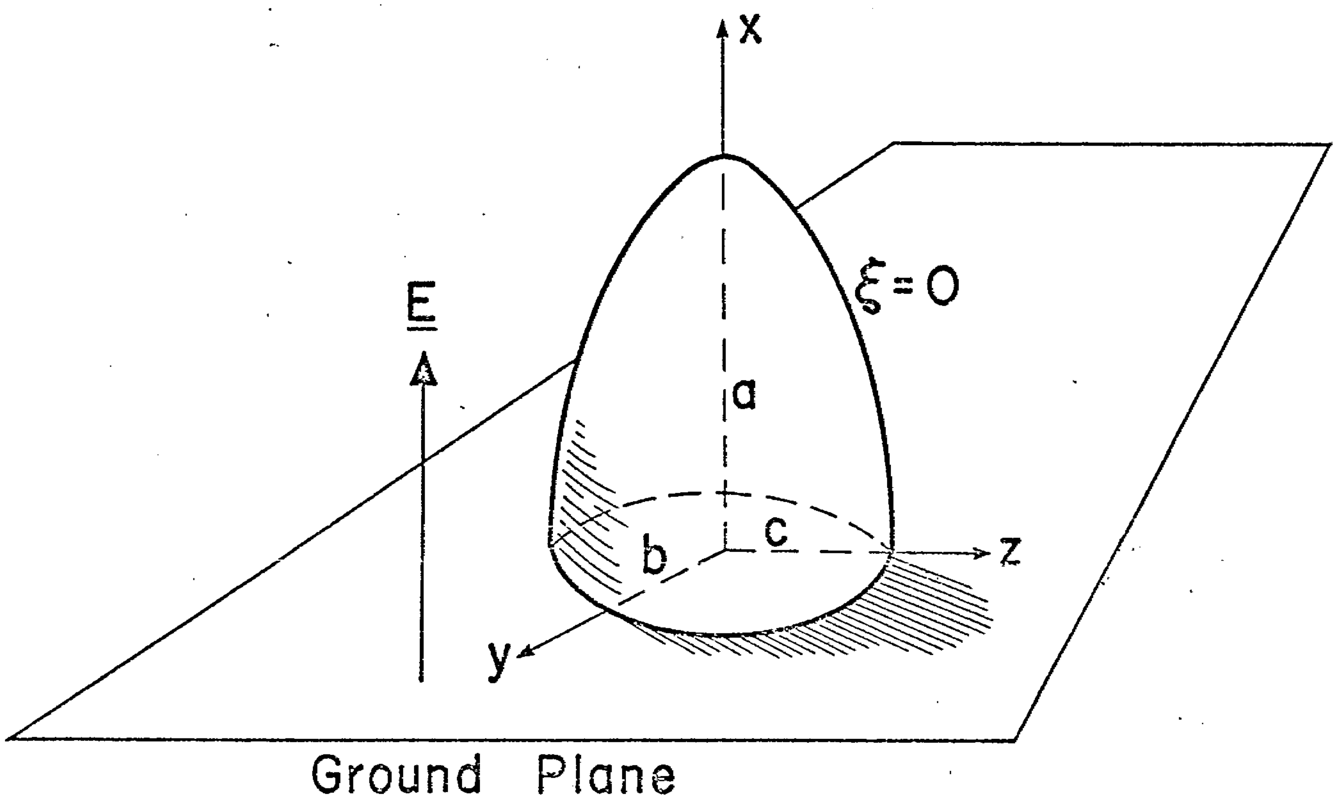

Figure 2. Infinite ground plane with an ellipsoidal boss in a uniform field.

as is obvious from the geometry of the problem. Substituting this form of ϕ_s in the Laplace equation one gets

$$\frac{d^2F}{d\xi^2} + \frac{dF}{d\xi}\frac{d}{d\xi}\ln[R_\xi(\xi + a^2)] = 0$$

the solution of which is found to be, by direct integration,

$$\phi_s = C\phi_o \int_\xi^\infty \frac{ds}{(s+a^2)R_s} \tag{5}$$

where C is a constant. At large distances (5) behaves as

$$\phi_s \sim -CEx\frac{2}{3r^3}, \qquad \text{as } r = \sqrt{x^2 + y^2 + z^2} \to \infty \tag{6}$$

which is a dipole field, as it should be. The constant C is determined from the boundary condition that $\phi = \phi_o + \phi_s = 0$ at $\xi = 0$. Thus

$$\phi = \phi_o \frac{\int_o^\xi [ds/(s+a^2)R_s]}{\int_o^\infty [ds/(s+a^2)R_s]}, \qquad R_s^2 = (s + a^2)(s + b^2)(s + c^2) \tag{7}$$

which vanishes at $\xi = 0$ as well as at $x = 0$. So (7) applies to an ellipsoid in free space as well as a half ellipsoidal boss on an infinite ground plane.

The induced surface charge density σ on the ellipsoid is obtained by differentiating (7):

$$\sigma = -\varepsilon\left[\frac{\partial\phi}{\partial n}\right]_{\xi=0} = -\varepsilon\left[\frac{1}{h_1}\frac{\partial\phi}{\partial\xi}\right]_{\xi=0}$$

$$= \frac{\varepsilon EbcN_a}{\sqrt{(a^2-b^2)(a^2-c^2)}}\sqrt{\frac{(\eta+a^2)(\zeta+a^2)}{\eta\zeta}}$$

$$= \varepsilon EN_a\frac{x/a^2}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}, \qquad \left(\frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \tag{8}$$

where

$$\frac{1}{N_a} = \frac{abc}{2}\int_o^\infty \frac{ds}{(s+a^2)R_s} = \frac{\beta\gamma}{2}\int_o^\infty \frac{dx}{(x+1)^{3/2}(x+\beta^2)^{\frac{1}{2}}(x+\gamma^2)^{\frac{1}{2}}}\ , \quad (\beta = \frac{b}{a}\ ,\ \gamma = \frac{c}{a}) \tag{9}$$

A quantity of interest is the electric-field enhancement factor f_E at the tip (a,0,0) of the ellipsoid. From (8) one immediately has

$$f_E = \left[\frac{\sigma}{\varepsilon E}\right]_{tip} = N_a \tag{10}$$

which will be graphed and tabulated in the next section.

Let us point out in passing an interesting observation. For a freely charged ellipsoid with total charge Q the surface charge density σ_f is given by[3,4,5,13]

$$\sigma_f = \frac{Q}{4\pi abc}\frac{1}{\sqrt{x^2/a^4+y^2/b^4+z^2/c^4}} \tag{11}$$

Writing Q = CV and using

$$\frac{8\pi\varepsilon}{C} = \int_o^\infty \frac{ds}{R_s}$$

for the capacitance C of an ellipsoid, one has the following interesting result upon comparing (8) and (11):

$$\frac{\sigma}{\sigma_f} = \frac{Ex}{V}\left[\int_o^\infty \frac{ds}{\sqrt{(s+1)(s+\beta^2)(s+\gamma^2)}}\Bigg/\int_o^\infty \frac{ds}{(s+1)^{3/2}\sqrt{(s+\beta^2)(s+\gamma^2)}}\right] \tag{12}$$

The quantity inside the square bracket is just a constant, symmetric with respect to β and γ. The simple looking formula given in (12) may be understood from the integral equation approach. For the forced problem one has, on the surface S of an ellipsoid,

$$\int_S \sigma G dA = Ex \qquad \text{(Forced)}$$

making the total potential on S zero. For the freely charged problem one has

$$\int_S \sigma_f G dA = V \qquad \text{(Free)}$$

making the total potential on S equal to V. The kernel G is, of course, the static Green's function. This way of looking at things is familiar to those who are well versed with the ideas of SEM.

III. Equivalent Area

In Fig. 3 we again show a half ellipsoid resting on a ground plane with a uniform electric field. But this time we also include in the figure an antenna gap together with a simple driving mechanism (feeding by a coaxial line). The short-circuit current that the gap "sees" is equal to the time rate of change of the total charge collected by that part of antenna above the gap. This is a simple consequence of the continuity equation and holds for all frequencies. The short-circuit current is, in turn, proportional to the time rate of change of the displacement current of the incident field, and this proportionality has the dimension of area and has been termed the equivalent area A_{eq}.[1] At low frequencies, A_{eq} approaches a definite limit depending only on the geometry of the antenna.

For LF antennas the antenna engineer often talks about a quantity called "sensitivity product," which is the product of the effective height h_e times the capacitance C_a (or the equivalent length times the inductance in the case of loop antennas) of the antenna.[6,7] This product is nothing more than the product of equivalent area A_{eq} times the permittivity ε of the ambient medium (see, Ref. 1). So all the desirable properties that the sensitivity product has from the antenna designer's viewpoint also apply to the equivalent area. Among them the most often quoted feature by the antenna engineer is that the sensitivity product has extremely simple transformation properties through some typical antenna matching networks, which always exist. For instance, $h_e C_a = h'_e C'_a$ across a shunt matching capacitance and $h_e \sqrt{C_a} = h'_e \sqrt{C'_a}$ across a transformer matching network, the primed and unprimed quantities referring to values measured at the terminals on the two different sides of a matching network. We may also add that the sensitivity product can be accurately calculated from a mathematically well-defined boundary-value problem because one need not worry about the local geometry of the feed point which has many unsettled theoretical questions (or questions which will never be settled to the satisfaction of those who have been concerned with the "gap problem" in the antenna theory).

