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Isolated Capacitance and Equivalent Radius of the Rectangular Parallelopiped

by

T. H. Shumpert
Auburn University
Auburn, Alabama 36830

and

D. J. Galloway
Dynetics, Inc.
Huntsville, Alabama

ABSTRACT

In order to predict the increase in the capacitance of a space system when inserted in a system-generated electromagnetic pulse (SGEMP) simulator, it is useful to have information concerning the isolated capacitance of the space system. This note attempts to provide pertinent data which may be used in obtaining preliminary values for the isolated capacitance of the often irregularly shaped space system. The isolated capacitances and equivalent radii of perfectly conducting parallelopipeds is calculated using the method of moments, and this data is presented and compared graphically with the capacitances of certain bounding geometries to indicate their validity. "It was related by Kirchhoff, that shortly before his death Dirichlet solved the problem of the distribution of electricity on a rectangular parallelopiped. If so, the solution has been lost."[1]

INTRODUCTION

In discussing the simulation of system generated EMP in exoatmospheric conditions, one important aspect is the electrical effect of the vacuum chamber utilized as a necessary part of the simulator. The capacitance to the chamber walls, cavity resonances, and reflection of higher frequencies from the cavity wall are some of the electromagnetic interactions of the space system with the test chamber which must be considered [2].

This note addresses itself to the problem of the system capacitance. In free space far from any other objects the space system has a certain capacitance to infinity. The simulator chamber inherently increases this capacitance thereby changing the low frequency characteristics of the space system. The first step towards obtaining a value for this capacitance between the space system and the test chamber is to have a knowledge of the capacitance of the space system to infinity. In this note, the isolated free space capacitance of perfectly conducting rectangular parallelopipeds is determined and appropriate equivalent radii are presented in graphical form. It is believed that this data would provide useful estimates of the isolated capacitance of the actual space systems to be tested.

Rectangular Plate: Formulation and Numerical Results

A charged rectangular conducting plate is shown in Figure 1. A cartesian coordinate system is shown centered on the plate, which has dimensions of 2a x 2b. Let $\rho(x',y')$ represent the surface charge density on the plate, assumed to have zero thickness. Following the same general procedures as did Harrington [3] for the charged conducting square, the isolated capacitance of this rectangular plate is

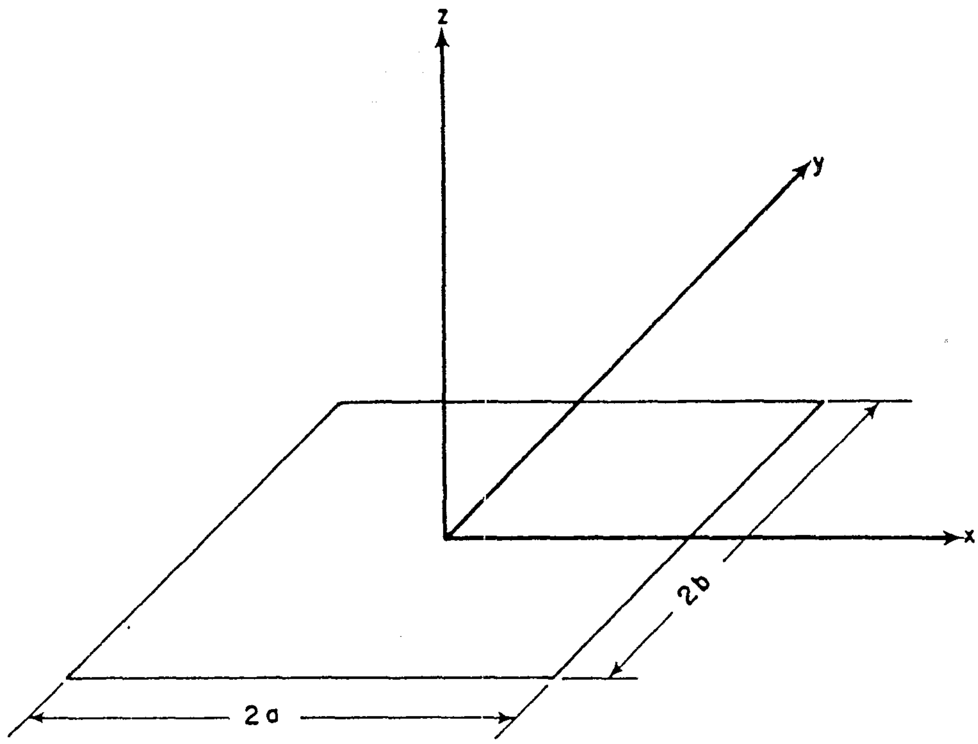

Figure 1. Charged Rectangular Conducting Plate.

