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A METHOD FOR TREATING THE SCATTERING OF RADIATION IN BODIES OF ARBITRARY CONVEX GEOMETRY

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ABSTRACT

This paper presents a method for formulating problems involving the passage of particles through material bodies of arbitrary convex geometry in which the interaction between a particle and the body depends on the length of the straight line path traveled. The radiation environment is described by a distribution function and the body is described by a projected cross-section and an angular chord distribution, permitting the effects of geometry to be treated independently of the physical process involved.

The method is first formulated for bodies scattering incident radiation, and then extended to include bodies containing uniformly distributed anisotropic radiation emitters.

Two sample problems are solved: the response of a scintillator to an X-ray flux, and the spectrum of recoil protons emerging from a hydrogenous body irradiated by a monoergic neutron beam.

Angular chord distributions are calculated for a sphere, an infinite slab, an infinite cylinder, and the face of a semi-infinite cylinder.

INTRODUCTION

This paper presents a method for formulating problems involving the passage of particles through material bodies of arbitrary convex geometry in which the interaction between a particle and the body depends, among other things, on the length of the straight line path traveled. In this method the particles are described by an angular distribution function and the body is described by a projected cross-section and an angular chord distribution, enabling the effects of geometry to be treated independently of the particular interaction involved. Because it clearly separates the geometry from the physical process, the method is a convenient one for estimating size and shape dependencies such as edge effects.

It is first formulated for bodies which scatter incoming particles, and then extended to include bodies which are themselves uniform (but not necessarily isotropic) sources of the particles. This extension can be shown to be equivalent to the method of Lewis[1] and Dirac[2] for cases where the emission is isotropic.

To demonstrate how it may be applied to both scattering and volume source problems, the method is used to calculate the response of a scintillator to a beam of X-rays and the spectrum of recoil protons emerging from a neutron-irradiated hydrogenous body. The results are obtained for various standard geometries, and their size and shape dependencies are pointed out.

As an aid to the calculations an appendix is included in which the angular chord distributions are derived for a sphere, an infinite slab, an infinite cylinder and the fece of a semi-infinite cylinder.

THEORY

We wish to find the response of a convex body to radiation under the assumption that the response depends only on the number of particles that enter the body and the path length each travels in it.

The number of particles of energy E that enter the body from a given direction $\underset{\sim}{\Omega}$ per unit time is

$$\Psi(E,\underset{\sim}{\Omega})\sigma(\underset{\sim}{\Omega})\,d\Omega dE,$$

where Ψ is the angular flux and $\sigma(\underset{\sim}{\Omega})$, the projected cross-section of the body in the direction $\underset{\sim}{\Omega}$ (see Fig. 1), is defined as

$$\sigma(\underset{\sim}{\Omega}) = \frac{1}{2}\int \left| \, dS\cdot\underset{\sim}{\Omega} \, \right| . \tag{1}$$

If we now define the angular chord distribution $\Phi(\ell,\underset{\sim}{\Omega})d\ell$ as the fraction of $\sigma(\underset{\sim}{\Omega})$ covered with tubes of length between ℓ and $\ell + d\ell$, and the interaction function $g(\ell,E)$ as the response expected when a radiation particle of energy E travels in a tube of length ℓ, we can write the body's total response as

$$I = \int \Psi(E,\underset{\sim}{\Omega})\sigma(\underset{\sim}{\Omega})\Phi(\ell,\underset{\sim}{\Omega})\,g(\ell,E)\,d\Omega dE d\ell \tag{2}$$

Once the interaction and environment are specified, we can write a formal solution to a scattering problem for any geometry whose angular chord distribution and projected cross-section have been calculated.

This method can be extended to problems involving a uniform distribution of anisotropic emitters. From the definitions of $\Phi(\ell,\underset{\sim}{\Omega})$ and $\sigma(\underset{\sim}{\Omega})$, the quantity $\sigma(\underset{\sim}{\Omega})\Phi(\ell,\underset{\sim}{\Omega})\ell d\ell$ is seen to be the volume of tubes of length between ℓ and $\ell + d\ell$ and direction $\underset{\sim}{\Omega}$. If the number of particles per unit volume emitted with energy E_o in the $\underset{\sim}{\Omega}$ direction is defined as $f(E_o,\underset{\sim}{\Omega})dE_o d\Omega$, then the number of particles which travel in such tubes is

$$f(E_o,\underset{\sim}{\Omega})\sigma(\underset{\sim}{\Omega})\Phi(\ell,\underset{\sim}{\Omega})\ell d\ell dE_o d\Omega .$$

