SENSOR AND SIMULATION NOTES

NOTE 195

# The Effect of a Center Conductor on the Resonant Modes of a Spherical Cavity with a Perfectly Conducting Wall

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## ABSTRACT

The effect of a spherically shaped conductor on the resonant modes of a spherical cavity is investigated. The cavity wall and center sphere are considered to be perfectly conducting. A graphic solution technique is used to solve the eigenvalue equations. For a small sphere at the center, the frequency perturbation is seen to vary as the ratio of the sphere volume to the cavity volume and, hence, be quite small. The results of this analysis can be applied to the problem of the effect of a satellite on the resonance modes of a simulator chamber. (This note was originally prepared as Tank Physics Memo #5, September 1972.)

This memo considers the effect that a conducting body at the center of the tank would have on the resonant mode frequencies of the tank. Actually, what we want to show is that the presence of a reasonably sized body will only perturb the frequencies slightly. This turns out to be true. Since both the tank wall and the central sphere are considered perfectly conducting, the frequencies and their shifts are real and no absorption is experienced.

The solutions to the transcendental equations are obtained rather simply through graphical methods for all ratios a/b, where a is the radius of the central sphere and b is the radius of the tank cavity. The variation for small a/b appears to be a volume effect so that the objects shape would not be a considerable factor in such cases.

Considering the electric modes first, the electric field inside of a spherical cavity is described by a summation of terms of the following form (Stratton, 1941; p. 557):

$$E_{\theta n}(\rho) = \frac{1}{\rho}\frac{\partial}{\partial \rho}[\rho j_n(\rho)] \tag{1}$$

where constants in $\rho$ are being neglected, and $\rho = kr$, where k is the complex wave number and r is the radial distance from the center of the cavity. The resonant modes are determined by setting $E_{\theta n}(\rho)$ equal to zero at the cavity surface ($r = b$) and solving for the roots of Equation 1 for all n, i.e., by solving for kb in

$$0 = \frac{\partial}{\partial\rho} [\rho j_n(\rho)]|_{\rho=kb} . \tag{2}$$

The introduction of a perfectly conducting sphere at the center, with radius r = a, complicates the problem only to the extent that the field must also equal zero at r = a. A second solution to the radial equation is required in addition to $j_n(\rho)$. This could be any of the remaining spherical Bessel functions (Abramowitz, 1964), i.e., $y_n(\rho)$, $h_n^{(1)}(\rho)$, or $h_n^{(2)}(\rho)$. We choose the spherical Bessel function of the second kind $y_n(\rho)$ because it has a sinusoidal type of behavior and the solution for the concentric spheres is expected to resemble that of two parallel plates as $a \to b$.

With the addition of this linearly independent solution, Equation 1 becomes

$$E_{n\theta}(\rho) = \frac{1}{\rho}\left\{\frac{\partial}{\partial\rho} [\rho j_n(\rho)] + K \frac{\partial}{\partial\rho} [\rho y_n(\rho)]\right\} \tag{3}$$

where K is a constant with respect to $\rho$. The boundary conditions at r = a, b yield

$$0 = \frac{\partial}{\partial\rho} [\rho j_n(\rho)]|_{\rho=ka} + K \frac{\partial}{\partial\rho} [\rho y_n(\rho)]|_{\rho=ka} \tag{4a}$$

$$0 = \frac{\partial}{\partial\rho} [\rho j_n(\rho)]|_{\rho=kb} + K \frac{\partial}{\partial\rho} [\rho y_n(\rho)]|_{\rho=kb} \tag{4b}$$

Eliminating K, we have

$$\frac{\frac{\partial}{\partial\rho} [\rho j_n(\rho)]|_{\rho=kb}}{\frac{\partial}{\partial\rho} [\rho y_n(\rho)]|_{\rho=kb}} = \frac{\frac{\partial}{\partial\rho} [\rho j_n(\rho)]|_{\rho=ka}}{\frac{\partial}{\partial\rho} [\rho y_n(\rho)]|_{\rho=ka}} \tag{5}$$

For simplicity, define

$$f_n(\rho) = \frac{\frac{\partial}{\partial\rho} [\rho j_n(\rho)]}{\frac{\partial}{\partial\rho} [\rho y_n(\rho)]} \tag{6}$$

then Equation 5 becomes

$$f_n(kb) = f_n(ka) \tag{7}$$

or, with $x = kb$,

$$f_n(x) = f_n\left(\frac{a}{b}x\right). \tag{8}$$

Equation 8 is in a form which permits a graphical solution once $f_n(x)$ is calculated. By plotting $f_n(x)$ on a graph with a logarithmic x axis, one can find the solution by placing an identical graph over it, displaced along the x axis by a/b, and reading the values of $x = kb$ corresponding to the points of intersection.

In order to calculate $f(x)$, the functions $j_n(\rho)$ and $y_n(\rho)$ must be known. They can be calculated from Rayleigh's formulas (Abramowitz, 1964):

$$j_n(\rho) = \rho^n\left(-\frac{1}{\rho}\frac{d}{d\rho}\right)^n \frac{\sin\rho}{\rho} \tag{9}$$

$$y_n(\rho) = -\rho^n\left(-\frac{1}{\rho}\frac{d}{d\rho}\right)^n \frac{\cos\rho}{\rho} \tag{10}$$

In general, $j_n(\rho)$ will be of the form

$$j_n(\rho) = P_n\left(\frac{1}{\rho}\right)\sin\rho - Q_n\left(\frac{1}{\rho}\right)\cos\rho \tag{11}$$

where $P_n\left(\frac{1}{\rho}\right)$ and $Q_n\left(\frac{1}{\rho}\right)$ are polynomials of $\left(\frac{1}{\rho}\right)$ and should not be confused with Legendre polynomials. The function $y_n(\rho)$ can be found from $j_n(\rho)$ by replacing $\sin\rho$ with $-\cos\rho$ and $\cos\rho$ with $\sin\rho$. Then $y_n(\rho)$ is of the form

$$y_n(\rho) = -P_n\left(\frac{1}{\rho}\right)\cos\rho - Q_n\left(\frac{1}{\rho}\right)\sin\rho \tag{12}$$

Equation 9 can be used to generate $j_0(\rho)$ and $j_1(\rho)$. The higher order functions can be generated from these through (Abramowitz; 1964)

$$j_{n+1}(\rho) = (2n+1)\frac{j_n(\rho)}{\rho} - j_{n-1}(\rho) \tag{13}$$

The first six functions are

$$j_0(\rho) = \frac{\sin\rho}{\rho} \tag{14}$$

$$j_1(\rho) = \left(\frac{1}{\rho^2}\right)\sin\rho - \left(\frac{1}{\rho}\right)\cos\rho \tag{15}$$

$$j_2(\rho) = \left(\frac{3}{\rho^3} - \frac{1}{\rho}\right)\sin\rho - \left(\frac{3}{\rho^2}\right)\cos\rho \tag{16}$$

$$j_3(\rho) = \left(\frac{15}{\rho^4} - \frac{6}{\rho^2}\right)\sin\rho - \left(\frac{15}{\rho^3} - \frac{1}{\rho}\right)\cos\rho \tag{17}$$

$$j_4(\rho) = \left(\frac{105}{\rho^5} - \frac{45}{\rho^3} + \frac{1}{\rho}\right)\sin\rho - \left(\frac{105}{\rho^4} - \frac{10}{\rho^2}\right)\cos\rho \tag{18}$$

$$j_5(\rho) = \left(\frac{945}{\rho^6} - \frac{420}{\rho^4} + \frac{15}{\rho^2}\right)\sin\rho - \left(\frac{945}{\rho^5} - \frac{105}{\rho^3} + \frac{1}{\rho}\right)\cos\rho \tag{19}$$