$$C \equiv \frac{Q}{V} = \frac{\int_{-a}^{a} \int_{-b}^{b} \rho(x',y')\, dx'\, dy'}{\int_{-a}^{a} \int_{-b}^{b} \frac{\rho(x',y')}{4\pi\varepsilon_0 r}\, dx'\, dy'} \quad , \tag{1}$$

where

$$r = \sqrt{(x-x')^2 + (y-y')^2} \quad . \tag{2}$$

Through the method of moments, divide the rectangular plate into N square subsections, Δs_n; represent the charge density by

$$\rho(x',y') = \sum_{n=1}^{N} \alpha_n f_n \quad , \tag{3}$$

where the α_n's are the unknown coefficients and

$$f_n = \left\{ \begin{array}{ll} 1 & \text{on} \quad \Delta s_n \\ 0 & \text{elsewhere} \end{array} \right\} \tag{4}$$

and satisfy equation (1) at the midpoints of the subsections, (x_m, y_m). These operations result in

$$V = \sum_{n=1}^{N} \ell_{mn}\, \alpha_n \qquad m = 1, 2, \ldots, N \tag{5}$$

$$Q = \sum_{n=1}^{N} \alpha_n\, \Delta s_n \quad , \tag{6}$$

with

$$\ell_{mn} = \int_{\Delta x_n} \int_{\Delta y_n} \frac{1}{4\pi\varepsilon_0 r_m}\, dx'\, dy' \tag{7}$$

$$r_m = \sqrt{(x_m - x')^2 + (y_m - y')^2} \quad . \tag{8}$$

ℓ_{mn} is the potential at the center of Δs_m due to a uniform charge density of unit amplitude over Δs_n. Thus,

$$C = \frac{Q}{V} = \sum_{n=1}^{N} \sum_{m=1}^{N} \left[\ell_{mn} \right]^{-1}_{mn} \left(\frac{4ab}{N} \right) \tag{9}$$

In order to evaluate ℓ_{mn}, treat the non-diagonal terms as those which would arise from point charges and analytically evaluate the self terms [2], such that

$$\ell_{nn} \simeq \frac{1.7628}{\pi \ \varepsilon_0} \sqrt{\frac{ab}{N}} \tag{10}$$

$$\ell_{mn} \simeq \frac{ab}{N\pi\varepsilon_0} \left(\frac{1}{r_{mn}} \right) , \tag{11}$$

where

$$r_{mn} = \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2} \quad . \tag{12}$$

Reitan and Higgins [4] applied numerical techniques to the charged plate many years ago. Figure 2 illustrates the excellent agreement between this moment method formulation and their crude "sub-areas" approach.

Rectangular Parallelopiped: Formulation

Consider the rectangular parallelopiped, or box, composed of six individual plates as shown in Figure 3. Let $\rho(s')$ be the surface charge density on the box, such that the electrostatic potential at any point in space is