The fraction of these particles which are produced at a distance between S and S + dS from their possible point of exit is dS/ℓ, so their number is given by

$$f(E_o,\underset{\sim}{\Omega})\sigma(\underset{\sim}{\Omega})\Phi(\ell,\underset{\sim}{\Omega})dS d\ell dE_o d\Omega .$$

Knowing the interaction function g, which involves only the initial properties and the distance S, enables us to write the entire response as

$$I = \int f(E_o,\underset{\sim}{\Omega})\sigma(\underset{\sim}{\Omega})\Phi(\ell,\underset{\sim}{\Omega})g(S,E_o)\Theta(\ell-S)dS d\ell dE_o d\Omega , \qquad (3)$$

where the function $\Theta(\ell-S)$ is zero whenever $S>\ell$ and unity otherwise.

In applying equations (2) and (3) to specific problems, the nature of the information desired will dictate which variables need be integrated over, as is illustrated in the examples below.

EXAMPLES

To illustrate how problems may be formulated with this method we will calculate the response of a scintillator to a monochromatic collimated beam of X-rays, and the spectrum of recoil protons emerging from a hydrogenous body irradiated by monoergic neutrons.

I. X-Ray Scattering

Let the X-ray photons have energy E_o and be traveling in the direction (θ_o, φ_o). The flux can then be written as

$$\Upsilon(E, \underset{\sim}{\Omega}) = Nc\delta(E-E_o)\delta(\theta-\theta_o)\delta(\varphi-\varphi_o)/\sin\theta, \qquad (4)$$

where N is the volume density of photons and c is the speed of light.

We assume that the response of the scintillator is simply proportional to the number of photons that scatter:

$$g(\ell, E) = I_o[1 - \exp(-\mu\ell)], \qquad (5)$$

where μ is the absorption coefficient of the material and I_o is the light expected on the average when an X-ray photon scatters. Depending on the details of a particular experiment I_o might actually be a function of the distance,[3] but we assume it to be proportional to the photon energy. The light output of the scintillator is obtained by substituting Eq. (4) and Eq. (5) into Eq. (2). In this case, it is simply

$$I = I_o Nc \int d\ell [1 - \exp(-\mu\ell)] \sigma(\theta_o, \varphi_o) \Phi(\ell, \theta_o, \varphi_o) \ . \qquad (6)$$

The output for particular shapes is gotten by inserting the expressions for σ and Φ given in the appendix and integrating. For example, that of a sphere is given by

$$I = I_o Nc \frac{\pi}{2} [D^2/2 - 1/\mu^2 + (D/\mu + 1/\mu^2)\exp(-\mu D)] \ .$$

The effect of the sphere's size can readily be seen by taking the asymptotic values of this expression:

$$I \approx I_o Nc \frac{\pi}{6} \mu D^3 \text{ for } D \ll 1/\mu \ ;$$

$$I \approx I_o Nc \frac{\pi}{4} D^2 \text{ for } D \gg 1/\mu \ .$$

Thus for small diameters the response is proportional to the volume, while for large diameters it is proportional to the surface and independent of μ.

II. Recoil Proton Spectrum

We will assume: (1) that the pertinent dimensions of the scatterer are small compared to the mean free path of the neutrons, (2) the neutron beam is collimated and monoergic, and (3) the nonrelativistic hard sphere approximation is valid for N-P scattering.[4]

If N neutrons/cm^2/sec are incident on a material with N_H protons/unit volume then the distribution of protons can be written as

$$f(E_o,\underset{\sim}{\Omega}) = N_H N \sigma_{np} \delta(E_o - E_n \cos^2\gamma)\cos\gamma, \tag{7}$$

where σ_{np} is the n-p cross-section and γ is the angle between the proton and neutron velocity. γ can be expressed in terms of the angle of neutron incidence (θ',φ') by the following relation:

$$\cos\gamma = \cos\theta'\cos\theta + \sin\theta'\sin\theta\cos(\varphi'-\varphi).$$