Let $Q(x_o)$ be the total charge collected by that part of antenna above the gap with surface area S_o (see Fig. 3). Then, from (8) we have

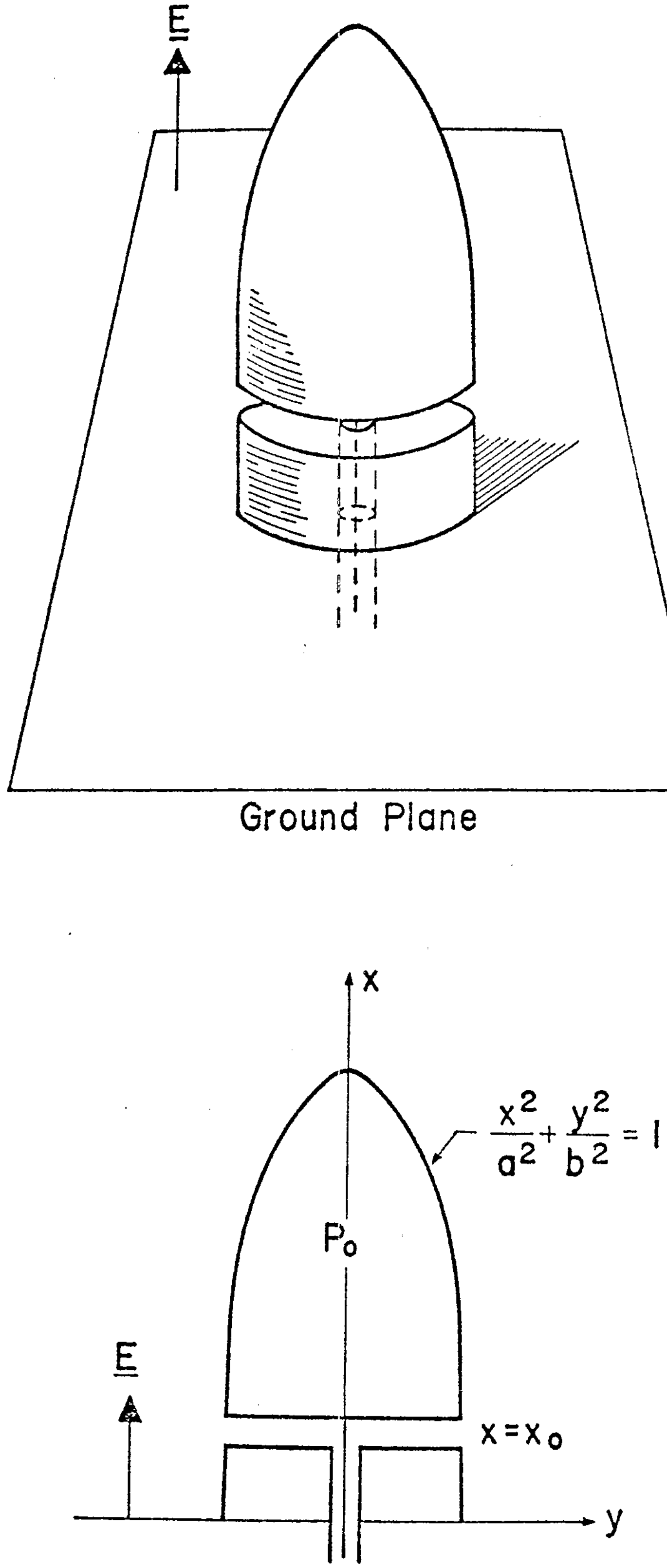

Figure 3. Ellipsoidal antenna with a driving gap at $x = x_o$.

$$Q(x_o) = \int_{S_o} \sigma dA = \frac{\varepsilon E N_a}{a^2} \int_{S_o} \frac{x dA}{\sqrt{x^2/a^4 + y^2/b^4 + z^2/c^4}} \tag{13}$$

Projecting dA onto the xy-plane we have

$$dA = \frac{dxdy}{n_z}$$

where

n_z = the z-component of the outward unit normal to S_o

$$= \left[\frac{1}{h_1} \frac{\partial z}{\partial \xi}\right]_{\xi=0} = \frac{z/c^2}{\sqrt{x^2/a^4 + y^2/b^4 + z^2/c^4}}$$

Thus, (13) becomes (Fig. 3)

$$\frac{Q}{\varepsilon E} = \frac{2c^2 N_a}{a^2} \int_{P_o} \frac{x}{z}\, dxdy \qquad (P_o = \text{projection of } S_o)$$

$$= \frac{2cN_a}{a^2} \int_{P_o} \frac{xdxdy}{\sqrt{1 - x^2/a^2 - y^2/b^2}} \qquad \left(z/c = \sqrt{1 - (x/a)^2 - (y/b)^2}\right)$$

$$= \frac{4cN_a}{a^2} \int_{x_o}^{a} \int_{0}^{b\sqrt{1-(x/a)^2}} \frac{xdxdy}{\sqrt{1-(x/a)^2-(y/b)^2}}$$

$$= \pi bcN_a\left(1 - \frac{x_o^2}{a^2}\right) \tag{14}$$

Let us define the equivalent area $A_{eq}(x_o)$ as

$$A_{eq}(x_o) = \pi bcN_a(1 - x_o^2/a^2) \tag{15}$$

Then N_a is simply the normalized equivalent area when the antenna gap is at the ground plane, i.e.,

$$N_a = \frac{A_{eq}(0)}{\pi bc}$$

The short-circuit current I_{sc} is given by, with time convention $e^{-i\omega t}$,

$$I_{sc}(x_o) = -i\omega\varepsilon E A_{eq}(x_o) \tag{16}$$

As evident from (15), the position of the gap is factored out as $(1 - x_o^2/a^2)$, which is so simple that it requires no further considerations.

We will now study in detail the normalized area N_a, which is also equal to the electric-field enhancement factor $f_{\bar{E}}$ according to (10). Let us rewrite (9) as

$$N_a = \frac{A_{eq}(0)}{\pi bc} = f_E$$

$$= \left[\frac{\beta\gamma}{2}\int_o^\infty \frac{dx}{(x+1)^{3/2}(x+\beta^2)^{\frac{1}{2}}(x+\gamma^2)^{\frac{1}{2}}}\right]^{-1} \tag{9'}$$

This integral will be evaluated in three different ranges for the values of β and γ: (a) $\beta \leq 1$, $\gamma \leq 1$; (b) $\beta \leq 1$, $\gamma \geq 1$; (c) $\beta \geq 1$, $\gamma \geq 1$. Because of $N_a(\beta,\gamma) = N_a(\gamma,\beta)$ these ranges cover all positive values of β and γ.

(a). $\beta \leq 1$, $\gamma \leq 1$ $(b \leq a,\ c \leq a)$

Substitute

$$z^2 = \frac{1-\gamma^2}{x+1}$$

in the integral (9). Then

$$\frac{1}{N_a} = \frac{\beta\gamma}{(1-\gamma^2)^{3/2}}\int_o^{\sqrt{1-\gamma^2}} \frac{z^2 dz}{\sqrt{(1-z^2)(1-mz^2)}} \qquad \left(m = \frac{1-\beta^2}{1-\gamma^2}\right)$$

$$= \frac{\beta\gamma}{m(1-\gamma^2)^{3/2}}\int_o^{\sqrt{1-\gamma^2}}\left[\frac{1}{\sqrt{(1-z^2)(1-mz^2)}} - \sqrt{\frac{1-mz^2}{1-z^2}}\right]dz$$

$$= \frac{\beta\gamma}{(1-\beta^2)(1-\gamma^2)^{\frac{1}{2}}}\left[F(\varphi\backslash\alpha) - E(\varphi\backslash\alpha)\right] \tag{17}$$

$$\varphi = \sin^{-1}\sqrt{1-\gamma^2}, \qquad \alpha = \sin^{-1}\sqrt{\frac{1-\beta^2}{1-\gamma^2}}$$

where F and E are respectively the incomplete elliptic integrals of the first and second kind.[8] Note that when $\beta \to 1$, equation (17) becomes

$$\frac{1}{N_a} = \frac{\gamma^2}{2(1-\gamma^2)}\left[\frac{\sin^{-1}\sqrt{1-\gamma^2}}{\gamma\sqrt{1-\gamma^2}} - 1\right]$$

(b). $\beta \le 1, \gamma \ge 1$ $(b \le a,\ c \ge a)$

Substitute

$$z^2 = \frac{\gamma^2-1}{x+1}$$

in the integral (9). Then

$$\frac{1}{N_a} = \frac{\beta\gamma}{(\gamma^2-1)^{3/2}}\int_0^{\sqrt{\gamma^2-1}} \frac{z^2\,dz}{\sqrt{(1-k^2z^2)(1+z^2)}} \qquad \left(k^2 = \frac{1-\beta^2}{\gamma^2-1}\right)$$

$$= \frac{\beta\gamma}{(\gamma^2-1)(1-\beta^2)^{\frac{1}{2}}}\left[\left(\frac{\gamma^2-\beta^2}{1-\beta^2}\right)^{\frac{1}{2}} E(\varphi\backslash\alpha) - \left(\frac{1-\beta^2}{\gamma^2-\beta^2}\right)^{\frac{1}{2}} F(\varphi\backslash\alpha) - \frac{\beta(\gamma^2-1)}{\gamma\sqrt{1-\beta^2}}\right] \qquad (18)$$

$$\varphi = \sin^{-1}(1-\beta^2/\gamma^2)^{\frac{1}{2}}, \qquad \alpha = \sin^{-1}\left(\frac{\gamma^2-1}{\gamma^2-\beta^2}\right)^{\frac{1}{2}}$$

where we have used Formula (3.153.2) in Ref. [9]. Note that when $\gamma \to 1$, equation (18) becomes

$$\frac{1}{N_a} = \frac{\beta}{2(1-\beta^2)^{3/2}}\left[\frac{\pi}{2} - \tan^{-1}\frac{\beta}{\sqrt{1-\beta^2}} - \beta\sqrt{1-\beta^2}\right]$$

and when $\gamma \to \infty$, equation (18) becomes

$$\frac{1}{N_a} = \frac{\beta}{1+\beta}$$

(c). $\beta \geq 1,\ \gamma \geq 1,\quad \beta \geq \gamma \quad (b \geq a,\ c \geq a,\ b \geq c)$

Use the same substitution as in case (b). Then

$$\frac{1}{N_a} = \frac{\beta\gamma}{(\gamma^2-1)^{3/2}} \int_0^{\sqrt{\gamma^2-1}} \frac{z^2 dz}{\sqrt{(1+mz^2)(1+z^2)}} \qquad \left(m = \frac{\beta^2-1}{\gamma^2-1}\right)$$

$$= \frac{\beta\gamma}{(\gamma^2-1)(\beta^2-1)^{\frac{1}{2}}} \left[\frac{\gamma}{\beta}\sqrt{\beta^2-1} - E(\varphi\backslash\alpha)\right] \tag{19}$$

$$\varphi = \tan^{-1}\sqrt{\beta^2-1}, \qquad \alpha = \sin^{-1}\left(\frac{\beta^2-\gamma^2}{\beta^2-1}\right)^{\frac{1}{2}}$$

where Formula (3.153.1) in Ref. [9] has been used. Note that when $\beta \to 1$, equation (19) becomes

$$\frac{1}{N_a} = \frac{\gamma^2}{2(\gamma^2-1)}\left[1 - \frac{\ln(\gamma+\sqrt{\gamma^2-1})}{\gamma\sqrt{\gamma^2-1}}\right]$$

Figures 4-6 and Tables 1-3 were obtained by numerically evaluating formulas (17), (18) and (19).