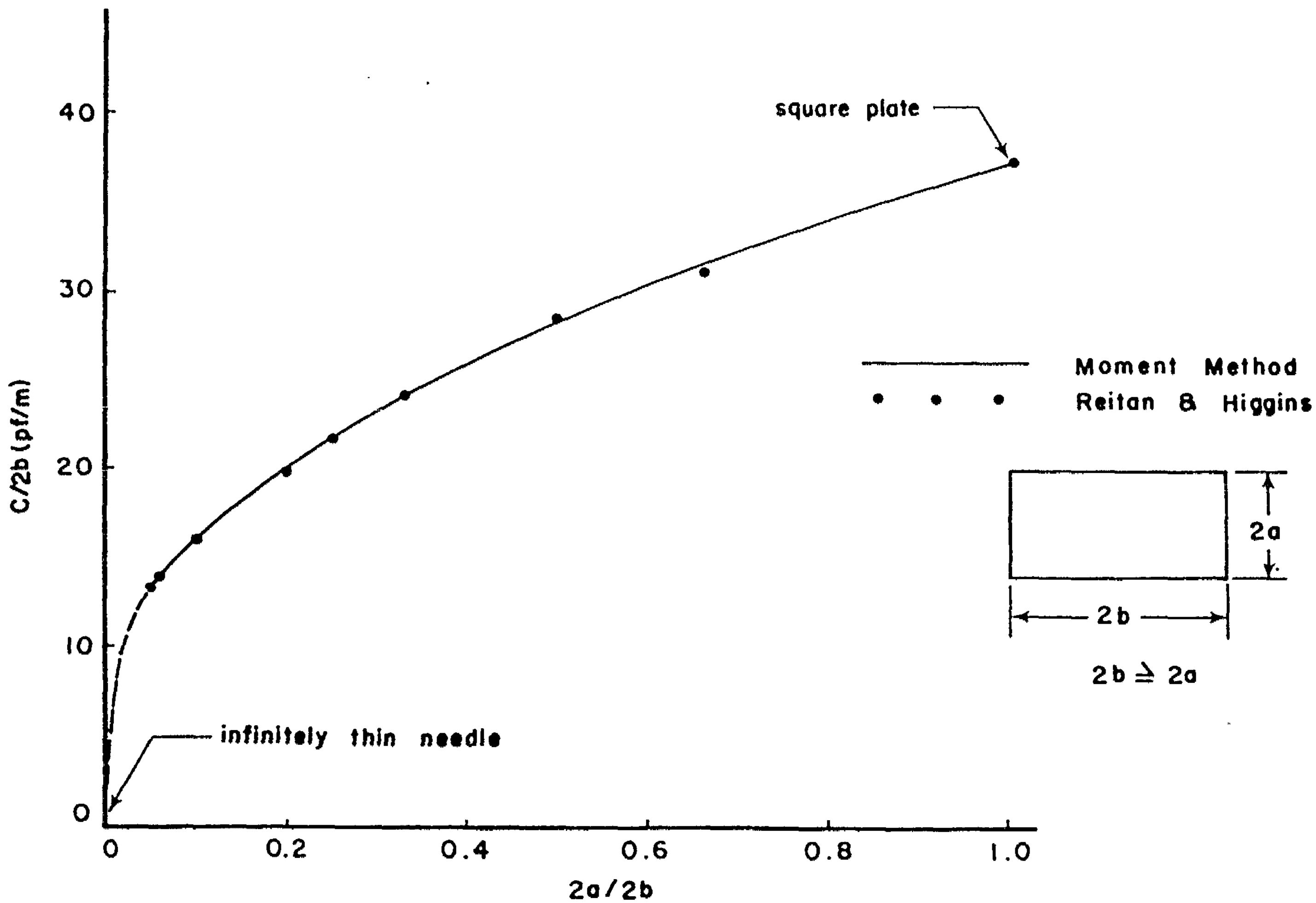

Figure 2. Normalized Capacitance of Rectangular Conducting Plate.

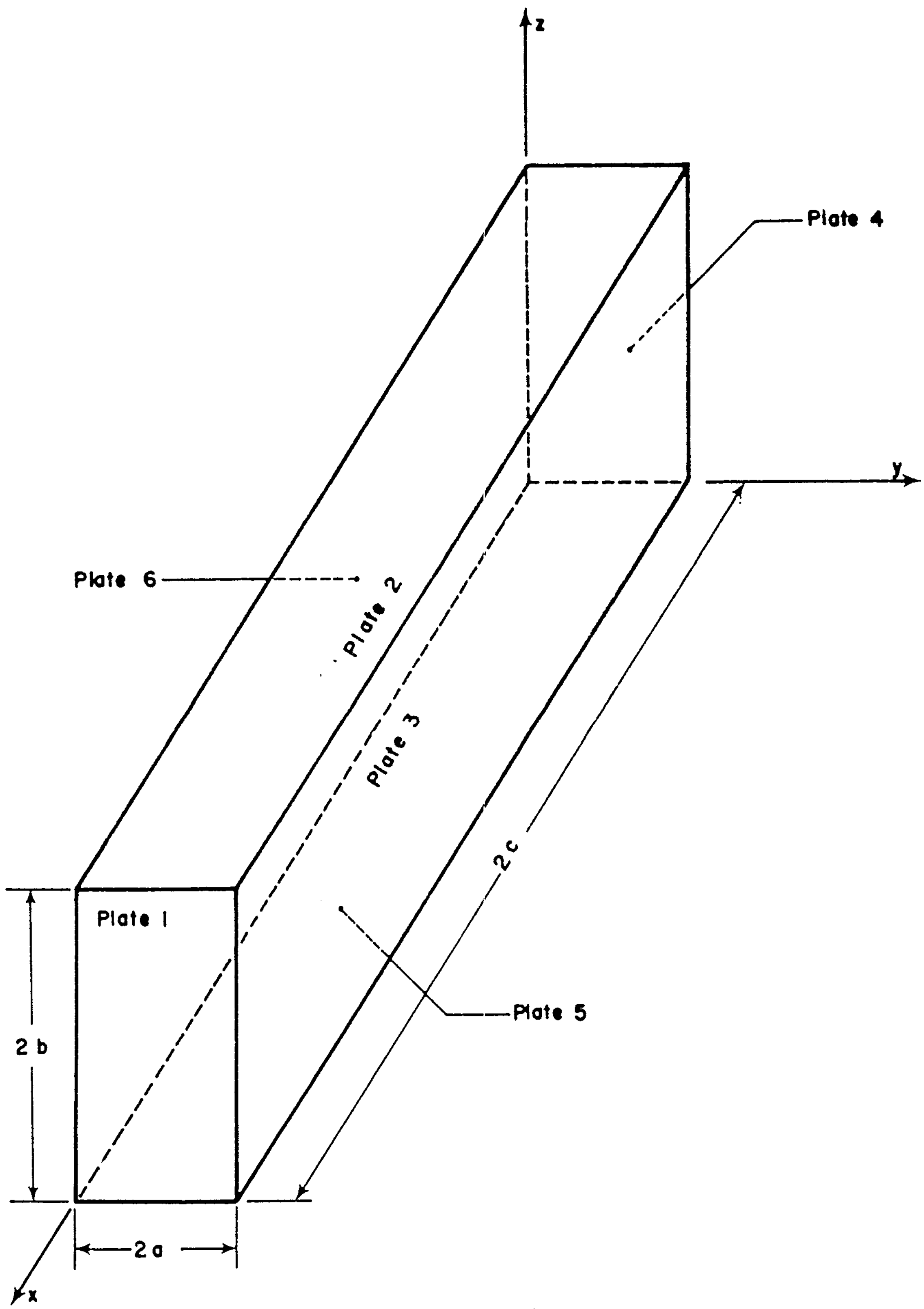

Figure 3. Charged Rectangular Conducting Parallelopiped.