The response function $g(S,E_o)$ in this case is just the probability that a proton whose initial energy was E_o will escape after traveling a distance S. If R(E) is the range of a proton of energy E, then we may write

$$g(S,E_o) = \Theta(R(E_o)-S).$$

Whenever g = 1, S can be regarded as the distance actually traveled by a proton emerging from the body with energy E:

$$S = R(E_o) - R(E);$$

$$dS = -\frac{dR}{dE}\,dE.$$

If we now define the quantity E_{min}^{-} as zero whenever $R(E_o)$ is less than ℓ, and otherwise that energy for which

$$R(E_{min}) = R(E_o) - \ell,$$

then we can write

$$\Theta(\ell - S)\Theta(R[E_o] - S) = \Theta(E - E_{min}). \tag{9}$$

Adding the obvious constraint that $E \leq E_o$ and substituting equations 8 and 9 into equation 3, we find that the spectrum of recoil protons is

$$n(E, \underset{\sim}{\Omega})\, dE\, d\Omega = N_H N\, \sigma_{np} \cos\gamma \frac{dR}{dE}\, dE\, d\Omega \left\{ \int_0^{E_n} dE_o\, \delta(E_o - E_n \cos^2\gamma)\, \Theta(E_o - E)\, dE_o \right.$$

$$\left. \int_0^{\infty} d\ell\; \Phi(\ell, \underset{\sim}{\Omega})\, \sigma(\underset{\sim}{\Omega})\, \Theta(E - E_{min}) \right\} \tag{10}$$

The integral over ℓ, defined as $\Lambda(E_o, E, \Omega)$, can be evaluated using the distributions given in the appendix to yield the following:

$$\frac{\pi}{4}\left\{D^2 - [R(E_o) - R(E)]^2\right\}$$

for a sphere,

$$\sigma_o \cos\theta\; \Theta\left\{\ell_o/\cos\theta - [R(E_o) - R(E)]\right\}$$

for slab,

$$L \sin\theta \left\{D^2 - [R(E_o) - R(E)]^2 \sin^2\theta\right\}^{\frac{1}{2}}$$

for an infinite cylinder, and

$$D^2\left\{\frac{\pi}{2} - \sin^{-1}\left[[R(E_o) - R(E)]\sin\theta/D\right]\right\}$$

for the face of a semi-infinite cylinder. The E_o integration consists only of replacing E_o with $E_n \cos^2\gamma$, so that a general expression for the recoil proton spectrum is

$$n(E, \underset{\sim}{\Omega}) = N_H N\, \sigma_{np} \cos\gamma \frac{dR}{dE}\, \Theta(E_n \cos^2\gamma - E)\, \Lambda(E_n \cos^2\gamma, E, \Omega).$$

To demonstrate the dependence of the spectrum on the shape of the body, $n(E,\Omega)$ has been calculated for each of the above solids using $R = KE^{3/2}$ for the range-energy relation. Fig. 3 shows the results when $\gamma = 0$, $E_n = 10$ Mev, and the characteristic dimension of each solid is set equal to the range of a 10 Mev proton. Note that under these conditions the semi-infinite cylinder and the slab yield practically identical results, implying that the former's edge effects can be neglected.

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APPENDIX

Calculations of Angular Functions for Various Geometries

Sphere

For a sphere of diameter D (see Fig. 2a) the projected area is simply given as $\sigma(\underset{\sim}{\Omega}) = \pi D^2/4$. All chords of length ℓ are found to intersect the disc, σ, at a distance x from the center. Therefore the fraction of σ covered by chords of length in the neighborhood of ℓ is:

$$\Phi(\ell,\underset{\sim}{\Omega})d\ell = 8x dx/D^2,$$

where $x = \frac{1}{2}(D^2 - \ell^2)^{\frac{1}{2}}$.

Slab

The functions for a slab (see Fig. 2b) can be written

$$\sigma(\underset{\sim}{\Omega}) = \sigma_o| \cos\theta |, \text{ and}$$

$$\Phi(\ell,\underset{\sim}{\Omega})d\ell = d\ell[\delta(\ell-\ell_o/\cos\theta) + \delta(\ell + \ell_o/\cos\theta)].$$

Infinite Cylinder

For a cylinder (see Fig. 2c) the projected area is given by

$$\sigma(\underset{\sim}{\Omega}) = LD\sin\theta.$$

The fraction $\Phi(\ell,\Omega)d\ell$ is simply

$$\Phi(\ell,\underset{\sim}{\Omega})d\ell = 2dx/D,$$

where $x = \frac{1}{2}(D^2 - \ell^2\sin^2\theta)^{\frac{1}{2}}$.

Semi-infinite Cylinder Circular Face

We now wish to calculate the angular chord distribution for a semi-infinite cylinder (see Fig. 2d). We assume that radiation can enter only over the forward face. The distribution obtained for this case is useful for estimating edge effects or for treating a cylinder with well shielded sides.