Similarly, the corresponding $y_n(\rho)$ are given by

$$y_0(\rho) = -\frac{\cos\rho}{\rho} \tag{20}$$

$$y_1(\rho) = -\left(\frac{1}{\rho^2}\right)\cos\rho - \left(\frac{1}{\rho}\right)\sin\rho \tag{21}$$

$$y_2(\rho) = -\left(\frac{3}{\rho^3} - \frac{1}{\rho}\right)\cos\rho - \left(\frac{3}{\rho^2}\right)\sin\rho \tag{22}$$

$$y_3(\rho) = -\left(\frac{15}{\rho^4} - \frac{6}{\rho^2}\right)\cos\rho - \left(\frac{15}{\rho^3} - \frac{1}{\rho}\right)\sin\rho \tag{23}$$

$$y_4(\rho) = -\left(\frac{105}{\rho^5} - \frac{45}{\rho^3} + \frac{1}{\rho}\right)\cos\rho - \left(\frac{105}{\rho^4} - \frac{10}{\rho^2}\right)\sin\rho \tag{24}$$

$$y_5(\rho) = -\left(\frac{945}{\rho^6} - \frac{420}{\rho^4} + \frac{15}{\rho^2}\right)\cos\rho - \left(\frac{945}{\rho^5} - \frac{105}{\rho^3} + \frac{1}{\rho}\right)\sin\rho$$

In order to calculate $f_n(x)$, we must have

$$\frac{\partial}{\partial\rho}\,[\rho j_n(\rho)] = p_n \sin\rho - q_n \cos\rho \tag{26}$$

and

$$\frac{\partial}{\partial\rho}\,[\rho y_n(\rho)] = -\,p_n \cos\rho - q_n \sin\rho \tag{27}$$

where

$$p_n = P_n + \rho\frac{\partial P_n}{\partial\rho} + \rho Q_n \tag{28}$$

$$q_n = Q_n + \rho\frac{\partial Q_n}{\partial\rho} - \rho P_n \tag{29}$$

Thus, $\partial/\partial\rho\ [\rho j_n(\rho)]$ has the same form as $j_n(\rho)$. The same is true for $\partial/\partial\rho\ [\rho y_n(\rho)]$ and $y_n(\rho)$. Equations 14 through 25 show that $P_n(1/\rho)$ and $Q_n(1/\rho)$ are polynomials of $1/\rho$, that the highest order term is contained in $Q_n(1/\rho)$ and the power is $n + 1$. The highest order term in $P_n(1/\rho)$ has the power $n$. $Q_n(1/\rho)$ and $P_n(1/\rho)$ each have either odd or even powers of $(1/\rho)$. The lowest power is unity, i.e., $1/\rho$. The same statements can be made for $p_n(1/\rho)$ and $q_n(1/\rho)$ with the exception that the lowest power is zero, i.e., a constant.

By Equations 26 and 27

$$f_n(x) = -\,\frac{p_n(1/x)\sin x - q_n(1/x)\cos x}{p_n(1/x)\cos x + q_n(1/x)\sin x} \tag{30}$$

or

$$f_n(x) = \frac{q_n(1/x) - p_n(1/x)\tan x}{q_n(1/x)\tan x + p_n(1/x)} \tag{31}$$

By Equations 28 and 29 we have for n = 1 through 5:

$$p_1(1/x) = -\,(1/x)^2 + 1 \tag{32}$$

$$q_1(1/x) = -\,(1/x) \tag{33}$$

$$p_2(1/x) = -6(1/x)^3 + 3(1/x) \tag{34}$$

$$q_2(1/x) = -6(1/x)^2 + 1 \tag{35}$$

$$p_3(1/x) = -45(1/x)^4 + 21(1/x)^2 - 1 \tag{36}$$

$$q_3(1/x) = -45(1/x)^3 + 6(1/x) \tag{37}$$

$$p_4(1/x) = -420(1/x)^5 + 195(1/x)^3 - 10(1/x) \tag{38}$$

$$q_4(1/x) = -420(1/x)^4 + 55(1/x)^2 - 1 \tag{39}$$

$$p_5(1/x) = -4722(1/x)^6 + 1205(1/x)^4 - 120(1/x)^2 + 1 \tag{40}$$

$$q_5(1/x) = -4722(1/x)^5 + 630(1/x)^3 - 15(1/x) \tag{41}$$

Using Equations 32 through 35, we have for n = 1 and 2,

$$f_1(x) = -\frac{(-1/x) - (-1/x^2 + 1)\tan x}{(-1/x)\tan x + (-1/x^2 + 1)} \tag{42}$$

$$f_2(x) = -\frac{(-6/x^2 + 1) - (-6/x^3 + 3/x)}{(-6/x^2 + 1)\tan x + (-6/x^3 + 3/x)} \tag{43}$$

or

$$f_1(x) = -\frac{x - (1 - x^2)\tan x}{x\tan x + (1 - x^2)} \tag{44}$$

$$f_2(x) = -\frac{x(x^2 - 6) - (3x^2 - 6)\tan x}{x(x^2 - 6)\tan x + (3x^2 - 6)} \tag{45}$$

With $x \ll 1$, these functions become

$$f_1(x) \simeq -x^3/3 \tag{46}$$

$$f_2(x) \simeq x^3/3 \tag{47}$$

and, for $x \gg 1$,

$$f_1(x) \simeq -\frac{1 - x\tan x}{\tan x - x} \tag{48}$$

$$f_2(x) \simeq -\frac{x - 3\tan x}{x\tan x + 3} \tag{49}$$

Figure 1 is a plot $2/\pi$ arctan $(f_1(x))$ for $1 \leq x < 20$. Figure 2 is a plot of $2/\pi$ arctan $(f_2(x))$ for the same range. Figure 3 shows these functions for $x < 1$. These functions are plotted instead of $f_n(x)$ since they are bounded between +1 and -1 instead of $\pm\infty$. They can be used more easily in the graphical solution of Equation 8 than $f_1(x)$ and $f_2(x)$. Figures 4 and 5 show some of these roots as a function of a/b for n = 1 and 2 respectively. The horizontal lines indicate the values of the roots for a = 0 (no center sphere). The first six roots are shown.

It can be seen that as the inner sphere's radius increases from small values, the magnitude of kb and, hence, k decreases for each root. Except for the first root, further enlargening of the ratio a/b causes the roots to pass through a minimum and approach infinity as $a \to b$. The first root, however, continues to decrease in magnitude and has a finite limit at a = b. We will show that this limit is $kb = \sqrt{n(n + 1)}$, where n is the order of the spherical Bessel functions involved. Before doing so, we will make some more observations. First, note that for n = 2, the magnitudes of the roots change more slowly with a/b than for n = 1. This property can probably be extended to higher values of n since the properties of the spherical Bessel functions of higher order n do not make any drastic changes. The n = 1 mode is then the most sensitive of the electric modes. For a given n, the higher roots are initially more sensitive to increasing a/b. The behavior of the roots for small a/b will be investigated in more detail in a later paragraph. The important thing to note is that the variations are small for small a/b and that the introduction of a sphere into the center of the cavity does not greatly change the mode frequencies unless a/b is on the order of 0.3.