$$\phi(X,Y,Z) = \int_{s'} \frac{\rho(s')}{4\pi\varepsilon_0 R} \, ds' \quad , \qquad (13)$$

where

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \quad .$$

After placing the field point on the surface of the box, the boundary condition can be expressed as

$$V = \int_0^{2b} \int_0^{2a} \frac{\rho^{(1)}(y',z')}{4\pi\varepsilon_0 R^{(1)}} dy'dz' + \int_0^{2a} \int_0^{2c} \frac{\rho^{(2)}(x',y')}{4\pi\varepsilon_0 R^{(2)}} dx'dy'$$

$$+ \int_0^{2b} \int_0^{2c} \frac{\rho^{(3)}(x',z')}{4\pi\varepsilon_0 R^{(3)}} dx'dz' + \int_0^{2b} \int_0^{2a} \frac{\rho^{(4)}(y',z')}{4\pi\varepsilon_0 R^{(4)}} dy'dz'$$

$$+ \int_0^{2a} \int_0^{2c} \frac{\rho^{(5)}(x',y')}{4\pi\varepsilon_0 R^{(5)}} dx'dy' + \int_0^{2b} \int_0^{2c} \frac{\rho^{(6)}(x',z')}{4\pi\varepsilon_0 R^{(6)}} dx'dz', \qquad (1$$

where superscripts refer to one of the six particular plates comprising the box and

$$\begin{aligned}
R^{(1)} &= \left[(x-2c)^2 + (y-y')^2 + (z-z')^2\right]^{1/2} \\
R^{(2)} &= \left[(x-x')^2 + (y-y')^2 + (z-2b)^2\right]^{1/2} \\
R^{(3)} &= \left[(x-x')^2 + (y-2a)^2 + (z-z')^2\right]^{1/2} \\
R^{(4)} &= \left[x^2 + (y-y')^2 + (z-z')^2\right]^{1/2} \qquad (15) \\
R^{(5)} &= \left[(x-x')^2 + (y-y')^2 + z^2\right]^{1/2} \\
R^{(6)} &= \left[(x-x')^2 + y^2 + (z-z')^2\right]^{1/2}
\end{aligned}$$

The isolated capacitance is

$$C = \frac{Q}{V} = \frac{\sum_{j=1}^{6} \iint \rho^{(j)}\left(s'^{(j)}\right) ds'^{(j)}}{V} \quad . \tag{16}$$

Define basis functions as

$$f_n^{(j)} = \begin{Bmatrix} 1 & \text{on all } \Delta s_n^{(j)} \\ 0 & \text{elsewhere} \end{Bmatrix} \tag{17}$$

where n represents the subsection and j the particular plate. The charge density is expanded as

$$\rho^{(j)}\left(s'^{(j)}\right) = \sum_{n=1}^{N^{(j)}} \alpha_n^{(j)} f_n^{(j)} \qquad j = 1, 2, 3, \ldots 6 \tag{18}$$

where $N^{(j)}$ is the number of square zones on rectangular plate j. Substitute this into equation (14) and satisfy the equation at the midpoints of each zone. The matrix representation can be put into the form of

$$\bar{V} = \left[\ell_{mn}^{(pq)}\right] \bar{\alpha}_n \quad \text{or} \quad V = \sum_{q=1}^{6} \ell_{mn}^{(pq)} \alpha_n^{(q)} \qquad p = 1,2,\ldots,6 \tag{19}$$

where

$$\overline{V} = \begin{bmatrix} V^{(1)} \\ V^{(2)} \\ \vdots \\ V^{(6)} \end{bmatrix} \qquad V^{(1)} = V^{(2)} = \ldots V^{(6)} \tag{20}$$

$$\overline{\alpha}_n = \begin{bmatrix} \alpha_n^{(1)} \\ \alpha_n^{(2)} \\ \vdots \\ \alpha_n^{(6)} \end{bmatrix} \tag{21}$$

$$\left[\ell_{mn}^{(pq)}\right] = \begin{bmatrix} \ell_{mn}^{(11)} & \ell_{mn}^{(12)} & \cdots & \ell_{mn}^{(16)} \\ \ell_{mn}^{(21)} & & & \\ \vdots & & & \\ \ell_{mn}^{(61)} & \cdot \quad \cdot & \cdot \quad \cdot & \ell_{mn}^{(66)} \end{bmatrix} \tag{22}$$