Before proceeding further let us make some pertinent remarks about equations (18) and (19). Equation (18) applies to the case where $c \geq a \geq b$, while equation (19) applies to the case where $b \geq c \geq a$. It is also important to remember that in these cases the electric field $\underline{E}$ is always directed along the a-axis of the ellipsoid. From geometry considerations, these two cases must correspond to other cases where $\underline{E}$ is directed along the other two axes of the ellipsoid if one maintains $a \geq b \geq c$ for ellipsoids. Indeed, the case where $c \geq a \geq b$ with $\underline{E}$ parallel to the a-axis corresponds exactly to the case where $a \geq b \geq c$ with $\underline{E}$ parallel to the b-axis, and the case where $b \geq c \geq a$ with $\underline{E}$ parallel to the a-axis corresponds precisely to the case where $b \geq c \geq a$ with $\underline{E}$ parallel to the c-axis. Of course, the correspondance holds only for ellipsoids in free space because an infinite ground plane does not allow a parallel static

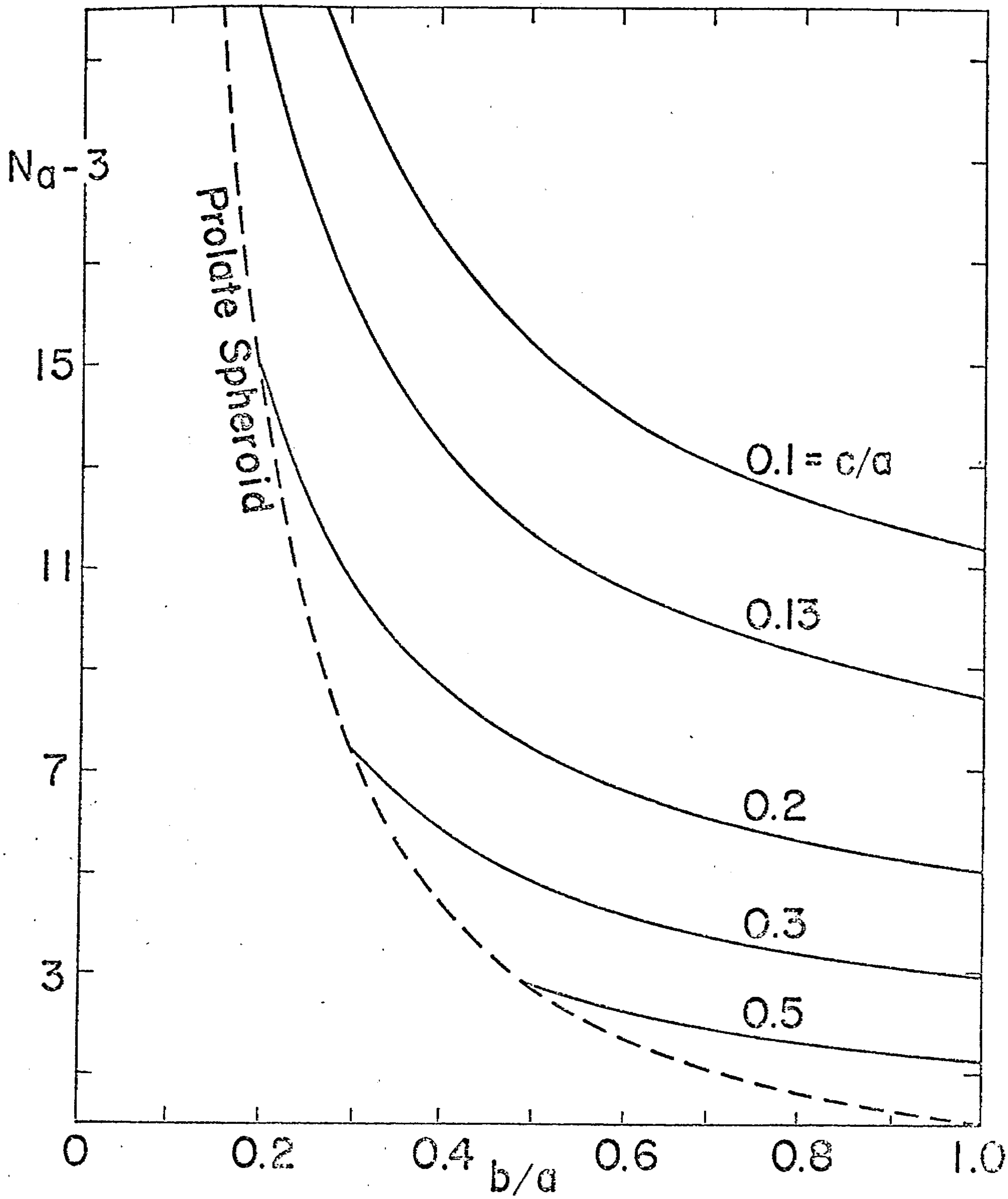

Figure 4. Normalized equivalent area for $b/a \leq 1$, $c/a \leq 1$.
$N_a = A_{eq}(0)/(\pi bc)$

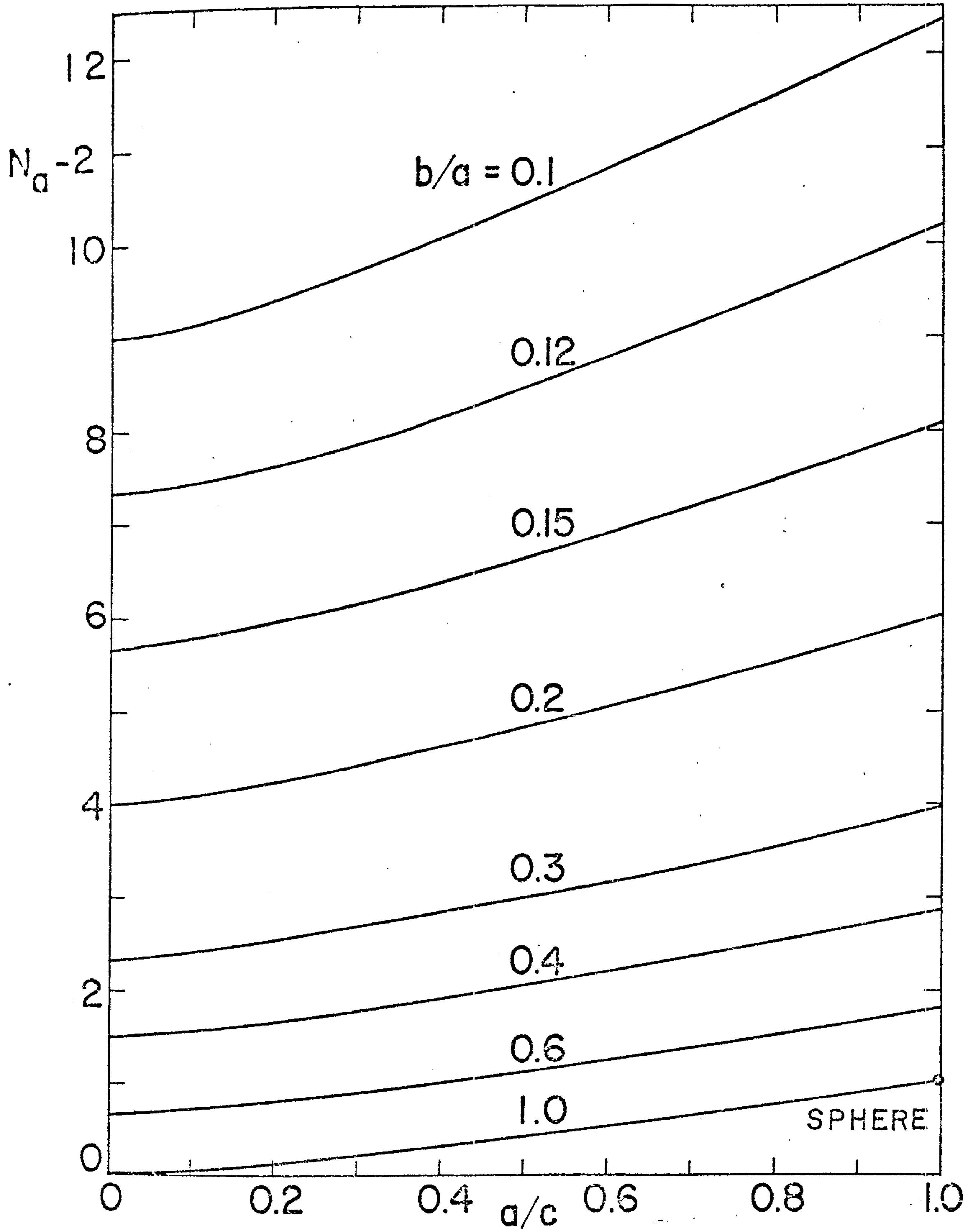

Figure 5. Normalized equivalent area for $b/a \leq 1$, $c/a \geq 1$.
$N_a = A_{eq}(0)/(\pi bc)$

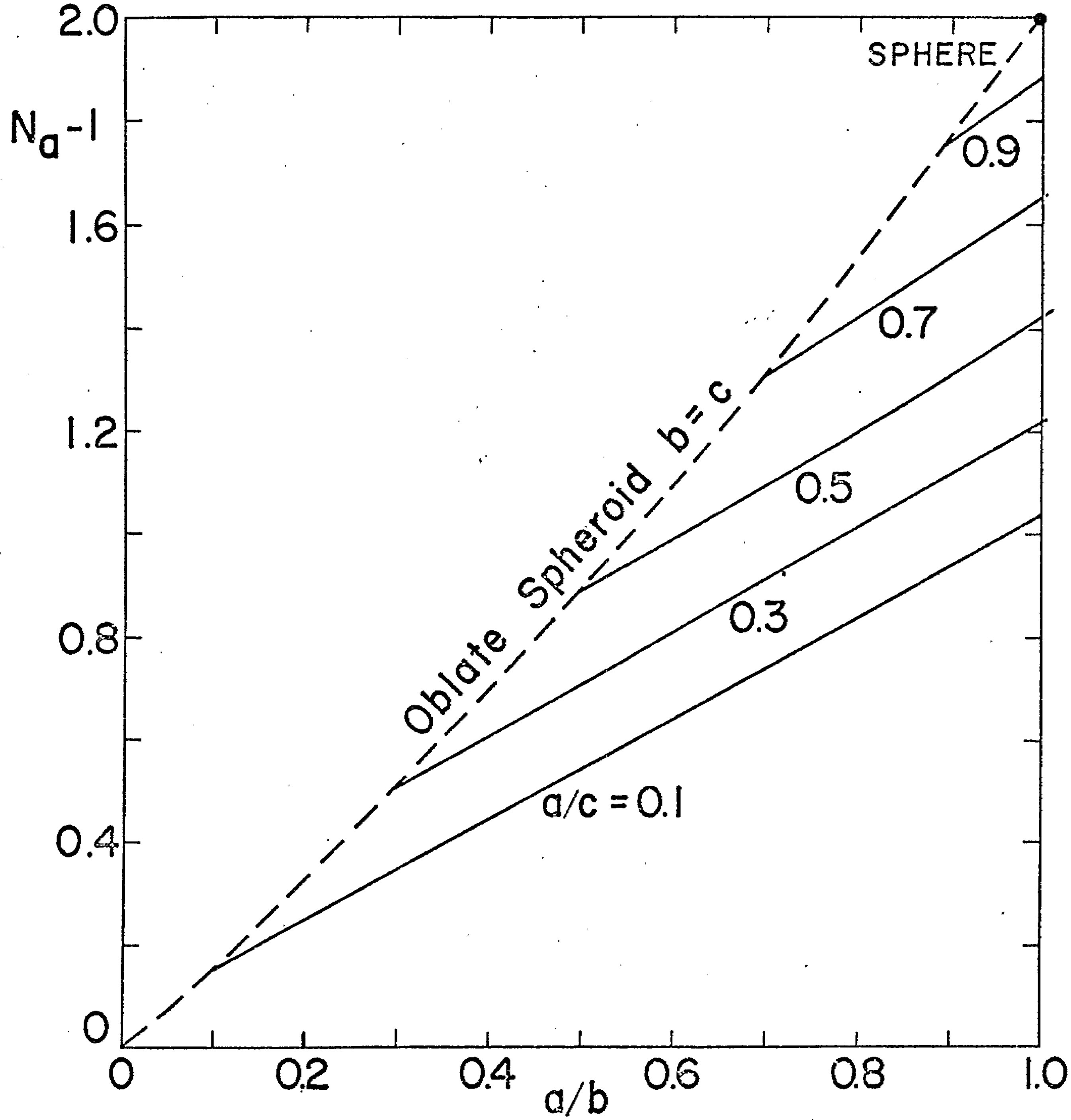

Figure 6. Normalized equivalent area for b/a ≥ 1, c/a ≥ 1.