Just as in the case of the plane, the projected area is

$$\sigma(\Omega) = \sigma_0 \cos\theta = \pi D^2 \cos\theta/4.$$

The fraction of tubes of length

$$\ell = \eta c/\sin\theta \qquad \text{(A1)}$$

covering the horizontal chord c in Fig. (2d) is given by

$$\Phi(\ell, c, \Omega)\, d\ell = d\eta.$$

We now eliminate the c dependence from $\Phi(\ell, c, \Omega)$ by noting that the maximum length of a chord ℓ on the horizontal chord c is

$$\ell_{max} = c/\sin\theta.$$

Let x be the distance from the axis to the horizontal chord c:

$$c^2 = D^2 - 4x^2.$$

The element of area can be written

$$dA = c dx.$$

The fraction of the face area covered by a tube of length ℓ can be written

$$d[\Phi(\ell, \Omega)\, d\ell] = c d\eta dx/(\pi D^2/4).$$

Substituting from Eq. (A1) and integrating over x, we obtain

$$\Phi(\ell, \Omega)\, d\ell = 4\sin\theta/(\pi D^2) \int_{-x_{max}}^{+x_{max}} dx,$$

where x_{max} is given by

$$x_{max} = \frac{1}{2}(D^2 - \ell^2 \sin^2\theta)^{\frac{1}{2}}.$$

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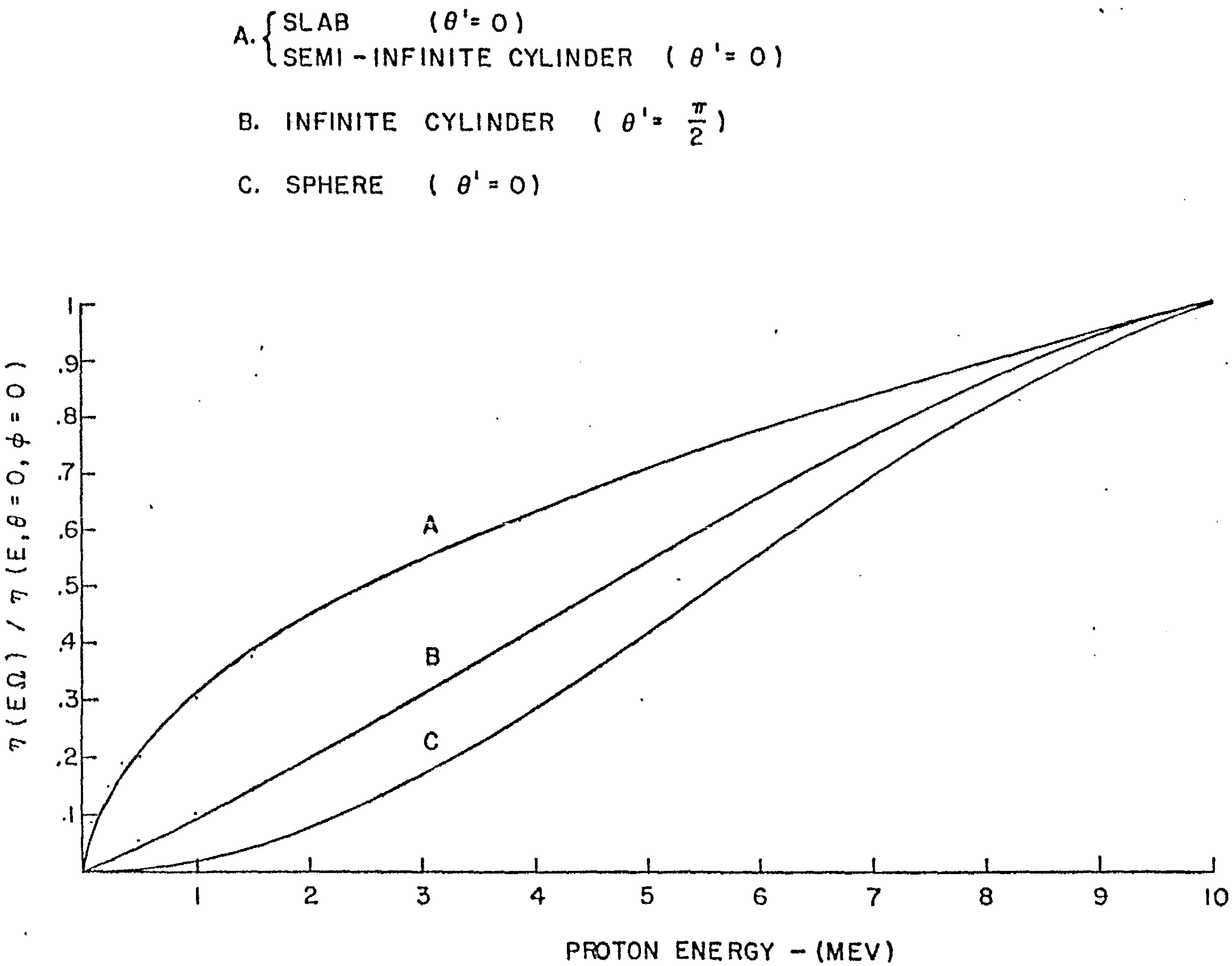