Note that $\ell_{mn}^{(pq)}$ is the potential at the center of Δs_m on plate p due to a uniform charge density of unit amplitude over Δs_n on plate q. As before, subsections were chosen to be squares, Harrington's approximations in the evaluation of $\ell_{mn}^{(pq)}$ were used, and full advantage was taken of the resulting symmetry of the equations. It was found that the isolated capacitance of the rectangular parallelopiped could be expressed simply as

$$C = 8w^2 \sum_{n=1}^{N} \sum_{m=1}^{N} \left[\ell_{mn}^{(pq)}\right]^{-1}_{mn} \tag{23}$$

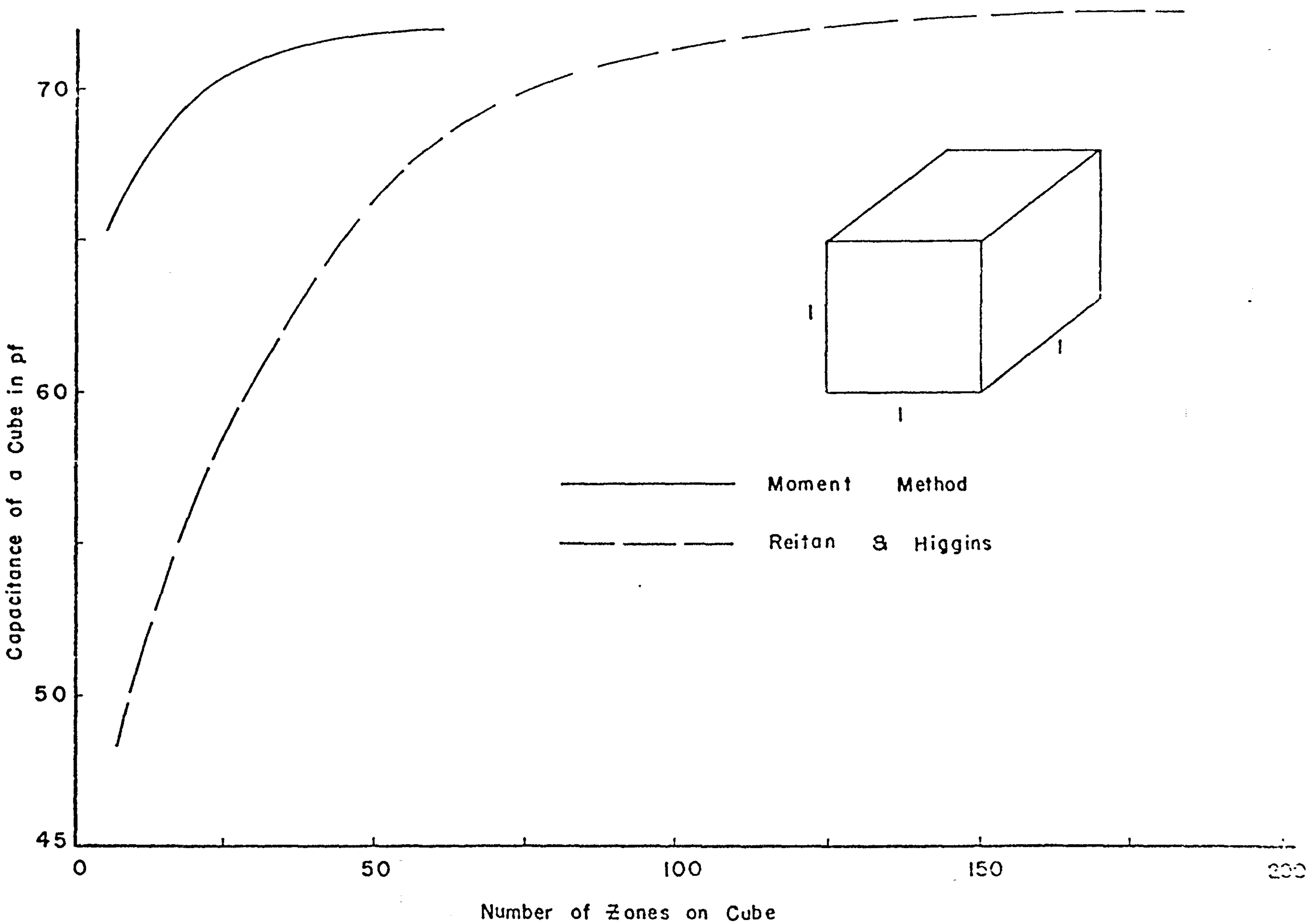

Figure 4. Capacitance of a Conducting Cube of Unit Sides.