$N_a = A_{eq}(0)/(\pi bc)$

Table 1

Normalized Equivalent Area, $A_{eq}(0)/(\pi bc)$, of an Ellipsoidal Antenna
for case (b): $b \leq a$, $c \leq a$

c/a \ b/a	.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
.05	148.1685	86.7434	54.7057	43.5257	37.7722	34.2552	31.8828	30.1765	28.8920	27.8922	27.1646
0.1		49.2954	30.2159	23.7060	20.4003	18.3976	17.0552	16.0942	15.3735	14.8141	14.3683
0.2			17.9144	13.7901	11.7217	10.4800	9.6534	9.0649	8.6254	8.2856	8.0155
0.3				10.4855	8.8363	7.8502	7.1959	6.7312	6.3851	6.1178	5.9058
0.4					7.3994	6.5418	5.9737	5.5708	5.2709	5.0396	4.8562
0.5						5.7616	5.2450	4.8789	4.6066	4.3965	4.2301
0.6							4.7628	4.4210	4.1668	3.9708	3.8155
0.7								4.0965	3.8551	3.6690	3.5215
0.8									3.6233	3.4444	3.3027
0.9										3.2713	3.1339
1.0											3.0000

The normalized equivalent area is a symmetric function of b/a and c/a

Table 2

Normalized Equivalent Area, $A_{eq}(0)/(\pi bc)$, of an Ellipsoidal Antenna for case (b): $b \leq a$, $c \geq a$

a/c \ b/a	.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
.05	21.0800	11.0438	6.0260	4.3536	3.5175	3.0160	2.6817	2.4430	2.2640	2.1249	2.0136
0.1	21.2470	11.1354	6.0803	4.3957	3.5539	3.0490	2.7127	2.4727	2.2929	2.1531	2.0414
0.2	21.7155	11.3925	6.2326	4.5141	3.6559	3.1417	2.7995	2.5555	2.3729	2.2312	2.1182
0.3	22.2852	11.7055	6.4185	4.6585	3.7802	3.2545	2.9049	2.6560	2.4699	2.3258	2.2108
0.4	22.9120	12.0505	6.6238	4.8182	3.9177	3.3791	3.0214	2.7669	2.5768	2.4297	2.3125
0.5	23.5737	12.4154	6.8415	4.9877	4.0638	3.5115	3.1449	2.8844	2.6900	2.5396	2.4200
0.6	24.2575	12.7933	7.0674	5.1640	4.2157	3.6492	3.2735	3.0065	2.8076	2.6538	2.5315
0.7	24.9557	13.1798	7.2992	5.3450	4.3718	3.7907	3.4055	3.1321	2.9283	2.7709	2.6459
0.8	25.6632	13.5723	7.5352	5.5295	4.5311	3.9352	3.5403	3.2601	3.0515	2.8904	2.7624
0.9	26.3768	13.9690	7.7743	5.7166	4.6928	4.0818	3.6771	3.3901	3.1764	3.0115	2.8806
1.0	27.0291	14.3683	8.0155	5.9058	4.8562	4.2301	3.8155	3.5215	3.3027	3.1339	3.0000

Table 3

Normalized Equivalent Area, $A_{eq}(0)/(\pi bc)$, of an Ellipsoidal Antenna
for case (c): $b \geq a$, $c \geq a$

a/b \ a/c	.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
.05	1.0797	1.1233	1.2180	1.3158	1.4146	1.5140	1.6137	1.7135	1.8135	1.9135	2.0136
0.1		1.1617	1.2508	1.3458	1.4432	1.5418	1.6411	1.7408	1.8408	1.9410	2.0414
0.2			1.3325	1.4234	1.5186	1.6162	1.7151	1.8151	1.9157	2.0168	2.1182
0.3				1.5121	1.6062	1.7036	1.8030	1.9037	2.0054	2.1079	2.2108
0.4					1.7002	1.7980	1.8983	2.0003	2.1036	2.2077	2.3125
0.5						1.8968	1.9984	2.1020	2.2070	2.3131	2.4200
0.6							2.1016	2.2070	2.3140	2.4223	2.5315
0.7								2.3145	2.4237	2.5342	2.6459
0.8									2.5352	2.6482	2.7624
0.9										2.7638	2.8806
1.0											3.0000

The normalized equivalent area is a symmetric function of a/b and a/c

$\underline{E}$ field. Thus, our numerical results just presented apply to all possible orientations of $\underline{E}$ field and all possible sizes of ellipsoids. Because the results for other field orientations are quite useful in other areas of application (such as the depolarization effects of an ellipsoid) it is worthwhile to see exactly how to interpret our results so as to apply to other field orientations.

(b'). *Conversion of* $c \geq a \geq b$ *and* $\underline{E} \parallel$ *a-axis to* $a \geq b \geq c$ *and* $\underline{E} \parallel$ *b-axis*

In equation (18) we replace a by b, b by c, and c by a. Then we have, calling N_a by N_b after conversion,

$$\frac{1}{N_b} = -\frac{\beta\gamma}{(1-\beta^2)(1-\gamma^2)^{\frac{1}{2}}}\left[F(\varphi\backslash\alpha) - E(\varphi\backslash\alpha)\right] + \frac{\beta\gamma}{(\beta^2-\gamma^2)(1-\gamma^2)^{\frac{1}{2}}}E(\varphi\backslash\alpha) - \frac{\gamma^2}{\beta^2-\gamma^2}$$

$$= \frac{\beta\gamma}{2}\int_o^\infty \frac{dx}{(x+1)^{\frac{1}{2}}(x+\beta^2)^{3/2}(x+\gamma^2)^{\frac{1}{2}}} \tag{20}$$

$$\varphi = \sin^{-1}\sqrt{1-\gamma^2}, \qquad \alpha = \sin^{-1}\sqrt{\frac{1-\beta^2}{1-\gamma^2}}$$

where $\beta = b/a$, $\gamma = c/a$ and $1 \geq \beta \geq \gamma$. The first term in (20) is equal to minus expression (17). It can be verified that equation (20) can be obtained directly by solving the problem depicted in Fig. 7a, i.e., by evaluating the integral in (20) directly. Our numerical results presented above for case (b) thus apply directly to case (b'), Fig. 7a, if the values β and γ in case (b) are to be interpreted, respectively, to be the values γ/β and $1/\beta$ in case (b').

(c'). *Conversion of* $b \geq c \geq a$ *and* $\underline{E} \parallel$ *a-axis to* $a \geq b \geq c$ *and* $\underline{E} \parallel$ *c-axis*

In equation (19) we simply replace c by b, b by a, and a by c and arrive at, with N_c in place of N_a,

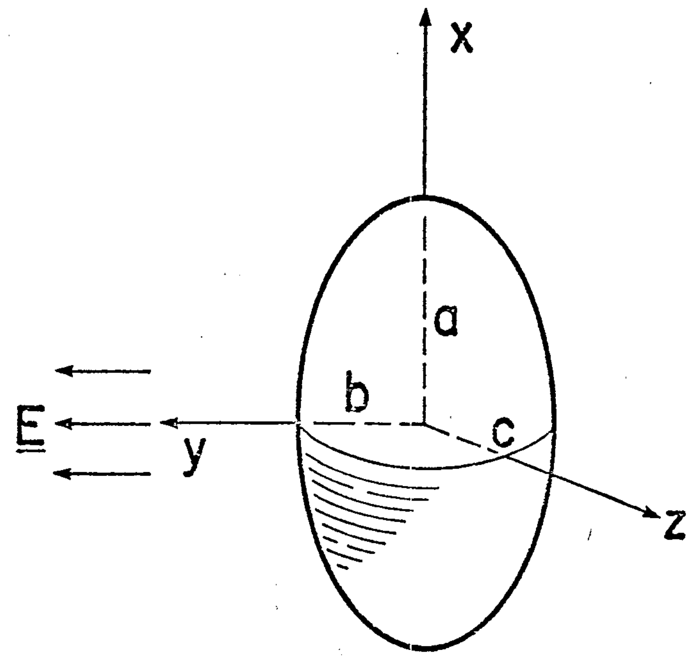

Figure 7a. $\underline{E} \parallel$ y-axis and $a \geq b \geq c$.

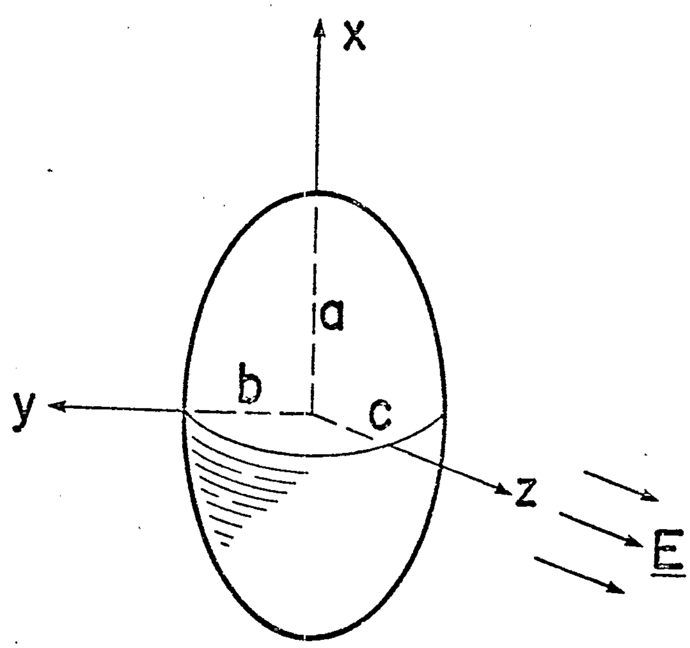

Figure 7b. $\underline{E} \parallel$ z-axis and $a \geq b \geq c$.

$$\frac{1}{N_c} = - \frac{\beta\gamma}{(\beta^2-\gamma^2)(1-\gamma^2)^{\frac{1}{2}}} E(\varphi\backslash\alpha) + \frac{\beta^2}{\beta^2-\gamma^2}$$

$$= \frac{\beta\gamma}{2} \int_o^\infty \frac{dx}{(x+1)^{\frac{1}{2}}(x+\beta^2)^{\frac{1}{2}}(x+\gamma^2)^{3/2}} \qquad (21)$$

$$\varphi = \sin^{-1}\sqrt{1-\gamma^2}, \qquad \alpha = \sin^{-1}\sqrt{\frac{1-\beta^2}{1-\gamma^2}}$$

where $\beta = b/a$, $\gamma = c/a$ and $1 \geq \beta \geq \gamma$. Again, it can be easily verified that expression (21) can be obtained directly by solving the problem depicted in Fig. 7b. Our extensive numerical results presented for case (c) thus apply directly to case (c'), Fig. 7b, if one simply interprets the values β and γ in case (c) to be, respectively, the values $1/\gamma$ and β/γ in case (c').