Fig. 1 $\sigma(\Omega)$, the projected area in the Ω direction, is intersected by tubes of length ℓ.

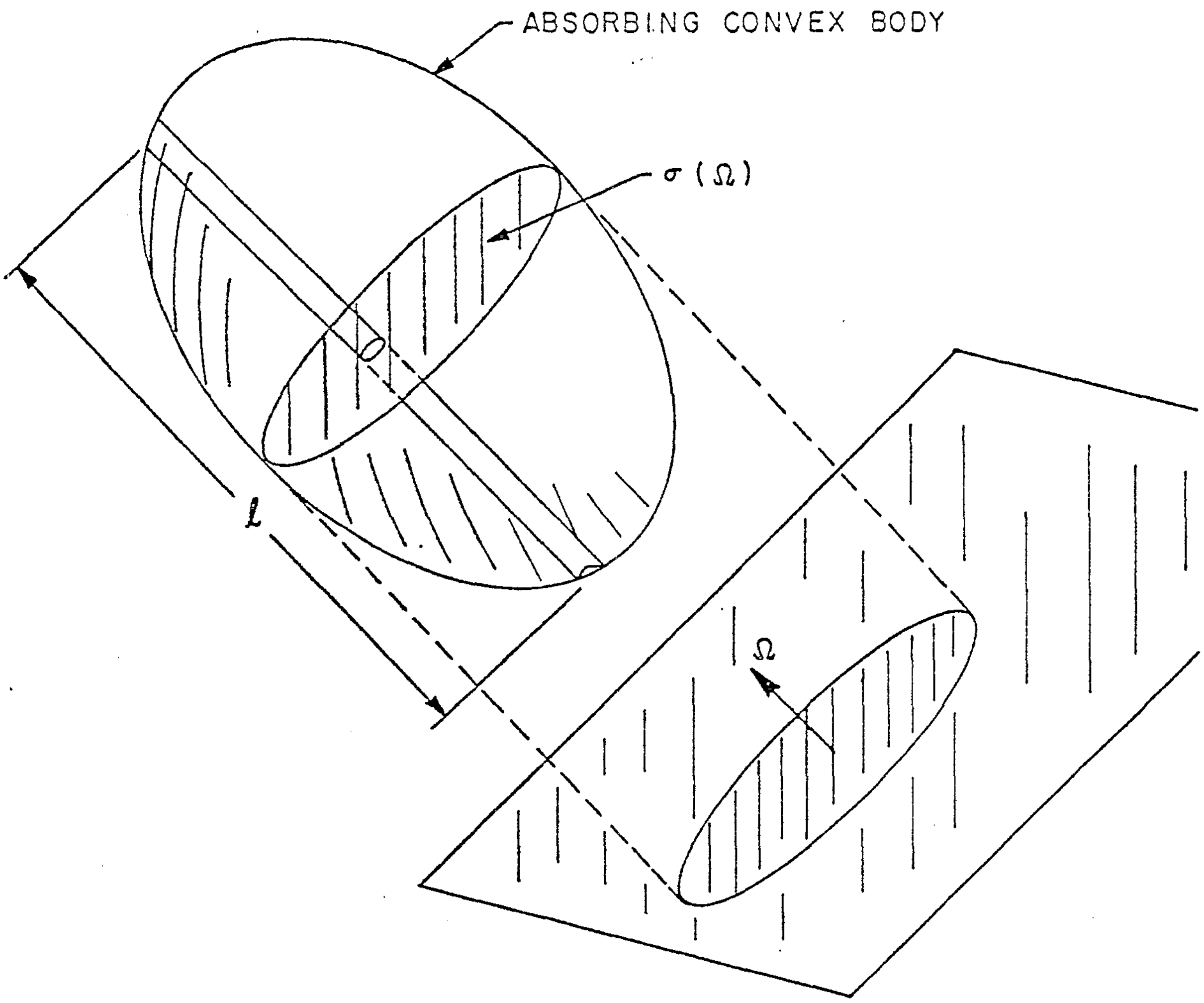

Fig. 2 Definition of angles and lengths used in appendix I.