with

$$\left[\ell_{mn}^{(pq)}\right] = \begin{bmatrix} \ell_{mn}^{(11)}+\ell_{mn}^{(14)} & \ell_{mn}^{(12)}+\ell_{mn}^{(15)} & \ell_{mn}^{(13)}+\ell_{mn}^{(16)} \\ \ell_{mn}^{(21)}+\ell_{mn}^{(24)} & \ell_{mn}^{(22)}+\ell_{mn}^{(25)} & \ell_{mn}^{(23)}+\ell_{mn}^{(26)} \\ \ell_{mn}^{(31)}+\ell_{mn}^{(34)} & \ell_{mn}^{(32)}+\ell_{mn}^{(35)} & \ell_{mn}^{(33)}+\ell_{mn}^{(36)} \end{bmatrix} \tag{24}$$

and 2w is the length of a subsection side. With the total number of subsections on the box equal to N, this matrix is N x N, as opposed to one in which symmetry was not utilized, which would be 2N x 2N.

Rectangular Parallelopiped - Numerical Results

The only numerical results available for comparison are those of Reitan and Higgins [1] and then only for the cube. Figure 4 illustrates the convergence of data for this special case of the cube as the number of subareas increases. Both solutions approach a value of approximately 73pf, but the moment method formulation gives much better results for a small number of subsections.

In a previous note [5], Shumpert defined an equivalent radius as

$$r_{eq} = \frac{C}{4\pi\varepsilon_0} \quad . \tag{25}$$

Also, the note advanced the supposition that the isolated capacitances of the inscribed and circumscribed ellipsoids could be considered as approximate bounds on the capacitance of more complex shaped bodies.

Figure 5 presents these bounds, the moment method results, as well as a surface area approximation established by Polya and Szego [6], which is

$$C = 4\pi\varepsilon_0 \left(\frac{S}{4\pi}\right)^{\frac{1}{2}} \quad , \tag{26}$$

where S is the total surface area of the solid. The surface area approximation gives reasonable results for a box whose dimensions are comparable to a cube, but as the parallelopiped approaches dimensions comparable to a wire, this approximation breaks down. The inscribed ellipsoid yields a better approximation to the moment method solution than an average of the two ellipsoidal curves would give.

Figures 6 and 7 depict the normalized equivalent radius of the rectangular parallelopiped versus its dimensions. Note that the degenerate cases--infinitely thin needle, rectangular plate, and the cube--appear as boundary lines or points on these general plots.