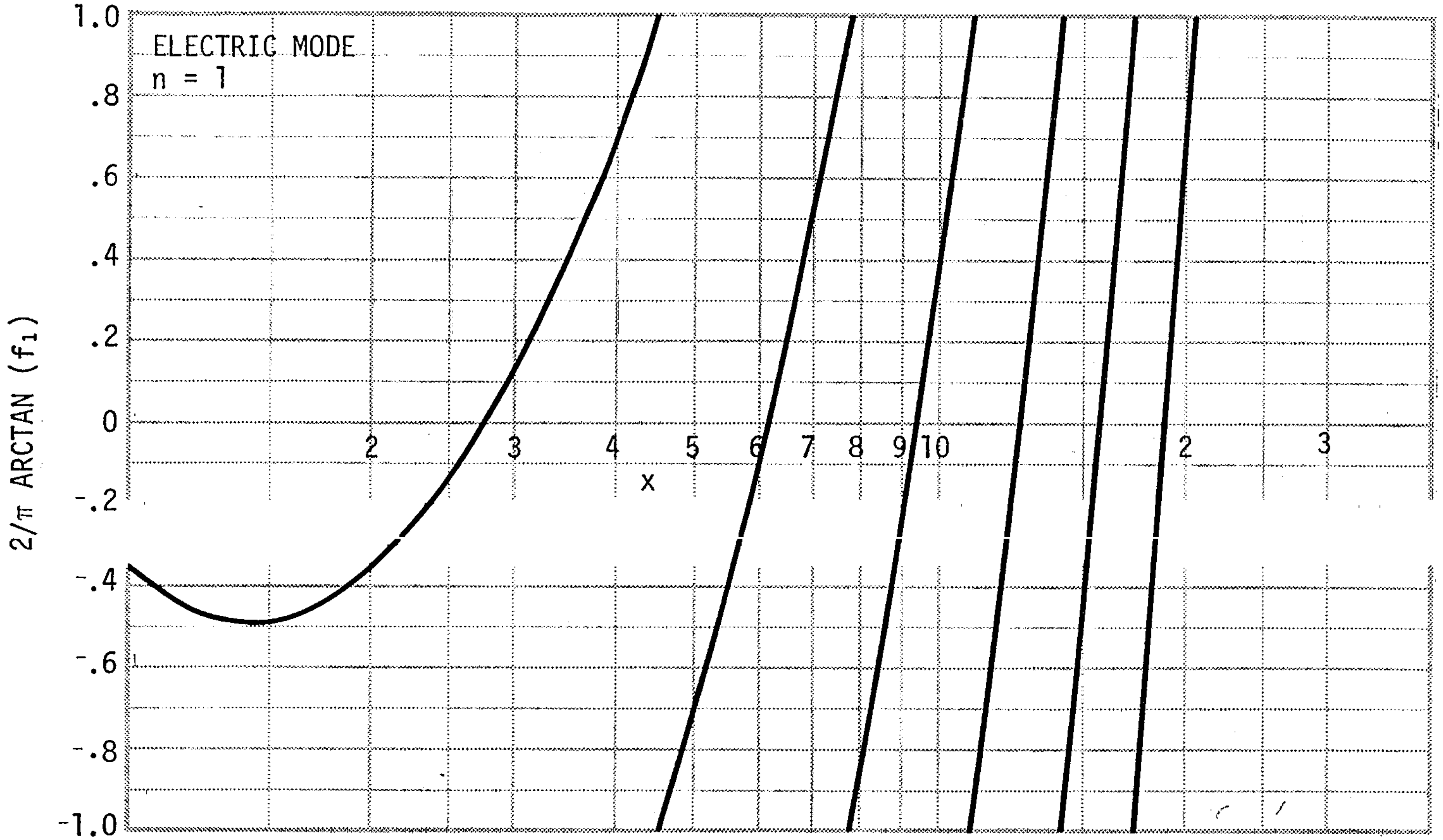


Figure 1. $2/\pi$ ARCTAN $(f_1(x))$ vs x for $x > 1$.

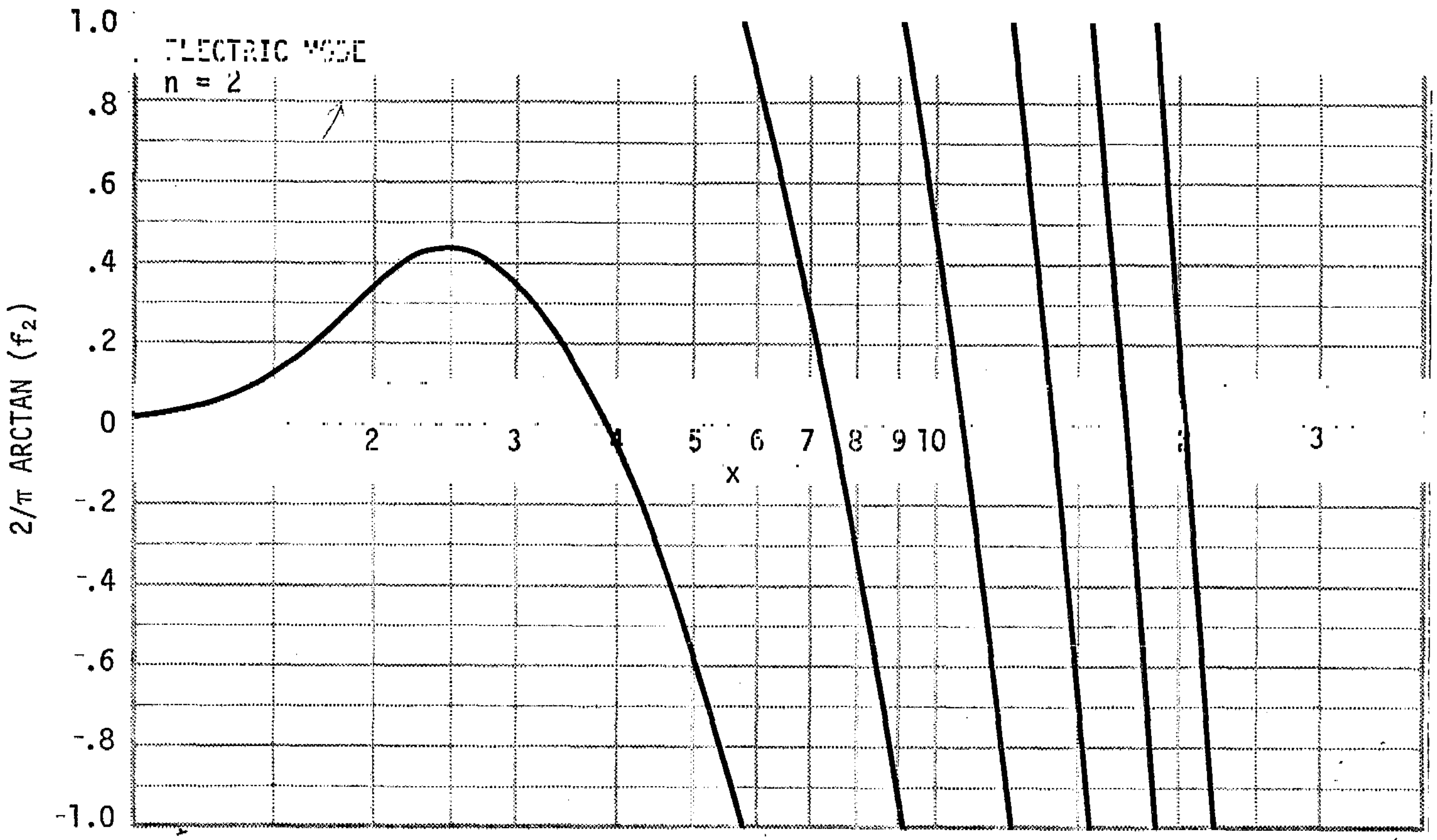


Figure 2. $2/\pi$ ARCTAN $(f_2(x))$ vs x for x > 1.