Before we conclude this section, let us discuss some important, special, degenerate cases of an ellipsoid: sphere, prolate spheroid, oblate spheroid, elliptic disk (blade).

Firstly, for a sphere we set $\beta = \gamma = 1$ in (17), (18) and (19) and they all give

$$N_a = 3 \qquad \text{(sphere)} \qquad (22)$$

as is expected. This value is marked on the curves in Figs. 4-6.

Secondly, for a prolate spheroid we set $\beta = \gamma < 1$ in (17) and get

$$N_a = \frac{2m}{1-m}\left[\frac{1}{\sqrt{m}} \ln \frac{1+\sqrt{m}}{1-\sqrt{m}} - 2\right]^{-1}, \qquad m = 1 - \beta^2$$

In terms of the eccentricity ε, $\varepsilon = \sqrt{1 - b^2/a^2}$, this expression becomes

$$N_a = \frac{\varepsilon^2}{1-\varepsilon^2}\left[\frac{1}{2\varepsilon}\ln\frac{1+\varepsilon}{1-\varepsilon} - 1\right]^{-1} \qquad \text{(prolate spheroid)} \qquad (23)$$

$$\sim \frac{a^2}{b^2}\frac{1}{\ln(2a/b)-1} \qquad \text{(thin rod: } a >> b\text{)}$$

which have been obtained by Maxwell in his investigation of the demagnetization effects of a permeable body.[10] Expression (23) is shown in Fig. 4.

Thirdly, for an oblate spheroid we set $\beta = \gamma > 1$ in (19) and obtain

$$N_a = \frac{m}{1+m}\left[1 - \frac{1}{\sqrt{m}}\tan^{-1}\sqrt{m}\right]^{-1}, \qquad m = \beta^2 - 1$$

which, in terms of the eccentricity $\varepsilon = \sqrt{1 - a^2/b^2}$, becomes

$$N_a = \varepsilon^2\left[1 - \frac{\sqrt{1-\varepsilon^2}}{\varepsilon}\sin^{-1}\varepsilon\right]^{-1}, \qquad \text{(oblate spheroid)} \qquad (24)$$

a form originally due to Maxwell.[10] Expression (24) is plotted in Fig. 6.

Fourthly, for an elliptic disk or blade (Fig. 8) we set $\gamma = 0$ in (17) and introduce the normalized equivalent area N_a^{blade} defined as

$$N_a^{blade} = \gamma N_a = \frac{A_{eq}(0)}{\pi ab} \qquad (25)$$

Thus,

$$N_a^{blade} = \frac{m_1}{\sqrt{1-m_1}} \cdot \frac{1}{K(m_1)-E(m_1)} \qquad \text{(blade: } a \geq b\text{)} \qquad (26a)$$

$$\sim \frac{a/b}{\ln(4a/b)-1}, \quad \text{for } a >> b \qquad (26a')$$

where $m_1 = 1 - b^2/a^2$, $b \leq a$, K and E are respectively the complete elliptic integrals of the first and second kind. For $\beta \geq 1$, we first interchange the role of β and γ in (18) and then set $\gamma = 0$. Thus,

$$N_a^{blade} = \frac{m_2}{E(m_2)-(1-m_2)K(m_2)} \qquad \text{(blade: } a \leq b\text{)} \qquad (26b)$$

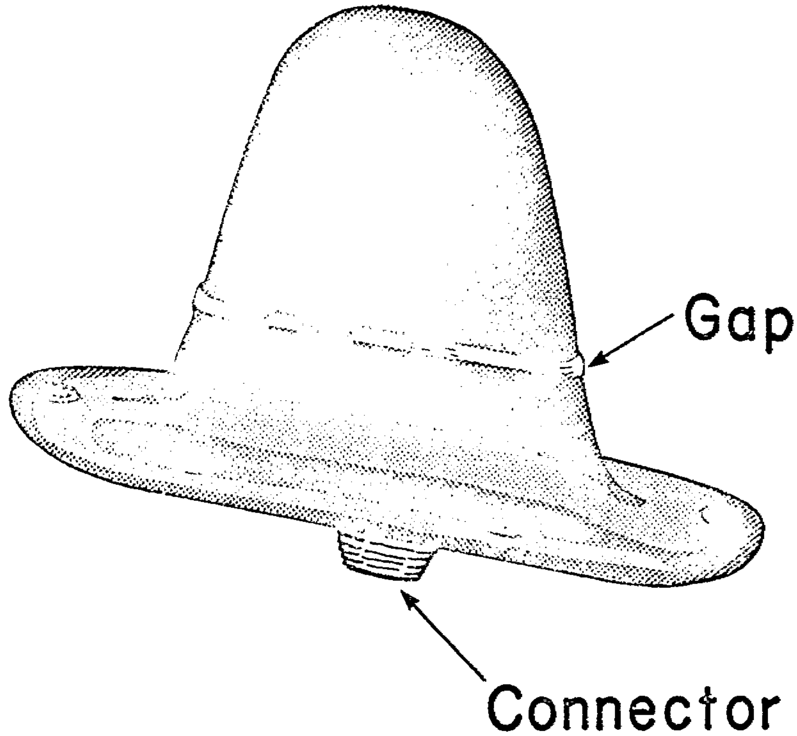

Figure 8. A typical aircraft blade antenna.

where $m_2 = 1 - a^2/b^2$ and $b \geq a$. Expressions (26a) and (26b) reduce to $4/\pi$ for a circular disk as they should. Since many aircraft antennas are of the form of a blade we present in Fig. 9 and Table 4 extensive numerical results based on equations (26a) and (26b). Notice that the curve in Fig. 9 is continuous and has a continuous first derivative at $a = b$, as can be verified by differentiating (26a) and (26b).

Finally, we examine expression (19) for the case where $\beta >> 1$ and $\gamma >> 1$. This limiting case corresponds to the problem of field distortions by a ground plane of finite thickness.[14] From (19) we have

$$N_a \approx 1 + \frac{a}{c} E(m), \qquad m = 1 - c^2/b^2 \tag{26c}$$

When $b = c$, this gives

$$N_a \approx 1 + \frac{\pi}{2}\frac{a}{c}, \tag{26d}$$

in agreement with the result in Reference [14].

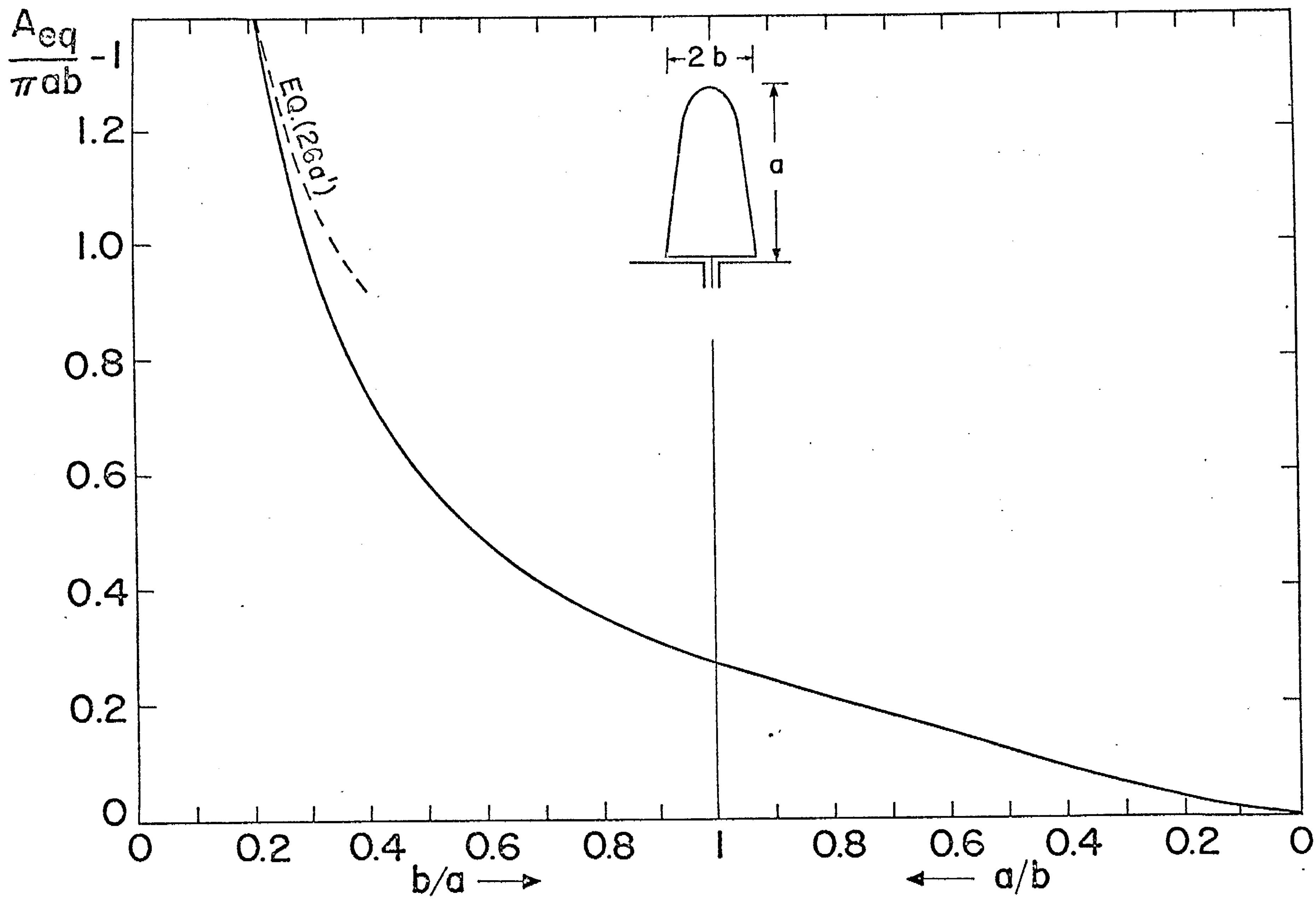

Figure 9. Normalized equivalent area of an elliptic disk (blade antenna).