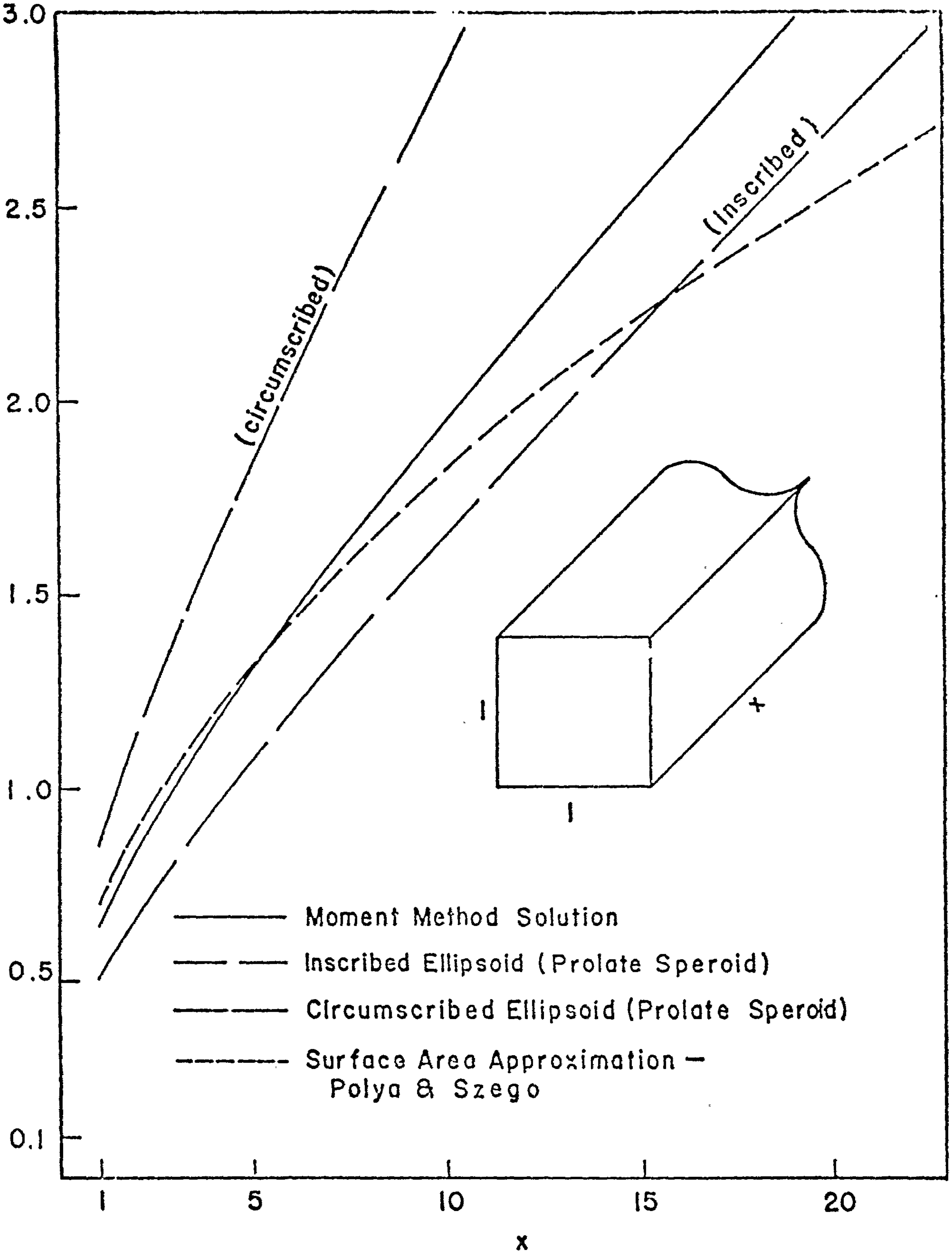

Figure 5. Equivalent Radius of a Rectangular Conducting Parallelopiped with Unit Cross-Sectional Area and Arbitrary Length.

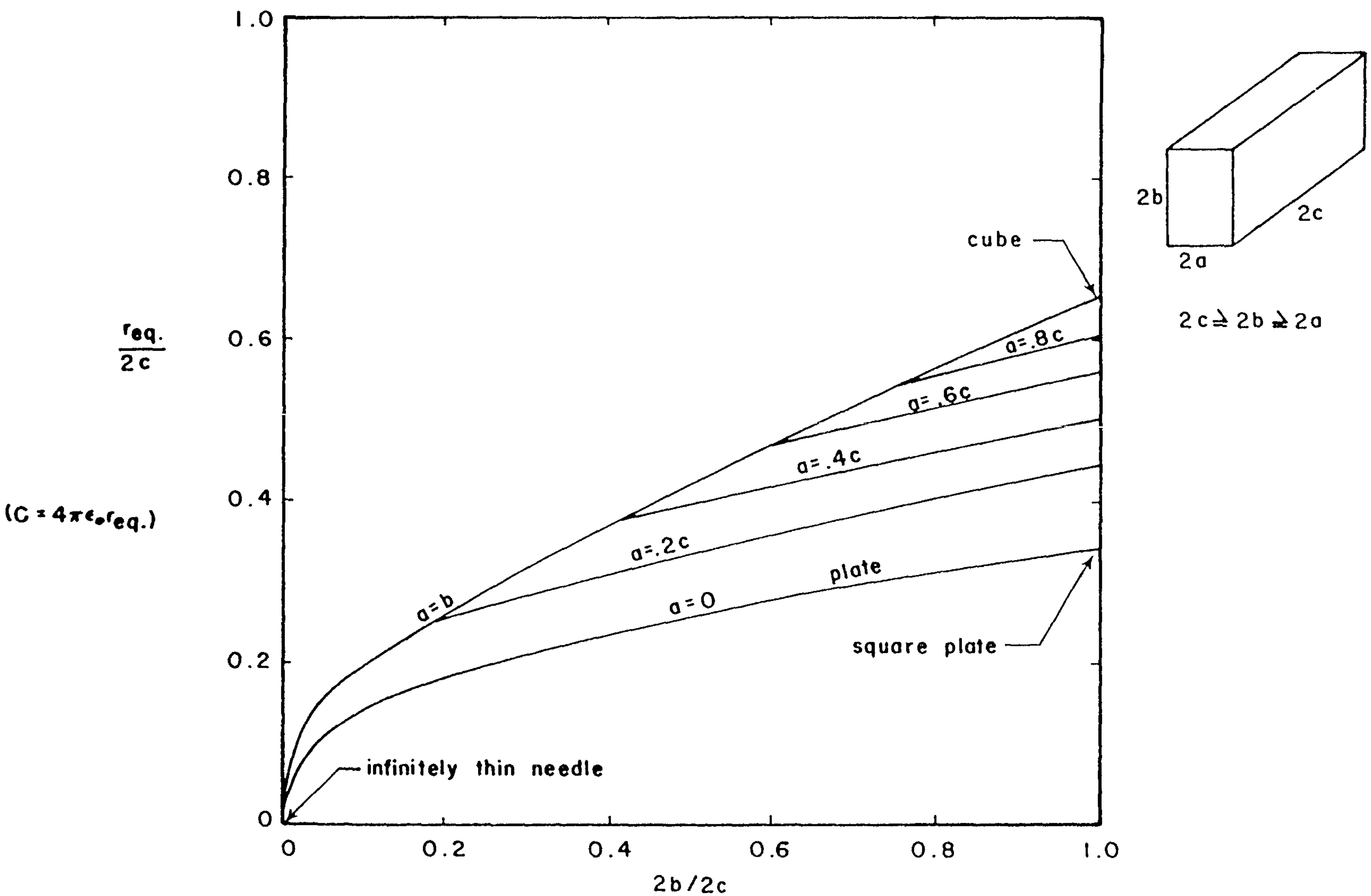

Figure 6. Normalized Equivalent Radius of Rectangular Conducting Parallelopiped of Arbitrary Dimensions.