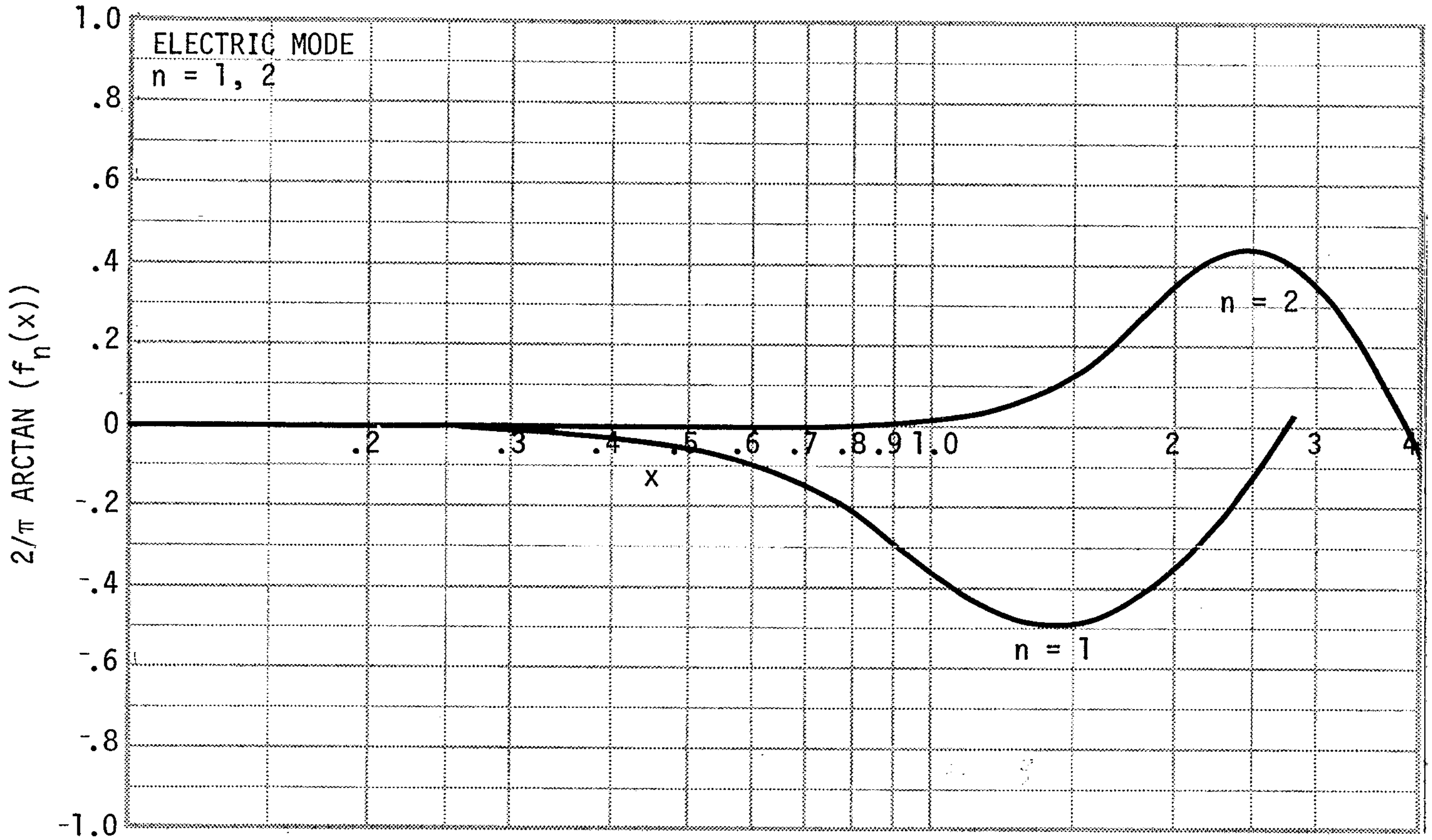


Figure 3. $2/\pi$ ARCTAN $(f_n(x))$ vs x for n = 1, 2 and x < 1.

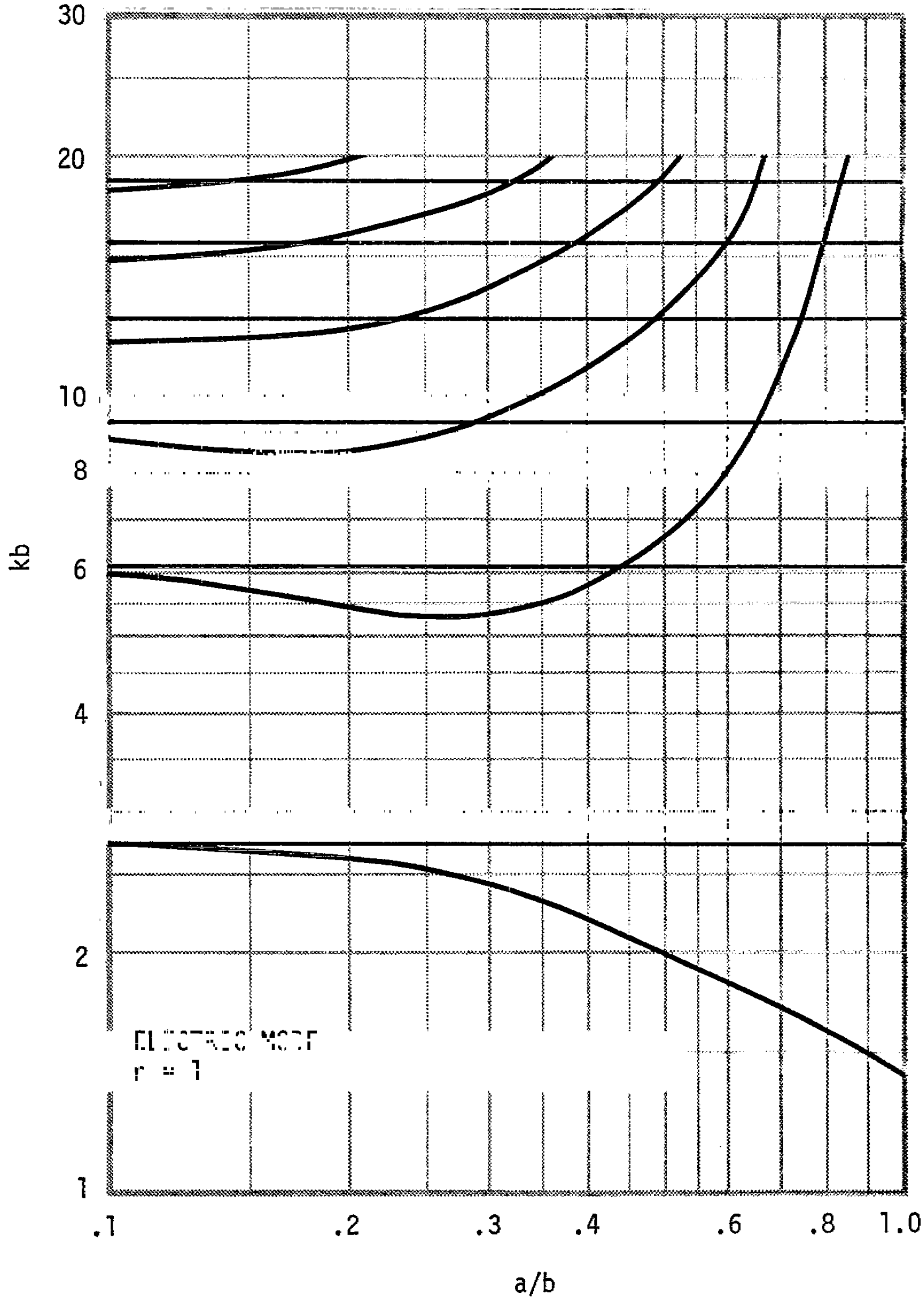


Figure 4. Roots of mode equation vs ratio of central sphere radius to tank radius.