Table 4

Normalized Equivalent Area of an Elliptic Disk (Blade Antenna)

$\frac{A_{eq}(0)}{\pi ab}$	$\frac{b}{a}$ | $\frac{a}{b}$	$\frac{A_{eq}(0)}{\pi ab}$
137.0980	10^{-3}	1.0000
20.0328	10^{-2}	1.0002
3.6945	0.1	1.0112
2.4420	0.2	1.0324
1.9890	0.3	1.0581
1.7376	0.4	1.0864
1.5865	0.5	1.1162
1.4836	0.6	1.1469
1.4091	0.7	1.1782
1.3527	0.8	1.2100
1.3087	0.9	1.2413
1.2732	1.0	1.2732

IV. Polarizabilities

The response of an electrically small body to an incident wave is predominantly of electric and magnetic-dipole type. Hence it is justified to characterize an electrically small object by its electric and magnetic polarizabilities. This characterization of small objects has been widely used in EMP interaction calculations. We will in this section calculate the electric and magnetic polarizabilities of a half ellipsoid on an infinite ground plane from the solutions of an electrostatic and a magnetostatic problem. The results will apply to surface protrusions on a complex structure such as an aircraft or a missile [11], and various "junk" inside an EMP simulator.

Let us first calculate the electric dipole $\underline{p}$ and, hence, the electric polarizability α_e for Fig. 2. Clearly, $\underline{p}$ is parallel to the external field $\underline{E}$ and we write

$$p = \alpha_e \varepsilon E$$

which defines α_e. To find p we use the induced charge density σ given by (8) and perform the following surface integral:

$$p = \int_S x\sigma dA = N_a \varepsilon E \int_S \frac{x^2/a^2 \; dA}{\sqrt{x^2/a^4+y^2/b^4+z^2/c^4}} \tag{27}$$

where S is the surface of the half ellipsoid (Fig. 2). Evaluating this integral along the same line as from (13) to (14) we get

$$p = V_h N_a \varepsilon E, \tag{28}$$

and so

$$\alpha_e = V_h N_a \tag{29}$$

where $V_h = 2/3\ \pi abc$, the volume of a half ellipsoid. The dipole moment p can also be obtained from the scattered potential at large distances, which is given by (6) for a whole ellipsoid in free space. We rewrite (6) as, with the appropriate expression for the constant C,

$$\phi_s \sim \left[\frac{8\pi\varepsilon E}{3} \Big/ \int_o^\infty \frac{ds}{(s+a^2)R_s}\right] \frac{x}{4\pi\varepsilon r^3} , \qquad r \to \infty$$

$$= (2V_h N_a \varepsilon E) \frac{x}{4\pi\varepsilon r^3} \tag{30}$$

from which we deduce that the dipole moment is twice the value given by (28). The factor 2 is obvious because when one uses (28) to calculate the field one must not forget the ground plane, whereas (30) applies to free space.

If one compares (29) with (10) and (15), one gets the following interesting relation

$$\frac{\alpha_e}{V_h} = \frac{A_{eq}(0)}{\pi bc} = f_E = N_a \tag{31}$$

that is, *the normalized electric polarizability, the normalized equivalent area, and the electric-field enhancement factor at the tip are all equal.* For an elliptic blade we have

$$\frac{\alpha_e}{(2a/3)\pi ab} = \frac{A_{eq}(0)}{\pi ab} = f_E = N_a^{blade} \tag{32}$$

Thus, the numerical results in Section III apply directly to the electric polarizability.

Let us now calculate the coefficients of the magnetic polarizability tensor $\underline{\underline{\alpha}}_m$:

$$\underline{m} = \underline{\underline{\alpha}}_m \cdot \underline{H}$$

where $\underline{m}$ is the magnetic dipole moment and $\underline{H}$ is the external magnetic field (Fig. 10). Let us first take $\underline{H}$ parallel to the y-axis and calculate the yy-component of the tensor $\underline{\underline{\alpha}}_m$. Then we will take $\underline{H}$ parallel to the z-axis and calculate the zz-component of $\underline{\underline{\alpha}}_m$. These two diagonal elements define $\underline{\underline{\alpha}}_m$ uniquely.

To solve our magnetostatic problem we introduce the magnetic scalar

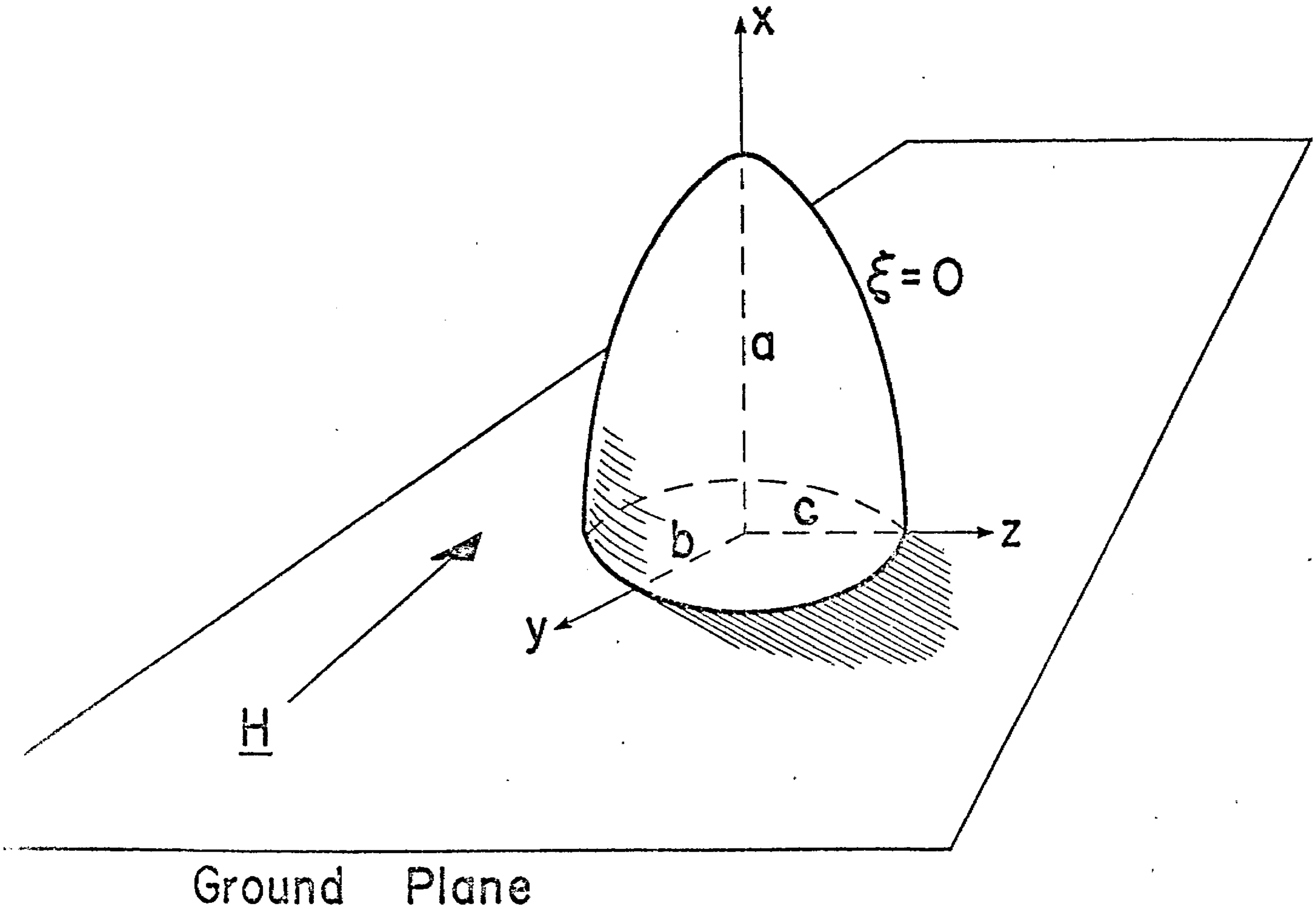

Figure 10. Infinite ground plane with an ellipsoidal boss in a uniform magnetic field.

potential Ω_2 such that

$$\underline{H} = -\nabla\Omega_2 \tag{33a}$$

$$\nabla^2\Omega_2 = 0 \tag{33b}$$

$$\frac{\partial\Omega_2}{\partial n} = 0, \qquad \text{on the plane } x = 0 \text{ and on the ellipsoid } \xi = 0 \tag{33c}$$

$$\Omega_2 \sim \Omega_o = -H_2 y = -H_2\sqrt{\frac{(\xi+b^2)(\eta+b^2)(\zeta+b^2)}{(c^2-b^2)(a^2-b^2)}}, \qquad \text{as } r \to \infty \tag{33d}$$

Following almost the same procedure as in Section II for the electrostatic case we find for the total potential Ω_2,

$$\Omega_2 = \Omega_o + \Omega_s = -H_2 y\left[1 + \frac{abc}{2}\frac{N_b}{N_b-1}\int_\xi^\infty \frac{ds}{(s+b^2)R_s}\right] \tag{34}$$

where

$$\frac{1}{N_b} = \frac{abc}{2}\int_o^\infty \frac{ds}{(s+b^2)R_s}$$

which has been evaluated in equation (20) in the previous section. At large distances the scattered potential Ω_s is given by

$$\Omega_s \sim -\frac{1}{3} H_2 \frac{abcN_b}{N_b-1}\frac{y}{r^3}, \qquad r \to \infty \tag{35}$$

Comparing this with the potential of a y-directed magnetic dipole just above a ground plane we then deduce the yy-component of $\underline{\underline{\alpha}}_m$, which will be denoted by α_m^b, to be

$$\alpha_m^b = -\frac{2\pi abc}{3}\frac{N_b}{N_b-1} \tag{36}$$

When the external magnetic field is along the z-axis, we find the total magnetic potential Ω_3 to be

$$\Omega_3 = -H_3 z\left[1+\frac{abc}{2}\frac{N_c}{N_c-1}\int_\xi^\infty \frac{ds}{(s+c^2)R_s}\right] \tag{37}$$

$$z = \sqrt{\frac{(\xi+c^2)(\eta+c^2)(\zeta+c^2)}{(a^2-c^2)(b^2-c^2)}}$$

$$\frac{1}{N_c} = \frac{abc}{2}\int_o^\infty \frac{ds}{(s+c^2)R_s}$$

$$\alpha_m^c = -\frac{2\pi abc}{3}\frac{N_c}{N_c-1} \tag{38}$$

where α_m^c denotes the zz-component of $\underline{\underline{\alpha}}_m$, and N_c has been evaluated in equation (21) of Section III.