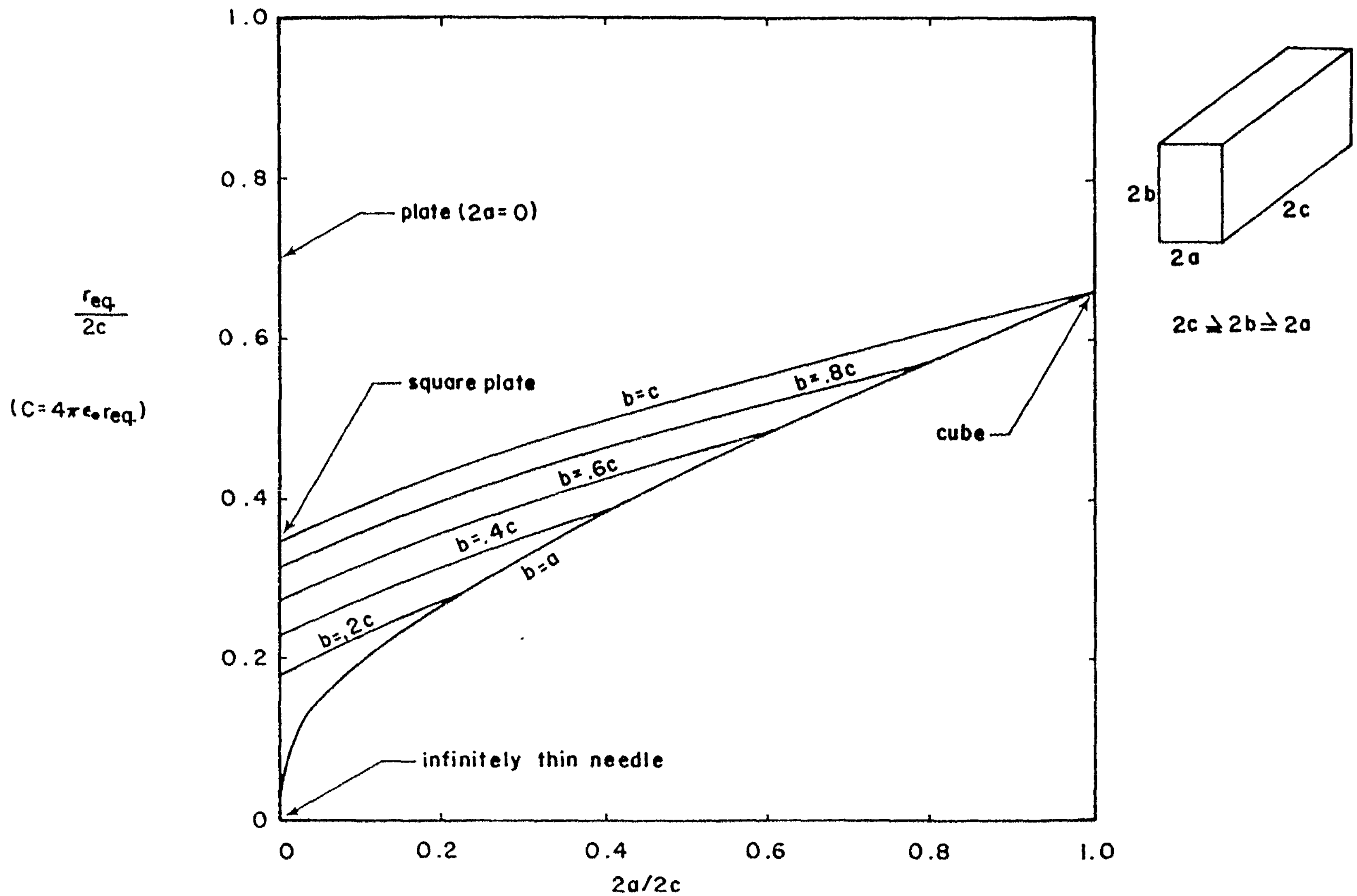

Figure 7. Normalized Equivalent Radius of Rectangular Conducting Parallelopiped of Arbitrary Dimensions.

Conclusions

The isolated capacitance of complex objects is a problem of interest. Approximations of and bounds for these isolated capacitances can be approached from many fronts. Inscribed and circumscribed ellipsoids yield bounds for many complex shaped objects. Perhaps, the rectangular parallelopiped could be used as a "building block" in order to arrive at estimations for the isolated capacitance of more complex bodies.

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