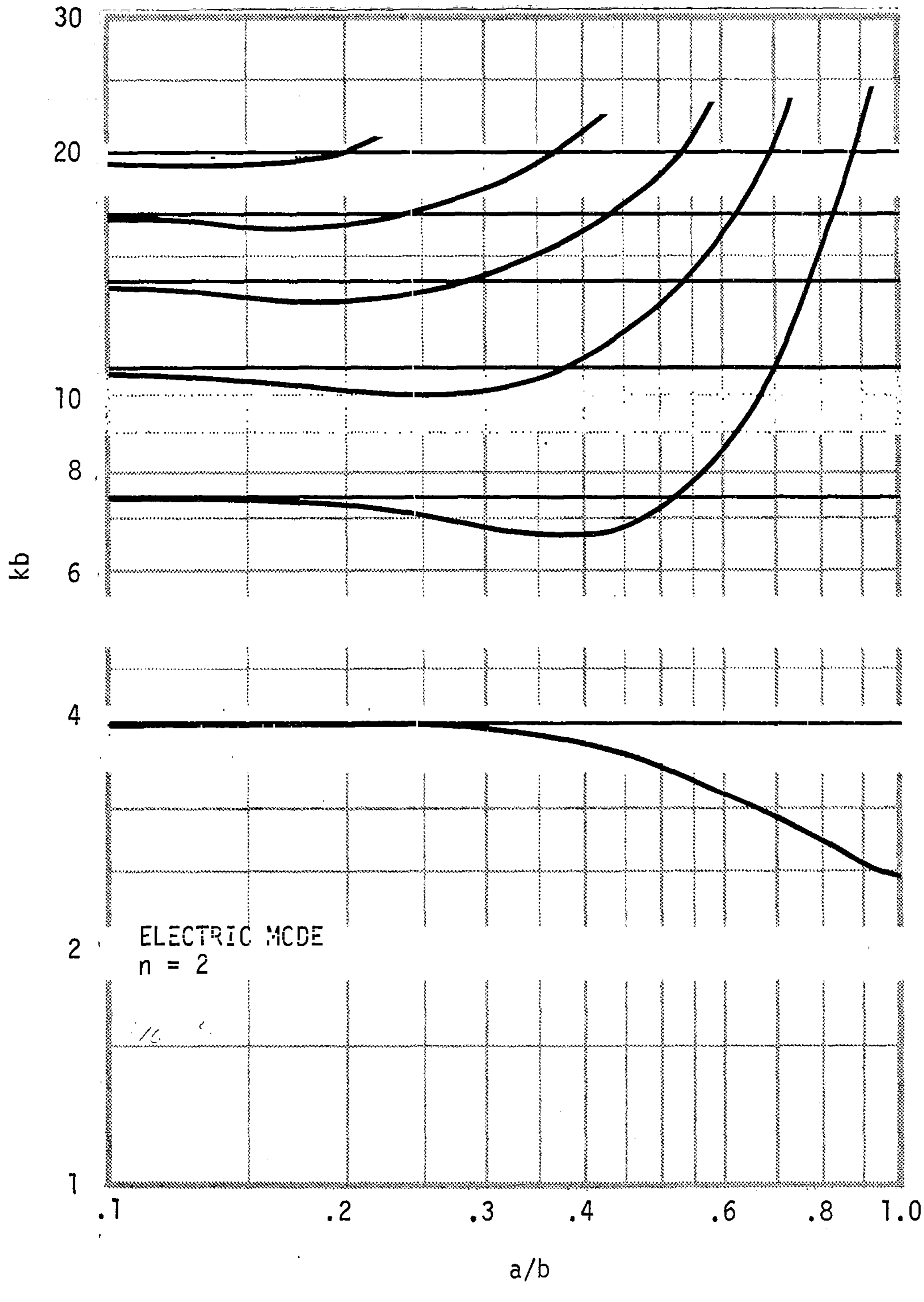


Figure 5. Roots of mode equation vs ratio of central sphere radius to tank radius.

We will now investigate the sensitivity of the roots of Equation 8 to the variation of a/b for small values of a/b. For $(a/bx) \ll 1$, Equation 8 reduces to

$$\frac{x - (1 - x^2)\tan x}{x\tan x + (1 - x^2)} \simeq \frac{1}{3}\left(\frac{a}{b}x\right)^3 \tag{50}$$

The smallest value of x is about 2.7. Assuming $x^2 \gg 1$, this equation reduces to

$$1 + x\tan x \simeq \frac{1}{3}\left(\frac{a}{b}x\right)^3 (\tan x - x) \tag{51}$$

Let $x_0$ denote the roots of

$$1 + x\tan x = 0 \tag{52}$$

These roots are approximately the solutions for the radial modes when a = 0. Since the roots are not expected to vary greatly for small a/b, define

$$x = x_0 + y \tag{53}$$

where $y \ll x_0$. Then, using a Taylor expansion

$$\tan x \simeq \tan x_0 + y\sec^2 x_0 \ , \tag{54}$$

we find

$$1 + (x_0 + y)(\tan x_0 + y\sec^2 x_0) \simeq \frac{1}{3}\left(\frac{a}{b}x_0\right)^3 (1 + 3y/x_0)(\tan x_0 - 1 + y\sec^2 x_0) \tag{55}$$

or, ignoring powers of y greater than one, and $y/x_0$,

$$y \simeq -\frac{1}{3}\left(\frac{a}{b}x_0\right)^3 \frac{1 - \tan x_0}{\tan x_0 + \sec^2 x_0\left[x_0 - \frac{1}{3}\left(\frac{a}{b}x_0\right)^3\right]} \tag{56}$$

But

$$\tan x_0 \simeq -1/x_0 \tag{57}$$

and

$$\sec^2 x_0 = 1 - \tan^2 x_0 \simeq 1 + 1/x_0^2 \simeq 1 \tag{58}$$

so that

$$y \simeq -\frac{1}{3}\left(\frac{a}{b}x_0\right)^3 \left[\frac{1}{-1/x_0 + x_0}\right]$$

$$y \simeq -\frac{1}{3}\left(\frac{a}{b}\right)^3 x_0^2 \tag{59}$$

This equation shows that, for the first electric mode (n = 1), the variation of the resonance frequency is a volume effect, i.e.,

$$\Delta kb = y$$

$$\Delta\omega = \frac{cy}{b} = -\frac{1}{3}\left(\frac{c}{b}\right)\left(\frac{a}{b}\right)^3 x_0^2 \tag{60}$$

where $\Delta\omega$ is the change in mode frequency and c is the velocity of light. The frequency change is quadratic in $x_0$, i.e., the higher roots change more quickly. This volume effect can be shown for the second electric mode (n = 2) and probably hold true for larger n.