A comparison of (29), (36) and (38) reveals some very interesting relationship between the electric and magnetic polarizabilities of an ellipsoid. Let us first introduce the normalized polarizabilities $\bar{\alpha}_e^i$ and $\bar{\alpha}_m^i$:

$$\bar{\alpha}_e^i = \frac{\alpha_e^i}{V_h}, \qquad \bar{\alpha}_m^i = \frac{\alpha_m^i}{V_h} \tag{39}$$

where, as before, $V_h = 2/3\ \pi abc$, i.e., the volume of the ellipsoidal boss under consideration. If one considers a whole ellipsoid in free space one simply uses the total volume, $4/3\ \pi abc$, as normalizing factor, and the following relations are unchanged. The superscript i can be any of the three principal axes of the ellipsoid. Then, we can rewrite (29), (36) and (38) as

$$\bar{\alpha}_e^i = N_i$$

$$\bar{\alpha}_m^i = -\frac{N_i}{N_i - 1} \qquad (40)$$

from which we get

$$\frac{1}{\bar{\alpha}_e^i} - \frac{1}{\bar{\alpha}_m^i} = 1 \qquad (41)$$

or, in un-normalized form,

$$\frac{1}{\alpha_e^i} - \frac{1}{\alpha_m^i} = \frac{1}{V} , \qquad (42)$$

a surprisingly simple relationship between the electric and magnetic polarizabilities! Also, summing over directions along the three principal axes a,b,c we have

$$\sum_i \frac{1}{\bar{\alpha}_e^i} = \frac{1}{N_a} + \frac{1}{N_b} + \frac{1}{N_c} = 1 \qquad (43)$$

which can be easily seen from the integral definitions of N_i as given in (9), (20), and (21). Equations (41) and (43) imply that

$$\sum_i \frac{1}{\bar{\alpha}_m^i} = -2 \qquad (44)$$

Equation (41) makes it evident that the numerical results of Section III are also applicable to the calculation of magnetic polarizabilities.

The electric polarizability coefficient we have just found for a perfectly conducting ellipsoid can be applied directly to a dielectric ellipsoid in a uniform electric field and to a permeable ellipsoid in a uniform magnetic field. In the case of a dielectric ellipsoid of permittivity ε in a uniform external field $\underline{E}_o$ (Fig. 11a) the interior field $\underline{E}_i$ can be expressed as [Ref. 4, p. 213]

$$\underline{E}_i = \underline{E}_o - \varepsilon_o^{-1} \underline{\underline{D}}_e \cdot \underline{P}_i \qquad (45)$$

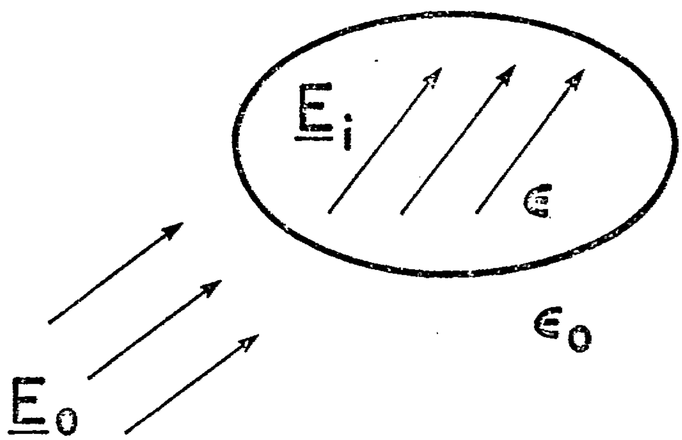

Figure 11a. Dielectric ellipsoid in uniform field.

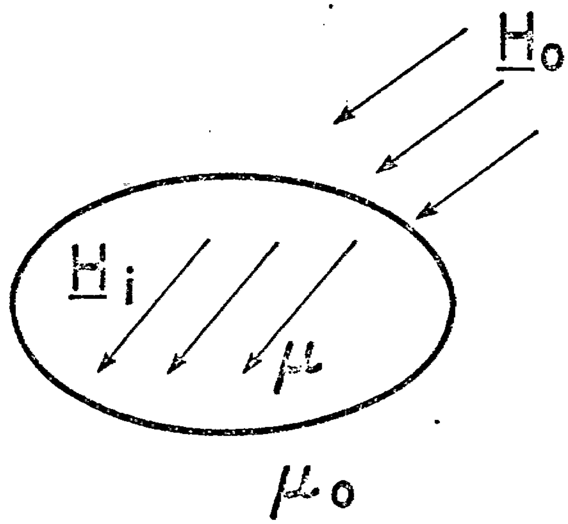

Figure 11b. Permeable ellipsoid in uniform field.

where $\underline{\underline{D}}_e$ is the so-called depolarization tensor, and $\underline{P}_i$ is the induced polarization and related to $\underline{E}_i$ in the usual way, that is,

$$\underline{P}_i = (\varepsilon - \varepsilon_o)\underline{E}_i$$

Elimination of $\underline{E}_i$ in (45) gives, with $\varepsilon_r = \varepsilon/\varepsilon_o$ and $\underline{\underline{U}}$ = unit dyad,

$$\underline{P}_i = \varepsilon_o\left[\frac{1}{\varepsilon_r - 1}\underline{\underline{U}} + \underline{\underline{D}}_e\right]^{-1}\cdot\underline{E}_o \tag{46}$$

Interestingly enough, it turns out that

$$\underline{\underline{D}}_e = [\overline{\underline{\underline{\alpha}}}_e]^{-1} \tag{47}$$

i.e., $\underline{\underline{D}}_e$ *is the inverse of the normalized electric polarizability tensor* $\overline{\underline{\underline{\alpha}}}_e$. When (47) is referred to the principal axes of the ellipsoid one has

$$D_e^i = \frac{1}{\overline{\alpha}_e^i}, \qquad (i = a,b,c) \tag{48}$$

In the case of an ellipsoid of permeability μ in a uniform external field $\underline{H}_o$ (Fig. 11b) the interior field $\underline{H}_i$ can be expressed as [Ref. 4, p. 258]

$$\underline{H}_i = \underline{H}_o - \underline{\underline{D}}_m\cdot\underline{M}_i \tag{49}$$

where $\underline{\underline{D}}_m$ is the so-called demagnetization tensor, and $\underline{M}_i$ is the induced magnetization. Elimination of $\underline{H}_i$ in (49) by means of

$$\underline{M}_i = (\mu_r - 1)\underline{H}_i, \qquad \mu_r = \mu/\mu_o$$

gives

$$\underline{M}_i = \left(\frac{1}{\mu_r - 1}\underline{\underline{U}} + \underline{\underline{D}}_m\right)^{-1}\cdot\underline{H}_o \tag{50}$$

Again, it turns out that

$$D_m = [\bar{\underline{\underline{\alpha}}}_e]^{-1} \tag{51}$$

Thus, *the electric polarizability tensor of a conducting ellipsoid is the inverse of the depolarization tensor of the corresponding dielectric ellipsoid and also is the inverse of the demagnetization tensor of the corresponding permeable ellipsoid.*

Before concluding this section let us calculate the magnetic-field enhancement factors from (34) for a y-directed field, and from (37) for a z-directed field (see Fig. 10). Let us differentiate (34) and evaluate the resulting expression on the surface of the ellipsoid: $\xi = 0$. We have

$$H_\eta = -\left[\frac{1}{h_2}\frac{\partial\Omega_2}{\partial\eta}\right]_{\xi=0} = H_2\left[1 + \frac{1}{N_b-1}\right]\left[\frac{1}{h_2}\frac{\partial y}{\partial\eta}\right]_{\xi=0} \tag{52}$$

Let us observe that H_η has a maximum at $(x = 0,\ y = 0,\ z = \pm c)$ which correspond to $(\xi = 0,\ \eta = -b^2,\ \zeta = -a^2)$ in ellipsoidal corrdinates, or at the tip. Let us first examine those points at the base (Fig. 12). At these points (Fig. 12a) we find

$$\left[\frac{1}{h_2}\frac{\partial y}{\partial\eta}\right]_{\substack{\xi=0\\ \eta=-b^2\\ \zeta=-a^2}} = 1$$

Hence, for the magnetic-field enhancement factor f_H^b we have

$$f_H^b = \left[\frac{H_\eta}{H_2}\right]_{max} = \frac{N_b}{N_b-1} = -\bar{\alpha}_m^b \tag{53}$$

where $\bar{\alpha}_m^b$ is the normalized magnetic polarizability as defined by (39).

Similarly, for a z-directed field (Fig. 12b) we find from (37)

$$f_H^c = \left[\frac{H_\eta}{H_3}\right]_{max} = \frac{N_c}{N_c-1} = -\bar{\alpha}_m^c \tag{54}$$

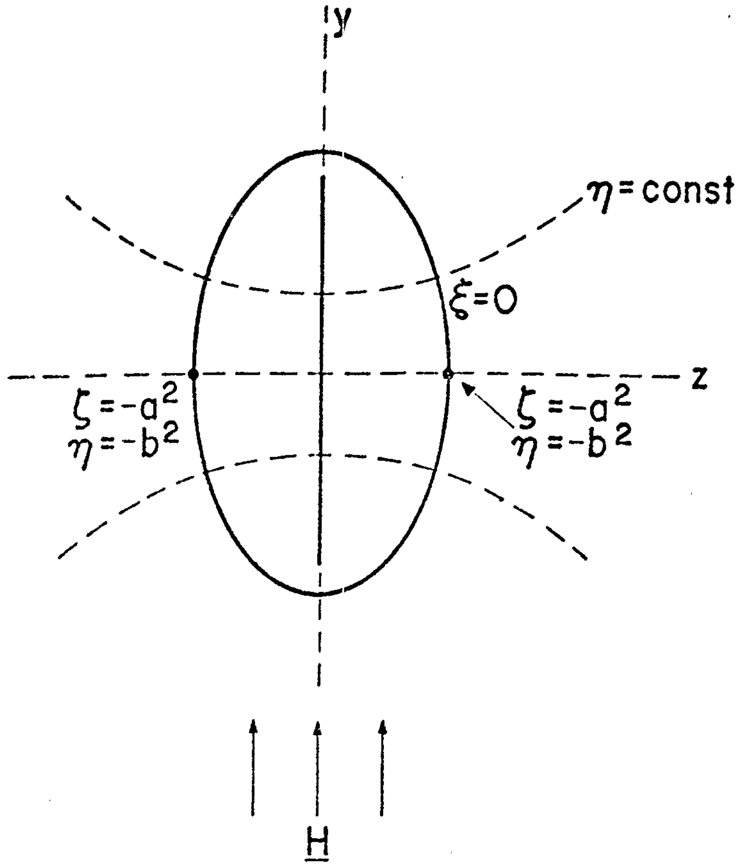

Figure 12a.