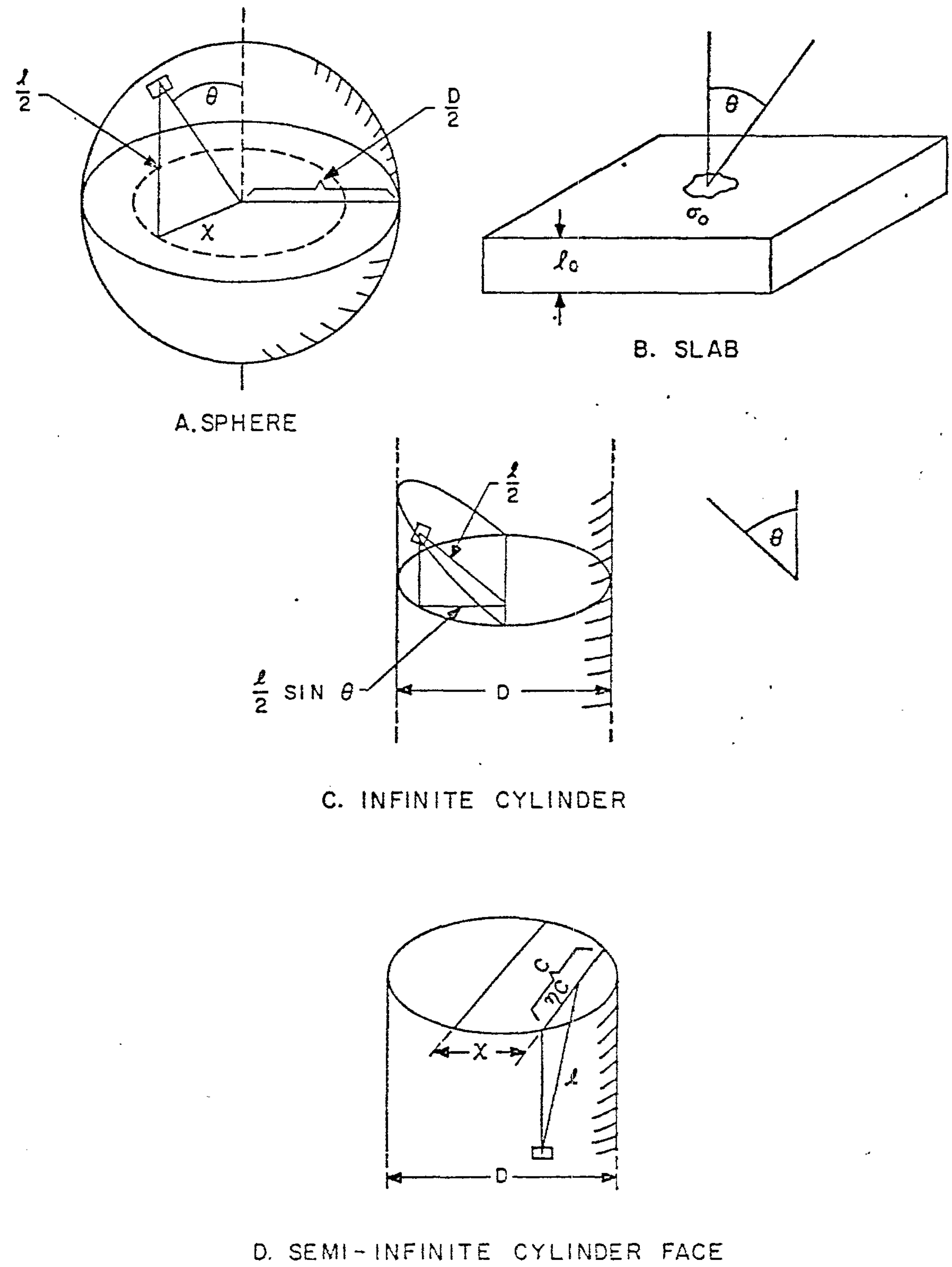

Fig. 3 Forward scattered recoil proton spectrum for a slab, a sphere and a cylinder.