We now proceed to show that the first root has a finite value in the limit that a = b. The wave equation for the radial part of the solution is (Stratton, 1941; p. 400).

$$r^2 \frac{d^2E}{dr^2} + 2r\frac{dE}{dr} + [k^2r^2 - n(n + 1)]E = 0 \tag{61}$$

As $a \rightarrow b$, the radial field becomes rather uniform, as between two parallel plates, and the derivative terms become negligible, leaving

$$kr = \sqrt{n(n + 1)} \tag{62}$$

Thus, for n = 1, 2, we expect the roots $kb = \sqrt{2}, \sqrt{6}$ to appear as a limit in Figures 4 and 5, which they do.

The magnetic mode can be analyzed in the same manner, and is, in fact, much simpler because no derivatives of the spherical Bessel functions are involved. The modes, kb, are defined by

$$E_{n\theta} = j_n(kr) + Ky_n(kr) = 0 \tag{63}$$

at r = a and b, leading to

$$\frac{j_n(kb)}{y_n(kb)} = \frac{j_n(ka)}{y_n(ka)} \tag{64}$$

Defining $g_n(x)$ in a manner analogous to $f_n(x)$ for the electric modes, Equation 64 becomes

$$g_n(x) = g_n\left(\frac{a}{b}x\right) \tag{65}$$

For n = 1 and 2, we have

$$g_1(x) = -\frac{\tan x - x}{1 + x\tan x} \tag{66}$$

$$g_2(x) = -\frac{(3 - x^2)\tan x - 3x}{(3 - x^2) + 3x\tan x} \tag{67}$$

Figure 6 shows $2/\pi$ arctan $(g_1(x))$ for $1 \leq x < 20$. Figure 7 shows the same for $x < 1$. Figure 8 shows $2/\pi$ arctan $(g_2(x))$. The roots of Equation 65 are shown in Figures 9 and 10 for n = 1 and 2 respectively, as a function of a/b. All of the roots asymptotically approach infinity as $a \to b$. An analysis similar to that performed for $f_1(x)$ shows that the roots vary as $(a/b)^3$ for small (a/b). The higher roots and functions with lower n are most sensitive, as with the electric mode.

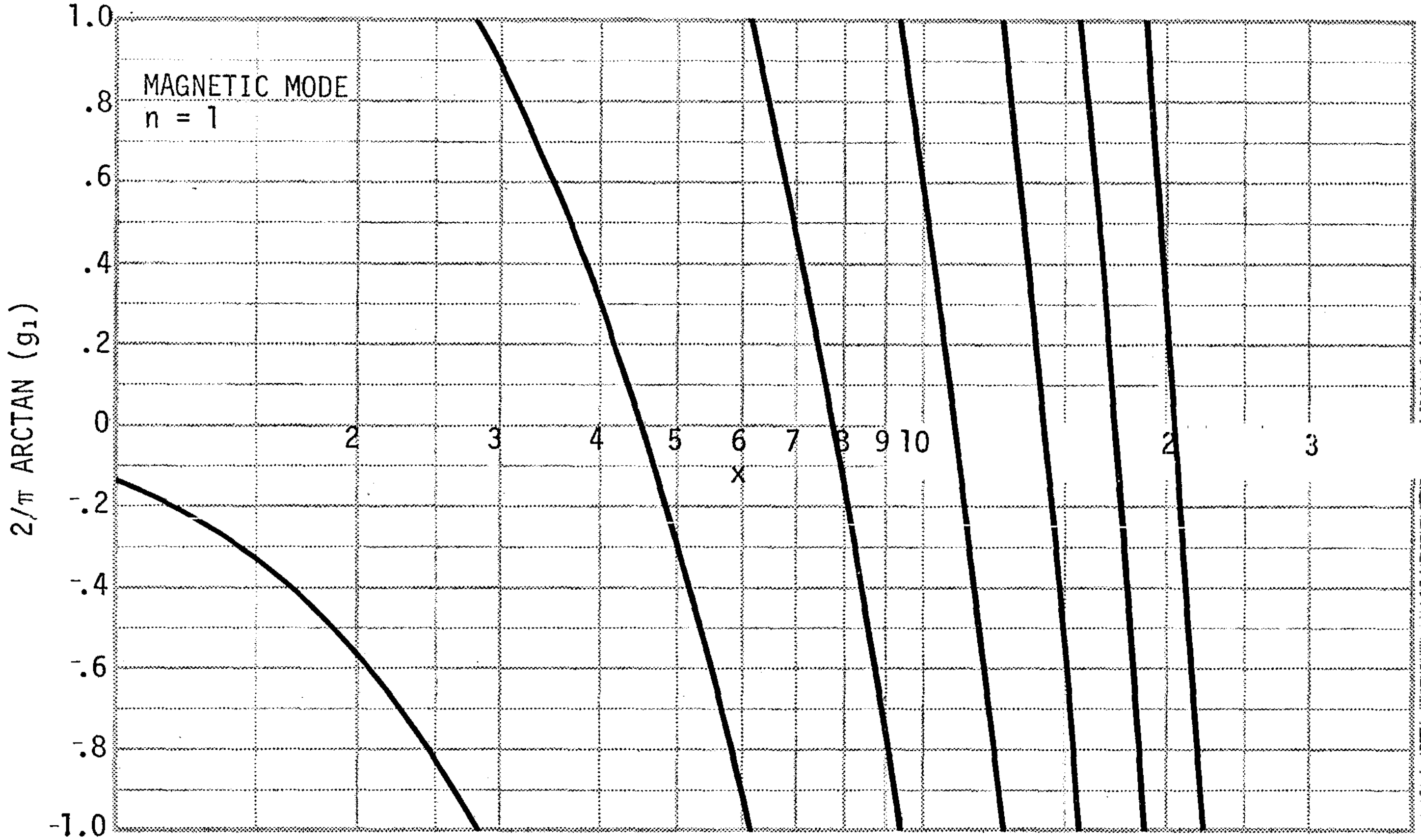


Figure 6. $2/\pi$ ARCTAN $(g_1(x))$ vs x for x > 1.