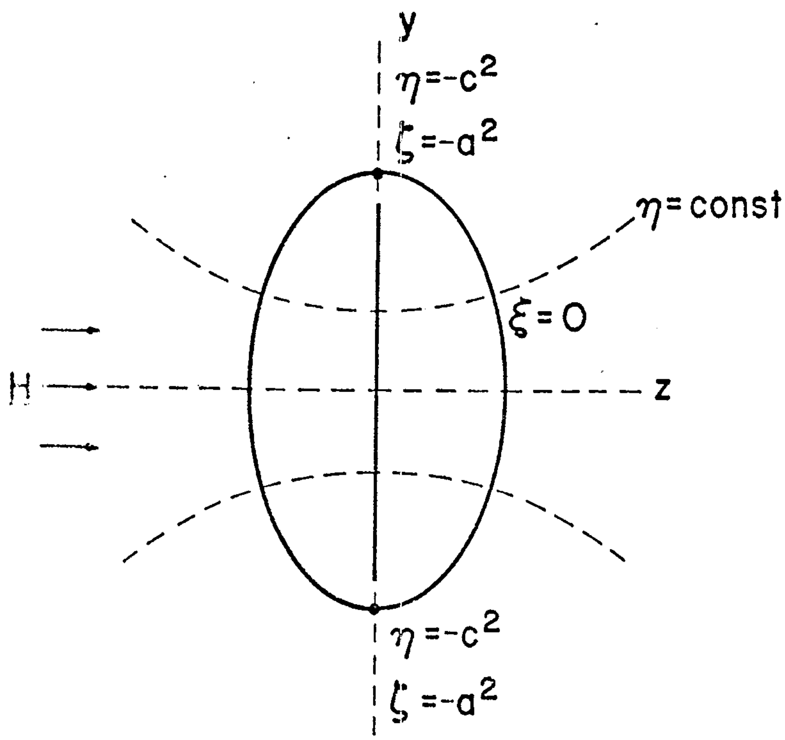

Figure 12b.

(x = 0 plane) of the ellipsoidal antenna of Fig. 10.

Just as in the electrostatic case, we find that the field enhancement factor is directly related to the polarizability in the magnetic case. Equations (10), (53) and (54) can be written in the concise form:

$$f_E^i = \overline{\alpha}_e^i, \qquad f_H^i = -\overline{\alpha}_m^i \tag{55}$$

where, as before, the superscript i denotes direction of external field along any of the three principal axes of an ellipsoid.

Finally, let us evaluate the magnetic field at the tip of the ellipsoid (Fig. 10). The total magnetostatic potential Ω is given by the sum of expressions (34) and (37). Differentiating Ω and evaluating the resulting expression at the tip $(x = a, y = 0, z = 0)$ we get

$$\underline{H}_{tip} = -\overline{\underline{\underline{\alpha}}}_m \cdot \underline{H}_o \tag{56}$$

where $\overline{\underline{\underline{\alpha}}}_m = \hat{y}\hat{y}\overline{\alpha}_m^b + \hat{z}\hat{z}\overline{\alpha}_m^c$ when referred to the b, c axes of the ellipsoid; $\underline{H}_o$ is the incident magnetic field. If one introduces the induced magnetic moment $\underline{m}$ into (56), one gets the extremely interesting result

$$\underline{H}_{tip} = -\frac{1}{V}\underline{m} \tag{57}$$

which is analogous to the electrostatic case, (10) and (28),

$$\varepsilon\underline{E}_{tip} = \frac{1}{V}\underline{p} \tag{58}$$

where V is the volume of the ellipsoid or the volume of the half ellipsoid if the latter is resting on a ground plane. Comparing (56) with (53) and (54) we have the following interesting result: *the magnetic-field enhancement factor at the tip is the same as those at the base (Fig. 12). In fact, the magnetic-field enhancement factor is the same for all points on the surface of the ellipsoid where the incident magnetic field is tangent to the ellipsoid.*

We will now evaluate the enhancement factors, f_H^b and f_H^c, for a flat ellipsoidal boss, i.e., $a \ll b$ and $a \ll c$ (Fig. 10), which may model a ground plane of finite thickness. Taking the limiting value of (17) and reinterpreting the symbols we get

$$f_H^b \approx 1 + \frac{ac}{b^2-c^2}\,[K(m_1) - E(m_1)], \qquad b > c, \quad m_1 = 1 - c^2/b^2$$

$$\approx 1 + \frac{ab}{c^2-b^2}\left[\frac{c^2}{b^2}\,E(m_2) - K(m_2)\right], \qquad c > b, \quad m_2 = 1 - b^2/c^2 \tag{59}$$

$$\approx 1 + \frac{\pi}{4}\frac{a}{b} \qquad \text{(flat oblate spheroid: } b = c\text{)}$$

For f_H^c one simply replaces b by c, and c by b in (59).

V. Radiation Resistance

We will make use of the results derived in the previous sections to obtain the radiation resistance of an ellipsoidal antenna with an infinite ground plane. It may sound surprising that we are able at all to calculate the radiation resistance from the low-frequency solution of the scattering problem of an ellipsoid, since radiation resistance is intimately associated with the transmitting antenna problem where the excitation of the ellipsoid is completely different from that of the scattering problem. One may then say that the current distributions on the antenna in the transmitting and scattering cases are so different that one can hardly expect to get a reliable answer for the radiation resistance solely from the scattering problem. But the difference in current distributions would not affect much the Poynting vector at large distances, especially when the antennas are electrically small. Indeed, a close relationship exists between the radiation resistance and the scattering cross section of an antenna as implied by the reciprocity theorem. This relationship has been discussed in great detail by Slater.[12] The following method of calculation is primarily his.

Let R_r be the radiation resistance when our ellipsoidal antenna is driven at its base with respect to an infinite ground plane (Fig. 3). Then the average scattered power P_{sc} is given by

$$P_{sc} = \frac{1}{2}|I_{sc}|^2 R_r = \frac{1}{2}\omega^2\varepsilon^2|E|^2 A_{eq}^2 R_r \tag{60}$$

where we have used equation (16). On the other hand, P_{sc} can be computed from the electric and magnetic moments $\underline{p}$ and $\underline{m}$ of Section IV. From the well-known dipole formulas one has

$$P_{sc} = \frac{1}{2} \cdot \frac{\omega^4\mu\sqrt{\mu\varepsilon}}{12\pi}\left[|p|^2 + Z_o^2\varepsilon^2|m|^2\right] \tag{61}$$

where $Z_o = 120\pi$ ohms and the factor ½ accounts for the presence of the infinite ground plane. The values for p and m in (61) should be twice the values in Section IV because of the images. Using (29), (36), (38) and taking the average of incident magnetic-field direction we find from (61)

$$P_{sc} = \frac{\omega^4 \mu\sqrt{\mu\varepsilon}}{24\pi} V^2 \varepsilon^2 |E|^2 \left[N_a^2 + \frac{N_b^2}{2(1-N_b)^2} + \frac{N_c^2}{2(1-N_c)^2} \right] \tag{62}$$

with $V = (4\pi abc)/3$. Equating (60) and (62) we get

$$R_r = \frac{16\pi}{27} \left(\frac{a}{\lambda}\right)^2 Z_o \left[1 + \frac{N_b^2}{2N_a^2(1-N_b)^2} + \frac{N_c^2}{2N_a^2(1-N_c)^2} \right] \tag{63}$$

where λ is the wavelength and (15) has been used for A_{eq} at $x_o = 0$. If the driving point is at $x_o \neq 0$, then

$$R_r(x_o) = \frac{1}{1-(x_o/a)^2} R_r(0) \tag{64}$$

where $R_r(0)$ is given by (63).

In the case of a prolate spheroid $(a > b = c)$ we find from (43)

$$N_b = N_c = \frac{2N_a}{N_a - 1},$$

and from (63)

$$R_r = \frac{16\pi}{27} \left(\frac{a}{\lambda}\right)^2 Z_o \left[1 + \left(\frac{2}{1+N_a}\right)^2 \right] \qquad \text{(prolate spheroid)} \tag{65}$$

When $a >> b$, equation (23), gives

$$\frac{1}{1+N_a} \approx \frac{\ln(2a/b)-1}{a^2/b^2}$$

and hence

$$R_r = \frac{16\pi}{27} \left(\frac{a}{\lambda}\right)^2 Z_o \qquad \text{(thin rod)} \tag{66}$$

One can find from any standard book on antennas the following approximate formulas for the radiation resistance of a thin short rod of length a above a ground plane:

$$R_r^{\Delta} = \frac{\pi}{3} \left(\frac{a}{\lambda}\right)^2 Z_o$$

for a triangular current distribution, and

$$R_r^u = \frac{4\pi}{3} \left(\frac{a}{\lambda}\right)^2 Z_o$$

for a uniform current distribution. By comparison, we have $R_r^u > R_r > R_r^\Delta$ which was somewhat expected, since the current distribution that we have used on the thin rod is parabola, as is shown by (16).

VI. Summary

Let us summarize what we have found in this note. We have calculated the following quantities of an electrically-small ellipsoid or a half ellipsoid symmetrically resting on a ground plane:

A_{eq} = equivalent area, the ratio of short-circuit current to the time derivative of the incident displacement current.

$\underline{\underline{\alpha}}_e$ = electric polarizability tensor.

$\underline{\underline{\alpha}}_m$ = magnetic polarizability tensor.

$\underline{p}$ = induced electric dipole moment.

$\underline{m}$ = induced magnetic dipole moment.

f_E = electric-field enhancement factor, the ratio of maximum electric field on the ellipsoid to the incident electric field.

f_H = magnetic-field enhancement factor, the ratio of maximum magnetic field on the ellipsoid to the incident magnetic field.

R_r = radiation resistance.

The results obtained for these quantities are applicable to (1) blade antennas on aircraft; (2) EMP electric-field sensors; (3) field distortions of a ground plane of finite thickness; (4) scattering from protrusions of aircraft or missiles or within an EMP simulator. With respect to the direction of any of the three principal axes of the ellipsoid the following relations exist among the calculated quantities:

$$\frac{A^i_{eq}}{A} = \frac{\alpha^i_e}{V} = f^i_E \tag{I}$$

$$\frac{\alpha^i_m}{V} = \frac{\alpha^i_e}{V-\alpha^i_e} = -f^i_H \tag{II}$$

$$\epsilon \underline{E}_{tip} = \frac{1}{V}\underline{p}, \qquad \underline{H}_{tip} = -\frac{1}{V}\underline{m} \tag{III}$$

$$\frac{R_r}{Z_o} = \frac{\pi}{3} \frac{1}{(\lambda A^i_{eq})^2} \left[2(\alpha^i_e)^2 + (\alpha^j_m)^2 + (\alpha^k_m)^2 \right], \qquad i \neq j \neq k \qquad \text{(IV)}$$

where i, j, k denote the directions of the principal axes; V is the total volume of the antenna (sensor); A is the cross-sectional area perpendicular to the i-axis; α^i_e, α^i_m, etc. are the components of the tensors $\underline{\underline{\alpha}}_e$ and $\underline{\underline{\alpha}}_m$ when the incidnet fields ($\underline{E}$ and $\underline{H}$) are parallel to the i-axis; $\underline{E}_{tip}$, $\underline{H}_{tip}$ are the fields at the tip of the ellipsoid at which point the external magnetic field is tangent to and the external electric field is perpendicular to the ellipsoid; λ is the wavelength; Z_o is the free-space impedance. In the case where the antenna is resting on a ground plane, V will be the volume of the ellipsoidal boss and equation (IV) should be divided by 2 and multiplied by 4 --- the factor 2 is due to radiation only in half-space, while the factor 4 is due to the images.

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6/2/d/85

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