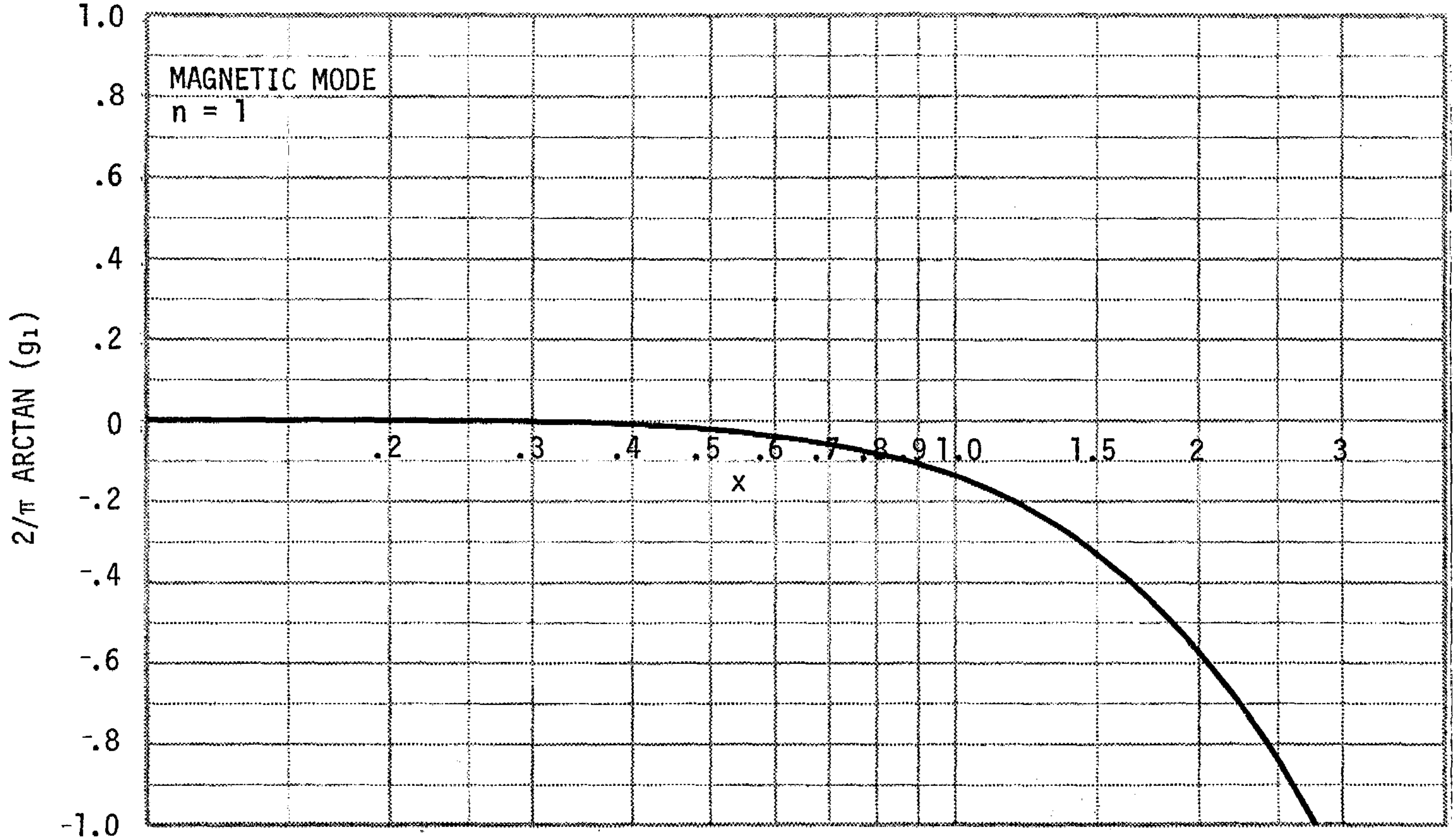


Figure 7. $2/\pi$ ARCTAN $(g_1(x))$ vs x for $x < 1$.

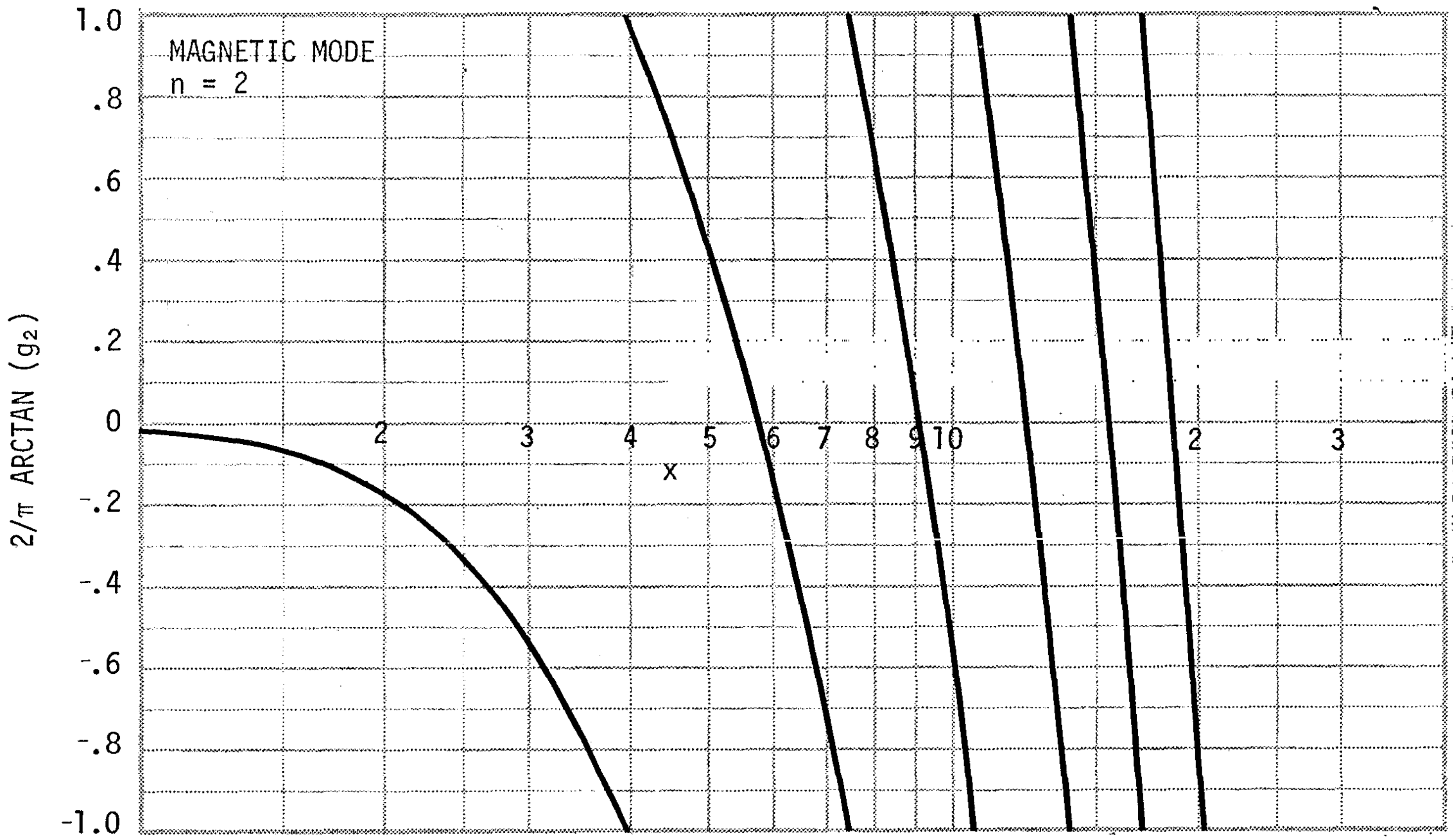


Figure 8. $2/\pi$ ARCTAN $(g_2(x))$ vs x for x > 1.

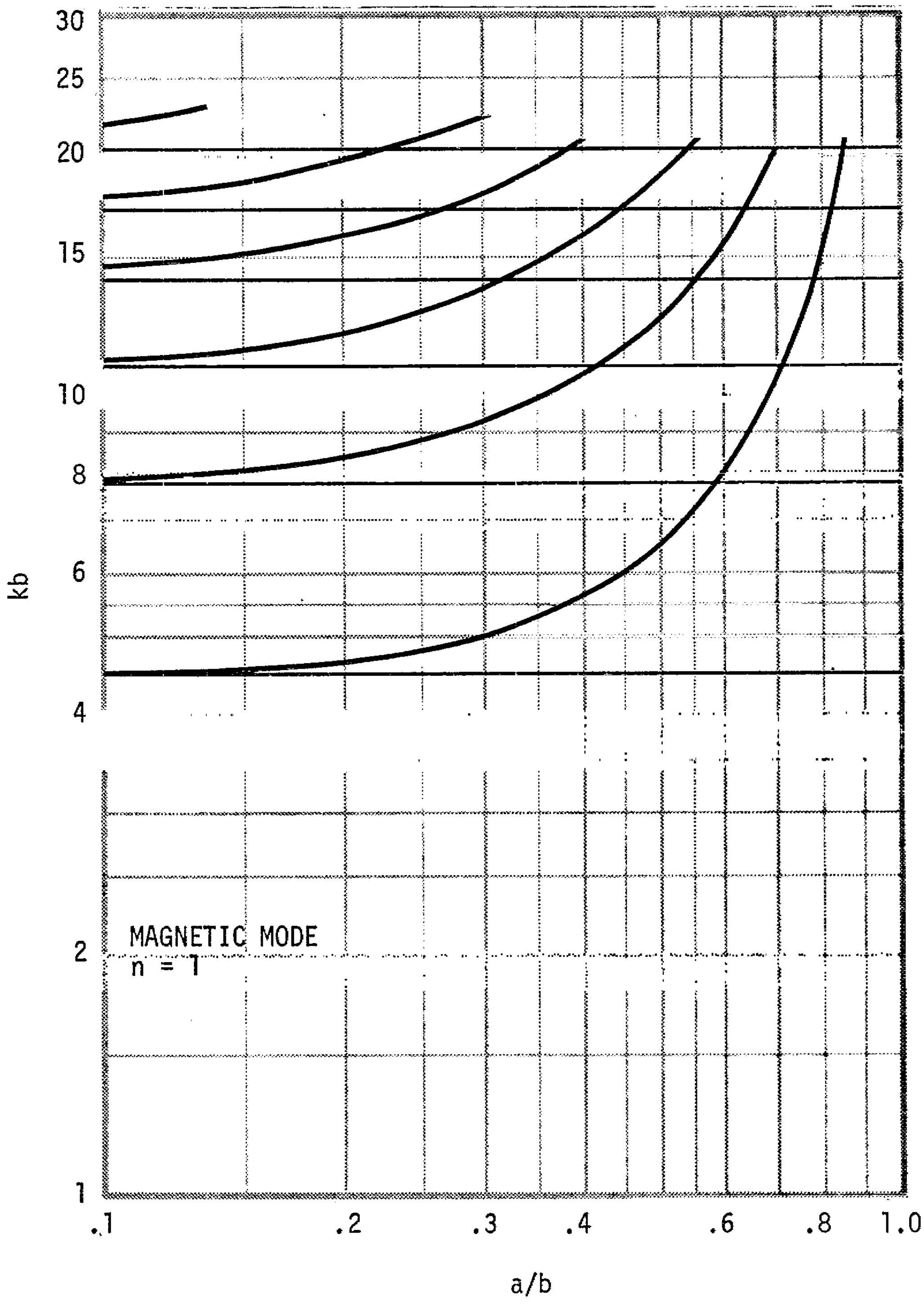


Figure 9. Roots of mode equation vs ratio of central sphere radius to tank radius.

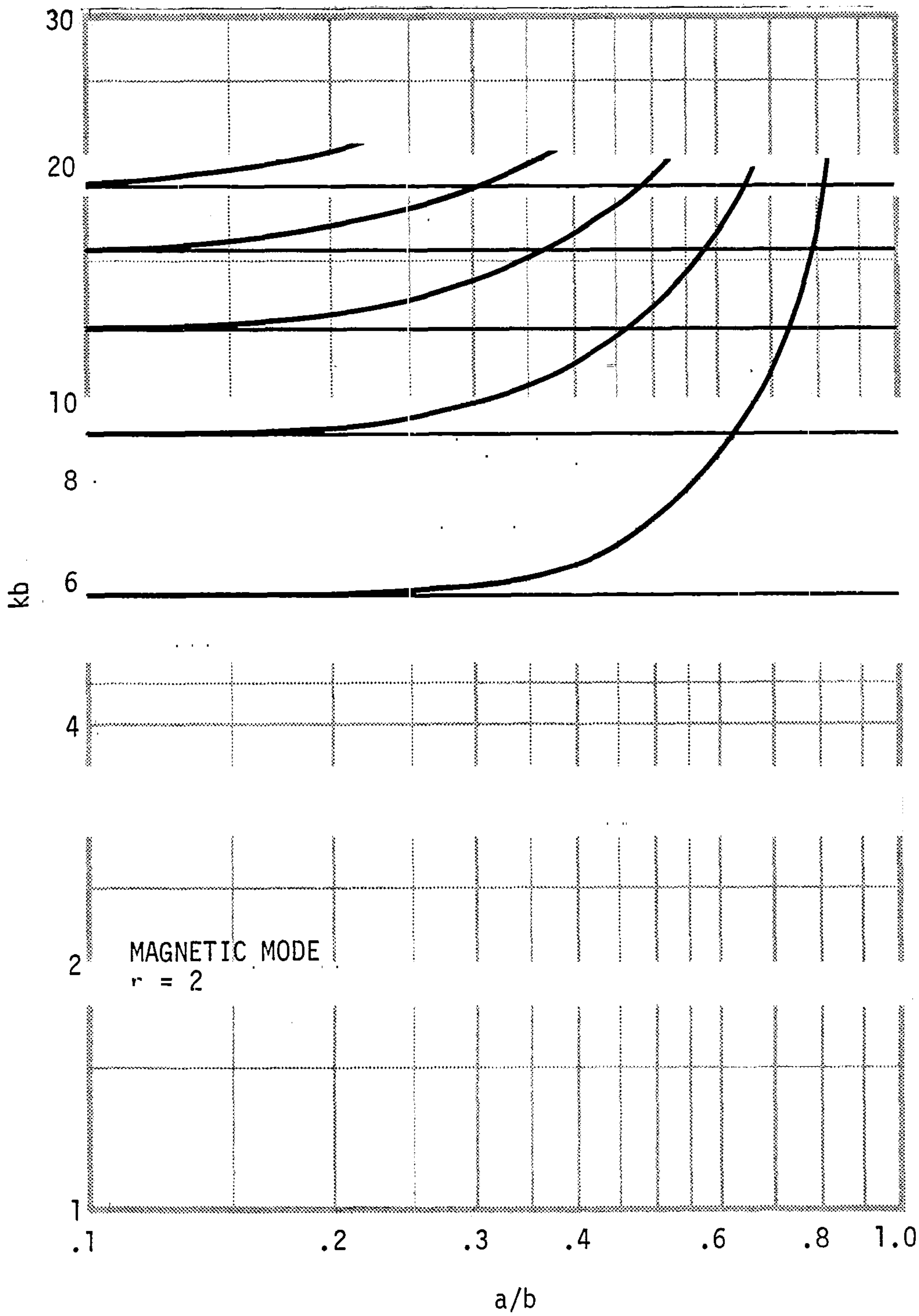


Figure 10. Roots of mode equation vs ratio of central sphere radius to tank radius.

# REFERENCES

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